Estimators of error in quantities of Summarized

interest:

In this document, one introduces the frame of the estimate of error in quantity of interest. This frame is used with explicit estimators in residue as well as with of the estimators in lissage. A certain number of quantities of interest are detailed, in particular their writing in term of loading for the dual problem.

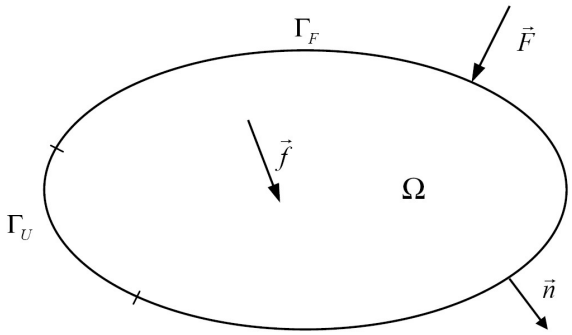
Contents

1 model Problem and notations

1.1 Equations

an elastic solid, related to a cartesian coordinate system of $\mathbb{R}^3$, occupies, in its natural state and before strain, a related limited open field Ω of $\mathbb{R}^3$ regular border $\partial\Omega$ of outgoing norm $\vec{n}$. This contour is the union of two parts disjoined Γ_U and Γ_F such as $\Gamma_U \cup \Gamma_F = \partial\Omega$ and $\Gamma_U \cap \Gamma_F = \emptyset$.

That is to say $\vec{u}(x) = u_1(x)\vec{e}_1 + u_2(x)\vec{e}_2 + u_3(x)\vec{e}_3$ the displacement of a point M of Ω coordinates $x = (x_1, x_2, x_3)$. On contour Γ_U, a displacement $\vec{u}(x) = \vec{0}$ is imposed and a surface force $\vec{F}(x)$ is imposed on Γ_F. The solid is also subjected to a voluminal force $\vec{f}(x)$. The study is placed in the frame of linear elasticity and the assumption of the small disturbances, without viscous damping.



Appear 1.1-a : Undistorted configuration

In the continuation, we adopt the indicielle notation for the vectors and the tensor, as well as the rule of summation of Einstein on the repeated dumb indices. Thus, the following linear system:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases} \qquad \textbf{éq. 1.1-1}$$

can be written:

$$a_{ij}x_j = b_i \qquad \textbf{éq. 1.1-2}$$

the derivative of $i^{ème}$ the component of a vector $\vec{v}$ compared to the $j^{ème}$ component of the coordinates of space will be written $v_{i,j}$.

The symmetric tensor of the stresses σ_{ij} is connected to the linearized symmetric tensor of the strains ε_{ij} by the relation:

$$\sigma_{ij} = a_{ijkl}\varepsilon_{kl} \qquad \textbf{éq. 1.1-3}$$

where a_{ijkl} is the tensor of the moduli of elasticity. It satisfies the relations with:

- symmetry: $a_{ijkh} = a_{jikh} = a_{ijhk} = a_{khij}$;

- positivity of the associated quadratic form: for any symmetric real tensor of the second order X_{ij} $\begin{cases} \exists \alpha_0 > 0, & \text{tel que} \quad X_{ij}\, a_{ijkh}(x)\, X_{kh} \geq \alpha_0\, X_{ij}\, X_{ij} \\ \forall\, X_{ij} = X_{ji}, & \forall\, x \in \Omega \end{cases}$.

The tensor of the strains is connected to displacement by the relation:

$$\varepsilon(u) = \frac{1}{2}\left(grad(u) + grad^T(u)\right) \qquad \textbf{éq. 1.1-4}$$

For a homogeneous and isotropic material, the constitutive law is the Hooke's law:

$$\sigma = \lambda\, \mathrm{tr}(\varepsilon)\, I + 2\,\mu\,\varepsilon \qquad \textbf{éq. 1.1-5}$$

where λ and μ are the coefficients of Lamé.

The balance equations, with the assumption of the small disturbances, are written:

$$div\,\sigma + \vec{f} = \vec{0} \qquad \textbf{éq. 1.1-6}$$

the boundary conditions defined previously are written:

$$\vec{u}(x) = \vec{0} \quad sur \quad \Gamma_U \qquad \textbf{éq. 1.1-7}$$

$$\sigma(x)\,\vec{n}(x) = \vec{F}(x) \quad sur \quad \Gamma_F \qquad \textbf{éq. 1.1-8}$$

all the equations éq. 1.1-4 with éq. 1.1-8 constitutes problem: (P_1)

$$\begin{cases} \sigma_{ij,\,j} + f_i = 0 & dans \quad \Omega \\ \sigma_{ij} = a_{ijkh}\,\varepsilon_{kh} & dans \quad \Omega \\ \varepsilon_{ij}(u) = \frac{1}{2}\left(u_{i,\,j} + u_{j,\,i}\right) & \\ u_i(x) = 0 & sur \quad \Gamma_U \\ \sigma_{ij}\,n_j(x) = F_i(x) & sur \quad \Gamma_F \end{cases} \qquad \textbf{éq. 1.1-9}$$

1.2 variational Formulation

space V is defined as the space of the acceptable v functions, sufficiently regular definite on Ω and with values in $\mathbb{R}^3$:

$$V = \left\{ v \in H^1(\Omega) \quad et \quad v = 0 \quad sur \quad \Gamma_U \right\}$$

The norms which are dependant for him are the following ones:

Norm L^2	$\|v\|^2_{L^2(\Omega)} = \int_\Omega v \cdot v \; d\,\Omega$
Pseudo norm H^1	$\|v\|^2_{H^1(\Omega)} = \int_\Omega \left(v \cdot v + \nabla\, v : \nabla\, v\right) d\,\Omega$
Normalizes H^1	$\|v\|^2_{H^1(\Omega)} = \int_\Omega \nabla\, v : \nabla\, v \; d\,\Omega$
in energy	$\|v\|^2_e = \int_\Omega \sigma(v) : \varepsilon(v) \; d\,\Omega$

the balance equation normalizes is multiplied by a sufficiently $v \in V$ regular function then integrated on the field Ω :

$$0 = \int_\Omega \sigma_{ij,j} v_i \, d\Omega + \int_\Omega f_i v_i \, d\Omega \quad \text{éq. 1.2-1}$$

$$0 = \int_\Omega \left(\sigma_{ij} v_i\right)_{,j} d\Omega - \int_\Omega \sigma_{ij} v_{i,j} \, d\Omega + \int_\Omega f_i v_i \, d\Omega \quad \text{éq. 1.2-2}$$

the application of the formula of Green gives:

$$\int_\Omega \left(\sigma_{ij} v_i\right)_{,j} d\Omega = \int_{\partial\Omega} \sigma_{ij} n_j v_i \, d\Gamma \quad \text{éq. 1.2-3}$$

the properties of symmetry of the tensor of the stresses imply:

$$\sigma_{ij} v_{i,j} = \sigma_{ij} \varepsilon_{ij}(v) \quad \text{éq. 1.2-4}$$

What makes it possible to write:

$$\int_{\partial\Omega} \sigma_{ij} n_j v_i \, d\Gamma - \int_\Omega a_{ijkh} \varepsilon_{kh}(u) \, \varepsilon_{ij}(v) \, d\Omega + \int_\Omega f_i v_i \, d\Omega = 0 \quad \text{éq. 1.2-5}$$

the first term of the second member is null on Γ_U and it remains:

$$\int_{\Gamma_F} F_i v_i \, d\Gamma - \int_\Omega a_{ijkh} \varepsilon_{kh}(u) \varepsilon_{ij}(v) \, d\Omega + \int_\Omega f_i v_i \, d\Omega = 0 \quad \text{éq. 1.2-6}$$

the variational formulation of the problem (P_1) is written [bib1]:

$$\begin{cases} Trouver \quad u \in V \quad tel\ que \\ \quad a(u,v) = l(v) \qquad \forall v \in V \end{cases} \quad \text{éq. 1.2-7}$$

with:

$$a(u,v) = \int_\Omega a_{ijkh} \, \varepsilon_{kh}(u) \, \varepsilon_{ij}(v) \, d\Omega \quad \text{éq. 1.2-8}$$

$$l(v) = \int_{\Gamma_F} F_i v_i \, d\Gamma + \int_\Omega f_i v_i \, d\Omega \quad \text{éq. 1.2-9}$$

1.3 Discretization by finite elements

Is Ω^h a partition of Ω in N elements. A space finite elements $V^h \subset V$ is built from continuous, polynomial functions by pieces, and of degree p_E on each element E. The discretization of the problem (P_1) by the method of Galerkin provides the problem (P_h) [bib1]:

$$\begin{cases} Trouver \quad u^h \in V^h \quad tel\ que \\ \quad a(u^h, v^h) = l(v^h) \qquad \forall v^h \in V^h \end{cases} \quad \text{éq. 1.3-1}$$

