Estimator of error of Summarized

ZHU-ZIENKIEWICZ:

One exposes in this document the method of estimate of the error of discretization suggested by ZHU - ZIENKIEWICZ.

This estimator leans on a continuous lissage of the calculated stresses allowing to obtain a better accuracy on the nodal stresses compared to the methods standards.

Two successive versions of this estimator are described, corresponding each one to a different lissage.

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1 Introduction

the search on estimators of error on the solutions obtained by computations finite elements and their coupling with procedures of adaptive mesh made these last years considerable great strides. The set aim is to mitigate the possible inadequacy of a modelization by adapting in an automatic way the mesh the solution sought according to certain criteria (equal distribution of the error of discretization, minimization amongst nodes to reach a given accuracy, lower costs).

One introduces here an estimator of error of the type a posteriori in the frame of linear and homogeneous elasticity. Historically, this estimator, proposed by ZHU-ZIENKIEWICZ [bib1] in 1987, was largely used because of his facility of establishment in his the existing low costs and computer codes. Nevertheless, the bad reliability of this estimator for the elements of even degree was noted empirically (undervaluation of the error) and led the authors to a modification of their method in 1992 [bib2], [bib3] with numerical checking of the asymptotic convergence of the estimator on all the element types.

Nevertheless, the scope of application of the version of 1992 being for the moment more reduced (see [3.2]), the two versions of this estimator were established in *Aster* and are the object of this note.

2 Principle of the method

2.1 Equations to solve

One considers the solution (u, s) of a linear elastic problem checking:

balance equations:

$$\begin{cases} \mathbf{Lu} = \boldsymbol{q} \quad \text{dans} \quad \Omega \\ \sigma_{ij} n_j = \bar{t}_i \quad \text{sur} \quad \Gamma_t \end{cases}$$

with $\boldsymbol{L} = {}^t\boldsymbol{BDB}$ operator of elasticity

compatibility equations:

$$\begin{cases} \boldsymbol{\varepsilon} = \mathbf{Bu} \\ \boldsymbol{u} = \bar{\boldsymbol{u}} \quad \text{sur} \quad \Gamma_u \end{cases}$$

with $\Gamma = \Gamma_u \cup \Gamma_t$

the constitutive law:

$$\boldsymbol{\sigma} = \boldsymbol{D}\,\boldsymbol{\varepsilon}$$

The problem discretized by finite elements consists in finding $(\boldsymbol{u}_h, \boldsymbol{\sigma}_h)$ solution of:

$$\boldsymbol{u}_h = N\,\bar{\boldsymbol{u}}_h \qquad \text{éq 2.1-1}$$

checking $\boldsymbol{K}\,\bar{\boldsymbol{u}}_h = \mathbf{f}$

with
$$\boldsymbol{K} = \int_\Omega {}^t(\boldsymbol{BN})\,\boldsymbol{D}\,(\boldsymbol{BN})\,d\,\Omega$$
$$\mathbf{f} = \int_\Omega {}^t\boldsymbol{Nq}\,d\,\Omega + \int_{G_t} {}^t\boldsymbol{N}\,\bar{t}\,dG$$

where:

- $\bar{\mathbf{u}}_h$ represent the nodal unknowns of displacement
- $\boldsymbol{N}$ the associated shape functions

the stresses are calculated starting from displacements by the relation:

$$\boldsymbol{\sigma}_h = \boldsymbol{DBu}_h \qquad \text{éq 2.1-2}$$

2.2 Estimator of error and index of effectiveness

One notes $e=u-u_h$ the error on displacements
$e_s=s-s_h$ the error on the stresses

the norm of the energy of the error e is written:

$$\|\mathbf{e}\|=\left(\int_{\Omega} {}^t\mathbf{eLe}\, d\,\Omega\right)^{1/2}$$

in the case of elasticity

$$=\left(\int_{\Omega} {}^t\mathbf{e}_{\sigma}\mathbf{D}^{-1}\mathbf{e}_{\sigma}\, d\,\Omega\right)^{1/2} \quad \text{éq 2.2-1}$$

the total error above breaks up into a sum of elementary errors according to:

$$\|\mathbf{e}\|^2=\sum_{i=1}^{N}\|\mathbf{e}\|_i^2$$

where N is the nombre total of elements.
$\|\mathbf{e}\|_i$ represent the local indicator of error on the element i.

