Error indicator spatial in residue for the transient thermal

Summarized

During computational simulations by finite elements, obtaining a gross profit is not sufficient any more. The user is increasingly petitioning of **spatial error analysis compared to his mesh**. **He needs for bearing methodological and tools "numériquo" - "data-processing" pointed to measure the quality of his studies and to improve.**
To this end, the error indicators spatial a posteriori make it possible to locate, on each element, a cartography of error on which the tools of mending of meshes will be able to rest: the first computation on a coarse mesh makes it possible to exhume the card of error starting from the data and of the solution discretized (from where the term "a posteriori"), refinement is carried out then locally by treating on a hierarchical basis this information.
The new indicator a posteriori (known as "in pure residue ") which has just been established **for post" - "to treat the thermal solvers of *the Code_Aster*** is based on their local residues extracted semi" - "discretizations in time. Via option "ERTH_ELEM" of CALC_ERREUR, it uses the thermal fields (EVOL_THER) emanating from THER_LINEAIRE and THER_NON_LINE. **It thus supplements the offer of the code in term of advanced tools making it possible to improve quality of the studies, their mutualisations and their comparisons**.
The goal of this note is to detail theoretical, numerical works and data processing which governed its establishment. With regard to the theoretical study we, initially, limited ourselves to the linear thermal of a motionless structure discretized by the standard isoparametric finite elements. But, in practice**, the perimeter of use of this option was partially extended to the nonlinear thermal**.
One gives to the reader properties and the theoretical and practical limitations of the exhumed indicator, while connecting these considerations, which can sometimes appear a little "ethereal", to a precise parameter setting of the accused operators and to the choices of modelization of the code. One tried constantly to bind different the items approached, while detailing, has minimum, of a little technical demonstrations seldom clarified in the specialized literature.

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During computational simulations by finite elements obtaining a gross profit is not sufficient any more. **The user is increasingly petitioning of spatial error analysis compared to his mesh**. **He needs for bearing methodological and pointed numerical tools to measure the quality of his studies and to improve.**
For example, the accuracy of the results is often degraded by local singularities (corners, heterogeneities…). One then seeks the good strategy to identify these critical areas and them to refine/déraffiner in order to optimize the compromise site/total error. And this, with the highhest possible degree of accuracy, in an automatic, reliable way (the error analysis must be itself the least approximate possible!) robust and with the lower costs.

For each type of finite elements, one in general has **estimates a priori of the spatial error** [bib1], [bib3]. But those are checked only asymptotically (when the size h of the elements tends towards zero) and they require a certain level of regularity which is precisely not reached in the zones with problem. Moreover, these increases underlie two types of strategies to improve computation:

- the "methods" -" p "which consist in increasing locally the order of the finite elements,
- "methods" -" h "which locally refines in order to decrease the geometrical characteristics of the elements.

We are interested here in the second strategy, but through another class of indicators: **error indicators a posteriori** . Since works founders of I. BABUSKA and W. RHEINBOLDT [bib18], the importance of this kind of indicator is well established and they arouse a growing interest, as well in pure numerical analysis [bib5], [bib6], [bib7] as in the field of the applications [bib4], [bib16]. They were in particular established and used in N3S, TRIFOU and *the Code_Aster* (for the linear mechanics cf [R4.10.01], [R4.10.02]). For a panorama of the existing indicators, one will be able to consult the reference work of R. VERFURTH [bib7] or, the ratio of X. DESROCHES [bib16].

To take again a sales leaflet of M.FORTIN (cf [bib17] pp468" - "469), the development of the estimate a posteriori is justified mainly by three reasons:

- the first is the need for establishing the accuracy of the results got by a computation finite elements: which credit to grant to them? All are the phenomena and all the data which intervene" - "they well taken into account in the modelization?
- the second purpose is to make it possible whoever to use a computer code without having to intervene in the construction of the mesh in order to obtain the necessary total accuracy,
- finally, the third direction of study is more particularly directed towards the three-dimensional problems for which the size of the meshes is limited by the core memory available and the cost of the resolution.

These specifications reveal two duaux problems : to estimate the accuracy of the solution obtained on the principal parameters of simulation and to propose layers to calculate a new solution which respects a minimal accuracy. The first problem is truly that of the estimate of error whereas the second relates to the associated adaptive methods (refinement/coarsening, mending of meshes, displacement of points, follow-up of border…).
Thus these indicators make it possible to locate on each element a cartography of error on which the tools of mending of meshes will be able to rest.

Note:

> *One prefers to him the denomination of "indicator" to the usual terminology of "estimator" (translation literal of English "the error estimator "). Count" - "held owing to the fact that it has the same theoretical limitations as those of the solver finite elements (as it "post - milked "), than he is him even often sullied with numerical approximations and than he is exhumed via relations of equivalence utilizing many constants dependant on the problem… the information which it under" - "tightens truly gives "only one order of magnitude "of the required spatial error. In spite of these restrictions, these cartographies of error a posteriori do not remain less important about it, and in any case, they constitute the only type of accessible information in this field.*

The first computation on a coarse mesh makes it possible to associate, with each element of the triangulation, an indicator calculated starting from the discretized data and first discrete solution. Refinement is carried out then locally by treating on a hierarchical basis this information.

In short, and in a nonexhaustive way, **the use of an indicator possibly coupled with a remaillor:**

- provides a certain estimate of the error of spatial discretization,
- gets a better frequency of errors due to the local singularities,
- allows to improve the modelization of the facts of the case (materials, loadings, sources…),
- allows to optimize (even accuracy with the lower costs) and to make reliable the process of convergence of the mesh,
- to estimate and qualify a computation for a class of mesh given.

These considerations show clearly that the computation of these estimators (which is finally only a post" - "processing of the problem considered) must:

- to be much less expensive than that of the solution,
- to require only the discretized data and the calculated solution,
- to be able to be localised,
- be equivalent (in a particular form) to the exact error.

We will see, that with the indicators in residue, one can obtain only one total increase of the exact error united with a local decrease of this same error. But these hight delimiters and lower of the error are complementary because, the first ensures us to have obtained a solution with a certain tolerance, while the second enables us to locally optimize the number of points to respect this accuracy and not to over-estimate it. They utilize constants which not depending on the discretizations spatial and temporal.

The goal of this note is to detail theoretical, numerical works and data processing which governed the error indicator installation of a posteriori making it possible "post-to treat" the thermal solvers of *Code_Aster* . It is about an indicator in pure residue initiated by option " `ERTH_ELEM` "of `CALC_ERREUR` .
With regard to the theoretical study we, initially, limited ourselves to the linear thermal of a motionless structure discretized by the standard isoparametric finite elements. But, in practice, the perimeter of use of this option was partially extended to the nonlinear thermal. For more details on the perimeter of use and functional of the thermal indicator and an example of use, one will be able to refer to [§6].

The indicator a posteriori that we propose is an indicator in pure residue based on the local residues of the semi strong equation" - "discretized in time. For certain elements of the theoretical study (and in particular its groundwork) we took as a starting point the innovative works of C. BERNARDI and B. METIVET [bib6]. They have extended they-even, of elliptic with parabolic, the results of R. VERFURTH [bib7]. They in particular were interested in computations of indicators on the model case of the equation of heat with homogeneous condition of Dirichlet, semi-discretized in time by an implicit diagram of Eulerian. We extended these results to with the problems really dealt by the operator linear thermal of the code, THER_LINEAIRE. They are mixed problems in extreme cases (Cauchy-Dirichlet-Neumann-Robin) inhomogeneous, linear, with coefficients variable and discretized by one θ- method.

A basic work was thus undertaken for encircling theoretical springs of the subjacent thermal problem well and to extrapolate the results of the problem models preceding. This in order to try to approach to the modelizations and the perimeter of the code while detailing often induced mathematical subtleties in the articles of Article a special effort was brought compared to to put in prospect the choices led in the Code_Aster search, passed and current, like clarifying the general philosophy of these indicators.
One gives to the reader properties and the theoretical and practical limitations of the indicators released while connecting these considerations to a precise parameter setting of operator CALC_ERREUR accused in this postprocessing. One tried constantly to bind different the items approached, to limit the recourse to long mathematical digressions, while detailing has minimum many "technical" demonstrations seldom clarified a little in the specialized literature.

This document is articulated around the following parts:

- Initially **, one leads a theoretical study** in order to underline holding them and the bordering subjacent thermal problem, and, their possible links with the choices of modelization of the code. First of all, one determines the Variational Frame Abstracted (CVA) minimum (cf [§2.1]) on which one will be able to rest to show the existence and the unicity of a field of temperature solution (cf [§2.2]). By recutting these pre-necessary theoretical with the practical stresses of the users, one from of deduced from the limitations as for the types of geometry and the licit loadings. Then one studies the evolution of the properties of stability of the problem (cf [§3]) during the process of semi-discretization in time and space.
- These results of controllability are very useful to create the standards, the techniques and the inequalities which intervene in the genesis of the indicator in residue. After having introduced the usual notations of this kind of problems (cf [§4.1]), **one exhumes a possible formulation of the indicator** as well as the increase of the total error (cf [§4.2]) and the decrease of the site error associated (cf [§4.4]). Various types of spatial indicators (cf [§4.3]) are evoked and one details several used strategies of construction of indicators into parabolic (cf [§4.5]). In this same paragraph, the temporal aspect of the problem is also examined through the contingencies of management of the spatial error with respect to that of time step.
- In a third part (cf [§5]), the principal contributions of these theoretical chapters and their restrains with the thermal solvers of the code are summarized.
- Finally, one concludes by approaching the practical difficulties from implementation (cf [§6.1]), the environment necessary (cf [§6.2]), the parameter setting (cf [§6.3]) and **the perimeter of use (cf [§6.4]) of the indicator actually established in the operator of postprocessing CALC_ERREUR**. An example of use extracted from a case official test (TPLL01J) is also detailed (cf [§6.5]).

Abstract:

The reader in a hurry and/or not very interested by theoretical springs genesis error indicator and subjacent thermal problem can, from the start, to jump to [§5] which recapitulates the principal theoretical contributions of the preceding chapters.

2 The problem in extreme cases

2.1 Context

One considers **a limited motionless body occupying open related** Ω of R^q (q =2 or 3) of **regular** $\partial\Omega := \Gamma := \bigcup_{i=1}^{3} \Gamma_i$ **border** characterized by its voluminal heat to constant pressure $\rho C_p(\mathbf{x})$ (the vectorial variable $\mathbf{x}$ symbolizes the couple ((x,y) resp here. (x, y, z)) for q =2 (resp. q =3))*)* and its coefficient of isotropic thermal conductivity $\lambda(\mathbf{x})$.

Note:

> *One will thus not take account of a possible displacement of the structure (cf* `THER_NON_LINE_MO` *[R5.02.04]).*

These data materials are supposed to be independent of time (modelization `THER` of *Code_Aster*) and constants by element (discretization P_0).

Note:

> *With modelization* `THER_FO` *these characteristics can depend on time. As of the first versions of the code and before the installation of* `THER_NON_LINE` *, it made it possible to simulate "pseudonym" non-linearities. Taking into account its rather marginal use, we will not be interested in this modelization.*

One is interested in the changes of the temperature in any point $\mathbf{x}$ of opened and at any moment $t \in [0,\tau[\; (\tau > 0)$, when the body is subjected to limiting conditions and loadings independent of the temperature but being able to depend on time. They are volumic source $s(\mathbf{x},t)$, boundary conditions of type imposed temperature $f(\mathbf{x},t)$ (on the external portion of surface Γ_1), normal flux imposed $g(\mathbf{x},t)$ (on Γ_2) and convective exchange $h(\mathbf{x},t)$ and $T_{ext}(\mathbf{x},t)$ (on Γ_3).
One places oneself thus in the frame of application of operator `THER_LINEAIRE` [R5.02.01] of *the Code_Aster* by retaining only the conductive aspects of this linear thermal problem.

Note:

> *Non-linearities pose serious theoretical problems [bib2] to show the existence, the unicity and the stability of the possible solution. Some are still completely open… But in practice, that by no means prevents from "stretching" the perimeter of use of the estimator of error which will be exhumed rigorously for the linear thermal, to the nonlinear thermal (operator* `THER_NON_LINE` *[R5.02.02]).*

This problem in extreme cases **mixed** (of type **Cauchy-Dirichlet-Neumann-Robin** (also called condition of Fourier) inhomogeneous, linear and with variable coefficients) is formulated

$$(P_0)\begin{cases} \rho\, C_p \dfrac{\partial T}{\partial t} - \mathrm{div}(\lambda\, \nabla T) = s & \Omega \times]0,\tau[\\ T = f & \Gamma_1 \times]0,\tau[\\ \lambda \dfrac{\partial T}{\partial n} = g & \Gamma_2 \times]0,\tau[\\ \lambda \dfrac{\partial T}{\partial n} + hT = hT_{ext} & \Gamma_3 \times]0,\tau[\\ T(\mathbf{x},0) = T_0(\mathbf{x}) & \Omega \end{cases} \qquad \text{éq 2.1-1}$$

Note:

- *In this theoretical study of the mixed problem* (P_0) *, one supposes that the border dissociates in portions on which a condition acts inevitably limits nonhomogeneous. This assumption is not in fact not paramount and one can suppose the existence of a portion* Γ_4 *, such as* $\Gamma_4 := \Gamma - \overset{3}{\underset{i=1}{\cup}} \Gamma_i \neq \emptyset$ *, on which intervenes a homogeneous condition of Neumann (thus, when one builds the variational formulation associated with the strong formulation* (P_0) *, the terms of edges related to this zone disappear. The problem remains well posed then since it is thermically unconstrained in this zone. By means of computer, it is besides well what occurs, since the terms of edges are initialized to zero). In practice, it is often the case besides.*
- *It will be supposed that the coefficient of heat exchange* $h(t, \mathbf{x})$ *is positive what is the case in Code_Aster (cf [U4.44.02 §4.7.3]). And that will facilitate a little the things to us in the demonstrations to come (cf for example property 5).*
- *The condition of Robin modelling the convective exchange (key word* `ECHANGE` *) on an edge portion of the field, can be duplicated to take account of exchanges between two under-parts of the border in opposite (key word* `ECHANGE_PAROI` *). This limiting condition models a thermal strength of interface*

$$\text{Avec}\,\Gamma_3=\Gamma_{12}\cup\Gamma_{21}\;,\;T_i=T_{|\Gamma_{ij}}\;\text{on a}\;\begin{cases}\lambda\dfrac{\partial T_1}{\partial n}+hT_1=hT_2 & \Gamma_{12}\times]0,\tau[\\ \lambda\dfrac{\partial T_2}{\partial n}+hT_2=hT_1 & \Gamma_{21}\times]0,\tau[\end{cases} \qquad \text{éq 2.1-2}$$

not to weigh down the writing of the problem and insofar as this option is similar to the condition of Robin with the external medium, we will not specifically mention it in computations which will follow.

- *The condition of Dirichlet can spread in the form of linear relations between the degrees of freedom (key word* `LIAISON_*` *) to simulate, in particular, of geometrical symmetries of structure.*

$$\text{Avec}\;\Gamma_1=\Gamma_{12}\cup\Gamma_{21}\;,\;T_i=T_{|\Gamma_{ij}}\;\text{on a}\,(\text{LIAISON\_GROUP})$$

$$\sum_i \beta_{1i}\,T_1^i(\mathbf{x},t)+\sum_j \beta_{2j}\,T_2^j(\mathbf{x},t)=\gamma(\mathbf{x},t)\;\;\text{sur}\;\Gamma_1\times]0,\tau[$$

of

$$\text{ou plus simplement}\;\;\sum_i \beta_i\,T_i(\mathbf{x},t)=\gamma(\mathbf{x},t)\;\;\text{sur}\;\Gamma_1\times]0,\tau[\;\;(\text{LIAISON\_DDL})$$

the same éq

2.1-3 functionalities `LIAISON_UNIF` *and* `LIAISON_CHAMNO` *make it possible to impose the same temperature (unknown) on a set of nodes. They constitute a surcouche of the preceding conditions by imposing particular* (β, γ) *couples. Not to weigh down the writing of the problem and insofar as these options are only typical cases of the generic condition of Dirichlet, we will not specifically mention them in computations which will follow.*

- *When the material is* ***anisotropic*** *conductivity is modelled by a diagonal matrix expressed in orthotropic reference of the material. That basically does not change following computations which take account only isotropic case. Care should just be taken not to commutate more, in the limiting conditions of Neumann and Robin, the scalar product with the norm and the multiplication by conductivity.*
- *For* ***a transient computation*** *, the initial temperature can be selected in three different ways: by carrying out a steady computation over the first time, by fixing it at a uniform or unspecified value created by a* `CREA_CHAMP` *and by*

carrying out a recovery from a preceding transient computation. This choice of the condition of Cauchy does not affect any the theoretical study which will follow.

- *We will not treat the case where (almost) all the loadings are multiplied by the same function dependant on time (option `FONC_MULT` , this functionality adapted well for certain mechanical problems is disadvised in thermal, because it can return in conflict with the temporal dependence of the loadings and, in addition, it applies selectively to each one of them. It was not taken again besides in `THER_NON_LINE`).*

It is shown that the functional frame most general and most convenient for "the catch in hand" of this parabolic problem is the following.

For the geometry:

Ω opened locally limited the only one with dimensions one of its border, (H1)

Γ variety of dimension $q-1$, lipschitzienne or C^1 per piece (H2)

For the data:

$$\begin{aligned} & s\in L^2(0,\tau ; H^{-1}(\Omega)) \quad T_0\in L^2(\Omega) \\ & f\in L^2\left(0,\tau ; H^{1/2}(\Gamma_1)\right),\ g\in L^2\left(0,\tau ; H^{-1/2}(\Gamma_2)\right),\ \ T_{ext}\in L^2\left(0,\tau ; H^{-1/2}(\Gamma_3)\right) \\ & \rho\ ,\ C_p\ ,\ \lambda\in L^\infty(\Omega) \quad h\in L^2(0,\tau ; L^\infty(\Gamma_3)) \end{aligned} \quad \text{(H3)}$$

which enables us to obtain a solution in the following intersection

$$T\in L^2(0,\tau ; H^1(\Omega))\cap C^0(0,\tau ; L^2(\Omega)) \quad \text{éq 2.1-4}$$

Note::

That is to say $(X,\|\|_X)$ *Banach, one notes* $L^p(0,\tau ; X)$ *the space of the strongly* $t\rightarrow v(t)$ *measurable functions for measurement* dt *such as* $\|v\|_{0,\tau ; p, X}=\left(\int_0^\tau \|v(t)\|_X^p dt\right)^{\frac{1}{p}}<+\infty$ *. It is Banach, therefore a space of Hilbert for the associated norm.*

The introduction as of these spaces of Hilbert particular "space times" comes from **the need for separating the variables** $\mathbf{x}$ and t. Any function $u:(\mathbf{x},t)\in Q_\tau:=\Omega\times]0,\tau[\rightarrow u(\mathbf{x},t)\in\Re$ can in fact of being identified (by means of the theorem of Fubini) with another function $\tilde{u}:t\in]0,\tau[\rightarrow\{\tilde{u}(t):\mathbf{x}\in\Omega\rightarrow\tilde{u}(t)(\mathbf{x})=u(\mathbf{x},t)\}$. The transformation $u\rightarrow\tilde{u}$ constituting an isomorphism, one will simplify the statements thereafter by noting *U* what should have been meant $\tilde{u}$.