1.4 Error of discretization

the numerical error e of the approximation finite elements u^h is defined by:

$$e = u - u^h$$ **éq. 1.4-1**

Like u and u^h are elements of V then $e \in V$. The error satisfies the equation with the residue obtained while replacing u by $u^h + e$ in the variational formulation of problem: (P_h)

$$a(e, v) = l(v) - a(u^h, v) = R_h^u(v) \quad \forall v \in V$$ **éq. 1.4-2**

As the bilinear form is definite positive, the residue R_h^u is a limited linear form on V which belongs to its dual V'. The norm of the residue is such as:

$$\|R_h^u\|_{V'} = \sup_{v \in V \setminus 0} \frac{|R_h^u(v)|}{\|v\|_e}$$ **éq. 1.4-3**

the residue measures the NON-checking of certain properties of the equations of the problem and characterizes the unbalances and thus the error of discretization. It is easy to deduce from equation 1.4-2 a new property. The function v^h is used as function test:

$$R_h^u(v^h) = l(v^h) - a(u^h, v^h) \quad \forall v^h \in V^h$$ **éq. 1.4-4**

Then by means of the discretized form of the variational formulation, the property of orthogonality or property of Galerkin is obtained:

$$a(e, v^h) = R_h^u(v^h) = 0 \quad \forall v^h \in V^h$$ **éq. 1.4-5**

equation 1.4-2 indicates that the error is solution of a problem of elasticity whose loading in force is the residue of equilibrium. This problem is as complicated to solve and expensive as the initial problem. Thus, instead of determining the error itself, in fact estimates will be sought.

1.5 Estimate of error *a priori*

As its name indicates it, the estimate *a priori* is done before computation finite elements because it does not utilize the approximation finite elements u^h. Thus the functional analysis and the numerical analysis allow, in many cases and under certain assumptions of regularity, to get results of estimate *a priori* ; i.e. the prediction of the asymptotic rate of convergence of the finite elements error. For more details, the works of Ciarlet [bib1] or Strang and Fix [bib2] could be consulted.

A function $\tilde{\eta}(h, d, u)$ is an estimate of error *a priori* if:

$$\|u - u^h\| \leq \tilde{\eta}(h, d, u)$$ (1.5.1)

where $\|\cdot\|$ is a norm on the fields of displacements, h the size of the elements, d a set of facts of the case and u the exact solution.

The convergence of the method finite elements used can be obtained if:

$$\lim_{h \to 0} \tilde{\eta}(h, d, u) = 0$$ (1.5.2)

and the rate of convergence, if there exists a reality $q>0$ such as:

$$\|u-u^h\|\leq h^q\ \bar{\eta}(d\,,u) \tag{1.5.3}$$

the estimators of error *a priori* do not allow to quantify the errors on the solution finite elements because they utilize the exact solution which is generally not known. In addition, these estimates are valid only in the asymptotic mode, mode which is with difficulty atteignable for computations 3D. For the formulations using of the elements of the isoparametric type [bib2], in the case of a regular solution and for the norm in energy of the error $q=p$, where p is the degree of interpolation of the finite elements. On the other hand, in the case of a singular solution, $q=\min(p\,,\alpha)$ where α is the order of the singularity of the solution of the problem (for a crack, for example, $\alpha=\frac{1}{2}$).

2 Estimators of errors *a posteriori*

2.1 Definition of the estimators *a posteriori*

In estimate *a posteriori*, the purpose is not to find an estimate of the function error e but to determine an estimate of a measurement of the error. Thus the estimators who were developed these thirty last years provide a reliable and precise estimate of the error in total norms, such as the norms H^1, norms L^2 or norms in energy, postprocessing of the solution finite elements. This choice of total norms is imposed by the bilinear form. When the form is definite positive and symmetric, as it is the case here, that induces a scalar product for which the associated norm is the norm in energy; it is thus natural to estimate the error in this norm.

The basic principle of this kind of estimate is to use the solution approached to estimate the error of discretization. In opposition to the estimate *a priori*, the estimates *a posteriori* can be made only once the calculated approximate solution.

A function $\eta(h\,,u^h,\,d)$ is an estimate of error *a posteriori* if:

$$\|u-u^h\|\leq\eta(h\,,u^h,\,d) \tag{2.1.1}$$

where $\|\cdot\|$ is a norm on the fields of displacements, h the size of the elements, d a set of facts of the case and u^h the approximate solution. Moreover, if $\eta(h\,,d\,,u^h)$ can be localised in the form:

$$\eta(h\,,u^h,\,d)=\left(\sum_E \eta_E(u^h\,,d)^2\right)^{\frac{1}{2}} \tag{2.1.2}$$

then the quantities $\eta_E(u^h\,,d)$, contributions elementary of the estimate of the total error $\eta(h\,,u^h,\,d)$, are called indicating buildings of error. They provide a base for the mesh adaptation.

Many estimators exist and can be classified in three categories:

- •Estimators of errors based on the unbalances resulting from works from Babuška and Rheinboldt [bib3];
- •Error indicators built from smoothed stresses resulting from works from Zienkiewicz and Zhu [bib4];
- •Measurements of errors based on the concept of error out of behavior model exits of works of Ladevèze [bib5].

These various estimators, controlled well today, will be introduced in a more or less detailed way. A more detailed review could be found in [bib6] with [bib10].

2.2 Quality of the estimators

When an estimator is defined, it is necessary to wonder about the criteria and the average making it possible to judge performances of this estimator. In a general way, it is necessary to seek to build an estimator whose asymptotic behavior (when the size h of the elements tends towards zero) follows that of the error. This behavior results in the existence of two constants C_1 and C_2, depend on the facts of the case and of the discretization but not of the size of the elements, checking the following relation (Ladevèze & Shovel, 2004):

$$C_1\eta \leq \|e\| \leq C_2\eta \text{ when } h \text{ tends towards zero} \qquad \textbf{éq. 2.2-1}$$

where η the estimate of the measurement of $\|e\|$ the error represents e on the field considered. Thus to judge performances of the estimators, there exist various intrinsic criteria.

2.2.1 Index of effectiveness

the index of effectiveness γ is defined like the relationship between the error calculated by an estimator $e_{estimée}$ and the true error e_{vraie} :

$$\gamma = \frac{e_{estimée}}{e_{vraie}} \qquad \textbf{éq. 2.2.1-1}$$

A less to have an analytical solution, the true error is calculated like the difference between a solution obtained on a very fine mesh (solution *overkill*) and the solution obtained on a given network. An index of effectiveness close to the unit characterizes a good estimator. If this property is reached when the size of the elements tends towards zero, the estimator is known as asymptotically exact. Nevertheless, the estimate must be sufficiently precise (γ near to 1) for rather coarse meshes in order to be usable for the current discretizations in engineering. Lastly, it is desirable to over-estimate the error ($\gamma > 1$) to be able to be usable like stopping criteria in an adaptive process.

Systematic studies were carried out by Strouboulis and Haque [bib11] and by Babuška [bib12] on various benchmarks having an analytical solution or for which a reference solution can be given on a very fine mesh. All in all, this reveals that the quality of an estimator depends on the topology of the mesh, of the regularity of the solution and the regularity (flatness) of the elements.

The only total evidence γ does not make it possible to give an account of the way in which the site error behaves: it may be, that in some cases, the estimator is satisfactory for this criterion without it not succeeding in locating the zones where the error in energy is important [bib13].

2.2.2 Index of robustness

In order to free itself from the local behavior of the estimator, another criterion was defined by Babuška and Rodriguez [bib14]. For a given triangulation T, the index of effectiveness local γ_ω relating to the subdomain ω is defined by:

$$\gamma_\omega = \frac{e^\omega_{estimée}}{e^\omega_{vraie}} \qquad \textbf{éq. 2.2.2-1}$$

Babuška and Strouboulis showed that one could reach numerically the asymptotic beach of variation of γ_ω [bib15]:

$$0 < C^\omega_{inf} \leq \gamma_\omega \leq C^\omega_{sup} < \infty \qquad \textbf{éq. 2.2.2-2}$$

the variation with 1 of γ_ω is measured by the index R_ω defined by:

$$R_\omega = \max\left\{\left|1-C_{sup}^{\omega}\right|+\left|1-C_{inf}^{\omega}\right| \; ; \; \left|1-\frac{1}{C_{sup}^{\omega}}\right|+\left|1-\frac{1}{C_{inf}^{\omega}}\right|\right\} \qquad \textbf{éq. 2.2.2-3}$$

the index of robustness R largest R_ω is obtained by varying the position of the cell ω of its vicinity on the meshes belonging to a class of triangulation. This approach makes it possible to highlight the error of discretization on a subdomain but also the distortion of the elements of the mesh and the regularity of the solution. Although having mathematical bases, this technique seems complicated to implement for complex meshes used in engineering.

