Dynamic Substructuring cyclic

Abstract:

This ratio rests on the notions of computation per modal synthesis described in the document [R4.06.02].

We approach the methods of under - cyclic dynamic structuring. Completely dedicated under investigation of structures to cyclic repetitivity, they benefit the best from the geometrical characteristics of structure. The methods of CRAIG-BAMPTON and MAC NEAL, developed in this frame, are exposed.

Contents

1 Introduction

Into this document, we make the synthesis of the methods of cyclic dynamic substructuring. We give a definition of the cyclic repetitivity (or cyclic symmetry) and we present the principal incidences of this property on the dynamic behavior of the structure (nodal circles and diameters, double modes). Then, we expose, in a rather detailed way, the two methods of cyclic dynamic substructuring, implemented in *Code_Aster*. Improvements were made to the classical methods, by the taking into account of the presence of the nodes of the axis.

These methods suppose that the mesh of the basic sector is such that its traces on the interfaces right-hand side and left are coïncidentes (compatible meshes).

General notations:

ω_m	:	Maximum pulsation of a system ($rad.s^{-1}$)
$\mathbf{M}$	:	Mass matrix resulting from the modelization finite elements
$\mathbf{K}$	:	Stiffness matrix resulting from the modelization finite elements
$\mathbf{q}$	:	Vector of the degrees of freedom resulting from the modelization finite elements
$\mathbf{f}_{\text{ext}}$	:	Vector of the external forces to the system
$\mathbf{f}_{\text{L}}$	:	Vector of the bonding strengths applied to one substructure
$\boldsymbol{\Phi}$	:	Stamp containing the vectors of a projection base organized in column
$\boldsymbol{\eta}$	:	Vector of the generalized degrees of freedom
$\mathbf{B}$	:	Stamp extraction of the degrees of freedom of interface
$\mathbf{L}$	:	Stamp connection
T	:	Kinetic energy
U	:	Strain energy
$\mathbf{Id}$	:	Stamp identity
$\boldsymbol{\lambda}$	:	Stamp diagonal generalized stiffness
$\mathbf{R}_{\text{e}}(\omega)$	:	Residual dynamic flexibility matrix
$\mathbf{R}_{\text{e}}(0)$	:	Residual static flexibility matrix

Notations specific to the cyclic substructuring:

N	=	many sectors
α	=	angle formed by the basic sector
β	=	phase shift AND element
Oz	=	axis of cyclic symmetry
θ	=	rotation of angleet α of axis Oz
$\text{Re}(Z)$	=	left real the complex Z
$\text{Im}(Z)$	=	left imaginary the complex Z
$\boldsymbol{\theta}$	=	transition matrix the nodes right to the nodes left
$\boldsymbol{\theta}_a$	=	matrix change sector for the nodes the axis

Note:

The index	*D*	*is*	*relating*	*to the degrees of freedom of right*
“	*G*	*”*	*“*	*to the degrees of freedom of left*
”	*h a s*	*“*	*“*	*with the degrees of freedom of the axis*
”	*1*	*“*	*“*	*with the identified eigen modes*

| ” 2 “ “ with the unknown eigen modes

2 cyclic Repetitivity

2.1 Definition

One says that a structure is with cyclic repetitivity of axis Oz , if there exists an angle $0<\alpha<\pi$ such as structure is geometrically and mechanically invariant by rotation around Oz this angle. If α is the smallest angle checking this property, then any angular portion of angle α of structure is called “basic sector” (or “irreducible sector”).

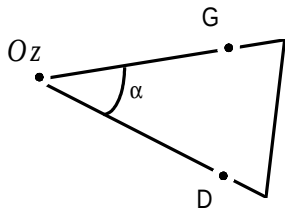
The total structure is then made up of N sectors:

$$N=\frac{2\Pi}{\alpha}$$

2.2 Wave propagation

One notes θ the rotation of axis Oz and angle α defined in R^3 .

Let us consider a basic sector of a structure with repetitivity of axis Oz , and two similar points of two contiguous sectors G and D :



One has the relation between the points G and D :

$$G=\theta(D)$$

It is noticed that the structure is left invariant by any rotation θ^m (with m integer).

One can note that all rotations leaving invariant structure (geometrically and mechanically) are of finished number:

$$\theta^m \quad m\in\{0,1,....N-1\}$$

Let us consider a scalar variable of state of the studied mechanical system U , and Z the associated complex:

$$U = \mathrm{Re}(Z) = \mathrm{Re}(U+jV)$$

It is possible to show, by the theory of the finished groups, the following relation for the points D and G [bib5]:

$$\exists\ m\in\{0,1,...,\frac{N}{2}\} \quad \text{tel que}\ \ Z(G)=e^{jm\alpha}Z(D) \qquad \textbf{éq 2.2-1}$$

Note:

the quantities are expressed in the cylindrical coordinate system (r,θ,z) ,
for an axisymmetric structure (cyclic typical case of repetitivity), m are called index of FOURIER,
in the case of an undamped plane wave, $e^{jm\alpha}$ are the complex phase shift between two contiguous sectors; the equation means that this phase shift can take only one finished number of known values,
it is possible to limit the number of the values of m to the values ranging between 0 and $N/2$; indeed, it is shown that the wave associated with the phase shift $N-m$ is identical to that associated with the phase shift m , but progresses in opposite direction [bib5].