Note:

- *The fact of separating, in first, the time of the variable of space makes it possible to be strongly inspired by the conceptual tools developed for the elliptic problems. It is completely coherent besides with the sequence "semi-discretization in time/total discretization in space" which usually chairs the determination of a formulation usable in practice.*
- *The assumptions on the geometry ensure us of the property of 1-prolongation of the open one* Ω *. Thus one will be able to confuse the space of Hilbert*

$$B^1(\Omega):=\left\{u\in L^2(\Omega)/\nabla\mathbf{u}\in\left(L^2(\Omega)\right)^q\right\}$$

on whom it is convenient to work, with space

$$H^1(\Omega):=\left\{u\in D'(\Omega)/\exists U\in H^1(\Re^q)\quad\text{avec } u=U_{|\Omega}\right\}$$

for which the standard theoretical results on the traces, the densities of space and the *equivalent norms are licit.*

- *Taking into account the character lipschitzien of the border the theoretical results which will follow will be able to apply to* ***structures comprising of the corners*** *(outgoing or returning). On the other hand the processing of* ***points*** *or* ***points of reflection*** *leaves this general theoretical frame. In the same way, the fact that the open one must locally be located the same with dimensions one of its border, prevents (theoretically) the processing of* ***crack*** *. To deal with this kind of problem rigorously, an approach consists in of the finite elements correcting the elementary functions by a suitable function centered on the internal end of the crack (cf P. GRISVARD. School of Analysis Numerical CEA-EDF-INRIA on the fracture mechanics, pp183-192, 1982).*
- ***The indicator in residue using the solution of the problem in temperature, its theoretical limitations are thus, at best, identical to those of the aforesaid problem.***

Taking into account the formulation [éq 2.1-1] one will thus be interested in a solution belonging to following functional space:

Note:

This space comprises also the possible conditions of Dirichlet "generalized" of linear relations type between ddls.

$$T \in W := \left\{ u \in H^1(\Omega) / \gamma_{0,1}\,\mathrm{u} := u_{|\Gamma_1} = f \right\} \quad \text{éq 2-1-5}$$

Moreover, thanks to the geometrical assumptions (H1) and (H2), there exists an operator of raising (compound of the operator of usual raising and the operator of prolongation by zero apart from Γ_1) $R : H^{\frac{1}{2}}(\Gamma_1) \to H^1(\Omega)$ linear, continuous and surjective such as:

$$\gamma_{0,1} Rf = f \quad \forall f \in H^{\frac{1}{2}}(\Gamma_1) \quad \text{éq 2-1-6}$$

One will thus be able **to make the problem initial homogeneous in Dirichlet** while being interested more that with the solution

$$u \in V := \left\{ u \in H^1(\Omega) / \gamma_{0,1} u := u_{|\Gamma_1} = 0 \right\} \quad \text{éq 2-1-7}$$

resulting from decomposition

$$T := u + Rf \quad \text{éq 2-1-8}$$

Note::

That is to say $(X, \| \|_X)$ *Banach, one notes* $L^p(0, \tau ; X)$ *the space of the strongly* $t \to v(t)$ *measurable functions for measurement* dt *such as* $\|v\|_{0,\tau ; p, X} = \left(\int_0^\tau \|v(t)\|_X^p \, dt \right)^{\frac{1}{p}} < +\infty$ *. It is Banach, therefore a space of Hilbert for the associated norm.*

This change of variable produces the problem simplified in u

$$(P_1)\begin{cases} \rho\, C_p \dfrac{\partial u}{\partial t} - div(\lambda\, \nabla u) = \hat{s} & \Omega \times]0,\tau[\\ u=0 & \Gamma_1 \times]0,\tau[\\ \lambda \dfrac{\partial u}{\partial n} = \hat{g} & \Gamma_2 \times]0,\tau[\\ \lambda \dfrac{\partial u}{\partial n} + hu = \hat{h} & \Gamma_3 \times]0,\tau[\\ u(0)=u_0 & \Omega \end{cases}$$ éq 2-1-9

with the new second member

$$\hat{s} := s - \rho\, C_p \frac{\partial Rf}{\partial t} + div(\lambda\, \nabla Rf) \;\in L^2(0,\tau; H^{-1}(\Omega)) \,, \qquad \text{éq 2-1-10}$$

the new loadings

$$\hat{g} := g - \lambda\, \frac{\partial Rf}{\partial n} \;\in L^2\left(0,\tau\,, H^{-1/2}(\Gamma_2)\right) \qquad \text{éq 2-1-11}$$

$$\hat{h} := h(T_{ext} - Rf) - \lambda\, \frac{\partial Rf}{\partial n} \;\in L^2\left(0,\tau\,, H^{-1/2}(\Gamma_3)\right) \qquad \text{éq 2-1-12}$$

and the new initial condition

$$u_0(.) := T_0(.) - Rf(.,0) \;\in L^2(\Omega) \qquad \text{éq 2-1-13}$$

Note::

- *This* ***theoretical raising*** *, which can appear a little "ethereal", a completely practical anchorage in the digital techniques has put in work to solve this kind of problem. It corresponds to* ***a substitution (this technique is not used in the Code_Aster, one prefers to him the technique of double dualisation via ddls of Lagrange [R3.03.01]) of the limiting conditions of Dirichlet*** *. By renumbering the unknowns so that these conditions appear in the last, the comparison can be schematized in the following matric form*

$$\begin{bmatrix} A & 0 \\ 0 & Id \end{bmatrix} \begin{bmatrix} T \\ T_{\Gamma_1} := Rf_{\Gamma_1} \end{bmatrix} = \begin{bmatrix} \hat{s} := s - \sum_{j>J} a_{ji} f_j \\ f \end{bmatrix}$$

the assumptions of regularity on the border also ensure us of the good following properties for the workspaces. One then will be able to place itself in the usual abstracted variational frame.

Lemma 1

Pennies the assumptions (H1) and (H2) the workspaces W and V Hilberts are provided with the induced norm par. $H^1(\Omega)$

Proof:

Result comes simply owing to the fact that the trace application $\gamma_{0,1} : H^1(\Omega) \rightarrow L^2(\Gamma_1)$ is the made up one of the application traces usual $\gamma_0 : H^1(\Omega) \rightarrow H^{\frac{1}{2}}(\Gamma) \subset L^2(\Gamma)$ linear, continuous and surjective (taking into account the assumptions selected) and of the operator of restriction on Γ_1 him so linear, continuous and surjective. Of share their definition, one from of deduced that W and V are sev closed of $H^1(\Omega)$. It of Hilberts is thus provided with the norm $\|\ \|_{1,\Omega}$.

Lemma 2

Pennies the assumptions (H_1) and (H_2), the norm and the pseudo norm induced by $H^1(\Omega)$ are equivalent on functional space V. One will note $P(\Omega) > 0$ the constant of Poincaré relaying this equivalence

$$\forall\, v \in V \;\; \|v\|_{1,\Omega} \leq P(\Omega)\, |v|_{1,\Omega}$$

Note:

One will note thereafter $\|u\|_{\infty,\Omega} := \underset{pp.\,t\,\in\,\Omega}{\mathrm{supess}} |u(t)|$ and $\forall (u,v) \in \left(H^m(\Omega)\right)^2 (u,v)_{m,\Omega} := \sum_{|\alpha|\le m} \left(\partial^\alpha u, \partial^\alpha v\right)_{L^2(\Omega)}$, $\|u\|^2_{m,\Omega} := \sum_{|\alpha|\le m} \|\partial^\alpha u\|^2_{L^2(\Omega)}$ et $|u|^2_{m,\Omega} := \sum_{|\alpha|=m} \|\partial^\alpha u\|^2_{L^2(\Omega)}$.

Proof:

This result is a corollary of the inequality of Poincaré checked by the open ones called of “Nikodym” of which been part Ω taking into account the assumptions selected. There are however two cases:

- either the problem is really mixed and comprises limiting conditions others that those of Dirichlet, $\mathrm{mes}(\Gamma - \Gamma_1) \neq 0$ (see the demonstration [bib1] §III.7.2 pp922-925),
- or one takes into account only conditions of type imposed temperature $\mathrm{mes}(\Gamma - \Gamma_1) = 0$, $V = H^1_0(\Omega)$ and one finds result standard equivalence of the norm and pseudo norm on this space (see for example the demonstration [bib3] pp18-19).

The compilation of the preceding results makes it possible to encircle **the Variational Frame Abstracted** (CVA) on which will rest the weak formulation:

- $H^1_0(\Omega) \subset V \subset H^1(\Omega)$,
- $V \subset H := L^2(\Omega) = H' \subset V' \subset H^{-1}(\Omega)$ while identifying H and its dual,
- one has a linear canonical injection continues V in H,
- V is dense in H and the injection is compact (he inherits in that the properties $H^1(\Omega)$ with respect to H),
- V is provided with the pseudo norm induced by $H^1(\Omega)$ and H of its usual norm.

Note:

According to a formulation of the theorem of compactness of Rellich adapted to spaces of Sobolev on open (for example, theorem 1.5.2 [bib3] pp29-30).

2.2 Strong formulation with weak

By multiplying the principal equation of the problem in extreme cases [éq 2.1-1] by a function test $v \in V$ and by means of the theorems of Green and Reynolds (to commutate the integral in space and derivative in time, with Ω fixed and of the characteristic materials independent of time), one obtains:

$$\frac{d}{dt}\int_\Omega \rho\, C_p u(t) v\, dx + \int_\Omega \lambda\, \nabla u(t) . \nabla v\, dx = \int_\Omega \hat{s}(t) v\, dx + \int_\Gamma \lambda \frac{\partial u(t)}{\partial n} v\, d\sigma \qquad \text{éq 2.2-1}$$

By introducing the limiting conditions into [éq 2.2-1], it occurs **the weak formulation** (within the meaning of **the distributions (in this general frame, the temporal derivative is thus to take with the weak meaning) temporal** of $D'(]0,\tau[)$) following:
One seeks the solution

$$u\in L^2(0,\tau\,;V)\cap C^0(0,\tau\,;H) \qquad \text{éq 2.2-2}$$

checking the problem

$$(P_2)\left\{\begin{array}{l} trouver\ u:t\in]0,\tau[\rightarrow u(t)\in V\ tel\ que \\ \forall v\in V\ \dfrac{d}{dt}(\rho\,C_p u(t),v)_{0,\Omega}+a(t\,;u(t),v)=(b(t),v) \\ u(0)=u^0 \end{array}\right\} \qquad \text{éq 2.2-3}$$

with

$$\begin{array}{l} a(t;u(t),v):=\int_{\Omega}\lambda\,\nabla u(t).\nabla v\,dx+\int_{\Gamma_3}h(t)\,\gamma_{0,3}u(t)\,\gamma_{0,3}v\,d\sigma \\ (b(t),v):=\langle\hat{s}(t),v\rangle_{-1\times 1,\Omega}+\langle\hat{g}(t),\gamma_{0,2}v\rangle_{-\frac{1}{2}\times\frac{1}{2},\Gamma_2}+\langle\hat{h}(t),\gamma_{0,3}v\rangle_{-\frac{1}{2}\times\frac{1}{2},\Gamma_3} \end{array} \qquad \text{éq 2.2-4}$$

by noting $\langle\,,\,\rangle_{p\times q,\Theta}$ the hook of duality between spaces $H^p(\Theta)$ and $H^q(\Theta)$.

Note:

- *The unknown field and the function test belong to the same functional space, which is more comfortable from a numerical and theoretical point of view.*
- *The hooks of duality will not be able to be transformed into integrals with the classical meaning (as for the surface term of (T has;. .)) that if one restricts the space of membership of the new source and of the new loadings with*

$$\hat{s}\in L^2(0,\tau\,;L^2(\Omega)),\ \hat{g}\in L^2(0,\tau\,;L^2(\Gamma_2))\ \text{et}\ \hat{h}\in L^2(0,\tau\,;L^2(\Gamma_3)) \qquad \textit{éq 2.2-5}$$

According to [éq 2-1-10] [éq 2-1-12] this restriction can be translated on the initial loadings in the form

$$f\in L^2\left(0,\tau\,;H^{\frac{3}{2}}(\Gamma_1)\right),\ s\in L^2(0,\tau\,;L^2(\Omega)),\ g\in L^2(0,\tau\,;L^2(\Gamma_2))\ \text{et}\ T_{ext}\in L^2(0,\tau\,;L^2(\Gamma_3))$$

éq 2.2-6

- *the formulation* (P_2) *has a meaning well, because it is shown that*

$$t\rightarrow a(t\,;u(t),v)\in L^2(]0,\tau[)\subset D'(]0,\tau[)$$

$$t\rightarrow \rho\ C_p u(t)\in L^2(0,\tau\,;V)\ et\ v\in V\Rightarrow t\rightarrow(\rho\ C_p u(t),v)_{0,\Omega}\in L^2(]0,\tau[)\subset D'(]0,\tau$$

$$t\rightarrow\hat{s}(t)\in L^2(0,\tau\,;H^{-1}(\Omega))\ \textbf{et}\ v\in H^1(\Omega)\subset H^{-1}(\Omega)$$

$$\Rightarrow t\rightarrow\langle\hat{s}(t),v\rangle_{-1\times 1,\Omega}\in L^2(]0,\tau[)\subset D'(]0,\tau[)$$

$$t\rightarrow\hat{g}(t)\in L^2\left(0,\tau\,;H^{-\frac{1}{2}}(\Gamma_2)\right)\ \textbf{et}\ \gamma_{0,2}v\in H^{\frac{1}{2}}(\Gamma_2)\subset H^{-\frac{1}{2}}(\Gamma_2)$$

$$\Rightarrow t\rightarrow\langle\hat{g}(t),\gamma_{0,2}v\rangle_{-\frac{1}{2}\times\frac{1}{2},\Gamma_2}\in L^2(]0,\tau[)\subset D'(]0,\tau[)$$

and one finds obviously the same thing for the term of exchange on Γ_3 *.*

- *In the surface integrals one will note henceforth* $u(t)$ *and* v *what should be noted (in any rigor)* $\gamma_{0,i}u(t)$ et $\gamma_{0,i}v$ *.*
- *The belonging from the solution with* $L^2(0,\tau\,;V)$ *rises from the assumptions on the data and the properties of the differential operators and trace. The fact that it must also belong to* $C^0(0,\tau\,;H)$ *comes just from the necessary*

| *justification of the condition of Cauchy.*

One can then be interested in **the existence and unicity of the solution of the initial problem** (P_0) by showing **his equivalence with** (P_2) and by applying to this last a parabolic alternative of the theorem of Lax-Milgram.

Theorem 3

In the variational frame abstracted (CVA) defined previously and by supposing that assumptions (H1), (H2) and (H3) are checked, then the problem (P_2) admits a solution and only one $u \in L^2(0,\tau;V) \cap C^0(0,\tau;H)$.

Proof:

This result comes from theorems 1 & 2 of the "Dautray-Lions" (cf [bib3], §XVIII pp615-627). To use them it is necessary nevertheless to check

- the mesurability of the bilinear form $\forall (u(t),v) \in V^2 \quad t \rightarrow a(t;u(t),v) \text{ sur }]0,\tau[$
- Its continuity on $V \times V$

$$pp\ t \in]0,\tau[\quad |a(t;u(t),v)| \leq \|\lambda\|_{\infty,\Omega} |u(t)|_{1,\Omega} |v|_{1,\Omega} + \|h(t)\|_{\infty,\Gamma_3} \|u(t)\|_{\frac{1}{2},\Gamma_3} \|v\|_{\frac{1}{2},\Gamma_3}$$

$$\forall (u(t),v) \in V^2 \qquad \leq \max\left(\|\lambda\|_{\infty,\Omega}, \|h(t)\|_{\infty,\Gamma_3} C_3^2 P^2(\Omega)\right) |u(t)|_{1,\Omega} |v|_{1,\Omega}$$

with C_3 the constant of continuity of the operator of trace on Γ_3 and $P(\Omega)$ the constant of Poincaré.

- Its V- ellipticity compared to H

$$pp\ t \in]0,\tau[\quad a(t;v,v) + \frac{\beta}{2} \|v\|_{0,\Omega}^2 \geq C_0^{-2}\left(\|\lambda\|_{\infty,\Omega} - \|h(t)\|_{\infty,\Gamma_3} C_3^2\right) \|v\|_{0,\Omega}^2$$

$$\forall v \in V \;\Rightarrow\; a(t;v,v) + \underbrace{\beta}_{>0} \|v\|_{0,\Omega}^2 \geq \underbrace{\left\{\frac{\beta}{2} + C_0^{-2}\left(\|\lambda\|_{\infty,\Omega} - \|h(t)\|_{\infty,\Gamma_3} C_3^2\right)\right\}}_{\alpha > 0} \|v\|_{0,\Omega}^2$$

with C_0 the constant of continuity of the canonical injection of $H^1(\Omega)$ in $L^2(\Omega)$.

- The continuity of the linear form $b(t)$ on V

$$pp\ t \in]0,\tau[\quad |(b(t),v)| \leq \|\hat{s}(t)\|_{-1,\Omega} \|v\|_{1,\Omega_3} + \|\hat{g}(t)\|_{-\frac{1}{2},\Gamma_2} \|v\|_{\frac{1}{2},\Gamma_2} + \|\hat{h}(t)\|_{-\frac{1}{2},\Gamma_3} \|v\|_{\frac{1}{2},\Gamma_3}$$

$$\forall v \in V \qquad \leq P(\Omega) \max\left(\|\hat{s}(t)\|_{-1,\Omega}, \|\hat{g}(t)\|_{-\frac{1}{2},\Gamma_2} C_2, \|\hat{h}(t)\|_{-\frac{1}{2},\Gamma_3} C_3\right) |v|_{1,\Omega}$$

with C_2 the constant of continuity of the operator of trace on Γ_2 .

Theorem the 4

problems (P_0) and (P_2) are equivalent and thus the initial problem admits a solution and only one $u \in L^2(0,\tau;V) \cap C^0(0,\tau;H)$.

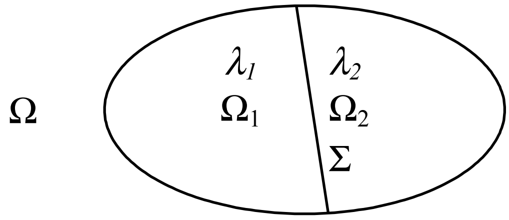
Proof:

The existence and the unicity of the solution of the problem $\left(P_0\right)$ of course result from the preceding theorem, once the equivalence of the two problems was shown. It thus remains to prove the opposite implication $\left(P_2\right)\Rightarrow\left(P_0\right)$ who is very tough to exhume "not formally". In particular the limiting conditions of Neumann, Robin and the condition of Cauchy are difficult to obtain rigorously. The "Dautray - Lions" proposes a very technical demonstration ([bib1] §XVIII pp637-641). By adapting his results one shows that in our case, the limiting conditions on Γ_i in fact are checked, not on $L^2\left(0,\tau,H^{-\frac{1}{2}}\left(\Gamma_i\right)\right)$, but on space $\left(B_i\right)'\supset H_{00}^{-\frac{1}{2}}\left(\Gamma_\tau^i\right)$ (while noting $\Gamma_\tau^i:=\Gamma_i\times\left]0,\tau\right[$) definite as being the dual topological one of

$$B_i:=\left\{w\in H^{\frac{1}{2}}\left(\partial\Omega_\tau\right)\cap L^2\left(\Gamma_\tau^i\right)/\exists\, v\in L^2\left(0,\tau;V\right)\text{ avec } v_{|\Omega\times\{0\}}=v_{|\Omega\times\{\tau\}}=0\text{ et } v_{|\Gamma_i}=w\right\}$$

Note:

- *Because of **low regularity** imposed on **thermal conductivity** $\lambda\in L^\infty\left(\Omega\right)$, one cannot claim with **the "standard"** regularity $u\in H^2\left(\Omega\right)$. Indeed in the case, for example, of **a bi--material** (with $\Omega=\Omega_1\cup\Sigma\cup\Omega_2$) from which the characteristics are distinct on both sides of the border Ω, [éq 2-1-9] and the theorem of the divergence Figure*



2.2-a imposes: Example of bi--material

$$\lambda_1\frac{\partial u\left(t\right)}{\partial n}\Big|_{\Omega_1}=\lambda_2\frac{\partial u\left(t\right)}{\partial n}\Big|_{\Omega_2}\ \textbf{dans}\ H_{00}^{-\frac{1}{2}}\left(\Sigma\right)\quad pp\ t\in\left]0,\tau\right[$$

But $\lambda_1\neq\lambda_2$, therefore the condition of transmission cannot be carried out on the border interns Σ

$$\frac{\partial u\left(t\right)}{\partial n}\Big|_{\Omega_1}\neq\frac{\partial u\left(t\right)}{\partial n}\Big|_{\Omega_2}\quad pp\ \Sigma\quad pp\ t\in\left]0,\tau\right[$$

*Thus $u\left(t\right)\in H^2\left(\Omega_1\right)\cup H^2\left(\Omega_2\right)$ does not involve $u\left(t\right)\in H^2\left(\Omega\right)$. This restriction will not enable us to exhume, as in [bib6], of **increases of the "strong" type** of the total spatial error and the local error indicator. In our frame of more general work one will have to be satisfied **with estimates of the "weak" type**.*

- *This kind of problem also meets when open polyhedric the nonconvex ones are treated (for example comprising a returning corner). Open polyhedric (known as polygonal into two-dimensional) is a finished meeting of polyhedrons. A polyhedron is a finished intersection of closed half spaces.*
- *To obtain estimates of the type "strong", it is necessary to concede more regularity on geometry and on loadings*

Γ *variety of dimension* $q-1$, C_2 *by piece (property of 2-prolongation)* *(H4)*

$$s \in L^2\left(0,\tau ; L^2(\Omega)\right) \quad T_0 \in H^1(\Omega)$$
$$f \in L^2\left(0,\tau ; H^{\frac{3}{2}}(\Gamma_1)\right),\ g \in L^2\left(0,\tau ; H^{\frac{1}{2}}(\Gamma_2)\right),\ \ T_{ext} \in L^2\left(0,\tau ; H^{\frac{1}{2}}(\Gamma_3)\right) \quad \text{(H5)}$$
$$\rho\ ,\ C_p\ ,\ \lambda \in L^\infty(\Omega) \quad h \in L^2\left(0,\tau ; L^\infty(\Gamma_3)\right)$$

What Now allows obtaining a solution in the following

$$u \in L^2\left(0,\tau ; H^2(\Omega)\right) \cap C^0\left(0,\tau ; H^1(\Omega)\right) \quad \textit{intersection éq}$$

2.2-7 that we made sure of the existence and the unicity of the solution in the functional frame required by the operators of *Code_Aster*, we **semi-will discretize in time (** P_0 **)** then **to spatially discretize the whole by a method of finite elements**. In parallel, we will study its properties of stability. They we will be very useful to create the standards, the techniques and the inequalities which will intervene in the genesis of the error indicator in residue.

3 Discretization and controllability

3.1 Controllability of the continuous problem

By not making no concession on the assumptions of regularity seen in the preceding paragraph, there is increase known as "weak" (to take again a terminology in force in the article which was used as a basis for our study [bib6]) following.