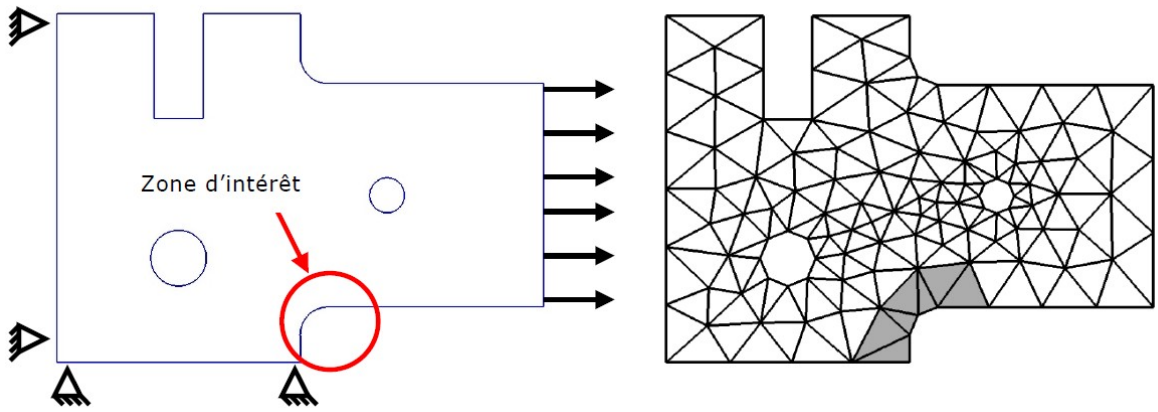
3 Site error

3.1 Error of pollution

3.1.1 Definition of the error of pollution

to define the error of pollution, we will provide a pragmatic definition while trying to understand which can be its behavior and thus its influence during an adaptive process. For that, we will study the convergence of the error in norm of energy in various situations:

- Convergence of the total error for a total uniform refinement;
- Convergence of the error in the zone of interest for a total refinement;
- Convergence of the total error for a local refinement;
- Convergence of the error in the zone of interest for a local refinement.

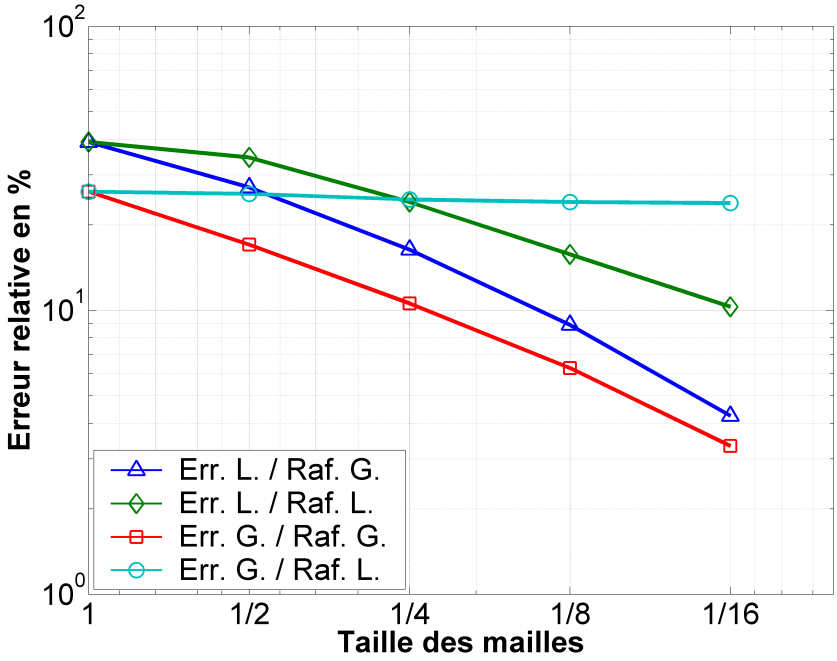


Appear 3.1.1-a : Geometry and mesh (192 elements).

The problem to be solved is the problem whose geometry and mesh are given by the figure3.1.1-a. The imposed force is unit, the Young modulus is equal to $E=1.0$ Pa and the Poisson's ratio is worth $\nu=0.3$. Finally the modelization employed is a modelization in plane stresses. A series of approximations finite elements will be given on a succession of linear meshes obtained by uniform refinement of structure or uniform refinement of the zone of interest ω. The error in norm of energy is then calculated (thus without estimate) from a solution obtained on a very fine network (solution *overkill*) which will be our reference solution. Finally the absolute error is normalized by the norm in energy of the solution on the structure or on the zone of interest according to the case. The results are presented in the figure3.1.1-b.

For a total refinement, the total error converges "a little less quickly" than $p=1$; into dividing the size of meshes by 10, the error is not divided by 10 (even into dividing the size by 16 the error is not divided by 10) (square markers). However this velocity of convergence is reached only asymptotically. It should be noted that the curve represented seems to approach theoretical convergence starting from the last segment (for sizes meshes between 1/8 and 1/16). It is thus only the first part of a curve of convergence which is traced, when convergence in $p=1$ is not reached yet (pre-asymptotic mode).

When only the zone of interest is refined, the solution is not improved in the rest of structure. The total error remains practically unchanged (round markers). This indicates that the error in the zone of interest contributes very little to the total error.



Appear 3.1.1-b : Description of the error of pollution: convergence of the error.

On the other hand by studying the curves relating to the error in the zone of interest, it is visible that significantly to improve the solution in this zone it is not enough to refine only this zone. Indeed the two curves (triangular markers rhombuses and markers) show a decrease of the site error but that obtained by local refinement shows a slower decrease: the two curves deviate. It thus seems that the error in the rest of structure affects also the quality of the solution in the zone of interest. What means that the sum of the local contributions of the total estimators on under field ω does not represent the site error exactly. It thus seems that the true site error is made up of two parts: a local component and a complementary component resulting moreover from structure. The consequence of this is that by refining only the zone of interest, the error in the rest of structure does not decrease (i.e the complementary component remains constant). One can then imagine that the curve representing the site error with a local refinement (markers rhombuses) converges towards a non-zero value if refinement is continued: this non-zero value will represent the complementary component (the local component tending towards zero). The value of the complementary component is then called “error of pollution”.

A more formal definition can be given. The error of pollution was introduced by Babuška and *al. [feeding-bottle* 16], [bib17], [bib18]. They suppose that the error of discretization under field of ω structure is made up: of

- a site error on, ω disregarding moreover structure; of
- an error of pollution or transported, resulting from the discretization in the rest of structure.

The fundamental relation between the total error and e the solution of Φ_E the local problem of the error (solution resulting from Equilibrated *Residual Method for example*) is pointed out: éq

$$a(e,v)=\sum_{E\in\Omega} a_E(\Phi_E,v) \quad \forall v\in V \qquad \text{. 3.1 3.1.1-1}$$

a site error on e_ω^{loc} a subdomain can ω be defined by: éq

$$a\left(e_{\omega}^{loc},v\right)=\sum_{E\subset\omega}a_E\left(\varphi_E,v\right)\quad\forall v\in V$$ **. 3.1 3.1.1-2**

the indicating function of site error φ_E depends only on the local residues of the approximation finite elements evaluated on the patch of elements, E * the site error e^{loc} depends only on the local residues on the patch, $\tilde{\omega}$ made up by the elements of and ω the close elements. The component complementary to the true error is the error of pollution: e_{ω}^{pol} éq

$$a\left(e_{\omega}^{pol},v\right)=\sum_{E\not\subset\omega}a_E\left(\varphi_E,v\right)\quad\forall v\in V$$ **. 3.1 3.1.1-3**

part of the true error comes from the residues located apart from under field and is transported until under field, thus affecting the accuracy. The error in e_{ω} a subdomain is ω thus the sum of the two components defined previously: éq

$$e_{\omega}=e_{\omega}^{loc}+e_{\omega}^{pol}$$ **. 3.1 3.1.1-4**

figure 3.1 3.1.1-b of the preceding definition, it becomes obvious that the control of the error of pollution is paramount. Indeed, apart from the zone of interest, when the mesh is coarse, it perhaps an important source of error of pollution. Consequently, a local refinement on under field must be counterbalanced by an adequate refinement apart from under field in order to controlling the error of pollution. We will see in the continuation that the estimators of error in quantity of interest allow to ensure this equilibrium between local refinement and refinement apart from under field. Estimate

3.1.2 of the error of pollution

to estimate the error of pollution, various methods were proposed. The method suggested by Babuška (Babuška and *al., 1995*) is related to the Green's functions which describe the interaction between various points of the field. The error of pollution is estimated with the error of approximation of the Green's function associated with certain points. Another method was used by Huerta and Díez (Huerta & Díez, 2000) and consists in finding an approximation from e_{ω}^{pol} resolution of the problem defining this error. Other works can be also quoted (Ainsworth, 1999), (Oden & Feng, 1996). The industrial interest of the evaluating of this error being limited, we will not more develop these methods of estimate. Many works on the computation of limits of the error make useless the estimate of the error of pollution because it is understood in the limits. Moreover, for the estimate of a site error, as we indicated already, the estimators in quantity of interest allow to free itself from the error of pollution thanks to the dual problem. Error

