Estimator of error in residue

Summarized

the estimator of error in residue allows to estimate the error of discretization due to the finite element method on the elements of a 2D mesh or 3D. It is an explicit estimator of error utilizing the residues of the balance equations and the jumps of the normal stresses to the interfaces, contrary to the estimator of Zhu - Zienkiewicz, which uses a technique of lissage of the stresses a posteriori [R4.10.01] and [bib5].

This estimator is established in *Code_Aster* in elastoplasticity 2D and 3D.

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1 Introduction

the estimator of error in residue was developed in 1993 by Bernardi-Métivet-Verfurth [bib1]. It is an explicit estimator of error utilizing the residues of the balance equations (from where its name). It applies to elliptic problems (Fish, Stokes, or linear elasticity) in dimension 2 or 3. These problems are supposed to be discretized by finite elements associated with a regular triangulation.

Historically, the first estimator of explicit error relating to the unbalances is due to Babuska and Rheinbolt [bib2] for the problems 1D with linear elements. Gago extended this estimator to 2D and added to the formulas the jumps of tension to the interfaces of the elements [bib3] and [bib4]. New estimators were proposed then, in whom the defaults of surface tension at the borders of the field were also taken into account thus that an improvement of the estimate of the jumps inter - elements giving of the more reliable results.

One is interested here in the estimator in residue applied to the case of linear elasticity. The set aim is, at the conclusion of an elastic design, to determine the card of error on the mesh in sight to possibly adapt this one (by refinement and/or coarsening) or simply for information. The adaptation can be done by sequence with the software of cutting Homard.

2 Formulation of the estimator in residue

Is Ω open of R^n, $n=2$ or 3, of border Γ, and T a regular triangulation of Ω.

In linear elasticity, the continuous problem is written:

to find (u,σ) such as:

$$\begin{cases} div\,\sigma+f=0 & \text{dans } \Omega \\ u=u_D & \text{sur } \Gamma_D \\ \sigma.\mathbf{n}=g_N & \text{sur } \Gamma_N \end{cases}$$

Γ_D is the border of Dirichlet of the mesh
u_D is the displacement imposed on this border
Γ_N is the border of Neumann
$\mathbf{n}$ the unit norm with Γ_N

g_N is the loading applied to this border; it can be continuous or discretized.
f is force of a voluminal type (gravity, rotation); it can be continuous or discretized.

σ_h is the stress obtained by the resolution of the discrete problem:

$$\begin{cases} div\,\sigma_h+f=0 & \text{dans } \Omega \\ u_h=u_D & \text{sur } \Gamma_D \\ \sigma_h.\boldsymbol{n}=g_N & \text{sur } \Gamma_N \end{cases}$$

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with the relation $\sigma_h = \mathbf{DB}\, u_h$ where:

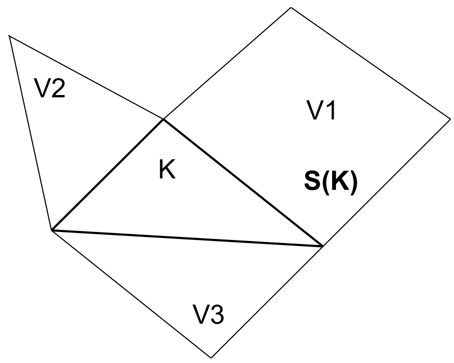
$\boldsymbol{D}$ is the matrix of Hooke
$\boldsymbol{B}$ is the linearized operator of the strains

If K an element running of the mesh indicates, the estimator of error (noted $\eta(\Omega)$) is defined as being the quadratic average of the local error indicators, noted $\eta(K)$:

$$\eta(\Omega) = \left[\sum_{K \in T} \eta(K)^2 \right]^{1/2}$$

The indicator by local residue

the indicator is composed of three terms; the first represents the residue of the balance equation on each mesh, the second term the jump of the normal stresses on the interfaces, the third term the difference between the normal stresses and the imposed loading on Γ_N if the element intersects Γ_N .



K : Elément courant où l'on souhaite calculer l'erreur,
V1 à V3 : Eléments ayant un bord commun avec l'élément courant,
S(K) : Ensemble des bords de l'élément courant ayant des voisins.

Appear 2-a: Internal elements with a mesh

the first term of the estimator is the norm L^2 of the residue of the balance equation on the mesh K, multiplied by h_K which is, either the diameter of the circle circumscribed for a triangular finite element, or the maximum diagonal for a quadrangle,
the second term is the integral, on $S(K)$ definite [Figure 2-a], of the jumps of normal stresses integrated on each edge F of the element which has a neighbor, and multiplied by the root of h_F , who is the length of the edge F,

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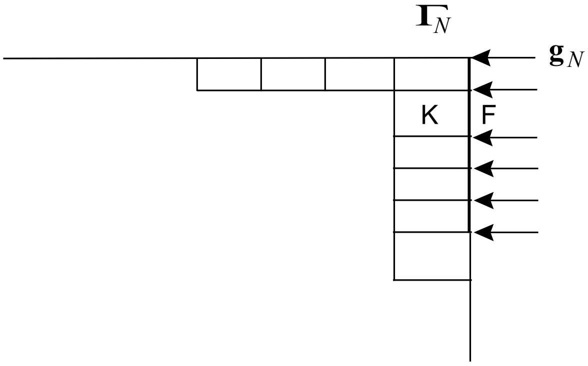
Γ_N : Frontière de Neumann

g_N : Force appliquée sur la frontière de Neumann



Appears 2-b: Elements located on the border of a mesh

the third term is the integral, on the intersection of each edge F of edges ∂K of the element running K with the border of Neumann Γ_N, of the jumps between the normal stresses of the element and the force of Neumann g_N, multiplied by the root of h_F, length of the edge F.