Property 5

In the variational frame abstracted (CVA) defined previously and by supposing that assumptions (H1), (H2) and (H3) are checked, one has **the "weak" controllability of the continuous problem** (with $K_1\left(\|\lambda\|_{\infty,\Omega}, mes(\Gamma_i), \gamma_{0,i}, P(\Omega)\right)>0$)

$$pp\ t \quad \|\sqrt{\rho\, C_p}u(t)\|^2_{0,\Omega}+\int_0^t \|\sqrt{\lambda}\,\nabla u(\xi)\|^2_{0,\Omega}d\xi \le \|\sqrt{\rho\, C_p}u_0\|^2_{0,\Omega}+ K_1\left\{\int_0^t \|\hat{s}(\xi)\|^2_{-1,\Omega}+\|\hat{g}(\xi)\|^2_{-\frac{1}{2},\Gamma_2}+\|\hat{h}(\xi)\|^2_{-\frac{1}{2},\Gamma_3}d\xi\right\} \quad \textbf{éq 3.1-1}$$

Proof:

One here will detail this a little technical demonstration because, on the one hand, the specialized literature seldom returns in this level of details and, on the other hand, one will re-use same methodology to exhume all increases which will follow one another in this theoretical part of the document. First of all, by multiplying the equation of [éq 2.1-1] by $u(t)$, while integrating spatially on Ω, then temporally in ON position $[0,t]$ avec $t\in[0,\tau[$ obtains, like the characteristic materials are supposed to be independent of time,

$$\frac{1}{2}\int_0^t \frac{\partial}{\partial t}\left(\rho\, C_p u(\xi), u(\xi)\right)_{0,\Omega}d\xi-\int_0^t\left(div\left(\lambda\,\nabla u(\xi)\right), u(\xi)\right)_{0,\Omega}d\xi=\int_0^t\langle\hat{s}(\xi), u(\xi)\rangle_{-1\times 1,\Omega}d\xi \quad \text{éq the 3.1-2}$$

By means of formula of Green and the limiting conditions of [éq 2.1-1] one obtains

$$\frac{1}{2}\left(\|\sqrt{\rho\, C_p}u(t)\|^2_{0,\Omega}-\|\sqrt{\rho\, C_p}u_0\|^2_{0,\Omega}\right)+\int_0^t\left(\lambda\,\nabla u(\xi), \nabla u(\xi)\right)_{0,\Omega}d\xi+\int_0^t h(\xi)u^2(\xi)d\xi= \int_0^t\left\{\langle\hat{s}(\xi), u(\xi)\rangle_{-1\times 1,\Omega}+\langle\hat{g}(\xi), u(\xi)\rangle_{-\frac{1}{2}\times\frac{1}{2},\Gamma_2}+\langle\hat{h}(\xi), u(\xi)\rangle_{-\frac{1}{2}\times\frac{1}{2},\Gamma_3}\right\}d\xi \quad \text{éq 3.1-3}$$

One can oust the term of exchange of [éq 3.1-3] because it is supposed that $h(t)\ge 0$ pp t . By means of an argument of duality, the inequality of Cauchy-Schwartz, lemma 2 and the relation $2ab\le\left(\frac{a}{\alpha}\right)^2+(b\alpha)^2\ (\alpha>0)$, one obtains

$$\int_0^t\langle\hat{s}(\xi), u(\xi)\rangle_{-1\times 1,\Omega}d\xi\le\frac{1}{2}\left(\frac{1}{\alpha^2}\int_0^t\|\hat{s}(\xi)\|^2_{-1,\Omega}d\xi+\frac{P^2(\Omega)}{\|\lambda\|_{\infty,\Omega}}\alpha^2\int_0^t\|\sqrt{\lambda}\,\nabla u(\xi)\|^2_{0,\Omega}d\xi\right) \quad \text{éq 3.1-4}$$

One carries out same work on the loadings, thus defining the parameters β and γ by taking again the notations of theorem 3 (for $C_i \ldots$), then one inserts these inequalities in [éq 3.1-3]

$$|||\bar{\rho}\, C_p u(t)||^2_{0,\Omega}+\left(2-\frac{P^2(\Omega)}{\|\lambda\|_{\infty,\Omega}}\left(\alpha^2+C_2^2\beta^2+C_3^2\gamma^2\right)\right)\int_0^t \|\sqrt{\lambda}\,\nabla u(\xi)\|^2_{0,\Omega} d\xi \leq$$

$$|||\bar{\rho}\, C_p u_0||^2_{0,\Omega}+\int_0^t \left\{\frac{\|\hat{s}(\xi)\|^2_{-1\times 1,\Omega}}{\alpha^2}+\frac{\|\hat{g}(\xi)\|^2_{-\frac{1}{2}\times\frac{1}{2},\Gamma_2}}{\beta^2}+\frac{\|\hat{h}(\xi)\|^2_{-\frac{1}{2}\times\frac{1}{2},\Gamma_3}}{\gamma^2}\right\} d\xi \quad \text{éq 3.1-5}$$

It now remains to seek a triplet of strictly positive realities (α,β,γ), not privileging no particular term, in order to reveal a constant independent of the solution and parameter setting in front of the term in gradient. One arbitrarily chooses to pose

$$2-\frac{P^2(\Omega)}{\|\lambda\|_{\infty,\Omega}}\left(\alpha^2+C_2^2\beta^2+C_3^2\gamma^2\right)=1 \quad \text{éq 3.1-6}$$

Is, for example,

$$\left|\begin{array}{l} \alpha^2=\dfrac{\|\lambda\|_{\infty,\Omega}\left(\text{mes}(\Gamma_1)+1\right)}{P^2(\Omega)\left(\text{mes}(\Gamma)+3\right)} \\ \beta^2=\dfrac{\|\lambda\|_{\infty,\Omega}\left(\text{mes}(\Gamma_2)+1\right)}{C_2^2 P^2(\Omega)\left(\text{mes}(\Gamma)+3\right)} \\ \gamma^2=\dfrac{\|\lambda\|_{\infty,\Omega}\left(\text{mes}(\Gamma_3)+1\right)}{C_3^2 P^2(\Omega)\left(\text{mes}(\Gamma)+3\right)} \end{array}\right. \quad \text{éq 3.1-7}$$

From where increase [éq 3.1-1] by taking

$$K_1=\frac{P^2(\Omega)\left(\text{mes}(\Gamma)+3\right)}{\|\lambda\|_{\infty,\Omega}}\max\left(\frac{1}{\left(\text{mes}(\Gamma_1)+1\right)},\frac{C_2^2}{\left(\text{mes}(\Gamma_2)+1\right)},\frac{C_3^2}{\left(\text{mes}(\Gamma_3)+1\right)}\right) \quad \text{éq 3.1-8}$$

Note:

- *The recourse to measurements of the external borders is a trick making it possible the inequality to support the transition with limit ($\Gamma_i \to 0$) when one or more limiting conditions have suddenly missed in this mixed problem.*
- *While placing oneself in the particular frame of a homogeneous problem of Cauchy-Dirichlet with characteristic materials constants equal to the unit*

$$\lambda=\rho\, C_p=1 \ , \ \Gamma_2=\Gamma_3=\varnothing \ \textbf{ et } \ \hat{s}=s \qquad \text{éq 3.1-9}$$

and by introducing particular norms on $V=H_0^1(\Omega)$ *and his dual*

$$\|\hat{s}(t)\|_{-1,\Omega}^{*}=\sup_{v\in V,\ v\neq 0}\frac{\langle \hat{s}(t),v\rangle_{-1\times 1,\Omega}}{\|v\|_{1,\Omega}^{*}} \text{ avec } \|v\|_{1,\Omega}^{*}=\frac{mes(\Gamma)+1}{(mes(\Gamma)+3)\,P^2(\Omega)}\|v\|_{1,\Omega} \qquad \text{éq 3.1-10}$$

one finds well the inequality (2) pp427 of [bib6].

- *If one allows more regularity on the geometry (H_4) and on the data (H_5), one can exhume* ***during, known as "extremely",*** *of the preceding property. The control of the solutions which it operates is of course more precise than with [éq 3.1-1] because it is carried out via stronger norms. Contrary to "weak" increase, it also utilizes* ***directly the infinite norm of the convective coefficient of heat exchange*** *. His obtaining here will not be detailed because this family of increase is not essential for the computation of the required indicator.*

3.2 Semi-discretization in time

One fixes time step Δt such as $\frac{\tau}{\Delta t}$ either an integer N and the temporal discretization or regular: $t_0=0,\ t_1=\Delta t,\ t_2=2\Delta t \cdots t_n=n\Delta t$.

Note:

This assumption of regularity does not have really importance, it just makes it possible to simplify the writing of the semi-discretized problem. To model an unspecified transient at time t_n *, it is just enough to replace* Δt *par.* $\Delta t_n=t_{n+1}-t_n$

the semi - discretization in times of [éq 2.1-1] by θ **- method** leads to the following problem:
One seeks the continuation

$$(u^n)_{0\le n\le N}\in V \qquad \text{éq 3.2-1}$$

such as

$$(P_1^{n+1})\begin{cases}\rho C_p\dfrac{u^{n+1}-u^n}{\Delta t}-\theta\,\mathrm{div}(\lambda\,\nabla u^{n+1})-(1-\theta)\,\mathrm{div}(\lambda\,\nabla u^n)=\theta\,\hat{s}^{n+1}+(1-\theta)\,\hat{s}^n & \Omega\ \ 0\le n\le N-1\\ u^{n+1}=0 & \Gamma_1\ \ 0\le n\le N-1\\ \lambda\,\dfrac{\partial u^{n+1}}{\partial n}=\hat{g}^{n+1} & \Gamma_2\ \ 0\le n\le N-1\\ \lambda\,\dfrac{\partial u^{n+1}}{\partial n}+h^{n+1}\,u^{n+1}=\hat{h}^{n+1} & \Gamma_3\ \ 0\le n\le N-1\\ u^0(.)=u_0 & \Omega\end{cases}$$

éq 3.2-2

while posing

$$\Xi^n = \Xi\left(\mathbf{x}, n\frac{\tau}{\Delta t}\right) \text{ avec } \Xi \in \{u, \hat{s}, \hat{h}, h, \hat{g}\} \text{ et } 0 \leq n \leq N$$

While multiplying [éq 3.2-2] by a function test v and while integrating on Ω, one of course finds (via the formula of Green) the variational formulation [éq 2.2-3] semi-discretized in time

$$\left(P_2^{n+1}\right)\begin{cases}\text{Etant donnés } u^n, \hat{s}^n, \hat{s}^{n+1}, \hat{g}^n, \hat{g}^{n+1}, \hat{h}^n, \hat{h}^{n+1}, h^n, h^{n+1} \\ \text{Calculer } u^{n+1} \in V \text{ tel que} \\ \left(\rho\, C_p u^{n+1}, v\right)_{0,\Omega} + \Delta t\; a\left(n\Delta t\theta\;;\; u_\theta^{n+1}, v\right) = \left(\rho\, C_p u^n, v\right)_{0,\Omega} + \Delta t\left(b_\theta^n, v\right)\quad \left(\forall v \in V\right)\end{cases}$$ éq 3.2-3

with

$$\begin{aligned}&\Xi_\theta^{n+1} := \theta\;\Xi^{n+1} + (1-\theta)\,\Xi^n \quad \text{où} \quad \Xi \in \{u, hu, b, \hat{s}, \hat{g}, \hat{h}\}\\ &a\left(n\Delta t\,\theta\;; u_\theta^{n+1}, v\right) := \int_\Omega \lambda\;\nabla u_\theta^{n+1}.\nabla v\;dx + \int_{\Gamma_3} (hu)_\theta^{n+1} v\;d\sigma\\ &\left(b_\theta^{n+1}, v\right) := \langle \hat{s}_\theta^{n+1}, v\rangle_{-1\times 1,\Omega} + \langle \hat{g}_\theta^{n+1}, \gamma_{0,2} v\rangle_{-\frac{1}{2}\times\frac{1}{2},\Gamma_2} + \langle \hat{h}_\theta^{n+1}, \gamma_{0,3} v\rangle_{-\frac{1}{2}\times\frac{1}{2},\Gamma_3}\end{aligned}$$ éq 3.2-4

This semi-discretization in time allowed to transform our parabolic problem into an elliptic problem to which one can apply the theorem of standard Lax-Milgram. The assumptions of this theorem are checked easily thanks to the results of continuity and ellipticity of the demonstration of theorem 3. From where the existence and the unicity of the required $\left(u^n\right)_{0\leq n\leq N} \in V$ continuation.

Note:

- *While posing $Rf=0$ one finds well the semi-discretized variational formulation of the Code_Aster (cf [R5.02.01 §5.1.3]). (Or them) the condition (S) of Dirichlet (generalized or not) are checked within the space of work W to which the solution must belong. Moreover, by implicitant it completely θ - method (Eulerian retrogresses) one finds the formulation of code SYRTHES [bib9].*
- *To be able semi-to discretize by θ - method one needs to restrict the belonging new source with $\hat{s} \in C^0\left(0, \tau\,; H^{-1}(\Omega)\right)$ (to be able to take a value in a given time). In addition, the initialization of the iterative process [éq 3.2-3] necessarily involves $u_0 \in H^1(\Omega)$.*
- *To simplify the statements one will not mention any more the temporal dependence of the bilinear form has (T;. .) (for the implicature of the term of exchange), it will remain implied by that of the solution.*

As for the continuous problem, by not making no concession on the assumptions of regularity, there is "weak" increase following:

Property 6

By supposing that the assumptions of property 5 are checked, that it θ - diagram is unconditionally stable ($\theta \geq \frac{1}{2}$), that $\hat{s} \in C^0\left(0,\tau ; H^{-1}(\Omega)\right)$ and $u_0 \in H^1(\Omega)$, one has **the "weak" controllability of the problem semi-discretized in time** (with $K_1\left(\|\lambda\|_{\infty,\Omega}, mes(\Gamma_i), \gamma_{0,i}, P(\Omega)\right) > 0$)

$$\begin{aligned} &\|\sqrt{\rho C_p} u^{n+1}\|_{0,\Omega}^2 + \Delta t \|\sqrt{\lambda} \nabla u_\theta^{n+1}\|_{0,\Omega}^2 \leq \frac{1}{2}\|\sqrt{\rho C_p} u^{n+1}\|_{0,\Omega}^2 + \frac{4\theta - 3}{2}\|\sqrt{\rho C_p} u^n\|_{0,\Omega}^2 \\ &\forall 0 \leq n \leq N-1 \qquad + \frac{\Delta t}{2}\|\sqrt{\lambda} \nabla u_\theta^{n+1}\|_{0,\Omega}^2 + \frac{K_1 \Delta t}{2}\left(\|\hat{s}_\theta^{n+1}\|_{-1,\Omega}^2 + \|\hat{g}_\theta^{n+1}\|_{-\frac{1}{2},\Gamma_2}^2 + \|\hat{h}_\theta^{n+1}\|_{-\frac{1}{2},\Gamma_3}^2\right) \end{aligned}$$

éq 3.2-5

Proof:

This inequality is obtained easily by taking again the stages described in the demonstration of property 5. It is necessary, on the other hand, to multiply [éq 3.2-2] by the particular function test

$$u_\theta^{n+1} := \theta\, u^{n+1} + (1-\theta)\, u^n \in V \qquad \text{éq 3.2-6}$$

and to oust the term of exchange by the argument

$$0 < \min\left(h^n, h^{n+1}\right) \|u_\theta^{n+1}\|_{0,\Gamma_3}^2 \leq \int_{\Gamma_3} (hu)_\theta^{n+1} u_\theta^{n+1} dx \leq \max\left(h^n, h^{n+1}\right) \|u_\theta^{n+1}\|_{0,\Gamma_3}^2 \qquad \text{éq 3.2-7}$$

In addition there is not that time the source term and the loadings which require the trick [éq 3.1-4], it should also be set up on the cross term $(2\theta - 1)\int_\Omega \rho C_p u^{n+1} u^n dx$. From where a fourth parameter η checking a system of the type [éq 3.1-6]

$$\begin{aligned} &|2 - \frac{P^2(\Omega)}{\|\lambda\|_{\infty,\Omega}}\left(\alpha^2 + C_2^2 \beta^2 + C_3^2 \gamma^2\right) = 1 \\ &|2\theta - \eta^2|1 - 2\theta| = 1 \end{aligned} \qquad \text{éq 3.2-8}$$

Note:

- *If a conditionally stable **diagram in the case of is not placed** , in addition to the numerical problems which are likely to occur at the time of implementation the effective of the operator, one will not be able to determine the parameters $(\alpha, \beta, \gamma, \eta)$ governing the equation [éq 3.2-8].*
- *While placing oneself in the particular frame [éq 3.1-9] of the article [bib6] and by taking again the equivalent norms [éq 3.1-10], like $\frac{4\theta - 3}{2} < \frac{1}{2}$, one finds well the inequality (5) pp428.*

While stating [éq 3.2-5] for the values of $m\in\{0,1\ldots,n\}$ and by adding these increases until n , one obtains the "weak" increase following which takes account of the history of the solutions and the data.

Corollary 7

Pennies the assumptions of property 6, one has increase

$$\|\sqrt{\rho\, C_p}u^n\|^2_{0,\Omega}+\Delta t\sum_{m=0}^{n-1}\|\sqrt{\lambda}\,\nabla u_\theta^{m+1}\|^2_{0,\Omega}+4(1-\theta)\sum_{m=0}^{n-1}\|\sqrt{\rho\, C_p}u^m\|^2_{0,\Omega}\leq(4\theta-3)\|\sqrt{\rho\, C_p}u_0\|^2_{0,\Omega}$$

$$\forall\, 0\leq n\leq N\qquad +K_1\Delta t\sum_{m=0}^{n-1}\left(\|\hat{s}_\theta^{m+1}\|^2_{-1,\Omega}+\|\hat{g}_\theta^{m+1}\|^2_{-\frac{1}{2},\Gamma_2}+\|\hat{h}_\theta^{m+1}\|^2_{-\frac{1}{2},\Gamma_3}\right)$$

éq 3.2-9

or more simply

$$\|\sqrt{\rho\, C_p}u^n\|^2_{0,\Omega}+\Delta t\sum_{m=0}^{n-1}\|\sqrt{\lambda}\,\nabla u_\theta^{m+1}\|^2_{0,\Omega}\leq\|\sqrt{\rho\, C_p}u_0\|^2_{0,\Omega}$$

éq 3.2-10

$$\forall\, 0\leq n\leq N\qquad +K_1\Delta t\sum_{m=0}^{n-1}\left(\|\hat{s}_\theta^{m+1}\|^2_{-1,\Omega}+\|\hat{g}_\theta^{m+1}\|^2_{-\frac{1}{2},\Gamma_2}+\|\hat{h}_\theta^{m+1}\|^2_{-\frac{1}{2},\Gamma_3}\right)$$

Proof:

Obtaining [éq 3.2-9] being already explained, it remains to be shown [éq 3.2-10]. This "coarse" inequality more comes simply owing to the fact that

$$4(1-\theta)\sum_{m=0}^{n-1}\|\sqrt{\rho\, C_p}u^m\|^2_{0,\Omega}\geq 0$$

$$(4\theta-3)\|\sqrt{\rho\, C_p}u_0\|^2_{0,\Omega}\leq\|\sqrt{\rho\, C_p}u_0\|^2_{0,\Omega}$$

éq 3.2-11

Note:

- *One can obviously pass the same remark as [bib6] by noting that the last term of [éq 3.2-9] is a sum of Riemann which tends towards the last term of [éq 3.1-1] when time step tends towards zero. In addition, if one introduces the function (*with $\chi_{[n\Delta t,(n+1)\Delta t]}$ the temporal function characteristic of the interval $[n\Delta t,(n+1)\Delta t]$) $u(t)=u_\theta^{n+1}\chi_{[n\Delta t,(n+1)\Delta t]}(t)$ *closely connected per pieces into [éq 3.1-1], one finds exactly [éq 3.2-9].*
- *As for [éq 3.1-1], by adopting the less restrictive approaches (H4) and (H5), one finds a version "strong" of properties 6 and 7.*

3.3 Error of temporal discretization

the preceding results on the continuous problem and its form semi-discretized in time are re-used jointly to study the controllability of the error of temporal discretization

$$\forall\, 0 \le n \le N \quad e^n := u^n - u(n\Delta t)$$
$$e^0 = 0 \qquad \text{éq 3.3-1}$$

One starts by revealing this error while withdrawing from the equation [éq 3.2-2] the relations

$$\frac{1}{\Delta t}\int_{n\Delta t}^{(n+1)\Delta t} \frac{\partial u(\xi)}{\partial t}\, d\xi = \frac{u((n+1)\Delta t) - u(n\Delta t)}{\Delta t}$$
$$\theta\,\rho\, C_p \frac{\partial u((n+1)\Delta t)}{\partial t} = \theta\, \mathrm{div}(\lambda\, \nabla u((n+1)\Delta t)) + \theta\, \hat{s}((n+1)\Delta t) \qquad \text{éq 3.3-2}$$
$$(1-\theta)\,\rho\, C_p \frac{\partial u(n\Delta t)}{\partial t} = (1-\theta)\, \mathrm{div}(\lambda\, \nabla u(n\Delta t)) + (1-\theta)\, \hat{s}(n\Delta t)$$

is

$$\rho\, C_p \frac{e^{n+1} - e^n}{\Delta t} - \mathrm{div}(\lambda\, \nabla e_\theta^{n+1}) = \frac{1}{\Delta t}\int_{n\Delta t}^{(n+1)\Delta t} \frac{\partial u(\xi)}{\partial t}\, d\xi + \rho\, C_p \left(\frac{\partial u}{\partial t}\right)_\theta \qquad \text{éq 3.3-3}$$

by noting

$$e_\theta^{n+1} := \theta\, e^{n+1} + (1-\theta)\, e^n$$
$$\left(\frac{\partial u}{\partial t}\right)_\theta := \theta\, \frac{\partial u}{\partial t}((n+1)\Delta t) + (1-\theta)\frac{\partial u}{\partial t}(n\Delta t) \qquad \text{éq 3.3-4}$$

From this statement one can describe, via the recourse to the formula of Taylor, the "weak" controllability of the error of temporal discretization. But to be able to use temporal derivatives of the solution one continues needs a minimum of regularity in t, for example by conceding that

$$u \in H^1(0,\tau\,;V) \cap H^2(0,\tau\,;H^{-1}(\Omega)) \qquad \text{éq 3.3-5}$$

Property 8

By supposing that the solution checks the additional assumption of temporal regularity [éq 3.3-5], one has **the "weak" controllability of the error of temporal discretization**

$$\forall\, 0 \le n \le N \quad \|\sqrt{\rho\, C_p}\, e^n\|_{0,\Omega}^2 + \Delta t \sum_{m=0}^{n-1} \|\sqrt{\lambda}\, \nabla e_\theta^{m+1}\|_{0,\Omega}^2 \le$$
$$\frac{K_1 (\Delta t)^3 (\rho\, C_p)^2}{4} \sum_{m=0}^{n-1} \left((1-\theta)\frac{\partial^2 u}{\partial t^2}(m\Delta t) - \theta\, \frac{\partial^2 u}{\partial t^2}((m+1)\Delta t) \right)$$

éq 3.3-6

Proof:

While evaluating [éq 3.3-3] by a formula of Taylor to order 2, one utilizes the derivative second temporal of the solution and one shows that the continuation of error $\left(e^n\right)_{0\leq n\leq N}\in V$ checks a problem similar to [éq 3.2-2] (by supposing that the temporal discretization of the limiting conditions are exact)

$$\left(P_3^{n+1}\right)\begin{cases}\rho C_p\dfrac{e^{n+1}-e^n}{\Delta t}-\operatorname{div}\left(\lambda\,\nabla e_\theta^{n+1}\right)= & \\ \quad\dfrac{\rho C_p\Delta t}{2}\left(\left(1-\theta\right)\dfrac{\partial^2 u}{\partial t^2}\left(n\Delta t\right)-\theta\,\dfrac{\partial^2 u}{\partial t^2}\left(\left(n+1\right)\Delta t\right)\right) & \Omega\quad 0\leq n\leq N-1\\ e^{n+1}=0 & \Gamma_1\quad 0\leq n\leq N-1\\ \lambda\,\dfrac{\partial e^{n+1}}{\partial n}=0 & \Gamma_2\quad 0\leq n\leq N-1\\ \lambda\,\dfrac{\partial e^{n+1}}{\partial n}+h^{n+1}\,e^{n+1}=0 & \Gamma_3\quad 0\leq n\leq N-1\\ e^0\left(.\right)=0 & \Omega\end{cases}$$

éq 3.3-7

One can then apply to him the second result of corollary 7 from where [éq 3.3-6] (one could, of course, just as easily to apply the gross profit of this corollary or that of property 6 from which it rises).