If N is even: $m=0$ and $m=N/2$ correspond to real modes:

$$m=0 \quad \Rightarrow \forall D \quad U(\theta(D))=U(D)$$

$$m=N/2 \quad \Rightarrow \forall D \quad U(\theta(D))=-U(D)$$

All the other values of m correspond to modes appearing per orthogonal pairs with a given frequency (one speaks then about degenerated modes):

$$U=\mathrm{Re}(Z) \quad et \quad V=\mathrm{Im}(Z)$$

If N is odd: $m=0$ corresponds to a real mode not degenerated:

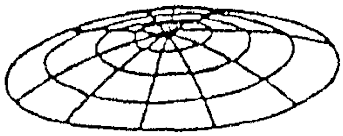
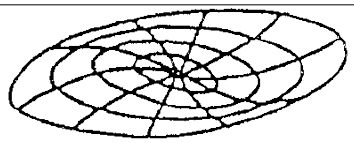
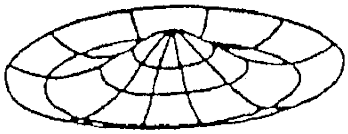
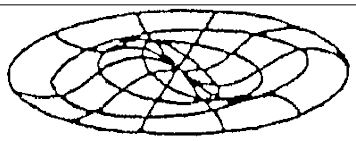
$$m=0 \quad \Rightarrow \quad \forall D \quad U(\theta(D))=U(D)$$

All the other values of m correspond to degenerated modes appearing per orthogonal pairs:

$$U=\mathrm{Re}(Z) \quad et \quad V=\mathrm{Im}(Z)$$

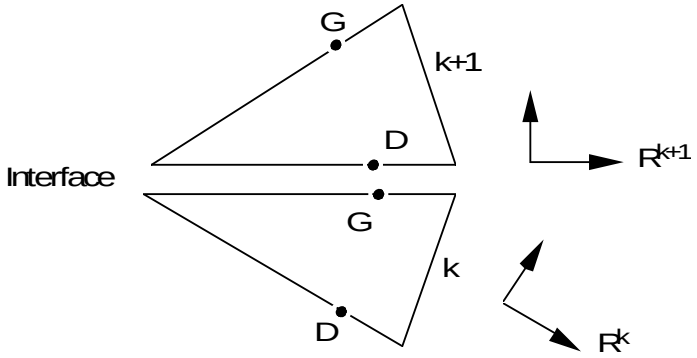
2.3 Notion of diameters and nodal circles

the cyclic property of repetitivity, translated by the equation [éq 2.2-1] makes it possible to know a priori the pace of the eigen modes of the structure, which strongly approaches what one can observe for axisymmetric structures. If one considers an eigen mode of a structure with cyclic symmetry, all the sectors have the same deformed shape but with an amplitude function of their angular position, which one can translate by a phase shift between substructures. This mode can be classified starting from the number of diameters and nodal circles which characterize it. A nodal diameter (which is confused with a diameter only if the structure is axisymmetric) is line of points of null motion passing by the axis of repetitivity; a nodal circle (which has the circular form only for axisymmetric structures) is line of points of null motion, it even with cyclic repetitivity. It is noted that it is the deformed shape of the mode of the substructure on which the mode of the complete structure is based which determines the number of circle (S) nodal (with). On the other hand, the number of diameter (S) nodal (with) is defined by the phase shift between two consecutive sectors.

Deformed sector	Phase between sector	Deformed overall	Family
Bending 1	N sectors in phase		0 rings 0 diameter
Bending 1	N/2 sectors in phase		0 rings 1 diameter
Bending 2	N sectors in phase		1 rings 0 diameter
Bending 2	N/secteurs in phase		1 rings 1 diameter

2.4 Boundary conditions

Consider a structure with cyclic repetitivity, and two successive basic sectors of this one:



Connections between sectors being regarded as perfect, there are the conditions between the sectors:

$$\boldsymbol{q}_g^k = \boldsymbol{q}_d^{k+1}$$ continuity of principle

$$\boldsymbol{f}_{L_g}^k = -\boldsymbol{f}_{L_d}^{k+1}$$ displacements of action-réactionéq 2.4-1

the exhibitor indicates the number of the sector considered. The preceding conditions of connection are expressed in the total reference.

By the formula [éq 2.2-1] relating to the wave propagation in structure and while posing: $\beta = m\alpha$, it a:

$$\{\boldsymbol{q}^{k+1}\}_{k+1} = e^{j\beta}\{\boldsymbol{q}^k\}_k$$
$$\{\boldsymbol{f}_L^{k+1}\}_{k+1} = e^{j\beta}\{\boldsymbol{f}_L^k\}_k$$

the index k is meant that the quantity is expressed in the reference related to the sector k : R^k .

The equations of connection [éq 2.4-1], written in the reference related to the sector k thus utilize the transition matrix of the sector k to the sector $k+1$. This matrix is not other than the matrix of rotation of the degrees of freedom of right towards those of left, that is to say the matrix of rotation of axis Oz and angle α , noted: θ .

We thus obtain the following system:

$$\begin{aligned}\{\boldsymbol{q}_{\boldsymbol{g}}^{\boldsymbol{k}}\}_k &= e^{j\beta}\boldsymbol{\theta}\{\boldsymbol{q}_{\boldsymbol{d}}^{\boldsymbol{k}}\}_k \\ \{\boldsymbol{f}_{\boldsymbol{L}_g}^{\boldsymbol{k}}\}_k &= -e^{j\beta}\boldsymbol{\theta}\{\boldsymbol{f}_{\boldsymbol{L}_d}^{\boldsymbol{k}}\}_k\end{aligned}$$ **éq the 2.4-2**

boundary conditions [éq 2.4-2] make it possible to calculate the eigen modes of the group of structure from one only basic sector.

This formalization can be generalized with the case of the nodes of the axis. One obtains then:

$$\begin{aligned}\{\boldsymbol{q}_{\boldsymbol{a}}^{\boldsymbol{k}}\}_k &= e^{j\beta}\boldsymbol{\theta}_{\boldsymbol{a}}\{\boldsymbol{q}_{\boldsymbol{a}}^{\boldsymbol{k}}\}_k \\ \{\boldsymbol{f}_{\boldsymbol{L}_a}^{\boldsymbol{k}}\}_k &= -e^{j\beta}\boldsymbol{\theta}_{\boldsymbol{a}}\{\boldsymbol{f}_{\boldsymbol{L}_a}^{\boldsymbol{k}}\}_k\end{aligned}$$

It is checked that if β is non-zero, the displacement of the nodes of the axis is null (in fact, one then notes the presence of one or several nodal diameters).