The goal is to consider the error exact from the equation [éq 2.2-1] formulated in stresses. The basic idea of the method is to build a new stress field noted $\boldsymbol{\sigma}^*$ from $\boldsymbol{\sigma}_{\mathbf{h}}$ and such as:

$$\mathbf{e}_{\sigma}\approx\mathbf{e}_{\sigma}^{*}=\boldsymbol{\sigma}^{*}-\boldsymbol{\sigma}_{\mathbf{h}}$$

The estimator of error will be written then:

$$^0\|\mathbf{e}\|=\left(\int_{\Omega} {}^t\mathbf{e}_{\sigma}^{*}\mathbf{D}^{-1}\mathbf{e}_{\sigma}^{*}\, d\,\Omega\right)^{1/2}$$

The quality of the estimator is measured by the quantity θ , called index of effectiveness of the estimator:

$$\theta=\frac{^0\|e\|}{\|e\|}$$

An estimator of error is known as asymptotically exact so $\theta\rightarrow 1$ when $\|e\|\rightarrow 0$ (or when $h\rightarrow 0$), which wants to say that the estimated error will always converge towards the exact error when this one decreases.

In an obvious way, the reliability of $^0\|e\|$ depends on “quality” on σ^* .

The two versions of the estimator of ZHU-ZIENKIEWICZ are different on this level (see [§3]).

2.3 Construction of an estimator asymptotically exact

the characterization of such an estimator is provided by the following theorem (see [feeding-bottle 2]).

Theorem

If $\|\mathbf{e}^*\|=\|\mathbf{u}-\mathbf{u}^*\|$ is the error norm of the rebuilt solution, then the estimator of error $^0\|e\|$ defined previously is asymptotically exact

so $\frac{\|\mathbf{e}^*\|}{\|\mathbf{e}\|}\to 0$ when $\|e\|\to 0$

This condition is carried out if the rate of convergence with h $\|\mathbf{e}^*\|$ is higher than that of $\|e\|$. Typically, if it is supposed that the exact error of the approximation finite element converges in $\|\mathbf{e}\|=0\left(h^p\right)$

and the error of the solution rebuilt in

$$\|\mathbf{e}^*\|=0\left(h^{p+\alpha}\right) \text{ with } \alpha>0$$

then a simple computation gives:

$$1-0\left(h^{\alpha}\right)\le\theta\le 1+0\left(h^{\alpha}\right)$$

and thus $\theta\to 1$ when $h\to 0$

3 Construction of the stress field recomputed

3.1 Version 1987

the solution $\mathbf{u_h}$ resulting from the equation [éq 2.1-1] being C_0 on Ω (because of the choice of shape functions C_0), it follows that $\boldsymbol{\sigma}_\mathbf{h}$ calculated by [éq 2.1-2] is discontinuous with the interfaces of the elements.

To get acceptable results on the nodal stresses, one generally resorts to an average with the nodes or a method of projection. It is this last method which is adopted here.

It is supposed that $\boldsymbol{\sigma}^*$ is interpolated by the same shape functions that $\mathbf{u_h}$, that is to say:

$$\boldsymbol{\sigma}^* = N\,\bar{\boldsymbol{\sigma}}^* \qquad \text{éq 3.1-1}$$

and one carries out a total lissage by least squares of $\boldsymbol{\sigma}_\mathbf{h}$, which amounts minimizing the functional calculus $J(\boldsymbol{\tau})=\int_\Omega {}^t(\boldsymbol{\tau}-\boldsymbol{\sigma}_\mathbf{h})(\boldsymbol{\tau}-\boldsymbol{\sigma}_\mathbf{h})\,d\,\Omega$ in space generated par. N

By derivative, $\boldsymbol{\sigma}^*$ must check $\int_\Omega {}^tN(\boldsymbol{\sigma}^*-\boldsymbol{\sigma}_\mathbf{h})\,d\,\Omega=0$

by means of the equation [éq 3.1-1], one obtains the linear system:

$$\mathbf{M}\{\bar{\boldsymbol{\sigma}}^*\}=\{\mathbf{b}\}$$

with $\mathbf{M}=\int_\Omega {}^t\mathbf{N}\mathbf{N}\,d\,\Omega$ and $\{\mathbf{b}\}=\int_\Omega {}^t\mathbf{N}\,\boldsymbol{\sigma}_\mathbf{h}\,d\,\Omega$

This total system is to be solved on each one of the components of the tensor of the stresses. The matrix M is calculated and reversed only once.

3.2 Version 1992

the stress of the field $\boldsymbol{\sigma}^*$ differs version 1987 compared to in the following way:

polynomial $\boldsymbol{\sigma}^*$ of the same degree is supposed than displacements on all the elements having a top node interns S jointly.

One notes $S_K=\bigcup_{S\in K} K$ this whole called patch.