3.2 in quantity of interest Quantities

3.2.1 of interest

the estimates of errors in norm of energy are too abstract for providing to the users computer codes sufficient information on the various aspects of the sought solution. It is then more useful to estimate an error in terms of quantities of interest; quantities having a physical meaning (an average of displacements in an under-area of the physical field or an average of stresses on an interface or even, in the case of vibrations, one of the eigenfrequencies). Mathematically, they are characterized by linear functional calculuses or not within the space of functions to which the solutions belong. Several

examples are given: the average of a component of displacement on a field: ω éq

$$Q(v)=\frac{1}{|\omega|}\int_{\omega}v_x\,d\,\Omega$$ **. 3.2 3.2.1-1**

the average of a component of the tensor of the stresses on a field: ω éq

$$Q(v)=\frac{1}{|\omega|}\int_{\omega}\sigma_{xx}\,d\,\Omega \qquad \text{. 3.2 3.2.1-2}$$

then seeks to calculate the quantity who ε^{Q} can be expressed in the following form: éq

$$\varepsilon^{Q}=Q(u)-Q(u^{h}) \qquad \text{. 3.2 3.2.1-3}$$

is Q linear, the following equality is checked: éq

$$\varepsilon^{Q}=Q(u-u^{h})=Q(e) \qquad \text{. 3.2 3.2.1-4}$$

the continuation, only the case of the linear functional calculuses will be considered because that corresponds to the frame of the simplified approach presented in the continuation. In a general way, the processing used for the nonlinear quantities (like the von Mises stress or the integral in J fracture mechanics) is the linearization [bib19] and [bib20]. Dual

3.2.2 problem and fundamental relation

the error in quantity of interest can be estimated by means of properties of duality of the operators intervening in the weak formulation of the problem of reference. From a mechanical point of view, that corresponds to the existence of a function solution of a dual problem, known as function of influence, representing the influence of the forces of the problem of reference on total energy. The function of influence is then used as weighting of the residues of equilibrium of the problem of reference.

A simplified approach is presented, insofar as the frame is restricted with linear elasticity and the linear quantities of interest. This results in the fact that is $Q(\cdot)$ a linear form and that is $a(\cdot,\cdot)$ a bilinear form. This approach makes it possible to better understand the need for using the error on the function of influence and not only the function of influence.

The relation between the error in quantity of interest and the residue can be expressed by the introduction of an unknown function, ω called function of influence: éq

$$Q(e)=\omega(R_{h}^{u}) \qquad \text{. 3.2 3.2.2-1}$$

, one shows in [bib21] that the preceding relation becomes: éq

$$Q(e)=R_{h}^{u}(\omega) \qquad \text{. 3.2 3.2.2-2}$$

combining the preceding relationship to the equation of the residues: éq

$$a(e,v)=R_{h}^{u}(v) \qquad \text{. 3.2 3.2.2-3}$$

a relation between the error on the problem of reference, the function of influence and the error in quantity of interest is obtained: éq

$$a(e,\omega)=Q(e) \qquad \text{. 3.2 3.2.2-4}$$

equality is necessarily checked when is $\omega \in V$ solution of the following problem: éq

$$\begin{cases} Trouver\ \omega \in V\ tel\ que \\ a(v,\omega) = Q(v) \qquad \forall\ v \in V \end{cases}$$ **. 3.2 3.2.2-5**

problem is called dual or assistant problem. It is noticed that if could ω be calculated in an exact way, then could $Q(u)$ be directly given starting from the facts of the case because: éq

$$Q(u) = a(u,\omega) = l(\omega)$$ **. 3.2 3.2.2-6**

unfortunately, it is as difficult to determine the function of influence as ω to determine in u the primal problem. This is why an approximation finite elements with $\omega^h \in V^h$ which ω satisfies éq

$$a(v^h, \omega^h) = Q(v^h) \quad \forall\ v^h \in V^h$$ **. 3.2 3.2.2-7**

is introduced. It is also noted that in substituent with e in v^h the dual problem, one obtains: éq

$$Q(e) = a(e,\omega)$$ **. 3.2 3.2.2-8**

by ω in ω^h the preceding relation is not sufficient to obtain an estimate of because $Q(e)$ (property $a(e,\omega^h) = 0$ of orthogonality). This is why it is necessary to evaluate the error on the function of noted influence. $\varepsilon = \omega - \omega^h$ By introducing this error into the preceding relation, a representation of the error is obtained: éq

$$Q(e) = a(e,\varepsilon)$$ **. 3.2 3.2.2-9**

nonlinear problems and/or of the nonlinear quantities of interest, which precedes does not apply. In this case, the primal problem is written: éq

$$\begin{cases} Trouver\ u \in V\ tel\ que \\ A(u;v) = L(v) \qquad \forall\ v \in V \end{cases}$$ **. 3.2 3.2.2-10**

is $A(\cdot;\cdot)$ a semi-linear form such as the arguments which follow the semicolon are linear and $L(\cdot)$ a continuous linear form on. V

The approach consists in regarding the primal problem as a problem of minimization under stress [bib19], [bib20]: éq

$$\begin{cases} Trouver\ u \in V\ tel\ que \\ Q(u) = \inf_{z \in S} Q(z) \\ où \\ S = \{ z \in V;\ A(z;v) = L(v),\ \forall\ v \in \in V \} \end{cases}$$ **. 3.2 3.2.2-11**

to solve the problem of minimization, the couple $(u,\omega) \in V \times V$ is sought such as it satisfies the primal problem and the dual problem: éq

$$\begin{cases} A(u;\tilde{v}) = L(\tilde{v}) & \forall\ \tilde{v} \in V \\ A'(u;v,\omega) = Q'(u;v) & \forall\ v \in V \end{cases}$$ **. 3.2 3.2.2-12**

, the relation between the error in quantity of interest and ε^Q the residue are given by: éq

$$\varepsilon^Q = Q(u) - Q(u^h) = R(u^h;\omega) + \Delta A + \Delta Q$$ **. 3.2 3.2.2-13**

and ΔA utilize ΔQ the second derivative and third of and A . Q Estimate

3.2.3 of the error in quantity of interest From

the fundamental relation 3.2 3.2.2-9 techniques were developed in order to estimate the error in quantity of interest.

The simplest technique and most direct is to use the total estimators in norm of energy. By applying the theorem of Cauchy-Schwartz, a relation between the error in quantity of interest and the norms in energy of the errors to the primal problem and the dual problem can be found [bib22]: éq

$$|Q(e)| \leq \|e\|_e \|\varepsilon\|_e$$ **. 3.2 3.2.3-1**

statement thus enables us by means of to estimate easily the error in quantity of interest any estimator of error in norm in energy, this without additional development in a computer code laying out total estimators. Although this estimate is simplest, it is also coarsest: she does not take into account the local interactions between the primal error and the dual error (schematically that wants to say that an error on displacement does not result in an error into quantity of interest). This can lead to heavy overestimations of the error in quantity of interest.

A a little different approach, called Dual *Weighted Residual (DWR)* [bib19], consists in considering the exact statement of the error in quantity of interest: éq

$$Q(e) = a(e, \omega - \omega^h) = R_h^u(\omega - \omega^h)$$ **. 3.2 3.2.3-2**

the dual problem is solved by means of a method of high approximation of order. Another method, less expensive, consists in using high interpolation functions of order definite on elementary patchs of the field. This approach leads to a guaranteed higher limit of the error.

An approach “limits exact” (exact *bounds approach) [*biberon23] uses the properties of the estimate based on the dual analysis. Thus thanks to a formulation in displacements of the finite element method, a lower limit of exact strain energy is obtained; thanks to a stress formulation, a higher limit of exact complementary energy is obtained. From a practical point of view, this method is very expensive since it implies to twice solve the primal problem and the dual problem.