3 Methods of cyclic substructuring

3.1 Method of Craig-Bampton

One considers the problem with the eigenvalues of total structure expressed on the basic sector. This last is thus subjected to the bonding strengths which are applied to him by the contiguous sectors. In addition, the basic sector checks the equations of connection [éq 2.4-2]. We thus have:

$$(\boldsymbol{K}-\omega^2\boldsymbol{M})\boldsymbol{q}=\boldsymbol{f}_{\boldsymbol{L}}$$

$$\boldsymbol{q}_{\boldsymbol{g}}=e^{j\beta}\boldsymbol{\theta}\,\boldsymbol{q}_{\boldsymbol{d}}$$ **éq 3.1-1**

$$\boldsymbol{f}_{\boldsymbol{L}_g}=-e^{j\beta}\boldsymbol{\theta}\,\boldsymbol{f}_{\boldsymbol{L}_d}$$

We suppose that the base is made up of the dynamic eigen modes of the basic sector embedded with its interfaces, noted $\boldsymbol{\Phi}$, and of the constrained modes relating to the degrees of freedom of interfaces right and left, noted $\boldsymbol{\Psi}_{\boldsymbol{d}}$ and $\boldsymbol{\Psi}_{\boldsymbol{g}}$.

Taking into account the fact that the only contribution to displacements of a degree of freedom of interface comes from the constrained mode corresponding, the transformation of RITZ can be written:

$$\boldsymbol{q}=\begin{Bmatrix}\boldsymbol{q}_i\\ \boldsymbol{q}_d\\ \boldsymbol{q}_g\end{Bmatrix}=\begin{bmatrix}\boldsymbol{\Phi} & \boldsymbol{\Psi}_{\boldsymbol{d}} & \boldsymbol{\Psi}_{\boldsymbol{g}}\end{bmatrix}\begin{Bmatrix}\boldsymbol{\eta}_i\\ \boldsymbol{q}_d\\ \boldsymbol{q}_g\end{Bmatrix}=\boldsymbol{\Phi}\,\eta$$

Consequently, by means of the transformation of RITZ, the system of equations [éq 3.1-1] become:

$$(\overline{K}-\omega^2\overline{M})\begin{Bmatrix}\eta_i\\ q_d\\ q_g\end{Bmatrix}=\begin{bmatrix}\phi & \Psi_d & \Psi_g\end{bmatrix}^T\begin{Bmatrix}0\\ f_{L_d}\\ f_{L_g}\end{Bmatrix}$$

éq the 3.1-2

$$q_g=e^{j\beta}\theta\, q_d$$
$$f_{L_g}=-e^{j\beta}\theta\, f_{L_d}$$

surmounted matrixes of a bar are projections of the matrixes finite elements on the basis of the basic sector modal base (generalized matrixes).

One can show that the constrained modes are orthogonal with the normal modes with respect to the stiffness matrix [bib5]. Thus, the corresponding products are null.

Let us adopt the following notations:

m : index relating to the eigen modes of the sector
d : index relating to the constrained modes of the right interface
g : index relating to the constrained modes of the left interface.

One can thus write these matrixes in the form:

$$\overline{K}=\begin{bmatrix}\overline{K_{mm}} & 0 & 0\\ 0 & \overline{K_{dd}} & \overline{K_{dg}}\\ 0 & \overline{K_{gd}} & \overline{K_{gg}}\end{bmatrix}\qquad \overline{M}=\begin{bmatrix}\overline{M_{mm}} & \overline{M_{md}} & \overline{M_{mg}}\\ \overline{M_{dm}} & \overline{M_{dd}} & \overline{M_{dg}}\\ \overline{M_{gm}} & \overline{M_{gd}} & \overline{M_{gg}}\end{bmatrix}$$

Taking into account their definition, the constrained modes check:

$$\Psi_d=\begin{bmatrix}\Psi_{di}\\ \Psi_{dd}\\ \Psi_{dg}\end{bmatrix}=\begin{bmatrix}\Psi_{di}\\ Id\\ 0\end{bmatrix}\qquad \Psi_g=\begin{bmatrix}\Psi_{gi}\\ \Psi_{gd}\\ \Psi_{gg}\end{bmatrix}=\begin{bmatrix}\Psi_{gi}\\ 0\\ Id\end{bmatrix}$$

The second member of the matric equation [éq 3.1-2] becomes:

$$\begin{bmatrix}\phi_i & 0 & 0\\ \Psi_{di} & Id & 0\\ \Psi_{gi} & 0 & Id\end{bmatrix}\begin{Bmatrix}0\\ f_{L_d}\\ f_{L_g}\end{Bmatrix}=\begin{Bmatrix}0\\ f_{L_d}\\ f_{L_g}\end{Bmatrix}$$

By taking account of these notations, let us develop the matric equation checked by the basic sector:

$$\overline{\boldsymbol{K}_{mm}}\boldsymbol{\eta}_i-\omega^2(\overline{\boldsymbol{M}_{mm}}\eta_i+\overline{\boldsymbol{M}_{md}}\boldsymbol{q}_d+\overline{\boldsymbol{M}_{mg}}\boldsymbol{q}_g)=0$$
$$\overline{\boldsymbol{K}_{dd}}\boldsymbol{q}_d+\overline{\boldsymbol{K}_{dg}}\boldsymbol{q}_g-\omega^2(\overline{\boldsymbol{M}_{dm}}\boldsymbol{\eta}_i+\overline{\boldsymbol{M}_{dd}}\boldsymbol{q}_d+\overline{\boldsymbol{M}_{dg}}\boldsymbol{q}_g)=\boldsymbol{f}_{L_d}$$
$$\overline{\boldsymbol{K}_{gd}}\boldsymbol{q}_d+\overline{\boldsymbol{K}_{gg}}\boldsymbol{q}_g-\omega^2(\overline{\boldsymbol{M}_{gm}}\boldsymbol{\eta}_i+\overline{\boldsymbol{M}_{gd}}\boldsymbol{q}_d+\overline{\boldsymbol{M}_{gg}}\boldsymbol{q}_g)=\boldsymbol{f}_{L_g}$$

$$\boldsymbol{q}_g=e^{j\beta}\boldsymbol{\theta}\,\boldsymbol{q}_d$$
$$\boldsymbol{f}_{L_g}=-e^{j\beta}\boldsymbol{\theta}\,\boldsymbol{f}_{L_d}$$