For each component of $\boldsymbol{\sigma}^*$, one writes:

$$\boldsymbol{\sigma}^*|_{S_K}=\mathbf{P}\mathbf{a}_s \qquad \text{éq 3.2-1}$$

where $\mathbf{P}$ the suitable polynomial terms

$\mathbf{a_s}$ the unknown coefficients of the corresponding students' rag processions Example contains: $2D$

$P1$ $\mathbf{P}=[1,x,y]$ $\mathbf{a_s}^t=[a_1,a_2,a_3]$

Q1 $\mathbf{P}=[1,x,y,xy]$ $\mathbf{a_s}={}^t[a_1,a_2,a_3,a_4]$

The determination of the coefficients of the polynomial $\mathbf{a_s}$ is done by minimizing the functional calculus:

$$
\begin{aligned}
F(\mathbf{a}) &= \sum_{i=1}^{N} \left(\boldsymbol{\sigma}_{\mathbf{h}}(x_i, y_i) - \boldsymbol{\sigma}^*|_{S_K}(x_i, y_i) \right)^2 \\
&= \sum_{i=1}^{N} \left(\boldsymbol{\sigma}_{\mathbf{h}}(x_i, y_i) - \mathbf{P}(x_i, y_i)\, \mathbf{a}_{\mathbf{s}} \right)^2
\end{aligned}
$$

(discrete lissage local from $\boldsymbol{\sigma}_{\mathbf{h}}$ least squares)

where (x_i, y_i) are the coordinates of Gauss points on S_K.

N of Gauss points on all the elements of the patch the solution is

the nombre total $\mathbf{a}_{\mathbf{s}}$ checks:

$$
\sum_{i=1}^{N} {}^t\mathbf{P}(x_i, y_i)\, \mathbf{P}(x_i, y_i)\, \mathbf{a}_{\mathbf{s}} = \sum_{i=1}^{N} {}^t\mathbf{P}(x_i, y_i)\, \boldsymbol{\sigma}_{\mathbf{h}}(x_i, y_i)
$$

from where $\mathbf{a}_{\mathbf{s}} = \mathbf{A}^{-1}\mathbf{b}$ with $\mathbf{A} = \sum_{i=1}^{N} {}^t\mathbf{P}(x_i, y_i)\, \mathbf{P}(x_i, y_i)$

A can be very badly conditioned (in particular on the elements of high degree) and consequently, impossible to reverse in this form. To cure this problem, the authors [bib4] proposed a standardization of the coordinates on each patch, which amounts carrying out the change of variables:

$$
\begin{aligned}
\bar{x} &= -1 + 2\,\frac{x - x_{\min}}{x_{\max} - x_{\min}} \\
\bar{y} &= -1 + 2\,\frac{y - y_{\min}}{y_{\max} - y_{\min}}
\end{aligned}
$$

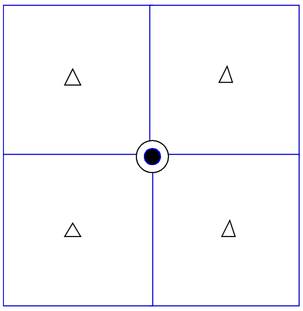
where $x_{\min}, x_{\max}, y_{\min}, y_{\max}$ represent the values minimum and maximum of x and y on the patch.

This method notably improves conditioning of A and removes the preceding problem completely.

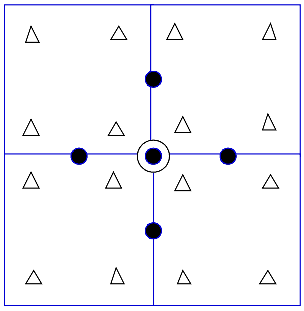
Once $\mathbf{a}_{\mathrm{s}}$ determined, the nodal values are deduced according to the equation [éq 3.2-1] only on the internal nodes with the patch, except in the case of patchs having edge nodes.

Patchs interns:

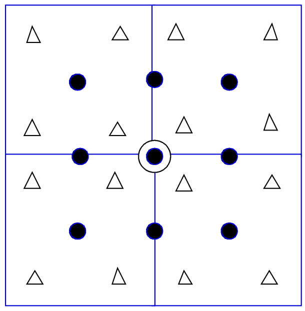
QUAD4

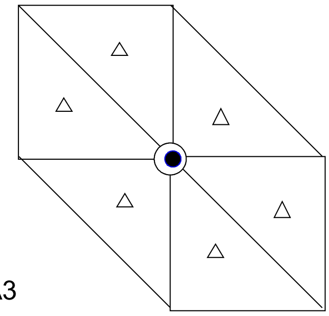


QUAD8

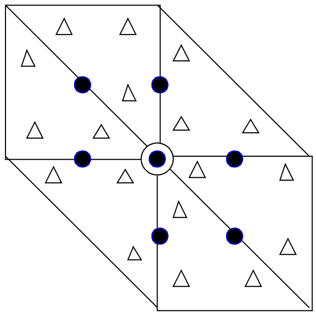


QUAD9





TRIA3



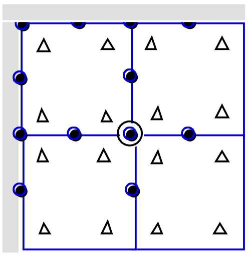
TRIA6

△ Gauss points where the stresses according to $\boldsymbol{\sigma}_{\mathbf{h}}$ the equation [éq 2.1-2] are calculated

● nodes of internal computation $\boldsymbol{\sigma}^{*}$ $\boldsymbol{\sigma}^{*}$

◉ of top defining the patch

Patchs edges:



The nodal values with the nodes mediums belonging to 2 patchs are realized, in the same way for the internal nodes in the case as of QUAD9.

Note:

In the case of finite elements of the different type, the choice of P in the equation [éq 3.2 -1] is delicate (problems of validity of a_s if space is too rich, loss super - convergence if it is not it enough). A thorough study seems essential.

This is why the estimator $ZZ2$ is limited for the moment to meshes comprising one type of element. This restriction does not exist for $ZZ1$.

The authors showed numerically [bib3] that with this choice of $\boldsymbol{\sigma}^*$, their estimator was asymptotically exact for elastic materials of which the characteristics are independent of the field and for all element types and whom rates of convergence with h $\|\mathbf{e}^*\|$ were improved compared to the previous model (especially for the elements of degree 2: to see case Manual test SSLV110 of Validation), from where a better estimate of the error.

One will find an illustration of these rates of convergence in the reference [feeding-bottle 5].

4 Establishment in *Code_Aster* and current limits of Establishment

4.1 use in *Code_Aster*

the two preceding estimators are established in *Code_Aster* in the ordering of postprocessing CALC_ERREUR [U4.81.04]. They are activated from options (ERZ1_ELEM for ZZ1 and ERZ2_ELEM for ZZ2) and enrich a data structure RESULTAT.

Moreover, the computation of the stress field smoothed by one or the other of the methods described with [paragraph 3] can be separately started computation of estimate of the error (option SIZ1_NOEU for ZZ1 and SIZ2_NOEU for ZZ2).

The estimator of error provides:

a field by element comprising 3 components:

- the estimate of the relative error on the element,
- the estimate of the absolute error on the element,
- the norm of the energy of the calculated solution $\boldsymbol{\sigma}_{\mathbf{h}}$.

leave-listing comprising same information at the total level (on all structure)

All the fields obtained are displayable via command IMPR_RESU.

4.2 Operational limits

the theoretical frame is homogeneous linear elasticity

For ZZ1, the modelization 2D (forced and plane strains, axisymmetric) and 3D are allowed whereas for ZZ1, only the modelization 2D (forced and plane strains, axisymmetric) are allowed.

Element types: triangles with 3 and 6 nodes,
quadrangles with 4,8 and 9 nodes.

For ZZ2, the mesh must comprise one type of elements.

5 Bibliography

1) ZIENKIEWICZ O.C., ZHU J.Z.: “A simple error estimator and adaptive procedure for practical engineering analysis” - Int. Newspaper for Num. Puts. in Eng., flight 24 (1987).

2) ZIENKIEWICZ O.C., ZHU J.Z.: “The superconvergent patch recovery and a posteriori error estimates - Share 1: the technical recovery” - Int. Newspaper for Num. Puts. in Eng., flight 33,1331 - 1364 (1992)

3) ZIENKIEWICZ O.C., ZHU J.Z.: “The superconvergent patch recovery and a posteriori error estimates - Share 2: error estimates and adaptivity” - Int. Newspaper for Num. Puts. in Eng., flight 33,1365-1382 (1992)

4) ZIENKIEWICZ O.C., ZHU J.Z., WU J.: “Superconvergent patch recovery techniques - Somme further tests” - Com. in Num. Puts. in Eng., flight 9,251 - 258 (1993)

5) DESROCHES X.: “Estimators of error in linear elasticity” - Notes HI-75/93/118.

6 Description of the versions of the document

Version Aster	Author (S) Organization (S)	Description of the modifications
06/02/09	***X. DESROCHES*** *(EDF/IMA/MMN)*	initial Text