Other methods exist to estimate the error in quantity of interest but from an industrial point of view, only the approach using of the total estimators in norm of energy seems to be convincing. It does not require an additional development to reach an indication of the error and it is also much less expensive. Limits

3.2.4 of the error

3.2.4.1 a new statement for the error Although

very simple to obtain, the relation 3.2 3.2.3-1 a consequent overestimation of the error. Various techniques were proposed to cure this problem. Prudhomme and Oden employ the relation of the

parallelogram for the problems with symmetric bilinear forms which gives access to a higher limit and a lower limit of the error [bib24], [bib21]. If

and e are ε the errors of the approximations finite elements, respectively of the solutions of the problems primal and dual then: éq

$$Q(e)=a(e,\varepsilon)=\frac{1}{4}\|e+\varepsilon\|_e^2-\frac{1}{4}\|e-\varepsilon\|_e^2 \qquad \text{. 3.2.4.1}$$
3.2.4.1-1

introduces the scalar factor éq $s\in R$

$$Q(e)=a(e,\varepsilon)=\frac{1}{4}\|s\,e+s^{-1}\varepsilon\|_e^2-\frac{1}{4}\|s\,e-s^{-1}\varepsilon\|_e^2 \qquad \text{. 3.2.4.1}$$
3.2.4.1-2

the scalar is s selected so that the quantities and $\|s\,e\|_e$ have $\|s^{-1}\varepsilon\|_e$ the same amplitude, which implies to choose. $s=\sqrt{\|\varepsilon\|_e/\|e\|_e}$ Indeed

and e can ε be of very different order of magnitude. The scalar aims s at "normalizing" the two terms, in order to avoid making a difference of two very close terms because that can induce an important numerical error. Formally

, any existing estimator of errors can be used to evaluate the norms in energy of the relation of the parallelogram; more this estimate will be "just" better will be the quality of the estimator of error in quantity of interest. Construction

3.2.4.2 of the limits

the total estimators η_{inf}^{+}, η_{sup}^{+} and η_{inf}^{-} are η_{sup}^{-} defined such as: éq

$$\eta_{inf}^{+}\le\|s\,e+s^{-1}\varepsilon\|_e\le\eta_{sup}^{+} \qquad \text{. 3.2.4.2}$$
3.2.4.2-1

$$\eta_{inf}^{-}\le\|s\,e-s^{-1}\varepsilon\|_e\le\eta_{sup}^{-} \qquad \text{. 3.2.4.2}$$
3.2.4.2-2

rewriting the two preceding inequalities as follows: éq

$$\frac{1}{4}\left(\eta_{inf}^{+}\right)^2\le\frac{1}{4}\|s\,e+s^{-1}\varepsilon\|_e^2\le\frac{1}{4}\left(\eta_{sup}^{+}\right)^2 \qquad \text{. 3.2.4.2}$$
3.2.4.2-3

$$\frac{1}{4}\left(\eta_{inf}^{-}\right)^2\le\frac{1}{4}\|s\,e-s^{-1}\varepsilon\|_e^2\le\frac{1}{4}\left(\eta_{sup}^{-}\right)^2 \qquad \text{. 3.2.4.2}$$
3.2.4.2-4

by adding them: éq

$$\frac{1}{4}\left(\eta_{inf}^{+}\right)^2-\frac{1}{4}\left(\eta_{sup}^{-}\right)^2\le\frac{1}{4}\|s\,e+s^{-1}\varepsilon\|_e^2-\frac{1}{4}\|s\,e-s^{-1}\varepsilon\|_e^2\le\frac{1}{4}\left(\eta_{sup}^{+}\right)^2-\frac{1}{4}\left(\eta_{inf}^{-}\right)^2 \qquad \text{. 3.2.4.2}$$
3.2.4.2-5

a framing of the error in quantity of interest is obtained. Then

the limits, η_{inf}^{Q} η_{sup}^{Q} are introduced such as: éq

$$\eta_{inf}^{Q}=\frac{1}{4}\left(\eta_{inf}^{+}\right)^2-\frac{1}{4}\left(\eta_{sup}^{-}\right)^2$$ **. 3.2.4.2 3.2.4.2-6**

$$\eta_{sup}^{Q}=\frac{1}{4}\left(\eta_{sup}^{+}\right)^2-\frac{1}{4}\left(\eta_{inf}^{-}\right)^2$$ **. 3.2.4.2 3.2.4.2-7**

the error in quantity of interest $Q(e)$ is limited by and η_{inf}^{Q} : η_{sup}^{Q} éq

$$\eta_{inf}^{Q}\leq Q(e)\leq\eta_{sup}^{Q}$$ **. 3.2.4.2 3.2.4.2-8**

is also possible to use the estimates and η_{eei}^{Q} definite η_{ees}^{Q} such as: éq

$$\eta_{eei}^{Q}=\frac{1}{4}\left(\eta_{inf}^{+}\right)^2-\frac{1}{4}\left(\eta_{inf}^{-}\right)^2$$ **. 3.2.4.2 3.2.4.2-9**

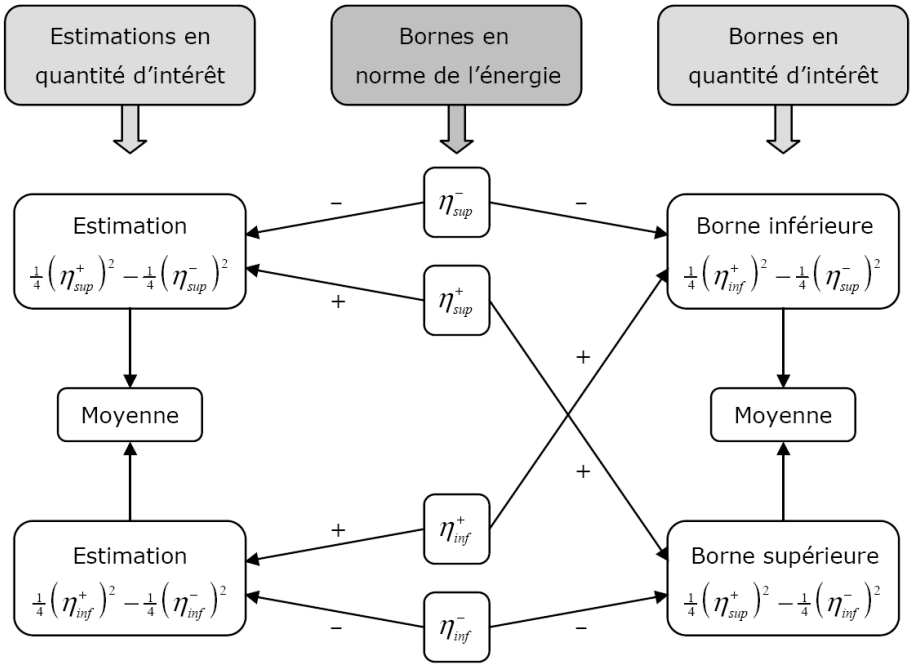
$$\eta_{ees}^{Q}=\frac{1}{4}\left(\eta_{sup}^{+}\right)^2-\frac{1}{4}\left(\eta_{sup}^{-}\right)^2$$ **. 3.2.4.2 3.2.4.2-10**

estimate of the error in quantity of interest. $Q(e)$ Lastly,

a last estimate can be obtained, by realising the limits and η_{inf}^{Q} or η_{sup}^{Q} the estimates and η_{eei}^{Q} , η_{ees}^{Q} noted: η_{moy}^{Q} éq

$$\begin{aligned}\eta_{moy}^{Q} &= \frac{1}{2}\left(\eta_{eei}^{Q}+\eta_{ees}^{Q}\right)\\ &= \frac{1}{2}\left(\eta_{inf}^{Q}+\eta_{sup}^{Q}\right)\\ &= \frac{1}{8}\left(\left(\eta_{inf}^{+}\right)^2+\left(\eta_{sup}^{+}\right)^2\right)-\frac{1}{8}\left(\left(\eta_{inf}^{-}\right)^2+\left(\eta_{sup}^{-}\right)^2\right)\end{aligned}$$ **. 3.2.4.2 3.2.4.2-11**

the following figure summarizes the various possible combinations of the estimate of error in norm of energy to obtain an estimate or limits of the error in quantity of interest. This shows which if one then has limits for an estimator in norm of energy one has limits for the error in quantity of interest. Appear



3.2.4.2 3.2.4.2-a Estimates and limits of the error in quantity of interest (Prudhomme and *al., 2003*). Estimates

3.2.4.3 of the limits of the error In

the preceding chapter, it was shown that limits of the total error were accessible by means of from the estimators in implicit residues. The broad outlines are given here by means of to obtain to limits of the error in quantity of interest the implicit estimators. It should be noted that this is only one example and that other methods of estimate can provide limits of the error in quantity of interest. With

an aim of obtaining the hight delimiters previously definite, the functions and ψ_E^u , ψ_E^ω corresponding respectively to the errors and e , ε are calculated by introducing the local problems relating to the problems primal and dual. The following estimates are then obtained: and

$$\eta_{sup}^{+}=\|s\psi_E^u+s^{-1}\psi_E^\omega\|_e \text{ éq } \eta_{sup}^{-}=\|s\psi_E^u-s^{-1}\psi_E^\omega\|_e \qquad \textbf{. 3.2.4.3 3.2.4.3-1}$$

the estimate of the lower limits, it is necessary to proceed in a similar way that for the estimate of the hight delimiters by building continuous functions than the interfaces from and ψ_E^u : ψ_E^ω and

$$\eta_{inf}^{+}=\frac{\left|\left(s\,R_h^u+s^{-1}\,R_h^\omega\right)\left(s\,\chi^u+s^{-1}\chi^\omega\right)\right|}{\|s\chi_E^u+s^{-1}\chi_E^\omega\|_e} \text{ éq } \eta_{inf}^{-}=\frac{\left|\left(s\,R_h^u-s^{-1}\,R_h^\omega\right)\left(s\,\chi^u-s^{-1}\chi^\omega\right)\right|}{\|s\chi_E^u-s^{-1}\chi_E^\omega\|_e} \qquad \textbf{. 3.2.4.3 3.2.4.3-2}$$