Let us introduce the two last equations of this system into the three first:

$$(\overline{\boldsymbol{K}_{mm}}-\omega^2\overline{\boldsymbol{M}_{mm}})\boldsymbol{\eta}_i-\omega^2(\overline{\boldsymbol{M}_{md}}+e^{j\beta}\overline{\boldsymbol{M}_{mg}}\boldsymbol{\theta})\boldsymbol{q}_d=0$$
$$(\overline{\boldsymbol{K}_{dd}}+e^{j\beta}\overline{\boldsymbol{K}_{dg}}\boldsymbol{\theta})\boldsymbol{q}_d-\omega^2\left(\overline{\boldsymbol{M}_{dm}}\boldsymbol{\eta}_i+(\overline{\boldsymbol{M}_{dd}}+e^{j\beta}\overline{\boldsymbol{M}_{dg}}\boldsymbol{\theta})\boldsymbol{q}_d\right)=\boldsymbol{f}_{L_d}$$
$$(\overline{\boldsymbol{K}_{gd}}+e^{j\beta}\overline{\boldsymbol{K}_{gg}}\boldsymbol{\theta})\boldsymbol{q}_d-\omega^2\left(\overline{\boldsymbol{M}_{gm}}\boldsymbol{\eta}_i+(\overline{\boldsymbol{M}_{gd}}+e^{j\beta}\overline{\boldsymbol{M}_{gg}}\boldsymbol{\theta})\boldsymbol{q}_d\right)=-e^{j\beta}\boldsymbol{\theta}\,\boldsymbol{f}_{L_d}$$

The association of the two last equations makes it possible to eliminate the terms from the bonding strengths. One leads then to a final problem with the eigenvalues which one can put in the form:

$$\left(\tilde{\boldsymbol{K}}(\beta)-\omega^2\tilde{\boldsymbol{M}}(\beta)\right)\tilde{\boldsymbol{q}}=0 \qquad \textbf{éq 3.1-3}$$

With: $\tilde{\boldsymbol{q}}=\begin{Bmatrix}\boldsymbol{\eta}_i\\ \boldsymbol{q}_d\end{Bmatrix}$

$$\tilde{\boldsymbol{K}}=\begin{bmatrix}\overline{\boldsymbol{K}_{mm}} & 0\\ 0 & \overline{\boldsymbol{K}_{dd}}+e^{i\beta}\overline{\boldsymbol{K}_{dg}}\boldsymbol{\theta}+e^{-i\beta}\boldsymbol{\theta}^{\boldsymbol{T}}\overline{\boldsymbol{K}_{gd}}+\theta^T\overline{\boldsymbol{K}_{gg}}\boldsymbol{\theta}\end{bmatrix}$$

$$\tilde{\boldsymbol{M}}=\begin{bmatrix}\overline{\boldsymbol{M}_{mm}} & \overline{\boldsymbol{M}_{md}}+e^{i\beta}\overline{\boldsymbol{M}_{mg}}\theta\\ \overline{\boldsymbol{M}_{dm}}+e^{-i\beta}\boldsymbol{\theta}^{\boldsymbol{T}}\overline{\boldsymbol{M}_{gm}} & \overline{\boldsymbol{M}_{dd}}+e^{i\beta}\overline{\boldsymbol{M}_{dg}}\boldsymbol{\theta}+e^{-i\beta}\boldsymbol{\theta}^{\boldsymbol{T}}\overline{\boldsymbol{M}_{gd}}+\theta^T\overline{\boldsymbol{M}_{gg}}\boldsymbol{\theta}\end{bmatrix}$$

The mass matrixes and of stiffness of the final problem are hermitian. The eigenvalues solutions are thus real. In addition, the problem is of reduced size.

The resolution of the problem to the complex eigenvalues [éq 3.1-3] makes it possible to determine the complex generalized coordinates of the eigen modes of total structure. The complex values of displacements of the basic sector in the total mode are given, starting from the generalized coordinates, by the following formula:

$$\boldsymbol{q}'=\left[\boldsymbol{\phi}\quad\boldsymbol{\Psi}_d+e^{j\beta}\boldsymbol{\theta}\boldsymbol{\Psi}_d\right]\tilde{\boldsymbol{q}} \qquad \text{éq 3.1-4}$$

to determine the actual values of displacements, it is necessary to distinguish three cases according to the values from the phase shift AND element:

Case n° 1 : $\beta=0$:

The displacements $\mathbf{q}'$ given by the formula [éq 3.1-4] are then with actual values. All the sectors deformed even and vibrate in phase. There is then only one real eigen mode:

$$\mathbf{q}=\mathrm{Re}(\mathbf{q}')=\mathbf{q}' \qquad \textbf{éq 3.1-5}$$

Cases n°2 : $0<\beta<(N+1)/2$:

The displacements provided by the formula [éq 3.1-4] are with complex values. A each one of these complex modes correspond two orthogonal degenerated real modes:

$$\mathbf{q_1}=\mathrm{Re}(\mathbf{q}') \qquad \mathbf{q_2}=\mathrm{Im}(\mathbf{q}') \qquad \textbf{éq 3.1-6}$$

Cases n°3 : $\beta=N/2$ (=> N is even):

The displacements provided by [éq 3.1-4] are then with complex values. There are nodal $N/2$ diameters, two contiguous sectors vibrate then in opposition of phase. Each complex mode is at the origin of only one real mode:

$$\mathbf{q}=\mathrm{Re}(\mathbf{q}')=-\mathrm{Im}(\mathbf{q}') \qquad \text{éq 3.1-7}$$

Taking into account of the nodes of the axis:

One supposes in this paragraph that the degrees of freedom carried by the nodes of the axis, as well as the nodes of interfaces right-hand side and left, were blocked for the computation of the dynamic modes of the basic sector and were the object of computations of constrained modes.

Projection base is thus made up of the dynamic eigen modes of the basic sector embedded with its interfaces, noted $\boldsymbol{\phi}$, and of the constrained modes relating to the degrees of freedom of interfaces right, left and axis, noted $\boldsymbol{\Psi}_d$, $\boldsymbol{\Psi}_g$ and $\boldsymbol{\Psi}_a$.

As we saw in the chapter [§2.4], if β is non-zero, the displacement of the nodes of the axis is null (presence of at least a nodal diameter). The taking into account of the nodes of the axis thus has meaning only if $\beta=0$. In this demonstration, we will limit ourselves to this case.