Note:

- *While placing oneself in the particular frame [éq 3.1-9] of the article [bib6] with an implicit scheme* $(\theta=1)$ *and by taking again the equivalent norms [éq 3.1-10] one finds well the inequality (8) pp429. It is enough to make tend* $\Delta t\rightarrow 0$ *and to approximate the integral by the sum of Riemann which constitutes the second member of [éq 3.3-6].*
- *The existence and the unicity of the continuation* (e^n) *rise of course from that otherwise* (u^n) *one can also the redémontrer by applying the theorem of Lax-Milgram to the weak formulation rising from [éq 3.3-7].*

3.4 Total discretization in time and space

One supposes that the field Ω is **polyhedric or not** and that it is discretized spatially by **a regular family** $(T_h)_h$ **of triangulations**. Because of this regularity the finite element method applied to $\left(P_2^{n+1}\right)$ converges when the largeest diameter of the elements K of $(T_h)_h$ tightens towards zero

$$h:=\max_{K\in T_h} h_K\rightarrow 0 \qquad \text{éq 3.4-1}$$

Note:

- *The finite elements* (K, P_K, Σ_K) *are closely connected equivalents with same elements of reference, they check relations of compatibility on their common borders and the forced geometrical [éq 3.4-1] and [éq 3.4-2].*
- *It is pointed out that the diameter of K is reality* $h_K:=\max_{(\mathbf{x},\mathbf{y})\in K^2}|\mathbf{x}-\mathbf{y}|$ *.*

By noting ρ_K the roundness (it is pointed out that the roundness of K is reality $\rho_K := \max\{\text{diamètre des sphères } \subset K\}$) associated with K, the finite elements of $(T_h)_h$ satisfy also the stress

$$\exists \sigma > 0 / \ \frac{h_K}{\rho_K} \leq \sigma \qquad \text{éq 3.4-2}$$

In the usual triplet (K, P_K, Σ_K) one defines polynomial space as being that of the polynomials of total degree lower or equal to k on K

$$P_K := P_k(K) \qquad \text{éq 3.4-3}$$

and spaces it approximation (with the "weak" meaning) associated

$$V_h := \{v_h \in V / \forall K \in T_h \ \ v_{h_{|K}} \in P_k(K)\} \subset V \qquad \text{éq 3.4-4}$$

to conclude, one will note Π_h, the operator of projection which associates with the solution continues its V_h – interpolated

$$\begin{aligned} \Pi_h : V &\rightarrow V_h \\ v &\rightarrow v_h \end{aligned} \qquad \text{éq 3.4-5}$$

Note::

> *With a regular family of triangulations, this operator of interpolation is continuous and it can be written $\Pi_h v = \sum_i v(\mathbf{x_i}) N_i$ by noting $\mathbf{x_i}$ the tops of the mesh and N_i their associated shape function.*

He will be of a very particular importance when it is necessary to describe the increase which will exhume the error indicator.

Note:

> - ***In practice the meshes are often polygonal, the approximation Ω_h of Ω becomes more rudimentary then than in the polyhedric case. To preserve the convergence of the method it is then necessary to resort to isoparametric elements*** *(cf [bib3] pp113-123 or P. GRISVARD. Behavior of the solutions of year elliptic boundary problem in has polygonal gold polyhedral domain. Numerical solution of PDE, ED. Academic Near, 1976).*
> - *The indicator in residue was established in the Code_Aster only for the isoparametric elements (triangle, quadrangle, face, tetrahedron, pentahedron and hexahedron). Moreover, as they are **simplexes** or **parallélotopes** , **the associated triangulation is regular** (cf [bib3] pp108-112).*
> - *For the simplexes the relation [éq 3.4-2] results by the existence of a lower limit on the angles and, for the parallélotopes, in the existence of a higher limit controlling the relationship between the height, the width and the length.*
> - *In the definition [éq 3.4-4] of V_h, they are the intrinsic relations of compatibility to the family of elements which assures us*
>
> $$\forall h, K \ \ v_{h_{|K}} \in P_k(K) \subset H^1(K) \Rightarrow \ v_h \in H^1(\Omega := \cup \bar{K}) \qquad \textit{éq 3.4-6}$$
>
> *In the literature one often prefers the more regular definition to him*
>
> $$V_h^* := V_h \cap C^0(\Omega) \qquad \textbf{\textit{éq 3.4-7}}$$

By regaining the shape semi-discretized $\left(P_2^{n+1}\right)$ with functions tests in V_h one obtains the problem completely discretized in time and space (for h built-in) according to:
One seeks the continuation

$$\left(u_h^n\right)_{0\le n\le N}\in V_h \quad \text{éq 3.4-8}$$

initialized by

$$u_h^0:=\Pi_h u_0 \quad \text{éq 3.4-9}$$

checking the following problem

$$\left(P_2^{h,n+1}\right)\begin{cases}\text{Etant donnés } u_h^n, \hat{s}^n, \hat{s}^{n+1}, \hat{g}^n, \hat{g}^{n+1}, \hat{h}^n, \hat{h}^{n+1}, h^n, h^{n+1}\\ \text{Calculer } u_h^{n+1}\in V_h \text{ tel que}\\ \left(\rho\, C_p u_h^{n+1}, v_h\right)_{0,\Omega}+\Delta t\; a\left(u_{\theta,h}^{n+1}, v_h\right)=\left(\rho\, C_p u_h^n, v_h\right)_{0,\Omega}+\Delta t\left(b_\theta^{n+1}, v_h\right)\quad\left(\forall v_h\in V_h\right)\end{cases}$$

éq 3.4-10

Just as one supposed in the preceding paragraph as **the temporal discretization of the loadings was exact**

$$e_\chi^n:=\Xi^n-\Xi\left(n\nu t\right)=0 \text{ avec } \Xi\in\left\{\hat{s}, \hat{h}, h, \hat{g}\right\} \text{ et } 0\le n\le N \quad \text{(H6)}$$

, one supposes here moreover than their **spatial discretization is also**

$$\forall h\;\; \Xi_h^n:=\Pi_h\Xi^n=\Xi^n \text{ avec } \Xi\in\left\{\hat{s}, \hat{h}, h, \hat{g}\right\} \text{ et } 0\le n\le N \quad \text{(H7)}$$

In ***Code_Aster*****, these assumptions can not be checked** and one will see that they impact **the quality of the indicator in residue** and his relations between equivalence and the exact error (cf [§4.3]). In practice, even if one is obliged to compose with this approximation, it is not truly problematic as long as the loadings "are not kicked up a rumpus too much" in time and space.

By applying the theorem of standard Lax-Milgram following the groundwork developed in the demonstration of theorem 3, one shows the existence and the unicity of the continuation $\left(u_h^n\right)_n$ in the closed sev (it is thus Hilbert, pre-necessary essential for the use of the famous theorem) V_h of Hilbert V. Moreover, while applying second result of corollary 7 (one could, of course, just as easily have applied the gross profit of this corollary or that of property 6 from which it rises), **the "weak" controllability of the completely discretized problem** takes the following shape:

Property 9

By leaning on the triangulation defined previously and by supposing that the assumptions (H_6) and (H_7) are checked, one has increase

$$\begin{aligned}&\|\sqrt{\rho\, C_p}\,u_h^n\|_{0,\Omega}^2+\Delta t\sum_{m=0}^{n-1}\|\sqrt{\lambda}\,\nabla u_{\theta,h}^{m+1}\|_{0,\Omega}^2\le\|\sqrt{\rho C_p}\,\Pi_h u_0\|_{0,\Omega}^2\\ &\forall\, 0\le n\le N \qquad +K_1\Delta t\sum_{m=0}^{n-1}\left(\|\hat{s}_\theta^{m+1}\|_{-1,\Omega}^2+\|\hat{g}_\theta^{m+1}\|_{-\frac{1}{2},\Gamma_2}^2+\|\hat{h}_\theta^{m+1}\|_{-\frac{1}{2},\Gamma_3}^2\right)\end{aligned} \quad \textbf{éq 3.4-11}$$

while noting $u_{\theta,h}^{m+1}:=\theta\, u_h^{m+1}+\left(1-\theta\right)u_h^m$.

Note:

- *While placing oneself in the particular frame [éq 3.1-9] of the article [bib6] with an implicit scheme $(\theta=1)$ and by taking again the equivalent norms [éq 3.1-10] one finds well the inequality (14) pp430.*
- *By adopting the less restrictive approaches (H_4) and (H_5), one finds a version "strong" of this increase utilizing the norm H^1 of the field result.*

Now that we determined the functional frame ensuring us of the existence and the unicity of the continuation discrete solution and to study the evolution of the controllability of the problem during discretizations, we will pool these "ethereal" results to release increase a little where the indicator will intervene.

4 Indicator in pure residue

4.1 Notations

to build the local error indicator one will require **the following notations** :

- All the sides (resp. nodes) of the element K is indicated by $S(K)$ (resp. $N(K)$).
- All the nodes associated with one with its sides F (pertaining to $S(K)$) are noted $N(F)$.

Note:

To make simple, one will indicate under the term "face", with dimensions one of a finite element in 2D or one of his sides in 3D.

- The diameter of the element K (resp. of one of its sides F) (h_K resp is noted. h_F).
- The group of the triangulation (T_h) breaks up in the form

$$T_h := T_{h,\Omega} \cup T_{h,1} \cup T_{h,2} \cup T_{h,3}$$

  into noting ($T_{h,i}$) the group of the finite elements having a face contained in the border Γ_i .
- With same logic, all the sides of the triangulation (T_h) break up in the form

$$S_h := S_{h,\Omega} \cup S_{h,1} \cup S_{h,2} \cup S_{h,3}$$

  with

$$\forall i \in \{1,2,3\} \quad S_{h,i} := \{\partial K / K \in T_h \;\; \partial K \subset \Gamma_i\} = \bigcup_{K \in T_{h,i}} S(K)$$

- In the same way, all the nodes of the triangulation (□ $_H$) breaks up in the form

$$N_h := N_{h,\Omega} \cup N_{h,1} \cup N_{h,2} \cup N_{h,3}$$

- the function "bubble" associated with K (resp. F) (ψ_K resp is noted. ψ_F).

Note:

It is the function of $D(\Omega)$ (together indefinitely differentiable functions and with compact support) resulting from the theorem of truncation on compact: its support is restricted with compact in question (here K or F) and it is worth between 0 and 1 on its interior (with the topological meaning of the term). It is thus null on the border of compact and outside that - Ci.

- One notes P_F the operator of raising on K traces on F, built from an operator of built-in raising on the element of reference.
- The union of the finite elements of the triangulation dividing at least a face with K is noted

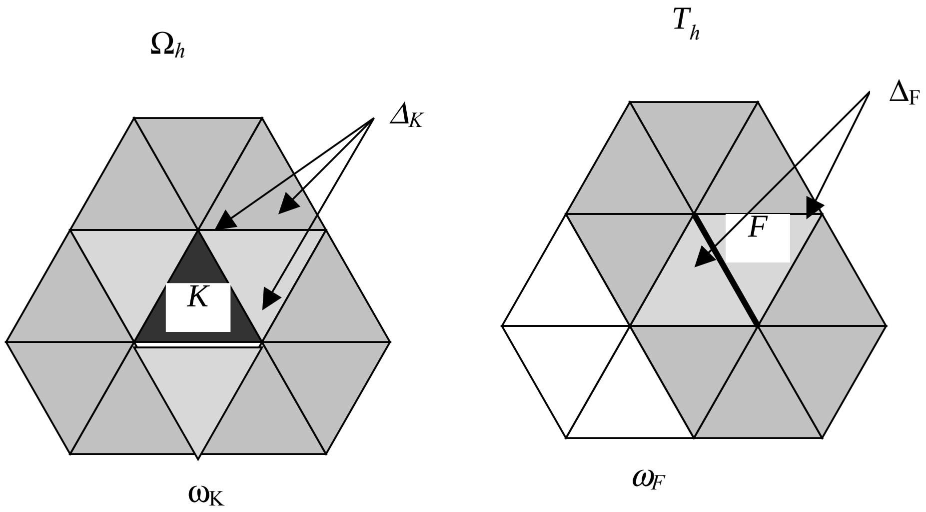
$$\Delta_K := \bigcup_{S(K) \cap S(K') \neq \emptyset} K'$$

- the union of the finite elements triangulation containing *F* in their border is noted

$$\Delta_F := \bigcup_{F \in S(K')} K'$$

- the union of the finite elements triangulation which shares at least a node with K (resp. with F) (

$\omega_K := \bigcup_{N(K) \cap N(K') \neq \emptyset} K'$ resp is noted. $\omega_F := \bigcup_{N(F) \cap N(K') \neq \emptyset} K'$).



Appear 4.1-a: Designation of the types of vicinities for K and F .

4.2 Increase of the total spatial error

We will thus see how computable **to obtain a local indicator of error starting from the data and discrete solution** $\left(u_h^n\right)_n$. As the discretized workspace is included in continuous space $V_h \subset V$, one can re-use [éq 3.2-3] with v_h. While withdrawing to him [éq 3.4-10] it occurs (with n and h built-in and while supposing (H_6) and (H_7))

$$\left(\rho\, C_p\left(u^{n+1} - u_h^{n+1}\right), v_h\right)_{0,\Omega} + \Delta t\; a\left(\left(u_\theta^{n+1} - u_{\theta,h}^{n+1}\right), v_h\right) = \left(\rho\, C_p\left(u^n - u_h^n\right), v_h\right)_{0,\Omega} \quad \left(\forall v_h \in V_h\right) \qquad \text{éq 4.2-1}$$

Note:

- *This relation V states the orthogonal character of the spatial error with respect to the elements of $_h$.*
- *It supposes in addition which the discretization is " **consistent** " i.e. there are no **additional errors introduced by the numerical integration of the integrals** . In practice it is of course not the case!*

Let us consider the following linear form

$$A(v):=\left(\rho\, C_p\left(u^{n+1}-u_h^{n+1}\right),v\right)_{0,\Omega}+\Delta t\; a\left(u_\theta^{n+1}-u_{\theta,h}^{n+1},v\right)\quad(\forall v\in V) \qquad \text{éq 4.2-2}$$

which will be used to us during as main idea this demonstration. By packing it via [éq 4.2-1], one obtains

$$\begin{aligned}A(v)&=\left(\rho\, C_p\left(u^n-u_h^n\right),v\right)_{0,\Omega}+\left(\rho\, C_p\left(u^n-u_h^n\right),\left(v_h-v\right)\right)_{0,\Omega}+\\ (\forall v\in V)\qquad&\left(\rho\, C_p\left(u^{n+1}-u_h^{n+1}\right),\left(v-v_h\right)\right)_{0,\Omega}+\Delta t\; a\left(\left(u_\theta^{n+1}-u_{\theta,h}^{n+1}\right),v-v_h\right)\end{aligned} \qquad \text{éq 4.2-3}$$

While taking [éq 3.2-3] after having replaced v_h by $v-v_h\in V$, one can build

$$\begin{aligned}&\left(\rho\, C_p\left(u^{n+1}-u_h^{n+1}-u^n+u_h^n\right),v-v_h\right)_{0,\Omega}+\Delta t\; a\left(u_\theta^{n+1}-u_{\theta,h}^{n+1},v-v_h\right)_{0,\Omega}=\\ &(\forall v\in V)\qquad \Delta t\left(b_\theta^{n+1},v-v_h\right)_{0,\Omega}-\Delta t\; a\left(u_{\theta,h}^{n+1},v-v_h\right)-\left(\rho\, C_p\left(u_h^{n+1}-u_h^n\right),v-v_h\right)_{0,\Omega}\end{aligned} \qquad \text{éq 4.2-4}$$

Then $A(v)$ becomes

$$\begin{aligned}A(v)&=\left(\rho\, C_p\left(u^n-u_h^n\right),v\right)_{0,\Omega}+\Delta t\left(b_\theta^{n+1},v-v_h\right)-\\ (\forall v\in V)\qquad&\left(\rho\, C_p\left(u_h^{n+1}-u_h^n\right),v-v_h\right)_{0,\Omega}-\Delta t\; a\left(u_{\theta,h}^{n+1},v-v_h\right)\end{aligned} \qquad \text{éq 4.2-5}$$

Then one breaks up the last three terms on each element K of the triangulation and one applies, with the last, the formula of Green

$$\begin{aligned}A(v)=&\left(\rho\, C_p\left(u^n-u_h^n\right),v\right)_{0,\Omega}+\Delta t\sum_{K\in T_h}\int_K\left(\hat{s}_\theta^{n+1}-\rho\, C_p\frac{u_h^{n+1}-u_h^n}{\Delta t}+\mathrm{div}\left(\lambda\,\nabla u_{\theta,h}^{n+1}\right)\right)\left(v-v_h\right)dx\\ &-\frac{\Delta t}{2}\sum_{F\in S_{h,\Omega}}\int_F\left[\lambda\frac{\partial u_{h,\theta}^{n+1}}{\partial n}\right]\left(v-v_h\right)d\sigma\\ \forall\; v-v_h\in V\qquad &+\Delta t\sum_{F\in S_{h,2}}\int_F\left(\hat{g}_\theta^{n+1}-\lambda\frac{\partial u_{h,\theta}^{n+1}}{\partial n}\right)\left(v-v_h\right)d\sigma\\ &+\Delta t\sum_{F\in S_{h,3}}\int_F\left(\hat{h}_\theta^{n+1}-\lambda\frac{\partial u_{h,\theta}^{n+1}}{\partial n}-\left(hu_h\right)_\theta^{n+1}\right)\left(v-v_h\right)d\sigma\end{aligned}$$

éq 4.2-6

Note::

- *One allowed oneself to replace the hooks of duality of [éq 3.2-4] by integrals and one can apply the formula of Green because on compact K the assumptions (H_4) and (H_5) are checked (while replacing Ω by K and Γ_i by $\partial K\cap\Gamma_i$). There is thus*

$$v-v_h\in H^1(K),\; u_h\in H^2(K),\; \hat{s}\in L^2(K),\; \hat{g}\in L^2\left(\partial K\cap\Gamma_2\right)\;\mathbf{et}\;\hat{h}\in L^2\left(\partial K\cap\Gamma_3\right) \qquad \text{éq 4.2-7}$$

Point out some properties of the operator Π_h of projection L^2- local introduces by P. CLEMENT [bib8]

$$\Pi_h : V \subset L^2(\Omega) \to V_h \qquad v \to v_h \qquad \text{éq 4.2-8}$$

It checks in particular increases of errors of projection

$$\begin{aligned} &\forall v \in V \quad && \|v - \Pi_h v\|_{0,K} := \|v - v_h\|_{0,K} \leq C_4\, h_K \|v\|_{1,\omega_K} \\ &\forall K \in T_h,\ \forall F \in S(K) \quad && \|v - \Pi_h v\|_{0,F} := \|v - v_h\|_{0,F} \leq C_5 \sqrt{h_F} \|v\|_{1,\omega_F} \end{aligned} \qquad \text{éq 4.2-9}$$

where the constants C_4 and C_5 depend on the smallest angles of the triangulation. By taking this operator of spatial projection and by applying the inequality of Cauchy-Schwartz to [éq 4.2-6] it thus occurs:

$$\begin{aligned} A(v) - \left(\rho\, C_p(u^n - u_h^n), v\right)_{0,\Omega} \leq & \Delta t C_4 \sum_{K \in T_h} h_K \left\| \hat{s}_\theta^{n+1} - \rho\, C_p \frac{u_h^{n+1} - u_h^n}{\Delta t} + \mathrm{div}\left(\lambda \nabla u_{\theta,h}^{n+1}\right) \right\|_{0,K} \|v\|_{1,\omega_K} \\ & + \frac{\Delta t}{2} C_5 \sum_{F \in S_{h,\Omega}} \sqrt{h_F} \left\| \left[\lambda \frac{\partial u_{h,\theta}^{n+1}}{\partial n} \right] \right\|_{0,F} \|v\|_{1,\omega_F} \\ \forall v \in V \quad & + \Delta t C_5 \sum_{F \in S_{h,2}} \sqrt{h_F} \left\| \hat{g}_\theta^{n+1} - \lambda \frac{\partial u_{h,\theta}^{n+1}}{\partial n} \right\|_{0,F} \|v\|_{1,\omega_F} \\ & + \Delta t C_5 \sum_{F \in S_{h,3}} \sqrt{h_F} \left\| \hat{h}_\theta^{n+1} - \lambda \frac{\partial u_{h,\theta}^{n+1}}{\partial n} - (h u_h)_\theta^{n+1} \right\|_{0,F} \|v\|_{1,\omega_F} \end{aligned}$$

éq 4.2-10

This inequality clearly lets show through a possible formulation of the indicator in pure residue:

Definition 10

In the frame of the linear operator of transient thermal of *the Code_Aster*, **the continuation** $\left(\eta^n(K)\right)_{0 \leq n \leq N}^{K \in T_h}$ **of theoretical local indicators** can be written in the form

$$\begin{aligned} \eta^{n+1}(K) := & h_K \left\| \hat{s}_\theta^{n+1} - \rho\, C_p \frac{u_h^{n+1} - u_h^n}{\Delta t} + div\left(\lambda \nabla u_{h,\theta}^{n+1}\right) \right\|_{0,K} + \frac{1}{2} \sum_{F \in S_\Omega(K)} \sqrt{h_F} \left\| \left[\lambda \frac{\partial u_{h,\theta}^{n+1}}{\partial n} \right] \right\|_{0,F} + \\ & \sum_{F \in S_2(K)} \sqrt{h_F} \left\| \hat{g}_\theta^{n+1} - \lambda \frac{\partial u_{h,\theta}^{n+1}}{\partial n} \right\|_{0,F} + \sum_{F \in S_3(K)} \sqrt{h_F} \left\| \hat{h}_\theta^{n+1} - \lambda \frac{\partial u_{h,\theta}^{n+1}}{\partial n} - (h u_h)_\theta^{n+1} \right\|_{0,F} \end{aligned}$$

éq 4.2-11

It is initialized by

$$\begin{aligned} \eta^0(K) := & h_K \left\| \hat{s}^0 + div\left(\lambda\ \nabla u_h^0\right) \right\|_{0,K} + \frac{1}{2} \sum_{F \in S_\Omega(K)} \sqrt{h_F} \left\| \left[\lambda \frac{\partial u_h^0}{\partial n} \right] \right\|_{0,F} + \\ & \sum_{F \in S_2(K)} \sqrt{h_F} \left\| \hat{g}^0 - \lambda \frac{\partial u_h^0}{\partial n} \right\|_{0,F} + \sum_{F \in S_3(K)} \sqrt{h_F} \left\| \hat{h}^0 - \lambda \frac{\partial u_h^0}{\partial n} - h^0 u_h^0 \right\|_{0,F} \end{aligned}$$

éq the 4.2-12

continuation $\left(\eta^n(\Omega)\right)_{0 \leq n \leq N}$ **of theoretical total indicators** is defined as being

$$\forall\, 0 \leq n \leq N \quad \eta^n(\Omega) := \left(\sum_{K \in T_h} \eta^n(K)^2 \right)^{\frac{1}{2}}$$

éq 4.2-13

Note:

- *While placing oneself in the particular frame [éq 3.1-9] of the article [bib6] with an implicit scheme ($\theta=1$) one finds well the definition (24) pp432.*
- *Whatever the initialization retained for thermal computation, one starts the temporal continuation of cartography of error indicators as if one were in hover: no the term in temporal finite difference, $n+1=0$ (in the Code_Aster a transitory field of temperature is initialized with index 0) and $\theta=1$.*
- *It should be stressed that this indicator is composed of four terms:* ***the term principal*** *, called* ***term of voluminal error*** *, controlling the good checking of the equation of heat, to which three* ***secondary terms are added*** *checking the good behavior of the limiting conditions (* ***terms of jump, flux and exchange*** *). In `2D-PLAN` or `3D` (resp. in `2D-AXI`), if the unit of the geometry is the meter, the unit of the first is W.m (resp .) $W.m.rad^{-1}$ and that of the other terms is it (resp $W.m^{\frac{1}{2}}$.). $W.m^{\frac{1}{2}}.rad^{-1}$ Attention thus with the units taken into account for the geometry when one is interested in the gross amount of the indicator and not in his relative value! While*
- *taking as a starting point the increases developed by R. VERFURTH (cf [bib7] pp84-94) for the Poisson's equation one could have taken as indicator the root of the sum of the squares of the terms quoted above. éq*

$$\tilde{\eta}^{n+1}(K):=\left\{\begin{array}{l} h_K^2\left\|\hat{s}_\theta^{n+1}-\rho C_p\dfrac{u_h^{n+1}-u_h^n}{\Delta t}+div\left(\lambda\ \nabla u_{h,\theta}^{n+1}\right)\right\|_{0,K}^2+\dfrac{1}{2}\sum_{F\in S_\Omega(K)}h_F\left\|\left[\lambda\ \dfrac{\partial u_{h,\theta}^{n+1}}{\partial n}\right]\right\|_{0,F}^2+\\ \sum_{F\in S_2(K)}h_F\left\|\hat{g}_\theta^{n+1}-\lambda\dfrac{\partial u_{h,\theta}^{n+1}}{\partial n}\right\|_{0,F}^2+\sum_{F\in S_3(K)}h_F\left\|\hat{h}_\theta^{n+1}-\lambda\dfrac{\partial u_{h,\theta}^{n+1}}{\partial n}-\left(hu_h\right)_\theta^{n+1}\right\|_{0,F}^2\end{array}\right\}^{\frac{1}{2}}$$

4.2-14 This

definition leads to an increase of the total error which is more optimal than that which will be released thereafter. But we preferred, to remain homogeneous B with the writings of. METIVET [bib6] and with the estimator in linear mechanics already installation in the code, to hold us with the version of definition 10. While leaning on [éq 4.2-10] and definition 10 one can then exhume the increase of the following total error: Property

11 Pennies

the assumptions of properties 6, of () and H_6 by means of definition 10, one has, at **the total level,** the " **weak" increase of the error (with**) $K_2\left(\|\lambda\|_{\infty,\Omega}, P(\Omega), C_4, C_5\right)>0$ via **the history of the indicators éq**

$$\begin{aligned}&\|\sqrt{\rho\, C_p}\left(u^n-u_h^n\right)\|_{0,\Omega}^2+4(1-\theta)\sum_{m=0}^{n-1}\|\sqrt{\rho\, C_p}\left(u^m-u_h^m\right)\|_{0,\Omega}^2+\Delta t\sum_{m=0}^{n-1}\|\sqrt{\lambda}\,\nabla\left(u_\theta^{m+1}-u_{\theta,h}^{m+1}\right)\|_{0,\Omega}^2\\ &\forall\, 0\le n\le N \qquad \le(4\theta-3)\|\sqrt{\rho\, C_p}\left(u_0-u_h^0\right)\|_{0,\Omega}^2+K_2\Delta t\sum_{m=0}^{n}\left(\eta^m(\Omega)\right)^2\end{aligned}$$

4.2-15 or

more simply éq

$$\begin{aligned}&\|\sqrt{\rho\, C_p}\left(u^n-u_h^n\right)\|_{0,\Omega}^2+\Delta t\sum_{m=0}^{n-1}\|\sqrt{\lambda}\,\nabla\left(u_\theta^{m+1}-u_{\theta,h}^{m+1}\right)\|_{0,\Omega}^2\\ &\forall\, 0\le n\le N \qquad \le\|\sqrt{\rho\, C_p}\left(u_0-u_h^0\right)\|_{0,\Omega}^2+K_2\Delta t\sum_{m=0}^{n}\left(\eta^m(\Omega)\right)^2\end{aligned}$$

4.2-16 Proof

:

The estimates [éq 4.2-15] [éq 4.2-16] are obtained by reiterating the same process as for properties 5,6 and 7. One takes in [éq 4.2-10] the particular function test éq

$$v:=u_\theta^{n+1}-u_{\theta,h}^{n+1}$$

4.2-17 One

ousts the term of exchange by the usual argument éq

$$\int_{\Gamma_3}\left(h\left(u-u_h\right)\right)_\theta^{n+1}\left(u_\theta^{n+1}-u_{\theta,h}^{n+1}\right)dx\;>0$$

4.2-18 It

is necessary to apply the trick [éq 3.1-4] to the cross term and $(2\theta-1)\int_\Omega \rho\, C_p\left(u^{n+1}-u_h^{n+1}\right)\left(u^n-u_h^n\right)dx$ the product utilizing the indicator. One has then to find the parameters and α checking β a system of the type [éq 3.2-8] éq

$$\begin{aligned}&|2-\frac{P^2(\Omega)}{\|\lambda\|_{\infty,\Omega}}\alpha^2=1\\ &|2\theta-\beta^2|1-2\theta|=1\end{aligned}$$

4.2-19 which

admits solution only if the diagram is unconditionally stable (). $\theta\ge\frac{1}{2}$ From where increase [éq 4.2-15] [éq 4.2-16] by taking éq

$$K_2=\frac{P^2(\Omega)}{\|\lambda\|_{\infty,\Omega}}\max\left(C_4^2, C_5^2\right)$$

4.2-20

the inequality [éq 4.2-16] more "coarse" results from the same sales leaflet as for corollary 7. Note:

While

- *placing oneself in the particular frame [éq 3.1-9] of the article [bib6] with an implicit scheme ()* $\theta=1$ *one finds well the inequality (25) pp432 (with).* $c=max(1,K2)$ *By*
- *adopting the less restrictive approaches ()* H_4 *and (),* H_5 *one finds a version "strong" of this property. This*
- *property can be shown more quickly while noticing than the inequation [éq 4.2-10] is similar to the equation of the problem semi-discretized in time [éq 3.2-3]: except for the inequality, while changing by* u *and* $u-u_h$ *by taking as term* $\left(b_\theta^n, v\right)$ *the second member of [éq 4.2-10]. One can then directly apply the corollary 7 to him who is during required estimate! Of*
- ***[éq 4.2-15] [éq 4.2-16] it appears that, at one time given, the error on the approximation of the condition of Cauchy and the history of the total indicators intervene on the total quality of the solution obtained. One will be able to thus minimize overall the error of approximation due to the finite elements in the course of time while re-meshing "advisedly", via the continuation of indicators, the structure. Because, in practice, one realizes that the refinement of meshes makes it possible to decrease their error and thus cause a drop in the temporal sum of the indicators. The total error will butt (and it is moral) against the value bottom of the error of approximation of the initial condition (which will tend it-also to drop of course!). The indicator "over-estimates" the spatial error overall. With***
- *the other alternative of indicator [éq 4.2-14] one finds the same type of increase. However the constant one changes* K_2 *. It is multiplied by the constant checking* C_6 *(cf [bib7] pp90) éq*

$$\sum_{K\in T_h}\|v\|_{1,\omega_K}^2+\sum_{F\in S_h}\|v\|_{1,\omega_F}^2\le C_6\|v\|_{1,\Omega}^2 \quad 4.2\text{-}21 \text{ éq}$$

$$\tilde{K}_2:=C_6K_2 \quad 4.2\text{-}22 \text{ According to}$$

the definitions [éq 2-1-8], [éq 2-1-10] with [éq 2-1-13] if the taking into account of the limiting conditions of Dirichlet (generalized or not), via the ddls of Lagrange, is exact (what is the case in Code_Aster *) (*H

$$\forall h \quad Rf_h^n:=\Pi_h Rf^n=Rf^n=Rf(n\Delta t) \quad 0\le n\le N \qquad 8)$$

the preceding property produces the following corollary then: Corollary

11bis Under

the assumptions of property 11 while supposing (H8), one by means of has the increase **of the total spatial error expressed in temperature**

$$\|\sqrt{\rho\,C_p}\left(T^n-T_h^n\right)\|_{0,\Omega}^2+\Delta t\sum_{m=0}^{n\text{-}1}\|\sqrt{\lambda}\,\nabla\left(T_\theta^{m+1}-T_{\theta,h}^{m+1}\right)\|_{0,\Omega}^2$$

$$\forall 0\le n\le N \qquad \le\|\sqrt{\rho\,C_p}\left(T_0-T_h^0\right)\|_{0,\Omega}^2+K_2\Delta t\sum_{m=0}^{n}\left(\eta^m(\Omega)\right)^2 \qquad \textbf{éq 4.2-23}$$

definition 10 of the indicator also expressed in temperature éq

$$u\Rightarrow T,\ \hat{s}\Rightarrow s,\ \hat{g}\Rightarrow g \text{ et } \hat{h}\Rightarrow hT_{ext} \qquad \textbf{4.2-24 Various}$$

4.3 types of possible indicators By

extrapolating a remark of [bib5] (pp194-195) it appears that increases of property 11 can be exhumed while taking as indicator éq

$$\eta_{p,t}^{n+1}(K):=h_K^r\|\hat{s}_\theta^{n+1}-\rho\, C_p\frac{u_h^{n+1}-u_h^n}{\Delta t}+div\left(\lambda\ \nabla u_{h,\theta}^{n+1}\right)\|_{L^p(K)}+\frac{1}{2}\sum_{F\in S_\Omega(K)}h_F^s\|\left[\lambda\frac{\partial u_{h,\theta}^{n+1}}{\partial n}\right]\|_{L^t(K)}+$$

4.3-1 where

$$\sum_{F\in S_2(K)}h_F^s\|\hat{g}_\theta^{n+1}-\lambda\frac{\partial u_{h,\theta}^{n+1}}{\partial n}\|_{L^t(K)}+\sum_{F\in S_3(K)}h_F^s\|\hat{h}_\theta^{n+1}-\lambda\frac{\partial u_{h,\theta}^{n+1}}{\partial n}-\left(hu_h\right)_\theta^{n+1}\|_{L^t(K)}$$

the constants and r are worth s éq

$$t\geq 1,\ \ p>1\ (\text{2D}\ \ q=2)\ \text{ou}\ \ p\geq\frac{6}{5}\ (\text{3D}\ \ q=3)$$

$$\left|\begin{array}{l} r(q,p):=q+1-\frac{q}{2}-\frac{q}{p} \\ s(q,t):=q-\frac{1}{2}-\frac{q-1}{2}-\frac{q-1}{t}\end{array}\right.$$

4.3-2 Note:

: Just

> *to introduce this generic shape of indicators, one passes from the notation hilbertienne of the norms of spaces to the notation of Lebesgue It*

is parameterized by the types of norms voluminal and surface which intervene for its obtaining. Contrary to indicator that we chose (who $\eta_{2,2}^{n+1}(K)$ corresponds to), $p=t=2$ some use the voluminal norm () L^1 or $p=t=2$ on the contrary the infinite norm. This

last formulation, just like its simplified form of definition 10 (or [éq 4.2-14]), constitutes **an error indicator well a posteriori because** its computation requires only the knowledge of the materials, the loadings, the geometrical data, and θ the approximate field solution of u_h the accused thermal problem. However **the exact estimate of the indicator is not always possible when one has complicated loadings. Two approaches are then possible: Either**

- one approximates **the integrals which** return in the composition of definition 10 by a formula **of squaring. Either**
- one **approximates the loadings by** a linear combination of simpler functions which will be able to allow an exact integration. Generally one uses same architecture as that which was installation for the finite elements modelling the field of temperature. Note:

In

> - *both cases the loadings are "prisoners of the vision finite elements" chosen to model the field solution. These*

two strategies are equivalent and in Code_Aster *it* is the first which was retained: the voluminal integral is calculated by a formula of Gauss, those surface by a formula of Newton-Dimensions.
Both introduce a skew **into the computation of the estimator who** can be represented by introducing the approximate versions of the loadings and the source (in the initial problem in and T in the problem transformed into) u éq

$$s_{\theta,h}^{n+1},\ g_{\theta,h}^{n+1},\ T_{ext,\theta,h}^{n+1}\ \text{et}\ \ h_{\theta,h}^{n+1}$$ 4.3-3 éq

$$\hat{s}_{\theta,h}^{n+1},\ \hat{g}_{\theta,h}^{n+1},\ \hat{h}_{\theta,h}^{n+1}\ \text{et}\ \ h_{\theta,h}^{n+1}$$ 4.3-4 into

spaces of voluminal approximation (for the source) and surface (for the loadings) éq

$$X_h(\Omega):=\left\{v_h\in L^2(\Omega)/\forall K\in T_h\ \ v_{h_{|K}}\in P_{l_1}(K)\right\}$$
$$X_h(\Gamma_i):=\left\{v_h\in L^2(\Gamma_i)/\forall F\in S_{h,i}\ \ v_{h_{|F\cap\Gamma_i}}\in P_{l_i}(F\cap\Gamma_i)\right\}$$ 4.3-5 In fact

, one introduces two **types of numerical errors during the computation of the indicator:** that inherent in the formulas **of squaring (for** polynomial loadings of a high nature) and that due at the end **voluminal. Indeed**, this last requires a double derivative which one carries out in three stages because in Code_Aster *one* does not recommend the use of second derivative of the shape functions. Note:

They

> *were recently introduced to treat derivative of rate of energy restitution (cf [R7.02.01 § Annexe 1]). On the one hand*

, one calculates (in the thermal operator) heat flux at the points of gauss, then one before you calculate extrapolates the values with the nodes corresponding by lissage local (cf [R3.06.03] `CALC_CHAMP` with `THERMAL = ' FLUX_ELNO')` the divergence of the vectors flux to Gauss points. With of the finite elements quadratic the intermediate operation is only approximate (one assigns like value to the median nodes the half the sum of their values to the extreme nodes). However numerical tests (restricted) showed that, even in, P_2 this approach does not provide results very different from those obtained by a direct computation via good second derivative. Note:

- *The indices* l_1 , l_2 *of* l_3 *these polynomial spaces can be unspecified and different from that of the approximate solution: .* k *However, to prevent that these terms do not become prevalent (it* ***is a question of estimating the error on the solution rather only that on the modelization of the loadings)*** *one will tend to take.* $l_i\geq k-2\ \ (i=1,2,3)$

Definition 10 and the weak estimate 11 partner are rewritten then in the following form. This new definition, is $\eta_R^{n+1}(K)$ subscripted by one (one R takes again in that the usual notations of [bib6] and [bib7]) (for "reality") in order to notify well that it corresponds better to the values which are calculated indeed in the code. Definition

12 In

the frame of the linear operator of transient thermal of the Code_Aster , the continuation **of** $\left(\eta_R^n(K)\right)_{0\leq n\leq N}^{K\in T_h}$ **real local indicators can** be written in the form éq

$$\eta_R^{n+1}(K):=h_K\|\hat{s}_{\theta,h}^{n+1}-\rho\, C_p\frac{u_h^{n+1}-u_h^n}{\Delta t}+div\left(\lambda\ \nabla u_{h,\theta}^{n+1}\right)\|_{0,K}+\frac{1}{2}\sum_{F\in S_\Omega(K)}\sqrt{h_F}\|\left[\lambda\frac{\partial u_{h,\theta}^{n+1}}{\partial n}\right]\|_{0,F}+\sum_{F\in S_2(K)}\sqrt{h_F}\|\hat{g}_{\theta,h}^{n+1}-\lambda\frac{\partial u_{h,\theta}^{n+1}}{\partial n}\|_{0,F}+\sum_{F\in S_3(K)}\sqrt{h_F}\|\hat{h}_{\theta,h}^{n+1}-\lambda\frac{\partial u_{h,\theta}^{n+1}}{\partial n}-\left(h_h u_h\right)_\theta^{n+1}\|_{0,F}$$

4.3-6 It

is initialized by éq

$$\eta_R^0(K):=h_K\|\hat{s}_h^0+div\left(\lambda\ \nabla u_h^0\right)\|_{0,K}+\frac{1}{2}\sum_{F\in S_\Omega(K)}\sqrt{h_F}\|\left[\lambda\frac{\partial u_h^0}{\partial n}\right]\|_{0,F}+\sum_{F\in S_2(K)}\sqrt{h_F}\|\hat{g}_h^0-\lambda\frac{\partial u_h^0}{\partial n}\|_{0,F}+\sum_{F\in S_3(K)}\sqrt{h_F}\|\hat{h}_h^0-\lambda\frac{\partial u_h^0}{\partial n}-h_h^0u_h^0\|_{0,F}$$

4.3-7

the continuation **of** $\left(\eta^n(\Omega)\right)_{0\leq n\leq N}$ **real total indicators is** defined as being éq

$$\forall\, 0\leq n\leq N\quad \eta_R^n(\Omega):=\left(\sum_{K\in T_h}\eta_R^n(K)^2\right)^{\frac{1}{2}}$$

4.3-8 Note:

: One

- *can pass the same remarks as for his "theoretical" alter ego. They is also declined according to the formulations [éq 4.2-14] and* $\tilde{\eta}_R^n(K)$ *[éq 4.3-1], [éq 4.3-2].* $\eta_{R,p,t}^n(K)$ *By*

leaning on the results of property 11, definition 12 and the triangular inequality one can then exhume the increase of the following real total error (one began again that the simplified version): Property

13 Pennies

the assumptions of properties 6, of (H6) and by means of definition 12, one has, at the total level, the “ **weak” increase of the error (with**) $K_2\left(\|\lambda\|_{\infty,\Omega}, P(\Omega), C_4, C_5\right)>0$ via **the history of the real indicators éq**

$$\|\sqrt{\rho C_p}\left(u^n - u_h^n\right)\|_{0,\Omega}^2 + \Delta t \sum_{m=0}^{n-1} \|\sqrt{\lambda}\,\nabla\left(u_\theta^{m+1} - u_{\theta,h}^{m+1}\right)\|_{0,\Omega}^2$$

$$\forall\, 0\leq n\leq N \quad \leq \|\sqrt{\rho C_p}\left(u_0 - u_h^0\right)\|_{0,\Omega}^2 + K_2 \Delta t \sum_{K\in T_h}\left(\eta_R^0(K)\right)^2 + \sum_{m=0}^{n-1}\left\{\left(\eta_R^{m+1}(K)\right)^2 + h_K^2 \|\hat{s}_{\theta,h}^{m+1} - \hat{s}_\theta^{m+1}\|_{0,K}^2\right\} +$$

$$K_2 \Delta t \sum_{K\in T_h}\sum_{m=0}^{n-1}\left\{\sum_{F\in S_2(K)} h_F \|\hat{g}_{\theta,h}^{m+1} - \hat{g}_\theta^{m+1}\|_{0,F}^2 + \sum_{F\in S_3(K)} h_F \|\hat{h}_{\theta,h}^{m+1} - \hat{h}_\theta^{m+1} - \left(h_h u_h\right)_\theta^{m+1} + \left(h u_h\right)_\theta^{m+1}\|_{0,F}^2\right\}$$