4 of the error in quantity of interest Estimator

4.1 of error in quantity of interest based on the explicit residues

the error in quantity of interest can be estimated from any estimator of the total error. Thus, an estimator of error in quantity of interest is built from the relation of the parallelogram and the estimator of error based on the explicit residues whose η statement on each element is η_E [R4.10.02]: éq

$$\eta_E^2 = h_E^2 \|r_E\|_{L^2(E)}^2 + \frac{1}{2}\sum_{\Gamma \not\subset \Gamma_F} l_\Gamma \|t_\Gamma\|_{L^2(\Gamma)}^2 + \sum_{\Gamma \subset \Gamma_F} l_\Gamma \|t_\Gamma\|_{L^2(\Gamma)}^2 \qquad \text{. 4.1 4.1-1}$$

the use of the statement given by the inequality of Cauchy-Schwartz: éq

$$|Q(e)| \leq \|e\|_e \|\varepsilon\|_e \qquad \text{. 4.1 4.1-2}$$

the estimator of type explicit residue easily gives an estimate of the error in quantity of interest while estimating and $\|e\|_e$ but $\|\varepsilon\|_e$ this estimate is coarse.

The relation of the parallelogram given by: éq

$$Q(e) = a(e,\varepsilon) = \frac{1}{4}\|e+\varepsilon\|_e^2 - \frac{1}{4}\|e-\varepsilon\|_e^2 = \frac{1}{4}\|s\,e+s^{-1}\varepsilon\|_e^2 - \frac{1}{4}\|s\,e-s^{-1}\varepsilon\|_e^2 \qquad \text{. 4.1 4.1-3}$$

(cf $s = \sqrt{\|\varepsilon\|_e / \|e\|_e}$ §3.2.4), provides a more precise estimate because it allows a real coupling between the primal error and the dual error. The use of the estimate of the error of each of the two problems is not possible, the norm of a sum is not equal to the sum of the norms: éq

$$\|e+\varepsilon\|_e^2 = a(e+\varepsilon, e+\varepsilon) = \|e\|_e^2 + \|\varepsilon\|_e^2 + 2\,a(e,\varepsilon) \qquad \text{. 4.1 4.1-4}$$

$$\|e-\varepsilon\|_e^2 = a(e-\varepsilon, e-\varepsilon) = \|e\|_e^2 + \|\varepsilon\|_e^2 - 2\,a(e,\varepsilon) \qquad \text{. 4.1 4.1-5}$$

it is necessary to set out again of the definition of the estimator of error in residue in order to evaluate directly and $\|s\,e+s^{-1}\varepsilon\|_e^2$. $\|s\,e-s^{-1}\varepsilon\|_e^2$ The error in solution being $e = u - u^h$ solution of a problem of elasticity of which the data in force are the residues of equilibrium, it thus acts to formulate the problems of which and $(s\ e+s^{-1}\varepsilon)$ are $(se-s^{-1}\varepsilon)$ solutions.

The primal problem and the dual problem are given by the following relations: éq

$$\begin{cases} a(u,v) = l(v) & \forall\ v \in V \\ a(v,\varepsilon) = Q(v) & \forall\ v \in V \end{cases} \qquad \text{. 4.1 4.1-6}$$

residues associated with these problems are respectively given by: éq

$$\begin{cases} R_h^u(v) = l(v) - a(u^h, v) \\ R_h^\omega(v) = Q(v) - a(v, \omega^h) \end{cases} \qquad \text{. 4.1 4.1-7}$$

replacing by u and $u^h + e$ by ω and $\omega^h + \varepsilon$ by means of the symmetry of, $a(\cdot,\cdot)$ one obtains the equations controlling the errors and e : ε éq

$$\begin{cases} a(e,v)=R_h^u(v) \\ a(\varepsilon,v)=R_h^\omega(v) \end{cases}$$ **. 4.1 4.1-8**

linear combination, there are the two following problems: éq

$$\begin{cases} a(s\,e+s^{-1}\varepsilon,v)=(s\,R_h^u+s^{-1}R_h^\omega)(v) \\ a(s\,e-s^{-1}\varepsilon,v)=(s\,R_h^u-s^{-1}R_h^\omega)(v) \end{cases}$$ **. 4.1 4.1-9**

the numerical error (and $(s\ e+s^{-1}\varepsilon)$) $(se-s^{-1}\varepsilon)$ is generated by the total residue (respectively $(s\,R_h^u+s^{-1}R_h^\omega)$). $(s\,R_h^u-s^{-1}R_h^\omega)$ In a way similar to the estimator in residue, there exists a constant (and C^+), C^- independent of the size of the elements, such as: éq

$$\|s\,e+s^{-1}\varepsilon\|_e^2\le C^+\left[\sum_E h_E^2\|s\,r_E^u+s^{-1}r_E^\omega\|_{L^2(E)}^2+\sum_{\Gamma\not\subset\Gamma_U} l_\Gamma\|s\,t_\Gamma^u+s^{-1}t_\Gamma^\omega\|_{L^2(\Gamma)}^2\right]$$ **. 4.1 4.1-10**

$$\|s\,e-s^{-1}\varepsilon\|_e^2\le C^-\left[\sum_E h_E^2\|s\,r_E^u-s^{-1}r_E^\omega\|_{L^2(E)}^2+\sum_{\Gamma\not\subset\Gamma_U} l_\Gamma\|s\,t_\Gamma^u-s^{-1}t_\Gamma^\omega\|_{L^2(\Gamma)}^2\right]$$ **. 4.1 4.1-11**

the estimator of error in quantity of interest is ${}^{REQI}\eta_E$ defined such as: éq

$${}^{REQI}\eta_E=\frac{1}{4}(\eta_E^+)^2-\frac{1}{4}(\eta_E^-)^2$$ **. 4.1 4.1-12**

and η_E^+ defines η_E^- by: éq

$$(\eta_E^+)^2=c^+\left[h_E^2\|s\ r_E^u+s^{-1}r_E^\omega\|_{L^2(E)}^2+\frac{1}{2}\sum_{\Gamma\not\subset\Gamma_F} l_\Gamma\|s\ t_\Gamma^u+s^{-1}t_\Gamma^\omega\|_{L^2(\Gamma)}^2+\sum_{\Gamma\subset\Gamma_F} l_\Gamma\|s\ t_\Gamma^u+s^{-1}t_\Gamma^\omega\|_{L^2(\Gamma)}^2\right]$$ **. 4.1 4.1-13**

$$(\eta_E^-)^2=c^-\left[h_E^2\|s\ r_E^u-s^{-1}r_E^\omega\|_{L^2(E)}^2+\frac{1}{2}\sum_{\Gamma\not\subset\Gamma_F} l_\Gamma\|s\ t_\Gamma^u-s^{-1}t_\Gamma^\omega\|_{L^2(\Gamma)}^2+\sum_{\Gamma\subset\Gamma_F} l_\Gamma\|s\ t_\Gamma^u-s^{-1}t_\Gamma^\omega\|_{L^2(\Gamma)}^2\right]$$ **. 4.1 4.1-14**

the definitions of the limits of the error in quantity of interest given in part 1, the estimator which we have just defined do not limit the error but gives some an approximate value: éq

$$Q(e)\approx\sum_E {}^{REQI}\eta_E$$ **. 4.1 4.1-15**

estimator thus determined is thus connected more with an error indicator allowing to guide a strategy of mesh. This indicator is programmed in Code_Aster *, it* is available for all element types 2D and 3D (except the pyramids) like for the linear and quadratic elements. Estimator

4.2 in quantity of interest based on the lissages of field

an estimator of error is built directly starting from the techniques of lissage [R4.10.01] and of the fundamental relation between the error in quantity of interest, error on the solution of the primal problem and that on the solution of the dual problem: éq

$$Q(e)=a(e,\varepsilon)=\int_\Omega Tr\left[{}^T(\sigma(u)-\sigma(u^h))\ K^{-1}\ (\sigma(\omega)-\sigma(\omega^h))\right]d\,\Omega$$ **. 4.2 4.2-1**

the approach consists in building from the solutions finite elements and u^h of ω^h the stress fields continuous and $\tilde{\sigma}(u^h)$ $\tilde{\sigma}(\omega^h)$ nearest possible to the exact solutions. This operation of lissage is carried out by the various techniques described in the first part. A higher limit is thus obtained [bib9]: éq

$$a(e,\varepsilon) \leq C \int_\Omega Tr\left[\left(\tilde{\varepsilon}(u)-\varepsilon(u^h)\right) K \left(\tilde{\varepsilon}(\omega)-\varepsilon(\omega^h)\right)\right] d\Omega \quad \textbf{. 4.2 4.2-2}$$

local contributions of error in quantity of interest can be as follows defined: éq

$$^{ZZQI}\eta_E = \int_E Tr\left[\left(\tilde{\varepsilon}(u)-\varepsilon(u^h)\right) K \left(\tilde{\varepsilon}(\omega)-\varepsilon(\omega^h)\right)\right] d E \quad \textbf{. 4.2 4.2-3}$$

the estimator is the sum on the elements of the contributions and is written: éq

$$^{ZZQI}\eta = \sum_E {}^{ZZQI}\eta_E \quad \textbf{. 4.2 4.2-4}$$

estimator is very simple to implement and practically does not require a development for a code equipped with techniques of lissage. This indicator is programmed in Code_Aster *, it* is available for all element types 2D. Estimator