The problem with the eigenvalues of total structure and the equations of connection, expressed on this basis are worth then:

$$(\overline{\boldsymbol{K}}-\omega^2\overline{\boldsymbol{M}})\begin{Bmatrix}\boldsymbol{\eta}_i\\ \boldsymbol{q}_d\\ \boldsymbol{q}_g\\ \boldsymbol{q}_a\end{Bmatrix}=\begin{bmatrix}\boldsymbol{\phi} & \boldsymbol{\Psi}_d & \boldsymbol{\Psi}_g & \boldsymbol{\Psi}_a\end{bmatrix}\begin{Bmatrix}0\\ \boldsymbol{f}_{L_d}\\ \boldsymbol{f}_{L_g}\\ \boldsymbol{f}_{L_d}\end{Bmatrix} \qquad \textbf{éq 3.1-8}$$

$$\boldsymbol{q}_g=\theta\,\boldsymbol{q}_d \;\; et \;\; \boldsymbol{q}_a=\boldsymbol{\theta}_a\,\boldsymbol{q}_a,$$
$$\boldsymbol{f}_{L_g}=-\theta\,\boldsymbol{f}_{L_g} \;\; et \;\; \boldsymbol{f}_{L_a}=-\boldsymbol{\theta}_a\,\boldsymbol{f}_{L_a}$$

Let us adopt the following notations:

m : index relating to the eigen modes of the sector
d : index relating to the constrained modes of the right interface
g : index relating to the constrained modes of the left interface
a : index relating to the constrained modes of the interface centers.

One can thus write the matrixes in the form:

$$\overline{K}=\begin{bmatrix}\overline{K_{mm}} & 0 & 0 & 0\\ 0 & \overline{K_{dd}} & \overline{K_{dg}} & \overline{K_{da}}\\ 0 & \overline{K_{gd}} & \overline{K_{gg}} & \overline{K_{ga}}\\ 0 & \overline{K_{ad}} & \overline{K_{ag}} & \overline{K_{aa}}\end{bmatrix} \qquad \overline{M}=\begin{bmatrix}\overline{M_{mm}} & \overline{M_{md}} & \overline{M_{mg}} & \overline{M_{ma}}\\ \overline{M_{dm}} & \overline{M_{dd}} & \overline{M_{dg}} & \overline{M_{da}}\\ \overline{M_{gm}} & \overline{M_{gd}} & \overline{M_{gg}} & \overline{M_{ga}}\\ \overline{M_{am}} & \overline{M_{ad}} & \overline{M_{ag}} & \overline{M_{aa}}\end{bmatrix}$$

Taking into account their definition, the constrained modes check:

$$\Psi_d=\begin{bmatrix}\Psi_{di}\\ \Psi_{dd}\\ \Psi_{dg}\\ \Psi_{da}\end{bmatrix}=\begin{bmatrix}\Psi_{di}\\ Id\\ 0\\ 0\end{bmatrix} \qquad \Psi_g=\begin{bmatrix}\Psi_{gi}\\ \Psi_{gd}\\ \Psi_{gg}\\ \Psi_{ga}\end{bmatrix}=\begin{bmatrix}\Psi_{gi}\\ 0\\ Id\\ 0\end{bmatrix} \qquad \Psi_a=\begin{bmatrix}\Psi_{ai}\\ \Psi_{ad}\\ \Psi_{ag}\\ \Psi_{aa}\end{bmatrix}=\begin{bmatrix}\Psi_{ai}\\ 0\\ 0\\ Id\end{bmatrix}$$

The second member of the matric equation [éq 3.1-8] becomes:

$$\begin{bmatrix}\phi_i & 0 & 0 & 0\\ \Psi_{di} & Id & 0 & 0\\ \Psi_{gi} & 0 & Id & 0\\ \Psi_{ai} & 0 & 0 & Id\end{bmatrix}\begin{Bmatrix}0\\ f_{L_d}\\ f_{L_g}\\ f_{L_a}\end{Bmatrix}=\begin{Bmatrix}0\\ f_{L_d}\\ f_{L_g}\\ f_{L_a}\end{Bmatrix}$$

By taking account of these notations, let us develop the matric equation checked by the basic sector:

$$\overline{K_{mm}}\eta_i-\omega^2(\overline{M_{mm}}\eta_i+\overline{M_{md}}q_d+\overline{M_{mg}}q_g+\overline{M_{ma}}q_a)=0$$
$$\overline{K_{dd}}q_d+\overline{K_{dg}}q_g+\overline{K_{da}}q_a-\omega^2(\overline{M_{dm}}\eta_i+\overline{M_{dd}}q_d+\overline{M_{dg}}q_g+\overline{M_{da}}q_a)=f_{L_d}$$
$$\overline{K_{gd}}q_d+\overline{K_{gg}}q_g+\overline{K_{ga}}q_a-\omega^2(\overline{M_{gm}}\eta_i+\overline{M_{gd}}q_d+\overline{M_{gg}}q_g+\overline{M_{ga}}q_a)=f_{L_g}$$
$$\overline{K_{ad}}q_d+\overline{K_{ag}}q_g+\overline{K_{aa}}q_a-\omega^2(\overline{M_{am}}\eta_i+\overline{M_{ad}}q_d+\overline{M_{ag}}q_g+\overline{M_{aa}}q_a)=f_{L_a}$$
$$q_g=\theta q_d \quad \text{and} \quad f_{L_d}=-\theta f_{L_d}$$
$$q_a=\theta_a q_a \qquad f_{L_g}=-\theta_a f_{L_a}$$

Let us replace, in the first four equations of the system $\mathbf{q_g}$ and $\mathbf{f_{L_g}}$ by their respective statements, according to $\mathbf{q_d}$ and $\mathbf{f_{L_d}}$ rewrite, in another form, the two last equations of the system, relating to the nodes of the axis.