4.3-9 Pennies

(H8), one by means of has the same statement in **temperature**

$$\|\sqrt{\rho C_p}\left(T^n - T_h^n\right)\|_{0,\Omega}^2 + \Delta t \sum_{m=0}^{n-1} \|\sqrt{\lambda}\,\nabla\left(T_\theta^{m+1} - T_{\theta,h}^{m+1}\right)\|_{0,\Omega}^2$$

$$\forall\, 0\leq n\leq N \quad \leq \|\sqrt{\rho C_p}\left(T_0 - T_h^0\right)\|_{0,\Omega}^2 + K_2 \Delta t \sum_{K\in T_h}\left(\eta_R^0(K)\right)^2 + \sum_{m=0}^{n-1}\left\{\left(\eta_R^{m+1}(K)\right)^2 + h_K^2 \|s_{\theta,h}^{m+1} - s_\theta^{m+1}\|_{0,K}^2\right\} +$$

$$K_2 \Delta t \sum_{K\in T_h}\sum_{m=0}^{n-1}\left\{\sum_{F\in S_2(K)} h_F \|g_{\theta,h}^{m+1} - g_\theta^{m+1}\|_{0,F}^2 + \sum_{F\in S_3(K)} h_F \|\left(h_h\left(T_{ext,h} - T_h\right)\right)_\theta^{m+1} - \left(h\left(T_{ext} - T_h\right)\right)_\theta^{m+1}\|_{0,F}^2\right\}$$

éq 4.3-10

definition 12 of **the indicator also expressed in temperature éq**

$$u\Rightarrow T,\ \hat{s}\Rightarrow s,\ \hat{g}\Rightarrow g \text{ et } \hat{h}\Rightarrow hT_{ext}$$

4.3-11 Note:

: As

- *for the theoretical value there is a morals with the history because, when one will refine, the total error will butt against the value bottom due to the approximations of the initial condition, the limiting conditions and the source. One cannot get results of better quality that the data input of the problem! Decrease*

4.4 of the local spatial error Before

exhuming the decrease of the spatial error, one will have to introduce some complementary results: Lemma

14 One

shows that there exist constants strictly positive Ci (i =6… 11) checking éq

$$\forall v \in P_{\sup\{k,l_1,l_2,l_3\}}(K) \quad C_6 \|\psi_K v\|_{0,K} \leq \|v\|_{0,K} \leq C_7 \|\psi_{K^{\frac{1}{2}}} v\|_{0,K}$$

$$\|\nabla \psi_K v\|_{0,K} \leq C_8 h_K^{-1} \|\psi_K v\|_{0,K}$$

$$\forall v \in P_{\sup\{k,l_1,l_2,l_3\}}(F) \quad C_9 h_F^{-\frac{1}{2}} \|\psi_K P_F v\|_{0,\Delta_F} \leq \|v\|_{0,F} \leq C_{10} \|\psi_{F^{\frac{1}{2}}} v\|_{0,F}$$

$$\|\nabla \psi_K v\|_{0,\Delta_F} \leq C_{11} h_F^{-1} \|\psi_K v\|_{0,\Delta_F}$$

4.4-1 Proof

: One

passes to the element of reference then one uses the fact that the norms are equivalent on polynomial spaces considered, because they are of finished size (cf [bib5] pp196-98, [bib7] [§1]). These

preliminary results are crucial to determine a decrease of the site error by the real indicator. But one will see that one will be able to obtain only one opposite room of [éq 4.3-9], [éq 4.3-10]. Property

15 Pennies

the assumptions of property 6, () and H_6 by leaning on definition 12 and the lemma 14, one has, at **the local level, the "weak" decrease of the error (with**) $K_3(C_i, i=6\cdots 11)>0$ via **the real indicator éq**

$$\eta_R^{n+1}(K)\le K_3\left\{\begin{array}{r} h_K\|\sqrt{\rho\, C_p}\dfrac{u^{n+1}-u_h^{n+1}-u^n-u_h^n}{\Delta t}\|_{0,\Delta_K}+\|\sqrt{\lambda}\,\nabla\left(u_\theta^{n+1}-u_{\theta,h}^{n+1}\right)\|_{0,\Delta_K}+\\ h_K\|\hat{s}_\theta^{n+1}-\hat{s}_{\theta,h}^{n+1}\|_{0,\Delta_K}+h_F^{\frac{1}{2}}\|\hat{g}_\theta^{n+1}-\hat{g}_{\theta,h}^{n+1}\|_{0,\Delta_K\cap\Gamma_2}+\\ h_F^{\frac{1}{2}}\|\hat{h}_\theta^{n+1}-\hat{h}_{\theta,h}^{n+1}-\left(hu_h\right)_\theta^{n+1}+\left(h_h u_h\right)_\theta^{n+1}\|_{0,\Delta_K\cap\Gamma_3}\end{array}\right\}$$

$$\forall\, 0\le n\le N-1$$

4.4-2 Pennies

(H8), one by means of has the same statement in **temperature**

$$\eta_R^{n+1}(K)\le K_3\left\{\begin{array}{r} h_K\|\sqrt{\rho\, C_p}\dfrac{T^{n+1}-T_h^{n+1}-T^n-T_h^n}{\Delta t}\|_{0,\Delta_K}+\|\sqrt{\lambda}\,\nabla\left(T_\theta^{n+1}-T_{\theta,h}^{n+1}\right)\|_{0,\Delta_K}+\\ h_K\|s_\theta^{n+1}-s_{\theta,h}^{n+1}\|_{0,\Delta_K}+h_F^{\frac{1}{2}}\|g_\theta^{n+1}-g_{\theta,h}^{n+1}\|_{0,\Delta_K\cap\Gamma_2}+\\ h_F^{\frac{1}{2}}\|h_\theta^{n+1}\left(T_{ext,\theta}^{n+1}-T_\theta^{n+1}\right)-h_{\theta,h}^{n+1}\left(T_{ext,\theta,h}^{n+1}-T_{\theta,h}^{n+1}\right)\|_{0,\Delta_K\cap\Gamma_3}\end{array}\right\}$$

$$\forall\, 0\le n\le N-1$$

éq 4.4-3

definition 12 of **the indicator also expressed in temperature éq**

$$u\Rightarrow T,\ \hat{s}\Rightarrow s,\ \hat{g}\Rightarrow g \text{ et } \hat{h}\Rightarrow hT_{ext}$$

4.4-4 Proof

: This

a little technical demonstration comprises three stages which will consist in successively raising each term of the indicator [éq 4.3-6] (by means of inequalities of the property 14) and to gather increases obtained: Firstly

, one will replace in the equation [éq 4.2-6] the term in by $v-v_h$ the product utilizing w_K the function "bubble" of éq K

$$\forall K\in T_h\quad v_K:=\hat{s}_{\theta,h}^{n+1}-\rho\, C_p\frac{u_h^{n+1}-u_h^n}{\Delta t}+div\left(\lambda\,\nabla u_{\theta,h}^{n+1}\right)$$
$$w_K:=\psi_K v_K$$

4.4-5 From where

succession of increases, via [éq 4.4-1] and the inequality of Cauchy-Schwartz, éq

$$\|v_K\|_{0,K}^2 \leq C_7^2 \int_K w_K v_K\, dx \leq C_7^2 \left\{ \left(\frac{u^{n+1} - u_h^{n+1} - u^n - u_h^n}{\Delta t}, w_K \right)_{0,\Omega} + a\left(u_\theta^{n+1} - u_{\theta,h}^{n+1}, w_K\right) - \left(\hat{s}_\theta^{n+1} - \hat{s}_{\theta,h}^{n+1}, w_K\right) \right\}$$

$$\leq C_7^2 \max\left(1, C_8\right) \left\{ \left\| \frac{u^{n+1} - u_h^{n+1} - u^n - u_h^n}{\Delta t} \right\|_{0,K} + h_K^{-1} \left\| \sqrt{\lambda}\, \nabla\left(u_\theta^{n+1} - u_{\theta,h}^{n+1}\right) \right\|_{0,K} + \left\| \hat{s}_\theta^{n+1} - \hat{s}_{\theta,h}^{n+1} \right\|_{0,K} \right\} \|w_K\|_{0,K}$$

$$\Rightarrow \|v_K\|_{0,K} \leq \frac{C_7^2}{C_6} \max\left(1, C_8\right) \left\{ \left\| \frac{u^{n+1} - u_h^{n+1} - u^n - u_h^n}{\Delta t} \right\|_{0,K} + \left\| \sqrt{\lambda}\, \nabla\left(u_\theta^{n+1} - u_{\theta,h}^{n+1}\right) \right\|_{0,K} + \left\| \hat{s}_\theta^{n+1} - \hat{s}_{\theta,h}^{n+1} \right\|_{0,K} \right\}$$

4.4-6 Then

, one reiterates the same process for the surface terms éq $w_{F,i}$

$$\forall F \in S(K) \cap S_{h,\Omega} \quad v_{F,1} := \left[\lambda \frac{\partial u_{h,\theta}^{n+1}}{\partial n} \right] \qquad \text{4.4-7 éq}$$

$$w_{F,1} := \psi_K P_F v_{K,1}$$

$$\forall F \in S(K) \cap S_{h,2} \quad v_{F,2} := \hat{g}_{h,\theta}^{n+1} - \lambda \frac{\partial u_{h,\theta}^{n+1}}{\partial n} \qquad \text{4.4-8 éq}$$

$$w_{F,2} := \psi_F P_F v_{F,2}$$

$$\forall F \in S(K) \cap S_{h,3} \quad v_{F,3} := \hat{h}_{h,\theta}^{n+1} - \lambda \frac{\partial u_{h,\theta}^{n+1}}{\partial n} - \left(h_h u_h\right)_\theta^{n+1} \qquad \text{4.4-9 Is}$$

$$w_{F,3} := \psi_F P_F v_{F,3}$$

, for example, for $i=1$ the succession of increases, via [éq 4.4-1] and the inequality of Cauchy - Schwartz, éq

$$\|v_{F,1}\|_{0,F}^2 \leq C_{10}^2 \int_F w_{F,1} v_{F,1}\, d\sigma \leq C_{10}^2 \left\{ \begin{array}{l} \left(\frac{u^{n+1} - u_h^{n+1} - u^n - u_h^n}{\Delta t}, w_{F,1} \right)_{0,\Omega} + a\left(u_\theta^{n+1} - u_{\theta,h}^{n+1}, w_{F,1}\right) \\ - \left(\hat{s}_\theta^{n+1} - \hat{s}_{\theta,h}^{n+1}, w_{F,1}\right)_{0,\Omega} - \left(v_K, w_{F,1}\right)_{0,\Omega} \end{array} \right\}$$

$$\leq C_{10}^2 \max\left(1, C_{11}\right) \left\{ \begin{array}{l} \left\| \frac{u^{n+1} - u_h^{n+1} - u^n - u_h^n}{\Delta t} \right\|_{0,\Delta_F} + h_F^{-1} \left\| \sqrt{\lambda} \nabla\left(u_\theta^{n+1} - u_{\theta,h}^{n+1}\right) \right\|_{0,\Delta_F} \\ + \left\| \hat{s}_\theta^{n+1} - \hat{s}_{\theta,h}^{n+1} \right\|_{0,\Delta_F} + \|v_K\|_{0,\Delta_F} \end{array} \right\} \|w_{F,1}\|_{0,\Delta_F}$$

$$\Rightarrow \|v_{F,1}\|_{0,F} \leq \frac{C_{10}^2}{C_9} \max\left(1, C_{11}\right) \left\{ \begin{array}{l} h_F^{\frac{1}{2}} \left\| \frac{u^{n+1} - u_h^{n+1} - u^n - u_h^n}{\Delta t} \right\|_{0,\Delta_F} + h_F^{-\frac{1}{2}} \left\| \sqrt{\lambda}\, \nabla\left(u_\theta^{n+1} - u_{\theta,h}^{n+1}\right) \right\|_{0,\Delta_F} \\ + h_F^{\frac{1}{2}} \left\| \hat{s}_\theta^{n+1} - \hat{s}_{\theta,h}^{n+1} \right\|_{0,\Delta_F} + h_F^{\frac{1}{2}} \|v_K\|_{0,\Delta_F} \end{array} \right\}$$

4.4-10 Finally

it is enough to carry out the linear combination implying [éq 4.4-9] and [éq 4.4-10] to conclude (car). $h_F \leq h_K$ et $\forall v\ \|v\|_{0,\Delta_F} \leq \|v\|_{0,\Delta_K}$ avec $F \in S(K)$ Note:

This

- *local decrease of the error is also declined according to the formulations [éq 4.2-14] and $\tilde{\eta}_R^n(K)$ [éq 4.3-1], [éq 4.3-2]. $\eta_{R,p,t}^n(K)$ While*
- *placing oneself in the particular frame [éq 3.1-9] of the article [bib6] with an implicit scheme () $\theta=1$ one finds well the inequality (29) pp432. By*
- *adopting the less restrictive approaches () H_4 and (), H_5 one finds a version "strong" of this property. This*
- ***result only one opposite room of total increase [éq 4.3-9 provides], [éq 4.3-10] but in the frame of this kind of indicator one will not be able to obtain better compromise. These estimates are optimal within the meaning of [bib5].** They show the equivalence of the sum hilbertienne indicators with the spatial part of the total exact error. The constants of equivalence are independent of the parameters of discretizations in space and in time, they depend only on the smallest angle of the triangulation. This*
- ***increase of the real error indicator shows, which if one very locally refines (around) K in order to decrease, $\eta_R^n(K)$** one **is not ensured of a reduction in the error in an immediate vicinity of the zone concerned (in). Δ_K The indicator "underestimates" the spatial error locally and only a more macroscopic refinement carries out theoretically a reduction in the error (cf property 13). Complements***

4.5

the constant just like K_3 its preceding alter ego, K_2 depends **intrinsically on the type of limiting conditions enriching the equation by initial heat as well as type of temporal and spatial discretization.** To try to free itself from this last stress, SR. GAGO [bib10] proposes (on a problem models 2D) a dependence of the constant according to K_2 the type **of finite elements used. She** is written éq

$$K_2 := \frac{\tilde{K}_2}{\sqrt{24\,p^2}}$$ 4.5-1 where

is p the degree of the polynomial of interpolation used (for $p=1$ SORTED 3 and QUAD4 , for $p=2$ SORTED 6 and QUAD 8/9) . From where the idea, once the calculated error indicator total, to multiply it by this "corrective" constant. $\frac{1}{\sqrt{24\,p^2}}$ This strategy was implicitly retained for the computation of the error indicator in mechanics (option "ERME_ELEM" of CALC _ERREUR, cf [R4.10.02 §3]). We however did not adopt it for the thermal because this constant was given only empirically on the equation of Laplace 2D. We do not want to thus skew the values of the indicators. It

was question, until now, only of spatial cards of error indicators calculated at a given time of the transient of computation. But, in fact, there **exist several ways to build error indicators on a parabolic problem:** one

- can very well, first of all, semi-to discretize the strong formulation space some and to control its spatial error by error indicator adapted a posteriori to the steady case (in our elliptic case). Then one applies a solver, of step and of order variables, treating the ordinary differential equations (for example [bib10] [bib11] [bib12]),
- a second strategy consists in semi-discretizing in time then in space and determining the error indicator spatial one time given (for example [bib4] [bib6] [bib13]) starting from the local residues of the semi-discretized form. One applies a linear solver to the variational form allowing to repeatedly build the solution at one time given from the solution of previous time, another
- possibility consists in discretizing simultaneously in time and space on of the finite elements suitable and controlling their "space-time" errors in a coupled way (for example [bib14] [bib15]). This

last **scenario is most tempting from** a theoretical point of view because he proposes a complete control of the error and he makes it possible to avoid unfortunate decouplings as for possible refinements/coarsenings controlled by a criterion with respect to the other (cf following paragraph). He is however very heavy to set up in a large industrial code such as Code_Aster *. He* supposes indeed, to be optimal, nothing less than one separate management time step by finite elements. What from the point of view of architecture supporting the finite elements of the code is a true challenge! One **thus prefers the second scenario to him who** has the large advantage of being able to be established directly in a code D finite elements because this it is based above all on the resolution of the completely discretized system. These is the indicators which was set up in N3S, TRIFOU and Code_Aster *. In the frame of* a true **"space-time" discretization of the problem (scenario** 3), one obtains, in any rigor, a " **space-time" indicator for each element of discretization which** $K\times[t_n,t_{n+1}]$ is the balanced sum of three terms:

1) the residue of the calculated solution and the data discretized compared to the strong formulation of the problem evaluated (P_0) on, $K\times[t_n,t_{n+1}]$
2) the jump spatial through $\partial K\times[t_n,t_{n+1}]$ operator traces associated (who naturally connects the formulations weak and strong via the formula of Green),
3) the temporal jump through $K\times\partial[t_n,t_{n+1}]$ calculated solution.

The solution which was installation does not make it possible obviously to reveal explicitly the term **of temporal jump.** It **re-appears however** implicitly **, because of** method of particular temporal semi-discretization, in all **the terms into cubes** θ **definitions 10 and 12. On the other hand**

, the fact of being interested mainly only in the spatial discretization and its possible refinement/coarsening should not occult certain contigences with respect to the management of time step. Indeed, during transient computations **comprising of abrupt variations of loadings and/or sources during time, for example** of the thermal shocks, the fields of calculated temperatures can $T^n\ (0<n\leq N)$ oscillate **spatially and temporally. Moreover**, they can and the violate the "**principle of the maximum** " by taking values apart from the limits imposed by the condition of Cauchy limiting conditions. To overcome this numerical phenomenon one parasitizes shows, on a canonical case without condition of exchange (cf [R3.06.07 §2]), that time step must remain between two limits: éq

$$\Delta t_{\min}(h)<\Delta t<\Delta t_{\max}(\theta)$$ 4.5-2 In practice

, it is difficult to have an order of magnitude of these limits, one has thus difficulty, if oscillations are detected, modifying time step in order to respect [éq 4.5-2]. In addition, this kind of operation is not always possible sometimes because it is necessary to precisely take into account the abrupt variations of loadings (in particular when is Δt too small). When

is Δt **too large one** can function in implicit **Eulerian what** $(\theta=1)$ will cause to gum the higher limit. On the other hand

lorsqu **"it is too weak, two** palliative strategies S" offer to the user: diagonalizing

- **the mass matrix via the lumped elements (cf** [R3.06.07 §4] [§5]) proposed in the code (that requires installation to treat the elements P2 *or* modelization 2D `_AXI)`, to decrease
- **the size of meshes (that** increases complexities necessary computation and memory). It

is from this point of view that the refinements **/déraffinements practised on the faith of our indicator can have an incidence.** The fact of refining will not pose any problem on the other hand while déraffinant one can deteriorate very well the decrease of [éq 4.5-2]. It is necessary thus to be very circumspect if one uses the option coarsening of the software HOMARD (encapsulated for Code_Aster *in* MACR `_ADAP_MAIL option` "DERAFFINEMENT" [U7.03.01]) on case test comprising a thermal shock. We

now will summarize the principal contributions of the preceding and their holding and bordering theoretical chapters with respect to the thermal computation set up in Code_Aster . *Summary*

5 of the theoretical study Is

(P **0) the problem in extreme cases mixed (of** linear type inhomogeneous Cauchy-Dirichlet-Neumann-Robin and with variable coefficients) solved by the operator **THER_LINEAIRE éq**

$$(P_0)\begin{cases} \rho\, C_p \dfrac{\partial T}{\partial t} - \mathrm{div}(\lambda\, \nabla T) = s & \Omega \times]0,\tau[\\ T = f & \Gamma_1 \times]0,\tau[\\ \lambda \dfrac{\partial T}{\partial n} = g & \Gamma_2 \times]0,\tau[\\ \lambda \dfrac{\partial T}{\partial n} + hT = hT_{ext} & \Gamma_3 \times]0,\tau[\\ T(\mathbf{x},0) = T^0(\mathbf{x}) & \Omega \end{cases}$$

5-1 Taking into account

the choices of modelizations operated in Code_Aster *(by* AFFE_MATERIAU , AFFE_CHAR_THER ...) one determines the Abstracted **Variational Frame (CVA** cf [§2]) minimal on whom one will be able to rest to show **the existence and the unicity of a field of temperature solution (cf** [§2]). By recutting these pre-necessary theoretical a little "ethereal" with the practical stresses of the users, one from of deduced from the limitations as for the types of geometry and the licit loadings. Then
, **while semi-discretizing in time and space by the usual methods of the code (which** one makes sure of course of the cogency and owing to the fact that they preserve the existence and the unicity of the solution), one studies **the evolution of the properties of stability of the problem (cf** [§3]). These results of controllability are very useful for us to create the standards, the techniques and the inequalities which intervene in the genesis of the indicator in residue. In these stages of discretization we also briefly approach the influence of such or such theoretical assumption on the functional **perimeter of the operators of the code. Before**

summarizing the principal theoretical results concerning the error indicator, we go repréciser some notations: one

- fixes time step such as Δt is $\dfrac{\tau}{\Delta t}$ an integer N *and* that the temporal discretization is regular: , $t_0=0,\ t_1=\Delta t,\ t_2=2\Delta t \cdots t_n=n\Delta t$ Note

: This

> assumption of regularity does not have really importance, it just makes it possible to simplify the writing of the semi-discretized problem. To model an unspecified transient at time, t_n it is just enough to replace by Δt is $\Delta t_n = t_n + 1 - t_{n.}$

- θ the parameter of - method θ semi-discretizing temporally, (P_0) are
- and the T^n T_h^n fields of temperatures to time, t_n $(0 \le n \le N)$ exact solutions of the initial problem, (P_0) respectively semi-discretized in time and completely discretized in time and space. Taking into account

the modelizations installation in the code, we can suppose that the temporal **discretization of the loadings and the source is exact and** that the taking **into account, via Lagranges, of the limiting conditions (generalized or not) of Dirichlet is too. On the other hand**, one of the approaches to model the numerical approximations carried out during the integral calculus of the error indicator, consists in supposing inaccurate **the spatial discretization of the loadings and the source. Their** approximate values are noted éq

$$s^{n+1}_{\theta,h},\ g^{n+1}_{\theta,h},\ T^{n+1}_{ext,\theta,h} \text{ et } h^{n+1}_{\theta,h}$$ 5-2 by

posing éq

$$\chi^{n+1}_{\theta}=\theta\chi\left(\mathbf{x},(n+1)\frac{\tau}{\Delta t}\right)+(1-\theta)\chi\left(\mathbf{x},n\frac{\tau}{\Delta t}\right) \text{ avec } \chi\in\{T,s,T_{ext},g,h\} \text{ et } 0\le n\le N-1$$ 5-3 Note:

:

> *This kind of indicator installation of (in mechanics as in thermal) is also sullied with another type of numerical approximations related to computations of second derivative of the voluminal term (cf [§4.3]). Its effect can possibly feel when one is interested in the intrinsic value of the voluminal error for sources very kicked up a rumpus on a coarse mesh. They*

exist two constant then K2 *and* K3 *independent* of the parameters of discretization in time and space, depending only on the smallest angle of the triangulation and the type of problem, which make it possible to build:

- An increase **of the total spatial error (L** "historical of L" indicating total reality " **on - considers"** the error spatial total) éq

$$\|\sqrt{\rho C_p}(T^n-T^n_h)\|^2_{0,\Omega}+\Delta t\sum_{m=0}^{n-1}\|\sqrt{\lambda}\nabla(T^{m+1}_{\theta}-T^{m+1}_{\theta,h})\|^2_{0,\Omega}$$

$$\forall 0\le n\le N \quad \le\|\sqrt{\rho C_p}(T_0-T^h_0)\|^2_{0,\Omega}+K_2\Delta t\sum_{K\in T_h}\left(\eta^0_R(K)\right)^2+\sum_{m=0}^{n-1}\left\{\left(\eta^{m+1}_R(K)\right)^2+h^2_K\|s^{m+1}_{\theta,h}-s^{m+1}_{\theta}\|^2_{0,K}\right\}+$$

$$K_2\Delta t\sum_{K\in T_h}\sum_{m=0}^{n-1}\left\{\sum_{F\in S_2(K)}h_F\|g^{m+1}_{\theta,h}-g^{m+1}_{\theta}\|^2_{0,F}+\sum_{F\in S_3(K)}h_F\|\left(h_h(T_{ext,h}-T_h)\right)^{m+1}_{\theta}-\left(h(T_{ext}-T_h)\right)^{m+1}_{\theta}\|^2_{0,F}\right\}$$

a 5-4

- decrease **of the local spatial error (it** "underestimates**"** the local error spatial) éq

$$\eta^{n+1}_R(K)\le K_3\left\{\begin{array}{c} h_K\|\sqrt{\rho C_p}\dfrac{T^{n+1}-T^{n+1}_h-T^n-T^n_h}{\Delta t}\|_{0,\Delta_K}+\|\sqrt{\lambda}\,\nabla(T^{n+1}_{\theta}-T^{n+1}_{\theta,h})\|_{0,\Delta_K}+\\ h_K\|s^{n+1}_{\theta}-s^{n+1}_{\theta,h}\|_{0,\Delta_K}+h^{\frac{1}{2}}_F\|g^{n+1}_{\theta}-g^{n+1}_{\theta,h}\|_{0,\Delta_K\cap\Gamma_2}\quad+\\ h^{\frac{1}{2}}_F\|h^{n+1}_{\theta}(T^{n+1}_{ext,\theta}-T^{n+1}_{\theta})-h^{n+1}_{\theta,h}(T^{n+1}_{ext,\theta,h}-T^{n+1}_{\theta,h})\|_{0,\Delta_K\cap\Gamma_3}\end{array}\right\}$$

$$\forall 0\le n\le N-1$$

5-5 With

- **the continuation of** $\left(\eta^n_R(K)\right)^{K\in T_h}_{0\le n\le N}$ **local real indicators (by means of** notations of [§4.1]) éq

$$\eta_R^{n+1}(K):=\eta_{R,vol}^{n+1}(K)+\eta_{R,saut}^{n+1}(K)+\eta_{R,flux}^{n+1}(K)+\eta_{R,éch}^{n+1}(K)$$
$$:=h_K\|s_{\theta,h}^{n+1}-\rho C_p\frac{T_h^{n+1}-T_h^n}{\Delta t}+div\left(\lambda\nabla T_{h,\theta}^{n+1}\right)\|_{0,K}+\frac{1}{2}\sum_{F\in S_\Omega(K)}\sqrt{h_F}\left\|\left[\lambda\frac{\partial T_{h,\theta}^{n+1}}{\partial n}\right]\right\|_{0,F}+$$
$$\sum_{F\in S_2(K)}\sqrt{h_F}\|g_{\theta,h}^{n+1}-\lambda\frac{\partial T_{h,\theta}^{n+1}}{\partial n}\|_{0,F}+\sum_{F\in S_3(K)}\sqrt{h_F}\|\left(h\left(T_{ext}-T\right)\right)_{\theta,h}^{n+1}-\lambda\frac{\partial T_{h,\theta}^{n+1}}{\partial n}\|_{0,F}$$

5-6 which

is initialized by éq

$$\eta_R^0(K):=h_K\|s_h^0+div\left(\lambda\ \nabla T_h^0\right)\|_{0,K}+\frac{1}{2}\sum_{F\in S_\Omega(K)}\sqrt{h_F}\left\|\left[\lambda\ \frac{\partial T_h^0}{\partial n}\right]\right\|_{0,F}+$$
$$\sum_{F\in S_2(K)}\sqrt{h_F}\|g_h^0-\lambda\frac{\partial T_h^0}{\partial n}\|_{0,F}+\sum_{F\in S_3(K)}\sqrt{h_F}\|\left(h\left(T_{ext}-T\right)\right)_h^0-\lambda\frac{\partial T_h^0}{\partial n}\|_{0,F}$$

5-7 This

local continuation makes it possible to build the continuation **of** $\left(\eta^n(\Omega)\right)_{0\le n\le N}$ **total real indicators éq**

$$\forall\,0\le n\le N\quad \eta_R^n(\Omega):=\left(\sum_{K\in T_h}\eta_R^n(K)^2\right)^{\frac{1}{2}}$$

5-8 Of

[éq 5-4] (cf [§4.2]) it appears that, at one time given, the error on the approximation of the condition of Cauchy and the history of the total indicators intervenes on the total quality of the solution obtained. One **will be able to thus minimize overall the error of approximation due to the finite elements in the course of time while re-meshing "advisedly", via the continuation of indicators, the structure. Because**, in practice, one realizes that the refinement of meshes makes it possible to decrease their error and thus cause a drop in the temporal sum of the indicators. **The total error will butt (and it is moral) against the value bottom due to the approximations of the initial condition, the limiting conditions and the source (which** will tend it-also to drop of course!). One **cannot get results of better quality that the data input of the problem!**
Result [éq 5-5] (cf [§4.4]) only one opposite room **of total increase [éq 5-4** provides] ("must" would have been to reveal also an increase at the local level) but, in the frame of this kind of indicator, one will not be able to obtain better compromise. These **estimates are optimal within the meaning of [bib5]. They** illustrate the equivalence of the sum hilbertienne indicators with the spatial part of the total exact error. The constants of equivalence are independent of the parameters of discretizations in space and in time, they depend only on the smallest angle of the triangulation and the type of with dealt problem. According to
this increase of the indicator [éq 5-6], if one very locally refines (around the element) K in order to decrease, $\eta_R^n(K)$ one is not ensured of a reduction in the error in an immediate vicinity of the zone concerned (in). Δ_K **The indicator "underestimates" the spatial error locally and only a more macroscopic refinement carries out a reduction in the error theoretically. Only**

in pure residue, all a spatial **"zoology" of error indicators are permissible (cf** [§4.3]), we retained a type similar of it to that already set up for the mechanics of Code_Aster . Leaning on the solutions and the discrete loadings of time running and previous time (except with the first time step), its **theoretical limitations are thus, at best, those inherent in the resolution of the problem in temperature: no** zones comprising of crack, point or points of reflection not, problem to the interfaces of multi-material, - unconditionally θ stable diagram, regular family of triangulation, polygonal mesh discretized by of the finite elements isoparametric, oscillations and violation of the principle of the maximum (cf [§4.5]). Of course, in practice, one very often passes in addition to, and this without encumbers, this perimeter of "theoretical" use. But

it is necessary well to keep in mind, that as a "simple postprocessing" of, $(\mathbf{P_0})$ the indicator cannot unfortunately provide more reliable diagnosis in the zones where the resolution of the initial problem stumbles (close to crack, shock…). Its denomination prudently reserved of indicator (instead of the usual terminology of estimator) is in these typical cases more than ever of setting! But if, in these extreme cases, its gross amount perhaps prone to guarantee, its **utility as an effective and convenient supplier of cards of error for a mending of meshes or a refinement/coarsening remains completely justified. In**

the same vein, even if the formulation [éq 5-6] were established only in the transitory linear case, isotropic or not, defines by (), P_0 one could also stretch his perimeter of use to nonlinear (operator THER_NON_LINE), different limiting conditions (ECHANGE_PAROI for example) or to others element types finished (lumped isoparametric elements, structural elements…) (cf [§2.1]). For more information on the "data-processing" perimeter corresponding to his effective establishment in the code, one can refer to [§6.2] or the user's documentation of CALC _ERREUR [U 4.81.06]. It

was question, until now, only of spatial cards of error indicators calculated at a given time of the transient of computation. But, in fact, there **exist several ways to build error indicators on a parabolic problem (cf** [§4.5]). That that we retained does not allow a complete control of the error and it always requires a certain **vigilance when with problems of the type shocks are dealt (** the same one as for the problem post-treaty!). It reveals only implicitly the term of jump temporal in all the terms into cubes θ [éq 5-6].

To finish, it should be stressed that this indicator is thus composed of four terms:

- the principal **term, called** term **of voluminal error, controlling** the good checking of the equation of heat, to which
- three secondary **terms are added checking** the good behavior of the spatial **jumps and** the limiting conditions: terms **of flux and exchange. In**

2D - PLANE or 3D (resp . in 2D - AXI), if the unit of the geometry is the meter, the unit of the first is W.m (*resp* .) and $W.m.rad^{-}$ that of the other terms is it (resp $W.m^{\frac{1}{2}}$.). Attention $W.m^{\frac{1}{2}}.rad^{-}$ **thus with the units taken into account for the geometry when one is interested in the gross amount of the indicator and not in his relative value! We**

now will approach, after the practical difficulties of implementation in the code, the environment necessary and its perimeter of use. One will conclude for an example of use drawn from a case official test. Put

6 in work in the Code_Aster *particular*

6.1 Difficulties To compute:

this kind of indicator it is necessary to compose with the vision "elementary computation + assembly" generally deployed in all the codes finite elements. However the estimate, at the local level, of requires $\eta(K)$, not only the knowledge of its local fields, but also that of its meshes close. One thus **needs to carry out a "total computation" at the level of, in Δ_K local computation! A copied** strategy on what had been set up for the estimator in mechanics consists in transmitting this kind of information in the components **of wide cards which they** will be transmitted in argument of entry of CALCUL . It is this kind of contingency which explains the heterogeneity of processing during the overloads of loadings between the thermal solvers and the computation of our indicator (cf [§6.2]). Another

type of difficulty, more numerical this time, relates to the computation of the voluminal term. Indeed, it requires a double derivative which one carries out in three stages, because in the Code_Aster *one* does not recommend the use of second derivative of the shape functions. Note:

They

> *were recently introduced to treat derivative of rate of energy restitution (cf [R7.02.01 § Annexe 1]). On the one hand*

, one calculates (in the thermal operator) the vector flux at the points of gauss, then one extrapolates the corresponding values with the nodes by local lissage (cf [R3.06.03] CALC_CHAMP with THERMAL = ' FLUX_ELNO' and [§ 6.2]) in order to calculate his divergence with Gauss points. With of the finite elements quadratic the intermediate operation is only approximate (one assigns like value to the median nodes the half the sum of their values to the extreme nodes). However numerical tests (restricted) showed that this approach does not provide results very different from those obtained by a direct computation via good second derivative. Lastly,

it was necessary to determine various geometrical characteristics (diameters, norms, jacobians…), connector industries of the elements in opposite and to reach the data which they underlie in all the cases envisaged by the code (started from mesh symmetrized and/or heterogeneous, loading function or reality, nonlinear material, all the thermal isoparametric elements 2D/3D and all loadings). Beyond these fastidious developments, a large **effort of data-processing validation "géométrico -" was made** to try to track possible bugs in this entrelac of small formulas. These hard tests on small cases model tests (TPLL01A/H for the 2D_PLAN /3D and TPNA 01 A for the 2D_AXI) appeared profitable (including for the indicator in mechanics and the lumped elements!) and essential. Because one does not lay out, to my knowledge, of theoretical values allowing to validate in certain situations these indicators: "nothing **resembles any more one value of indicator… than another value of indicator!". For another thing and, although in a process of validation that is not the panacea, it is thus necessary to try to release a maximum of confidence in all these components. Environment**

6.2 necessary/parameter setting The computation

of this indicator is carried out, via option "ERTH_ELEM" of the operator of postprocessing CALC_ ERREUR, on an EVOL_THER (provides RESULTAT to the key word) coming from a former thermal computation (linear or not, transient or steady, isotropic or orthotropic, via THER_LINEAIRE or THER_NON_LINE , cf more precise perimeter [§6.4]). As
one already underlined, it requires **as a preliminary the recourse to option "FLUX_ELNO" of CALC_CHAMP** which determines the values of the vector heat flux to the nodes (cf example of use [§6.5]). This indicator
consists of fifteen components by elements and for a given time. In order to be able post-to treat them via RAISED POST_ or GIBI one needs to extrapolate these fields by element in fields at nodes by element. The addition of option "ERTH_ELNO" **(after** the call to "ERTH_ELEM") makes it possible to carry out this purely data-processing transformation. For one time and a given finite element, it does nothing but duplicate the fifteen components of the indicator on each nodes of the element. For

carrying out the integral postprocessing of desired thermal computation well, it is necessary: To carry out

- **on all the geometry, TOUT= "YES" (default value** , if not computation stops in ERREUR _FATALE). This provisional choice was led by data-processing and functional contingencies, because thus all the finite elements are seen affecting a homogeneous indicator calculated with the same number of terms (if not quid of the notion of term of jump and term of CL at the edge of the field considered?). In addition the tools of refinement/coarsening of the code (the software HOMARD encapsulated in MACR_ ADAP_MAIL), emerged natural of our cartographies of error, do not make it possible to treat only parts of meshes. Note:

  That

  > *poses problems of propagation of subdivisions to preserve the conformity of the triangulation. In fact, to divert this kind of contingency, it would be necessary, either to define a buffer zone making the junction between the “dead” zone of the mesh and the zone “activates” with treating, or of way more optimal but also much more difficult from a point of view structures, to reduce it to a layer of joined elements. To provide*

- **the same temporal parameter setting: value** of (default value θ equalizes to 0.57) provided to the key word PARM_THETA ; if necessary if with a transitory problem is dealt, it is necessary to inform the usual fields ANY NUMÉRIQUE/LISTE_ORDRE with licit values with respect to thermal computation. The computation of the history of the indicator can then be carried out from any time of a transient, knowing that with the first increment one carries out computation like in hover (, and $\theta=1$ not $n+1=0$ of term in finite difference cf [éq 5-7]). Moreover
  , in hover, if the user provides a value of different θ from 1, one imposes this last value to him after having informed some. In a related
  way, one detects the request for supply of cards of errors between noncontiguous sequence numbers (there are an ALARME) or the data of an EVOL_THER not comprising a field of temperature and vector flux to the nodes (computation stops in ERREUR _FATALE). The value of and θ the number of sequence number taken into account are traced in the message file [§6.3]. The sequence number and corresponding time accompany also each occurrence by error indicator in the results file ([§6.3]). To use
- **the same loadings and by complying with the rules of overloads particular to the computation options** of error of this operator. Thus, in the thermal solvers (and mechanics) one incorporates the limiting conditions of the same type, whereas in error analyses of CALC_ ERREUR (and thus also with our indicator) one cannot take into account, for a kind of limiting condition given, that last provided to the key word EXCIT . The order **of these loadings thus is a crucial importance! Note:**

  This

  > *restriction finds its base in the first remark of the preceding paragraph. For making well it would be necessary, either concatenate on the elements of skin concerned all the limiting conditions, or to provide to elementary computations variable cards of sizes containing all the loadings exhaustively. The first solution seems by far most optimal but also hardest to implement. It would then be necessary also to make the same thing for the indicator in residue of the mechanics (OPTION = ' ERRE_ELGA_NORE'). However*

, in the event of conflict between loadings of the same type, one often can and easily to find a solution palliative via the AFFE_CHAR_THER adequate . The user is informed presence of several occurrence of the same type of loading by an alarm message and the list of the loadings actually taken into account is traced in the message file ([§6.3]). The code stops on the other hand in ERREUR _FATALE if the provided loadings pose certain problems (interpolation of loadings function, access to the components, presence of the CHAMPGD of the coefficient of heat exchange and absence of the CHAMPGD of the outside temperature or vice versa….), In

- **the same general frame: value** of the model (parameter MODELS), the necessary materials (CHAM_MATER), structure EVOL_THER given (RESULTAT) and result (assignment of CALC_ ERREUR with possibly one "reuse" reentrant). They are traced in the message file ([§6.3]). If

the user does not respect this necessary homogeneity of parameter setting (with the rules of overload near) between the thermal solver and the tool for postprocessing, it is exposed to skewed results even completely false (without inevitably an alarm message or an ERREUR _FATALE stopping it , one cannot all control and/or prohibit!). There then remains only judge of the relevance of his results. Let us recapitulate

all this parameter setting of operator CALC_ ERREUR directly impacting the computation of the error indicator spatial in thermal. Factor key word

Key word	***Default value***	***Value***	***compulsory (O) or advised (C) MODELS***
	thermal computation		Idem (O) CHAM_MATER
	Idem		thermal computation (O) TOUT
	"YES	" "YES	" (O) TOUT /
	NUMÉRIQUE/LISTE_ORDRE "YES	" "YES	" (C) PARM_THETA
	0.57	Idem	thermal computation (O) RESULTAT
	EVOL_THER		of thermal computation (O) reuse
	EVOL_THER		of thermal computation (C) EXCIT
CHARGE	Idem		thermal computation + rule of overload (O) OPTION
	"ERTH_ELEM		**" "ERTH_ELNO** " INFO
	1 1 (	C)	Table

6.2-1: Summary of the parameter setting of CALC_ ERREUR impacting the computation of the indicator Note

: Out of transient

- *, it (strongly) is advised to calculate the history of the indicator over times of computations contiguous. If not, the postprocessing of the temporal semi-discretization will be distorted, and according to the devoted formula… the user will become only judge of the relevance of his results. /exploitation*

6.3 presentation of the results of the error analysis option

"ERTH_ELEM" provides in fact, not one, but fifteen **components by finite elements and** K **time step. Indeed** t_{n+1} , for **each of the four terms of [éq** 5-6], _the principal term voluminal and the three surface secondary terms _, one calculates not only the absolute error **, but** also a term **of standardization (the theoretical** value of the discretized loadings that one would have had to find) and the associated **relative error. By adding** these three families of four contributions, one establishes also the total absolute error, the total term of standardization and the total relative error. What makes the account well! The fact

of dissociating the contributions of each component of this indicator makes it possible to compare their relative importances and to target strategies of refinement/coarsening on a certain kind of error. Even if the voluminal term (representing the good checking of the equation of heat) and the term of jump (related to the modelization finite elements) remain the dominating terms, it can prove to be useful to measure the errors due to certain type of loading in order to refine their modelization or to re-mesh the accused frontier zones. Moreover

this kind of strategy can be easily diverted of its primary goal in order to make refinement/coarsening by zone: it is enough to impose, only in this zone, a kind of limiting condition fictitious (with very bad value in order to cause a large error). The mode
of computation of these components and the name of their component "of reception" in symbolic field "ERTH_ELEM_TEMP" of the EVOL_THER are recapitulated in the table below (by leaning on the nomenclature of [éq 5-6]). Absolute error

	relative	**Error (in %) Term**	**of voluminal standardization**
Term TERMVO	$\eta^{n+1}_{R,vol}(K)$ **TERMV**	$\frac{\eta^{n+1}_{R,vol}(K)}{N^{n+1}_{R,vol}(K)}\times 100.$ **2 TERMV**	$N^{n+1}_{R,vol}(K):=h_K\|s^{n+1}_{\theta,h}\|_{0,K}$ **1 Term**
of jump TERMSA	$\eta^{n+1}_{R,saut}(K)$ TERMS	$\frac{\eta^{n+1}_{R,saut}(K)}{N^{n+1}_{R,saut}(K)}\times 100.$ 2 TERMS	$N^{n+1}_{R,saut}(K):=\frac{h_F^{\frac{1}{2}}}{2}\left\|\left[\lambda \frac{\partial T^{n+1}_{\theta,h}}{\partial n}\right]\right\|_{0,F}$ 1 Term
of flux TERMFL	$\eta^{n+1}_{R,flux}(K)$ TERMF	$\frac{\eta^{n+1}_{R,flux}(K)}{N^{n+1}_{R,flux}(K)}\times 100.$ 2 TERMF	$N^{n+1}_{R,flux}(K):=h_F^{\frac{1}{2}}\|g^{n+1}_{\theta,h}\|_{0,F}$ 1 Term
of exchange TERMEC	$\eta^{n+1}_{R,éch}(K)$ TERM	$\frac{\eta^{n+1}_{R,éch}(K)}{N^{n+1}_{R,éch}(K)}\times 100.$ 2 TERM	$N^{n+1}_{R,éch}(K):=h_F^{\frac{1}{2}}\|(h(T_{ext}-T))^{n+1}_{\theta,h}\|_{0,F}$ 1 Total
ERTABS	$\eta^{n+1}_{R}(K):=\sum_i \eta^{n+1}_{R,i}(K$ **ERTREL**	$\frac{\eta^{n+1}_{R}(K)}{N^{n+1}_{R}(K)}\times 100.$ **TERMNO**	$N^{n+1}_{R}(K):=\sum_i N^{n+1}_{R,i}(K)$ **Table**