4.3 in quantity of interest based on the hierarchical bases

the starting point to build this estimator is still the fundamental relation of the error in quantity of interest: éq

$$Q(e) = a(e,\varepsilon) \quad \textbf{. 4.3 4.3-1}$$

the resolution of the primal problem and the dual problem with “rich” meshes more (practically they are uniformly refined meshes) makes it possible to obtain the solution improved of $u*$ the primal problem and the solution improved of the dual problem. $\omega*$ By

considering that the approximations and $u*$ are $\omega*$ sufficiently precise compared to the original approximations and u^h then ω^h the error in quantity of interest is given by: éq

$$Q(e) = a(e,\varepsilon) = a(u-u^h, \omega-\omega^h) \approx a(u*-u^h, \omega*-\omega^h) = Q(e*) \quad \textbf{. 4.3 4.3-2}$$

a local estimator of error in quantity of interest can be defined as follows: éq

$$^{BHQI}\eta_E = \int_E Tr\left[\left(\varepsilon(u*)-\varepsilon(u^h)\right) K \left(\varepsilon(\omega*)-\varepsilon(\omega^h)\right)\right] d E \quad \textbf{. 4.3 4.3-3}$$

the total estimator is the sum on the elements and is written as follows: This

$$^{BHQI}\eta = \sum_E {}^{BHQI}\eta_E$$

estimator can be of an interest limited for an industrial use taking into account the cost of generated computation. Indeed the primal problem and the dual problem will have to be each one solved twice: once on the initial mesh and once on a finer mesh thus with a overcost of computation. However this estimator can return the error in quantity of interest available for classes of problems for which no estimator available or is implemented (problems with the eigenvalues or of dynamics in Code_Aster *for example*). Formulation

5 of the quantity of interest As

we already saw in the first part, instead of evaluating a measurement of the error such as the norm in energy for example, it is more useful to consider the error on a quantity of interest having a physical meaning. This quantity of interest (quantity *of interest or* output *) can* be represented by a linear functional calculus definite $Q(\cdot)$ on the space of the functions tests. The goal of the estimate in quantity of interest is thus to make sure of quality of by $Q(u^h)$ estimating the quantity: éq

$$\varepsilon^Q = Q(u-u^h) = Q(e) \qquad \textbf{. 5 - 5-1}$$

to obtain the estimate, it is necessary to use the functional calculus like $Q(\cdot)$ the loading of the dual problem defined by: éq

$$\begin{cases} Trouver\ \omega \in V\ tel\ que \\ \quad a(v,\omega) = Q(v) \qquad \forall\ v \in V \end{cases} \qquad \textbf{. 5 - 5-2}$$

many physical quantities are used in design and must be controlled. These quantities can be, for examples, the average of a component of displacement or the average of a component of the stresses on a subdomain (zone of interest): éq

$$Q(v) = \frac{1}{|\omega|}\int_\omega v_x \, d\Omega \qquad \textbf{. 5 - 5-3}$$

$$Q(v) = \frac{1}{|\omega|}\int_\omega \sigma_{xx} \, d\Omega \qquad \textbf{. 5 - 5-4}$$

quantities are not usable directly like loading of the dual problem. It is necessary to express them in an adequate way in order to find the loading to be imposed on the local problem. The general form of these quantities is the following one [bib25]: éq

$$Q(v) = \int_\Omega f^Q \cdot v \, d\Omega + \int_{\Gamma_F} F^Q \cdot v \, d\Gamma - a(u^Q, v) \qquad \textbf{. 5 - 5-5}$$

, f^Q and F^Q u^Q are given. In the continuation, we will see that simple quantities of interest result directly from this general form to constitute of them typical cases in which one or more terms are null. Quantities

5.1 associated with Component displacements

5.1.1 with displacement in a field

the first quantity related to displacement is the mean value of a component of displacement under - field: ω éq

$$Q^{depl}(v) = \frac{1}{|\omega|}\int_\omega v_i \, d\Omega \quad i = 1,2 \text{ ou } 3 \qquad \textbf{. 5.1 5.1.1-1}$$

$Q^{depl}(v)$ be written: éq

$$\int_\Omega f^Q \cdot v \, d\Omega \qquad \textbf{. 5.1 5.1.1-2}$$

corresponds to the typical case where and $F^Q=0$ $u^Q=0$ where is f^Q a constant vector on and ω no one elsewhere.

The dual problem to solve is written in the following way: éq

$$\int_\Omega \sigma_{ij}(v)\varepsilon_{ij}(\omega)\ d\ \Omega=\int_\Omega f_i^Q\ v_i\ d\ \Omega \quad \forall\ v\in V$$ **. 5.1 5.1.1-3**

the property of symmetry of the tensor of the stresses and while deriving, one obtains: éq

$$\int_\Omega \left(\sigma_{ij}(\omega)\ v_i\right)_{,j}\ d\ \Omega-\int_\Omega \sigma_{ij,j}(\omega)\ v_i\ d\ \Omega=\int_\Omega f_i^Q\ v_i\ d\ \Omega$$ **. 5.1 5.1.1-4**

by means of the formula of Stokes, one a: éq

$$\int_\Omega \left(\sigma_{ij,j}(\omega)+f^Q\right)\ v_i\ d\ \Omega-\int_{\partial\Omega}\sigma_{ij}(\omega)\ n_j\ v_i\ d\ \Omega=0$$ **. 5.1 5.1.1-5**

enables us to write the local dual problem: éq

$$\begin{cases}\sigma_{ij,j}(\omega)+f_i^Q &= 0 \quad dans\ \ \Omega\\ u_i(x) &= 0 \quad sur\ \ \Gamma_U\\ \sigma_{ij}(\omega)\ n_j(x) &= 0 \quad sur\ \ \Gamma_F\end{cases}$$ **. 5.1 5.1.1-6**

notes thus that to impose as $Q^{depl}(v)$ loading of the dual problem amounts imposing a constant voluminal loading on. ω It should be noted that if one searches the error on the vector complete displacement, it will be necessary to solve as much dual problem than components (2 in dimension 2 and 3 in dimension 3). Component

5.1.2 of displacement on an edge Another

useful quantity related to displacement is the mean value of a component of displacement on an edge: $\gamma\subset\Gamma_F$ éq

$$Q^{depl}(v)=\frac{1}{|\gamma|}\int_\gamma v_i\ d\ \Gamma \quad i=1,2\ ou\ 3$$ **. 5.1 5.1.2-1**

$Q^{depl}(v)$ be written: éq

$$\int_{\Gamma_F} F^Q\cdot v\ d\ \Gamma$$ **. 5.1 5.1.2-2**

corresponds to the typical case where and $f^Q=0$ $u^Q=0$ where is F^Q a constant vector on and γ no one elsewhere and with for each v_i ; $F_i^Q=\frac{1}{|\gamma|}$ other components of being F^Q null.

The local dual problem is written in the following way: éq

$$\begin{cases}\sigma_{ij,j}(\omega) &= 0 \quad dans\ \ \Omega\\ u_i(x) &= 0 \quad sur\ \ \Gamma_U\\ \sigma_{ij}(\omega)\ n_j(x) &= F_i^Q \quad sur\ \ \Gamma_F\end{cases}$$ **. 5.1 5.1.2-3**

as $Q^{depl}(v)$ loading of the dual problem amounts imposing a linear in dimension 2 and surface loading in dimension 3 restricted on. Γ_F Normal

5.1.3 displacement on a board Finally

the last quantity related to displacement is the average of normal displacement on a board: $\gamma \subset \Gamma_F$ éq

$$Q^{depl}(v)=\frac{1}{|\gamma|}\int_\gamma v_n \, d\Gamma \qquad \text{. 5.1 5.1.3-1}$$

$Q^{depl}(v)$ be written: éq

$$\int_{\Gamma_F} pn\cdot v \, d\Gamma \qquad \text{. 5.1 5.1.3-2}$$

is n the normal vector with edge and γ on $p=\frac{1}{|\gamma|}$ and γ no one elsewhere.