$$(\overline{K_{mm}}-\omega^2\overline{M_{mm}})\eta_i-\omega^2\{(\overline{M_{md}}+\overline{M_{mg}}\theta)q_d+\overline{M_{ma}}q_a\}=0$$
$$(\overline{K_{dd}}+\overline{K_{dg}}\theta)q_d+\overline{K_{da}}q_a-\omega^2\{(\overline{M_{dm}}\eta_i+(\overline{M_{dd}}+\overline{M_{dg}}\theta)q_d)+\overline{M_{da}}q_a\}=f_{L_d}$$
$$(\overline{K_{gd}}+\overline{K_{gg}}\theta)q_d+\overline{K_{ga}}q_a-\omega^2\{(\overline{M_{gm}}\eta_i+(\overline{M_{gd}}+\overline{M_{gg}}\theta)q_d)+\overline{M_{ga}}q_a\}=-\theta f_{L_d}$$
$$(\overline{K_{ad}}+\overline{K_{ag}}\theta)q_d+\overline{K_{aa}}q_a-\omega^2\{(\overline{M_{am}}\eta_i+(\overline{M_{ad}}+\overline{M_{ag}}\theta)q_d)+\overline{M_{aa}}q_a\}=f_{L_a}$$

$$q_a=(1+\theta_a)\frac{q_a}{2}$$
$$(1+\theta_a^T)f_{L_a}=0$$

One replaces, in the first four equations of the system $\mathbf{q}_a$ by his statement determined in before last equation. In addition, the association of the second and the third equation makes it possible to eliminate the terms from the bonding strengths on the right. Lastly, the fourth equation is multiplied by $(1+\theta_a^T)$, which makes it possible to eliminate the terms from the bonding strengths to the axis. One leads then to a final problem with the eigenvalues which one can put in the form:

$$(\tilde{K}-\omega^2\tilde{M})\tilde{q}=0 \qquad \textbf{éq 3.1-9}$$

With:

$$\tilde{q}=\begin{Bmatrix}\eta_i\\ q_d\\ q_a/2\end{Bmatrix}$$

$$\tilde{K}=\begin{bmatrix}\overline{K_{mm}} & 0 & 0\\ 0 & \overline{K_{dd}}+\overline{K_{dg}}\theta+\theta^T\overline{K_{gd}}+\theta^T\overline{K_{gg}}\theta & (\overline{K_{da}}+\theta^T\overline{K_{ga}})(1+\theta_a)\\ 0 & (1+\theta_a^T)(\overline{K_{ad}}+\overline{K_{ag}}\theta) & (1+\theta_a^T)\overline{K_{aa}}(1+\theta_a)\end{bmatrix}$$

$$\tilde{M}=\begin{bmatrix}\overline{M_{mm}} & \overline{M_{md}}+\overline{M_{mg}}\theta & \overline{M_{ma}}(1+\theta_a)\\ \overline{M_{dm}}+\theta^T\overline{M_{gm}} & \overline{M_{dd}}+\overline{M_{dg}}\theta+\theta^T\overline{M_{gd}}+\theta^T\overline{M_{gg}}\theta & (\overline{M_{da}}+\theta^T\overline{M_{ga}})(1+\theta_a)\\ (1+\theta_a^T)\overline{M_{am}} & (1+\theta_a^T)(\overline{M_{ad}}+\overline{M_{ag}}\theta) & (1+\theta_a^T)\overline{M_{aa}}(1+\theta_a)\end{bmatrix}$$

Displacements of the axis, divided by two, make it possible to preserve at the stiffness matrixes and of mass their character hermitian. One restores modal complex displacements by the following formula:

$$\mathbf{q}'=\begin{bmatrix}\phi & \Psi_d+e^{j\beta}\theta\Psi_d & 2\Psi_a\end{bmatrix}\tilde{\mathbf{q}}$$

3.2 Method of Mac Neal

One considers the problem with the eigenvalues of total structure expressed on the basic sector. This last is thus subjected to the bonding strengths which are applied to him by the contiguous sectors. In addition, the basic sector checks the equations of connection [éq 2.4-2]. We thus have:

$$\begin{aligned}(\boldsymbol{K}-\omega^2\boldsymbol{M})\boldsymbol{q}&=\boldsymbol{f}_L\\ \boldsymbol{q}_g&=e^{j\beta}\boldsymbol{\theta}\,\boldsymbol{q}_d\\ \boldsymbol{f}_{L_g}&=-e^{j\beta}\boldsymbol{\theta}\,\boldsymbol{f}_{L_d}\end{aligned}$$ **éq 3.2-1**

modal base used to reduce dimensions of the problem to be solved, is a modal base with free interfaces including of the dynamic modes and the attach modes relating to the degrees of freedom of the interfaces right and left. Let us suppose that the degrees of freedom of the basic sector are ordered in the following way:

$$\boldsymbol{q} = \begin{Bmatrix}\boldsymbol{q}_i\\ \boldsymbol{q}_d\\ \boldsymbol{q}_g\end{Bmatrix} \begin{matrix}\text{degrés de liberté internes}\\ \text{degres de liberté de l'interface droite}\\ \text{degrés de liberté de l'interface gauche}\end{matrix}$$

Are $\boldsymbol{B}_d$ and $\boldsymbol{B}_g$, the rectangular matrixes of extraction such as:

$$\boldsymbol{q}_d=\boldsymbol{B}_d\boldsymbol{q} \quad et \quad \boldsymbol{q}_g=\boldsymbol{B}_g\boldsymbol{q}$$

The boundary condition on displacements becomes with these notations:

$$\boldsymbol{B}_g\boldsymbol{q}=e^{j\beta}\boldsymbol{\theta}\,\boldsymbol{B}_d\boldsymbol{q} \quad \Rightarrow \boldsymbol{B}_{dg}\boldsymbol{q}=0 \quad \text{avecéq} \quad \boldsymbol{B}_{dg}=e^{j\beta}\boldsymbol{\theta}\,\boldsymbol{B}_d-\boldsymbol{B}_g$$ **3.2-2**

For the forces, the boundary condition becomes:

$$\boldsymbol{f}_L=\boldsymbol{B}_g^T\boldsymbol{f}_{L_g}+\boldsymbol{B}_d^T\boldsymbol{f}_{L_d} \quad \Rightarrow \quad \boldsymbol{f}_L=(\boldsymbol{B}_g^T-e^{-j\theta}\boldsymbol{B}_d^T\boldsymbol{\theta}^T)\boldsymbol{f}_{L_g}=-\boldsymbol{B}_{dg}^T\boldsymbol{f}_{L_g}$$

Let us regard as base, for the transformation of RITZ, all the dynamic eigen modes of the basic sector, by distinguishing the modes identified and the unknown modes:

$$\boldsymbol{q}=\begin{bmatrix}\boldsymbol{\phi}_1 & \boldsymbol{\phi}_2\end{bmatrix}\begin{Bmatrix}\boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{Bmatrix}$$ **éq 3.2-3**

where index 1 (resp. 2) refers to the known modes (resp. unknowns). In the continuation, we will suppose that the eigen modes are normalized with the unit modal mass.