6.3-1: Components of the error indicator. For the absolute error and the term of standardization, in 2D-PLAN or 3D (resp . in 2D-AXI), if the unit of the geometry is the meter, the unit of the first term is W.m (resp .) and that $W.m.rad^{-}$ of the other terms is it (resp. $W.m^{\frac{1}{2}}$). Attention $W.m^{\frac{1}{2}}.rad^{-1}$

thus with the units taken into account for the geometry when one is interested in the gross amount of the indicator and not in his relative value! This information

is accessible in three forms: For each

- time of the transient, these fifteen values are added on all the mesh (one makes the same thing as in the table [Table 6.3-1] while replacing by) and K are traced Ω in a table of results file (.RESU). ***

```
THERMAL
:      ESTIMATOR      OF      ERREUR      IN      RESIDUAL
**********************************************
PRINTING

OF THE TOTAL NORMS: SD EVOL_THER

RESU_1 SEQUENCE NUMBER
3 TIME
5.0000E+           00 absolute errors
            /RELATIVE    /STANDARDIZATION TOTAL 0.5863
E              05 0.2005E-   04% 0.2923E+  02 VOLUMINAL
TERM
0.3539E-   05 0.0000E+   00% 0.0000E+  00 TERM SAUT
0.2217E-        05 0.1006E-   04% 0.2205E+  02 TERM FLUX
0.4384E-           06 0.3886E-    05% 0.1128E+   02 TERM
ECHANGE
0.4591E-           06  0.5755E-      05%  0.7977E+    01
**********************************************

Example
```

6.3-1: Traced L" option "ERTH_ELEM_TEMP " in the results file It

- is stored by means of computer in the fifteen components of symbolic field "ERTH_ELEM_TEMP" of the thermal SD_RESULTAT . The variables of access of this field are, for each mesh (in our M1 example), the sequence number (NUME_ORDRE) and time (INST). With option "ERTH_ELNO_ELEM" there is the same thing for each node of the element considered. FIELD PAR

```
ELEMENT WITH GAUSS POINTS OF SYMBOLIC NAME ERTH_ELEM_TEMP  SEQUENCE
NUMBER:
 3 INSTS: 5.00000E  +00 M1 ERTABS
       ERTREL       TERMNO TERMVO
          TERMV2       TERMV1 TERMSA
          TERMS2       TERMS1 TERMFL
          TERMF2       TERMF1 TERMEC
          TERME2       TERME1 1        0.5863
       E  05 0.2005E-   04 0.2923E+  02 0.3539E-
          05 0.0000E+   00 0.0000E+  00 0.2217E-
          05 0.1006E-   04 0.2205E+  02 0.4384E-
          06 0.3886E-   05 0.1128E+  02 0.4591E-
          06 0.5755E-   05 0.7977E+  01 ........
                                FIELD PAR

ELEMENT WITH GAUSS POINTS OF SYMBOLIC NAME ERTH_ELNO_ELEM  Sequence
number:
 3 INSTS: 5.00000E  +00 M1 ERTABS
       ERTREL       TERMNO TERMVO
          TERMV2       TERMV1 TERMSA
          TERMS2       TERMS1 TERMFL
          TERMF2       TERMF1 TERMEC
          TERME2       TERME1 N1       0.5863
 E       05 0.2005E-   04 0.2923E+  02 0.3539E-
          05 0.0000E+   00 0.0000E+  00 0.2217E-
          05 0.1006E-   04 0.2205E+  02 0.4384E-
          06 0.3886E-   05 0.1128E+  02 0.4591E-
          06 0.5755E-   05 0.7977E+  01 N3 0.5863
 E       05 0.2005E-   04 0.2923E+  02 ........
                                Example
```

6.3-2: Layouts, via IMPR_RESU , of the components of symbolic field "ERTH_ELEM_TEMP "/" ERTH_ELNO_ELEM " in the results file One can

- also trace the values of each one of these components in the message file (.MESS) by `initializing` key word INFO to `2. However` this functionality rather reserved for the developers (for maintenance or of the pointed diagnoses) also revealed complementary printings (documented but too exhaustive) on the elements constituting the indicator and the characteristic of the finite elements and of their vicinities. TE0003 **********

```
NOMTE/L2
D THPLTR3/T RHOCP            2.0000000000000
     ORIENTATION
 NETS 1.0000000000000     … --->
TERMVO
 /TERMV1 1.2499997764824    1.2499997764826    >>> CURRENT
MESH <<< 3 TRIA3  DIAMETRE
 3.5355335898314     D-02 EDGES OF
  TYPE SEG2 >>>>>>>>
 >>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>> NUMERO

 D'ARETE/HF 1 2.4999997764826      D-02 NOMBRE FROM
 TOPS 2 CONNECTOR INDUSTRY
 1 2 XN   0.59999992847442
     0.59999992847442   YN -0.80000005364418
    -0.80000005364418  JAC 1.2499998882413
     D-02 1.2499998882413    D-02 <<< CLOSE
 MESH 2 QUAD4   IGREL  /IEL
 1 2 INOV  LOCAL
 /GLOBAL 2 5….  *********************************************

TOTAL ON
      the MESH 2 absolute errors
             /RELATIVE    /MAGNITUDE     TOTAL 0.5900
 D             03 0.1079D-   03% 0.5466D-   03 VOLUMINAL TERM
 0.1768D-   01 0.1000D-   03% 0.1768D-   01 TERM SAUT
 0.5882D-         03 0.1080D-   03% 0.5448D-   03 TERM FLUX
 0.0000D+         00 0.0000D+   00% 0.0000D+   00 TERM ECHANGE
 0.0000D+     00 0.0000D+   00% 0.0000D+   00
*********************************************
 Example
```

6.3-3: Layout, via INFO=2, `in` the message file Note:

When

- *the term of standardization is null (a certain kind of loading or of source is null, as C" is the case in the examples [Example 6.3-1] and [Example 6.3-2] above with the voluminal term), one does not calculate the term of relative error associated. There remains initialized to zero. Moreover*
- *, to compute: indeed the absolute error relative to a condition limits null (a flux or a condition of exchange) it should be imposed as a function via AFFE_CHAR _THER_F adhoc. And this for simple data-processing contingencies, which make that with a constant loading, one cannot make the distinction between: condition*

1) *limits null: the user imposes zero on this portion of border and he wants to test the associated absolute error, condition*
2) *limits null: there are no limiting conditions on this edges, Of the tests*

- *of NON-regression "numérico-data processing" showed that the way model the loadings and the source, as constants or functions, could notably influence the values of very small terms of error (especially in relative error of course) and worry the user unnecessarily. This phenomenon is explained by differences in codings of the discretized loadings [éq 5-2]. This kind of behavior is also found as soon as one changes linear solver, preconditioner, method of renumbering, of platform… In hover*
- *, when one of the finite elements uses a non-zero source with linear, the term principal is very badly estimated since it requires a double derivative of the field of temperature. An ALARME thus warns the user and enjoins it to pass into quadratic. Perimeter*

6.4 of use This indicator

was developed, for time, only on the isoparametric elements (TRIA3/6, QUAD4/8/ 9, TETRA4/10 , PENTA6/13 /15 and HEXA 8/20 /27) and for modelizations PLANE, PLAN_DIAG , AXIS, AXIS_DIAG , 3D and 3D_DIAG . It thus does not calculate the contributions of the structural elements of type shell (COQUE_PLAN, COQUE_AXIS, COQUE), of the pyramids (PYRAM5 and PYRAM 13) and of the modelization of Fourier (AXIS_FOURIER). It does not make it possible either to calculate the terms of jumps of these elements with the authorized elements. However **, if a mesh comprises licit and illicit elements, computation does not stop and, via** the OPTION – 2 in the catalogs of elements idoines, one warns **the user of not taken into account of the aforesaid elements. Indeed**
to carry out this postprocessing, it is necessary as a preliminary to call, explicitly, option "FLUX_ELNO" (computation of the vector heat flux to the nodes) and, implicitly, "INIT_MAIL_VOIS" (determination of the characteristics of the vicinity of an element Δ_K). One is K thus **tributary of their respective perimeters of use. It is also necessary**

to keep in mind some **more minor rules but which** can be of a very particular importance for very precise studies: The computation

- with the indicator does not deal that the elements of the mesh belonging to models indicated by the key word MODELS CALC_ERREUR of the command . One can thus work with meshes (not cleaned) comprising "meshes of outline" to which one allots a different model. In
- a mesh **in dimension, one** q calculates the terms of jump and loading, only on elements **of skin of dimension. Therefore,** $q-1$ one and the treats the relations of the TRIA/QUAD with the SEG relations TETRA/PENTA /HEXA with the FACE. For example , in the event of presence of segments in a three-dimensional mesh, the option will not stop but she will not take into account their (possible) contributions. Option
- "ERTH_ELEM_TEMP" and its preliminary options do not know the PYRAM. Their **contributions** will be ignored. This gap comes from their introduction into Code_Aster *more recent* than those of the already quoted preliminary options. Note:

  In any case

  > *these elements are minority in a mesh 3D and are generated only by the voluminal free mesh generator of GIBI, which creates some locally to supplement portions of meshes hexahedral. In 2D,*

- one should not **accidentally intercalate a segment between two triangles or quadrangles, if not** the term of jump of these elements will not be calculated and one will enquérira oneself wrongly of the existence of a possible limiting condition. The computation will not stop but with this interface, the value of the indicator will be incomplete. However, for special needs (charging density internal and localised in a structure, fissures…), one can of course allow this kind of situation. In 3D, the problem also arises of course when one intercalates quadrangles or triangles between two FACE contiguous . the same
- type of imbroglio occurs when two points **of the mesh are superimposed geometrically** . There still, computation would have not to stop, but value of indicator will be incomplete for level of this zone of covering, If one works
- **with a mesh which results from operations of symmetrization, he is necessary** to try not to be in the two preceding cases. Moreover, on both sides of the axis of symmetry, the meshes **close ones do not have inevitably (with in particular mesh generator GIBI) of the directional senses which meet the standard of the Code_Aster *(they should*** be reversed). The computation of the indicator, for which this information is crucial (and to define the external norms in each mesh connector industries in opposite), detects the problem by calculating the jacobian of each mesh. In 2D, an algorithm of substitution makes it possible to circumvent the problem and to rebuild the arrays of connector industry "nodes of the element running nodes of its neighbors". In 3D, the problem is much more difficult and individual with each element, the code thus stops in ERREUR_FATALE in the event of problem. If one wants

•**to refine or déraffiner his mesh with MACR_ADAP _MAIL [U7.03.01], the mesh** should comprise only triangles or tetrahedrons. Concerning the surface or voluminal loadings, the "good practice" consists in using only mesh groups. If nodes groups are used, one must expect distorted computations, because after some refinements, other points will have probably formed part geometrically of the zone concerned with the GROUP_NO without seeing itself affecting any loading (one cannot modify the composition of a GROUP_NO in the course of session!). For

specific loadings or points of statement (on which go, for example, to rest POST_RELEVE _T) the GROUP_NO is licit . On the other hand, it is not advised to use directly meshes (MA) or (NO) nodes (apart from group), because in this case, to the liking of the renumberings, HOMARD probably will lose their trace. It can preserve the memory of meshes or the nodes only through one name of GROUP_MA or GROUP_NO . Thanks to this mechanism, it can adopt a Lagrangian vision of becoming of these meshes or these points! The computation

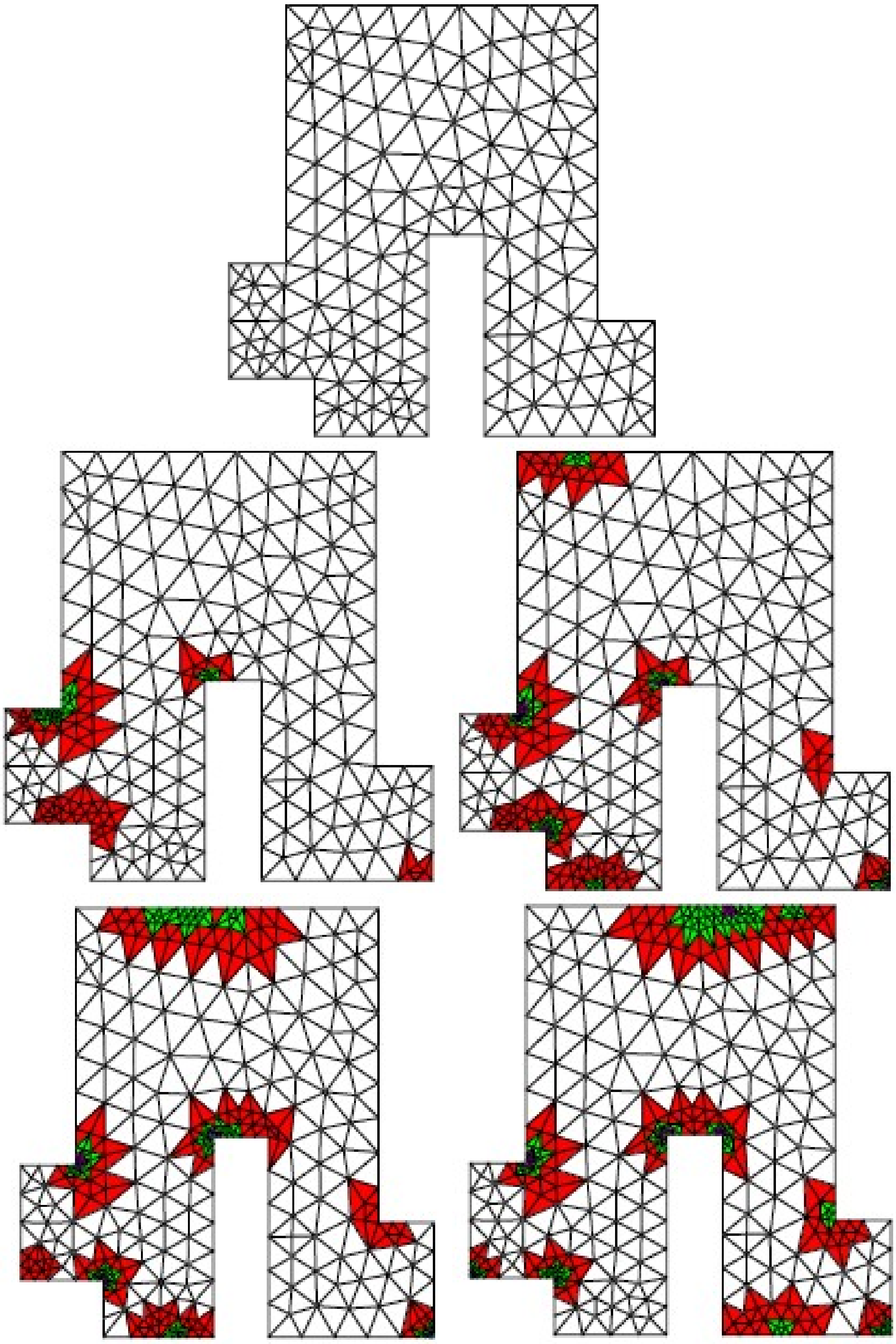
of the indicator takes place indifferently on an EVOL_THER coming from THER_LINEAIRE or THER_NON_LINE , steady or transitory, isotropic or orthotropic, and, on a motionless structure with a grid by elements answering the preceding criteria. Into nonlinear

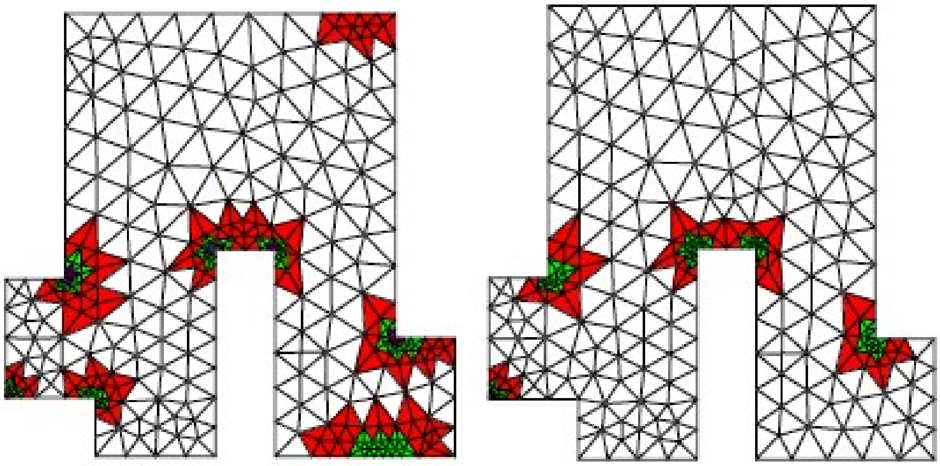
one takes into account non-linearities of the materials and the modification of the problem in enthalpy. However the possible contributions of nonlinear loadings are not treated (FLUX_NL and RAYONNEMENT). The user is informed by an ALARME, just like it by it is informed not taken into account of a limiting condition of type ECHANGE_PAROI . Indeed, into linear one recognizes, for time, only the contributions of loadings SOURCE, FLUX_REP and ECHANGE . For the taking into account of these loadings, particular rules of overload are applied (cf [§6.2]). Example

6.5 of use

to familiarize itself with the use of this indicator in thermal and its possible coupling with the HOMARD encapsulation (for more information, one will be able to consult the site http://www.code - aster.com/outils/homard) via MACR_ADAP _MAIL [U7.03.01] one can take as a starting point the case test TPLL01J [V4.02.001]. In this

other example extracted Internet site of HOMARD, coupling ERTH_ELEM /MACR_ADAP_MAIL [U7.03.01] simulates the circulation of a "hot" fluid on both sides of a bent metal part (in top and bottom, via a condition of ECHANGE depending on time in AFFE_CHAR _THER_F). The circulation of the fluid is carried out left towards the line. The accuracy
is especially necessary at the ends of structure, on the level of the propagation of the fluid: thanks to the coupling error indicator/tools of refinement-coarsening, the mesh thus remains fine out of edge of part, in the zone where concentrates the "hot" fluid. Finally it is déraffiné in the back, once the fluid passed. It is also noted
that, as envisaged by the theory (cf remarks [§2.2]), the resolution of the thermal problem "is blunted" in the returning corners and that the error indicator (although it is also penalized in these zones) announces this irrefutable fact (even when the part cooled). Example





6.5-2: Use of option "ERTH_ELEM " `coupled` with HOMARD Conclusion

7 – Prospect During

computational simulations by finite elements, obtaining result gross is not sufficient any more. The user **is increasingly petitioning of spatial error analysis compared to the mesh. He needs for bearing methodological and pointed tools "numériquo-data processing" to measure the quality of his studies and to improve. To this end**

, the error indicators spatial a posteriori make it possible to locate, on each element, a cartography of error on which the tools of mending of meshes will be able to rest: the first computation on a coarse mesh makes it possible to exhume the card of error starting from the data and of the solution discretized (from where the term "a posteriori"), refinement is carried out then locally by treating on a hierarchical basis this information. The new

indicator a posteriori which has just been established post-to treat the thermal problems of the Code_Aster *is based* on their local residues extracted the semi-discretizations in time. Via option "ERTH_ELEM" of CALC_ERREUR , it uses the thermal fields (EVOL_THER) emanating from THER_LINEAIRE and THER_NON_LINE . This new

indicator supplements the offer of the code in term of advanced tools making it possible to improve quality of the studies, their mutualisations and their comparisons. Indeed, error indicators in mechanics and macro of refinement/coarsening MACR_ADAP _MAIL [U7.03.02] are already available. It remains to supplement the perimeter of use of these tools and, to pack them, in particular for better managing non-linearities and the interactions spatial error/temporal error. Note:

Estimator

> *by lissage of stresses of Zhu & Zienkiewicz (CALC_ERREUR + OPTION " ERZ1/ ERZ2_ELEM" [R4.10.01]) and indicator in pure residue ("ERME_ELEM" [R4.10.02]). Thereafter*

, the prospects for this work are several orders: From a functional

- **point of view, the complétude** of this indicator could also improve in taking into account the possible ones nonlinear limit conditions (FLUX_NL and RAYONNEMENT) and exchanges between walls (ECHANGE_PAROI) . In the long term, it would also be necessary to be able to lean on of the finite elements structure (shell…), pyramids and power process of the problems of convection-diffusion (operator THER_NON _LINE_MO [R5.02.04]). From a theoretical
- **point of view, when** new limiting conditions are used and/or when one leans on new modelizations (shell, beam…), a study "numériquo - functional" similar to that of this document, should be carried out to consider limitations theoretical and practical (with respect to the Code_Aster *) of such* an indicator and to exhume its adhoc formulation. Let us recall
- finally that a string of **error indicators a posteriori are available, and, that** rather little were tested and validated on industrial cases. In order to refine diagnoses, to establish comparisons and to set up strategies of mending of meshes per class of problem, it would be interesting to pack the list of the indicators available. Various indicators in residue plus local problem thus appeared more effective (but also more expensive) during numerical tests (into elliptic) in N3S [bib5]. Note:

  The indicator

  > *is the norm of the solution of a local, of the same problem standard than the initial problem, but discretized on spaces moreover high degree and whose second member is the residue. According to the limiting conditions affixed with this local problem, one distinguishes some from various types. They thus mix the vision "bases hierarchical" and the aspects "residue" with the error indicators a posteriori. The ideal*

- consists in discretizing simultaneously in time and space on of the finite elements suitable and controlling their "space-time" errors in a coupled way. This " **space-time" indicator gives access** to a complete control of the error and it makes it possible to avoid unfortunate decouplings as for possible refinements/coarsenings controlled by a criterion with respect to the other (cf discussion [§4.5]). It is however very heavy to set up in a large industrial code such as the Code_Aster . *It supposes* indeed, to be optimal, nothing less than one separate management time step by finite elements. What from the point of view of architecture supporting the finite elements of the code is a true challenge! Bibliography

8 R. DAUTRAY

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