There is thus the following formula for the estimator in residue:

$$\eta(K)=h_K\|f+div\,\sigma_h\|_{L^2(K)}+\frac{1}{2}\sum_{F\in S(K)} h_F^{1/2}\|[\sigma_h.n]\|_{L^2(F)}+\sum_{F\subset\partial K\subset\Gamma_N} h_F^{1/2}\|g_N-\sigma_h.n\|_{L^2(F)} \quad \text{éq 2-1}$$

For the choice of the various terms of [éq 2-1], one returns to [bib1].

3 Properties of the estimator in residue

One notes $\eta_{EX}(K)$ the exact error $\|u-u_h\|_{H^1(K)}$ on the element K (unknown a priori)

and $\eta_{EX}(\Omega)$ the total exact error $\|u-u_h\|_{H^1(\Omega)}$

One has the following properties then ([bib1]):

some either element K, the elementary error $\eta(K)$ is raised by the exact site error (multiplied by a constant independent of the triangulation),

or $\forall K \quad \eta(K)\leq C_1\times\eta_{EX}(K)$

the exact total error is raised by the error considered total $\eta(\Omega)$ (multiplied by a constant independent of T)

or $\eta_{EX}(\Omega)\leq C_2\times\eta(\Omega)$

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the constants $C_1 \text{et} C_2$ depend a priori on the type of finite element and the boundary conditions of the problem. Kelly and Gago [bib3] proposed in 2D a constant C_2 depending only on the degree p of the polynomial of interpolation used:

$$C_2=\left(\frac{1}{24\,p^2}\right)^{1/2} \text{soit} \quad C_2=\frac{1}{2\text{p}\sqrt{6}} \text{pour les TRIA3 et QUAD4 (degré 1)}$$

$$C_2=\frac{1}{4\text{p}\sqrt{6}} \text{pour les TRIA6 et QUAD8 (degré 2)}$$

For 3D, one does not have evaluating of the constant. One can nevertheless say that the error considered total over-estimates the total exact error in all the cases. This result is not inevitably true at the local level.

4 Establishment in *Aster*

the estimator in residue is established in 2D and 3D on all the element types. It is calculated by the command `CALC_ERREUR` by activating option "`ERME_ELEM`".

This option calculates on each element:

the absolute error $\eta(K)$ (see [éq 2-1]),

the norm of the tensor of the stresses $\|\sigma_h\|_{L^2(K)}$ which is used to normalize the absolute error,

the relative error $\eta_{rel}(K)=100\times\frac{\eta(K)}{\sqrt{\eta(K)^2+\|\sigma_h\|_{L^2(K)}^2}}$.

Note :

This definition of the relative error implies that in the zones where the stresses are very low, the relative error can be important and nonsignificant. It is then the absolute error which it is necessary to consider.

It also calculates at the total level:

the absolute error $\eta(\Omega)=\left[\sum_{K\in T}\eta(K)^2\right]^{1/2}$,

the total norm of the tensor of the stresses $\|\sigma_h\|_{L^2(\Omega)}=\left[\sum_{K\in T}\|\sigma_h\|_{L^2(K)}^2\right]^{1/2}$,

the relative error $\eta_{rel}(\Omega)=100\times\frac{\eta(\Omega)}{\sqrt{\eta(\Omega)^2+\|\sigma_h\|_{L^2(\Omega)}^2}}$.

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According to the statement [éq 2-1], one sees that to compute: the error indicator on the mesh K, one must know:

- possible loadings f on K and g_N on $\partial K \cap \Gamma_N$ (or their discretization f_h and g_{Nh}),
- quantities h_K, h_F and $\mathbf{n}$ related to the geometry of the element,
- the stress field σ_h,
- the list of the neighbors K to recover the stresses on these elements, necessary to the computation of the 2nd term of [éq 2-1].

1 and 2 can be calculated or recovered easily.

3 must be calculated as a preliminary by one of options "`SIGM_ELNO`" or "`SIEF_ELNO`".

In the contrary case, an error message fatal is transmitted.

4 requires the computation of a particular connectivity mesh-meshes, besides standard connectivity mesh-nodes. This new object is stored in data structure of mesh type.

For the detail of the establishment in *Aster*, to see [bib6].
For the validation of the estimator, to see [bib7].

5 Bibliography

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[6] A. CORBEL: Establishment of an estimator of error in residue in the Code of mechanics Aster. Ratio of end of internship - June 94

[7] V. NAVAB: Validation of an estimator of error in residue in elasticity Bi and three-dimensional. Ratio of internship - Mars 95

6 Description of the versions of the document

Version Aster	Author (S) Organization (S)	Description of the modifications
04/07/09	*X.DESROCHES (EDF-R&D/AMA)*	

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