While replacing $\boldsymbol{q}$ by its statement according to the eigen modes, and while multiplying on the left by transposed of the matrix of the modes, the matric equations [éq 3.2-1] and [éq 3.2-2] become:

$$\begin{aligned}(\boldsymbol{\lambda}_1-\omega^2\boldsymbol{Id})\boldsymbol{\eta}_1&=\boldsymbol{\phi}_1^T\boldsymbol{f}_L\\ (\boldsymbol{\lambda}_2-\omega^2\boldsymbol{Id})\boldsymbol{\eta}_2&=\boldsymbol{\phi}_2^T\boldsymbol{f}_L\\ \boldsymbol{B}_{dg}\boldsymbol{\phi}_1\boldsymbol{\eta}_1+\boldsymbol{B}_{dg}\boldsymbol{\phi}_2\boldsymbol{\eta}_2&=0\end{aligned}$$ éq the 3.2-4

where $\boldsymbol{\lambda}$ is the matrix of the generalized stiffness (generalized masses are unit).

One can thus draw a formulation from it from η_2 :

$$\eta_2=(\lambda_2-\omega^2 Id)^{-1}\phi_2^T f_L \qquad \text{éq 3.2-5}$$

Consequently, one can eliminate η_2 from the system of equations [éq 3.2-4]. One then obtains the problem with the eigenvalues according to:

$$(\lambda_1-\omega^2 Id)\eta_1+\phi_1^T B_{dg}^T f_{Lg}=0$$
$$B_{dg}\phi_1\eta_1-B_{dg}\phi_2(\lambda_2-\omega^2 Id)^{-1}\phi_2^T B_{dg}^T f_{Lg}=0$$

The final system to solve can be written:

$$(\tilde{K}-\omega^2\tilde{M})\tilde{q}=0 \qquad \textbf{éq 3.2-6}$$

With:

$$\tilde{q}=\begin{Bmatrix}\eta_1\\ f_{L_g}\end{Bmatrix}$$

The forms of the stiffness matrixes and mass are:

$$\tilde{K}=\begin{bmatrix}\lambda_1 & \phi_1 B_{dg}^T\\ B_{dg}\phi_1 & -B_{dg}R_e(\omega)B_{dg}^T\end{bmatrix} \qquad \tilde{M}=\begin{bmatrix}Id & 0\\ 0 & 0\end{bmatrix}$$

The matrix $\left[R_e(\omega)\right]$ is the residual dynamic flexibility matrix of the not identified modes:

$$R_e(\omega)=\Phi_2(\lambda_2-\omega^2 Id)^{-1}\Phi_2^T$$

One approximates residual dynamic compliance by his static contribution, in taking into account the attach modes. Then, the formula of restitution which makes it possible to calculate the complex values of displacements starting from the generalized coordinates of the solutions modes of [éq 3.2 - 6] is the following one:

$$q'=\left[\phi_1 \quad -R_e(0)B_{dg}^T\right]\tilde{q}$$

The actual values of displacements are determined, as for the method of Craig - Bampton, by the formulas [éq 3.1-5], [éq 3.1-6], [and éq 3.1-7].

Taking into account of the nodes of the axis:

We suppose, in this paragraph, that the degrees of freedom carried by the nodes of the axis, as well as the nodes of interfaces right and left, was the object of computations of attach modes. We limit ourselves to $\beta=0$ who is the only case modified by the taking into account of the nodes of the axis ([§2.4] and [§3.1]). We thus have:

$$(K-\omega_2 M)q=f_L$$
$$q_g=\theta q_d \quad et \quad f_{L_g}=-\theta f_{L_d}$$
$$q_a=\theta_a q_d \Rightarrow (1-\theta_a)q_a=0 \qquad \textbf{éq 3.2-7}$$
$$f_{L_a}=-\theta_a f_{L_a}\Rightarrow f_{L_a}=(1-\theta_a^T)\frac{f_{L_a}}{2}$$

the organization of the degrees of freedom of the basic sector is similar to that of the preceding chapter:

$$\boldsymbol{q} = \begin{Bmatrix} \boldsymbol{q}_i \\ \boldsymbol{q}_d \\ \boldsymbol{q}_g \\ \boldsymbol{q}_a \end{Bmatrix} \begin{matrix} \text{degrés de liberté internes} \\ \text{degres de liberté de l'interface droite} \\ \text{degrés de liberté de l'interface gauche} \\ \text{degrés de liberté de l'interface axe} \end{matrix}$$

Are $\boldsymbol{B}_a$ the rectangular matrix of extraction of the degrees of freedom of the axis:

$$\boldsymbol{q}_a = \boldsymbol{B}_a \boldsymbol{q}$$

The boundary condition on displacements of the axis becomes with these notations:

$$(1-\boldsymbol{\theta}_a)\boldsymbol{B}_a \boldsymbol{q} = 0 \quad \Rightarrow \boldsymbol{B}_{aa}\boldsymbol{q} = 0 \quad \text{with} \quad \boldsymbol{B}_{aa} = (\theta_a - 1)\boldsymbol{B}_a$$

For the forces, the boundary condition becomes:

$$\boldsymbol{f}_L = \boldsymbol{B}_g^T \boldsymbol{f}_{L_g} + \boldsymbol{B}_d^T \boldsymbol{f}_{L_d} + \boldsymbol{B}_a^T \boldsymbol{f}_{L_a} \quad \Rightarrow \quad \boldsymbol{f}_L = -\boldsymbol{B}_{dg}^T \boldsymbol{f}_{L_g} + \boldsymbol{B}_a^T (1-\boldsymbol{\theta}_a^T)\frac{\boldsymbol{f}_{L_a}}{2}$$

$$\Rightarrow \quad \boldsymbol{f}_L = -\boldsymbol{B}_{dg}^T \boldsymbol{f}_{L_g} + \boldsymbol{B}_{aa}^T \frac{\boldsymbol{f}_{L_a}}{2}$$

The base of the transformation of RITZ made up of all the free eigen modes of the basic sector, by distinguishing the identified modes (index 1) and the unknown modes (index is 2) defined by the equation [éq 3.2-3].

The equation [éq 3.2-7], written in this base takes the following shape:

$$\begin{aligned} &(\lambda_1 - \omega^2 \boldsymbol{Id})\boldsymbol{\eta}_1 = \boldsymbol{\phi}_1^T \boldsymbol{f}_L \\ &(\lambda_2 - \omega^2 \boldsymbol{Id})\boldsymbol{\eta}_2 = \boldsymbol{\phi}_2^T \boldsymbol{f}_L \\ &\boldsymbol{B}_{dg}\boldsymbol{\phi}_1\boldsymbol{\eta}_1 + \boldsymbol{B}_{dg}\boldsymbol{\phi}_2\boldsymbol{\eta}_2 = 0 \\ &\boldsymbol{B}_{aa}\boldsymbol{\phi}_1\boldsymbol{\eta}_1 + \boldsymbol{B}_{aa}\boldsymbol{\phi}_2\boldsymbol{\eta}_2 = 0 \end{aligned} \qquad \textbf{éq 3.2-8}$$

the second equation makes it possible to determine $\boldsymbol{\eta}_2$ (cf [éq 3.2-5]), which can thus be eliminated from the system. One then obtains the problem with the eigenvalues according to:

$$(\lambda_1 - \omega^2 \boldsymbol{Id})\boldsymbol{\eta}_1 + \boldsymbol{\phi}_1^T \boldsymbol{B}_{dg}^T \boldsymbol{f}_{L_g} + \boldsymbol{\phi}_1^T \boldsymbol{B}_{aa}^T \frac{\boldsymbol{f}_{L_a}}{2} = 0$$

$$\boldsymbol{B}_{dg}\boldsymbol{\phi}_1\boldsymbol{\eta}_1 - \boldsymbol{B}_{dg}\boldsymbol{R}_e(\omega)\boldsymbol{B}_{dg}^T \boldsymbol{f}_{L_g} - \boldsymbol{B}_{dg}\boldsymbol{R}_e(\omega)\boldsymbol{B}_{aa}^T \frac{\boldsymbol{f}_{L_a}}{2} = 0$$

$$\boldsymbol{B}_{aa}\boldsymbol{\phi}_1\boldsymbol{\eta}_1 - \boldsymbol{B}_{aa}\boldsymbol{R}_e(\omega)\boldsymbol{B}_{dg}^T \boldsymbol{f}_{L_g} - \boldsymbol{B}_{aa}\boldsymbol{R}_e(\omega)\boldsymbol{B}_{aa}^T \frac{\boldsymbol{f}_{L_a}}{2} = 0$$

Thus, by defining the following unknown vector:

$$\tilde{\boldsymbol{q}} = \begin{Bmatrix} \boldsymbol{\eta}_1 \\ \boldsymbol{f}_{L_g} \\ \boldsymbol{f}_{L_a}/2 \end{Bmatrix}$$

The following final system is obtained:

$$(\tilde{\boldsymbol{K}}-\omega^2\tilde{\boldsymbol{M}})\tilde{\boldsymbol{q}}=0 \qquad \textbf{éq 3.2-9}$$

with:

$$\tilde{\boldsymbol{K}}=\begin{bmatrix} \lambda_1 & \boldsymbol{\phi}_1\boldsymbol{B}_{dg}^T & \boldsymbol{\phi}_1^T\boldsymbol{B}_{aa}^T \\ \boldsymbol{B}_{dg}\boldsymbol{\phi}_1 & -\boldsymbol{B}_{dg}\boldsymbol{R}_e(\omega)\boldsymbol{B}_{dg}^T & -\boldsymbol{B}_{dg}\boldsymbol{R}_e(\omega)\boldsymbol{B}_{aa}^T \\ \boldsymbol{B}_{aa}\boldsymbol{\phi}_1 & -\boldsymbol{B}_{aa}\boldsymbol{R}_e(\omega)\boldsymbol{B}_{dg}^T & -\boldsymbol{B}_{aa}\boldsymbol{R}_e(\omega)\boldsymbol{B}_{aa}^T \end{bmatrix} \qquad \tilde{\boldsymbol{M}}=\begin{bmatrix} \boldsymbol{Id} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Division by two of the bonding strengths applied to the axis makes it possible to preserve at the stiffness matrixes and of mass their character hermitian.

One restores modal complex displacements by the following formula:

$$\boldsymbol{q}'=\begin{bmatrix}\boldsymbol{\phi}_1 & -\boldsymbol{R}_e(0)\boldsymbol{B}_{dg}^T & -2\boldsymbol{R}_e(0)\boldsymbol{B}_{aa}^T\end{bmatrix}\tilde{\boldsymbol{q}}$$

4 Put in work in *Code_Aster*

the processing of the basic sector is identical to that of substructures in the classical substructuring. It utilizes the operators: MODE_ITER_SIMULT [U4.52.03] or MODE_ITER_INV [U4.52.04], DEFI_INTERF_DYNA [U4.64.01] and DEFI_BASE_MODALE [U4.64.02].

The eigen modes of structure with cyclic symmetry are calculated by the operator MODE_ITER_CYCL [U4.52.05] according to projection base of the basic sector previously definite and amongst sectors of complete structure.

The restitution of the results on physical base is identical to the classical substructuring. It utilizes the operator REST_SOUS_STRUC [U4.63.32] and possibly operator DEFI_SQUELETTE [U4.24.01].

5 Conclusion

the principles of under structuring make it possible to expose the transformation of RITZ and the modal recombination to lead to the modal synthesis which integrates these two techniques. The rules of liaisonnement between substructures are clarified.

Two methods were developed in *Code_Aster* : that of Craig-Bampton and that of Mac Neal. We present, here, their characteristics, as well in the definition of initial modal base, as in its operating.

After having exposed the definition of a structure to cyclic symmetry and the properties which result from this, we presented the methods of cyclic substructuring put in work in *Code_Aster*. They appear very interesting for the computation of the eigen modes of a structure with cyclic symmetry, such as the rotors of the revolving machines of which they benefit fully from the geometrical and mechanical characteristics.

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