The local dual problem is written in the following way: éq

$$\begin{cases} \sigma_{ij,j}(\omega) & = & 0 & dans \ \ \Omega \\ u_i(x) & = & 0 & sur \ \ \Gamma_U \\ \sigma_{ij}(\omega)\, n_j(x) & = & pn_i & sur \ \ \Gamma_F \end{cases} \qquad \text{. 5.1 5.1.3-3}$$

as $Q^{depl}(v)$ loading of the dual problem amounts imposing a loading of linear in dimension 2 and surface pressure in dimension 3. Quantities

5.2 associated with the stresses Component

5.2.1 with the stresses in a field We

are interested now in the estimate of the error on the mean value of a component of the stresses under - field: ω éq

$$Q^{sigma}(v)=\frac{1}{|\omega|}\int_\omega \sigma_{ij} \, d\Omega \quad i,j=1,2 \ ou \ 3 \qquad \text{. 5.2 5.2.1-1}$$

to determine, $Q^{sigma}(v)$ it is enough to determine: éq

$$-a(u^Q,v)=\int_\Omega \sigma(v):\Sigma(u^Q) \, d\Omega \qquad \text{. 5.2 5.2.1-2}$$

corresponds to the typical case where and $f^Q=0 \quad F^Q=0$ where is $\Sigma(u^Q)$ a constant operator symmetric on and ω no one elsewhere. For

that, it is enough to write that the component of the stresses is equal to the trace of the product between operators of which one is non-zero in: ω éq

$$Q^{sigma}(v)=\frac{1}{|\omega|}\int_{\omega}\sigma_{ij}\,d\,\Omega=\int_{\omega}\sigma:\Sigma\,d\,\Omega+\underbrace{\int_{\Omega\setminus\omega}\sigma:\Sigma\,d\,\Omega}_{=0}=\int_{\Omega}\sigma:\Sigma\,d\,\Omega\quad i,j=1,2\ ou\ 3$$ **. 5.2 5.2.1-3**

for each, σ_{ij} for $\Sigma_{ij}=\frac{1}{|\omega|}$; $i=j$ for $\Sigma_{ij}=\frac{1}{2|\omega|}$. $i\neq j$

The dual problem to solve is written in the following way: éq

$$\int_{\Omega}\sigma_{i}(v)\ \varepsilon_{ij}(\omega)\,d\,\Omega=\int_{\Omega}\sigma_{ij}(v)\ \Sigma_{ij}\,d\,\Omega$$ **. 5.2 5.2.1-4**

deriving, one obtains: éq

$$\int_{\Omega}\left(\left(\sigma_{ij}(\omega)-a_{ijkl}\Sigma_{kl}\right)v_{i}\right)_{,j}\,d\,\Omega-\int_{\Omega}\left(\sigma_{ij}(\omega)-a_{ijkl}\Sigma_{kl}\right)_{,j}\,v_{i}\,d\,\Omega=0$$ **. 5.2 5.2.1-5**

by means of the formula of Stokes, one a: éq

$$\int_{\Omega}\left(\left(\sigma_{ij}(\omega)-a_{ijkl}\Sigma_{kl}\right)v_{i}\right)_{,j}\,d\,\Omega-\int_{\Omega}\left(\sigma_{ij}(\omega)-a_{ijkl}\Sigma_{kl}\right)n_{j}\,v_{i}\,d\,\Omega=0$$ **. 5.2 5.2.1-6**

enables us to write the local dual problem: éq

$$\begin{cases}\left(\sigma_{ij}(\omega)-a_{ijkl}\Sigma_{kl}\right)_{,j} &= 0 \quad dans \quad \Omega\\ u_{i}(x) &= 0 \quad sur \quad \Gamma_{U}\\ \left(\sigma_{ij}(\omega)-a_{ijkl}\Sigma_{kl}\right)n_{j}(x) &= 0 \quad sur \quad \Gamma_{F}\end{cases}$$ **. 5.2 5.2.1-7**

notes thus that to impose as $Q^{sigma}(v)$ loading of the dual problem amounts imposing an initial strain. Σ It should be noted that if one searches the error on the complete tensor of the stresses, it will be necessary to solve as much dual problem than components (3 in dimension 2 and 6 in dimension 3). Normal stress

5.2.2 on an edge It

can be interesting to consider the error on the normal stress on an edge: γ éq

$$Q^{sigma}(v)=\int_{\gamma}n\cdot\sigma(v)n\,d\,\Omega$$ **. 5.2 5.2.2-1**

quantity does not arise in the general form. The introduction of an auxiliary function is necessary [bib25]. This function is χ defined such as and $\chi|_{\gamma}=1$ equal to zero on the rest of. $\partial\Omega$ If

, $n^{\gamma}=n|_{\gamma}$ the function checks χ the following equation: éq

$$\begin{aligned}a\left(u,\chi\ n^{\gamma}\right) &=\int_{\Omega}\sigma_{ij}(u)\varepsilon_{ij}(\chi\ n^{\gamma})\,d\,\Omega\\ &=\int_{\partial\Omega}\sigma_{ij}(u)\ n_{j}(\chi\ n^{\gamma})_{i}\,d\,\Omega-\underbrace{\int_{\Omega}\left(\sigma_{ij}(u)\right)_{,j}(\chi\ n^{\gamma})_{i}\,d\,\Omega}_{=0}\\ &=\int_{\gamma}\sigma_{ij}(u)\ n_{j}(\chi\ n^{\gamma})_{i}\,d\,\Omega\end{aligned}$$ **. 5.2 5.2.2-2**

shows that the quantity of interest can be written in an equivalent way: éq

$$Q^{sigma}(u)=\int_{\gamma} n\cdot\sigma(u)n\ d\ \Omega=a(u,\chi\ n^{\gamma})=:\tilde{Q}(u) \qquad \text{. 5.2 5.2.2-3}$$

as the preceding equality is not true that for a individual u , it should be noted that: éq

$$Q^{sigma}(v)=\int_{\gamma} n\cdot\sigma(v)n\ d\ \Omega\neq a(v,\chi\ n^{\gamma})=:\tilde{Q}(v) \qquad \text{. 5.2 5.2.2-4}$$

more facility, the quantity $\tilde{Q}(v)$ will be used instead of the quantity. $Q^{sigma}(v)$ Indeed it corresponds to use. $u^{Q}=-\chi\ n^{\gamma}$

The quantity of interest to use is thus written in the following way: éq

$$\tilde{Q}(v)=-a(u^{Q},v)=\int_{\Omega}\sigma(v):\Sigma(-\chi\ n^{\gamma})\ d\ \Omega \qquad \text{. 5.2 5.2.2-5}$$

corresponds to the typical case where and $f^{Q}=0$. $F^{Q}=0$ In

a way similar to the case of the average of the stresses, the local dual problem is written: éq

$$\begin{cases}\left(\sigma_{ij}(\omega)-a_{ijkl}\Sigma_{kl}\right)_{,j} &= 0 \quad dans \quad \Omega\\ u_i(x) &= 0 \quad sur \quad \Gamma_U\\ \left(\sigma_{ij}(\omega)-a_{ijkl}\Sigma_{kl}\right)n_j(x) &= 0 \quad sur \quad \Gamma_F\end{cases} \qquad \text{. 5.2 5.2.2-6}$$

as $\tilde{Q}(v)$ loading of the dual problem amounts imposing a strain field initial calculated starting from the field of displacement. $u^{Q}=-\chi\ n^{\gamma}$ This quantity of interest was not implemented in Code_Aster . *Von Mises stress*

5.2.3 in a field

the von Mises stress is a very useful mechanical quantity in term of design for the engineer. The difficulty to estimate the error on this quantity lies in the fact that its statement is not linear compared to displacement and thus unusable with the technique presented. The difficulty is overcome thanks to a linearization of the statement and under certain assumptions [bib21].

The tensor of the stresses can σ break up into the sum of a tensor deviatoric (whose σ^{d} trace is null) and of a spherical tensor (of which σ^{s} the diagonal extra terms are null): éq

$$\sigma=\sigma^{d}+\sigma^{s} \qquad \text{. 5.2 5.2.3-1}$$

and σ^{d} σ^{s} are defined by: éq

$$\sigma^{d}=\sigma-\frac{1}{3}tr(\sigma)\ \boldsymbol{I} \qquad \sigma^{s}=\frac{1}{3}tr(\sigma)\ \boldsymbol{I} \qquad \text{. 5.2 5.2.3-2}$$

the von Mises stress, noted, σ_{vm} is defined by: éq

$$\sigma_{vm}=\sqrt{\frac{3}{2}\sigma^{d}:\sigma^{d}} \qquad \text{. 5.2 5.2.3-3}$$

linear elasticity, the stress depends linearly on displacement, which enables us to write: éq

$$\sigma^d(u)=\sigma^d(u^h)+\sigma^d(e) \qquad \text{. 5.2 5.2.3-4}$$

this in the definition of the von Mises stress, one obtains: éq

$$\sigma_{vm}(u)=\sqrt{\frac{3}{2}\left(\left(\sigma^d(u^h)+\sigma^d(e)\right):\left(\sigma^d(u^h)+\sigma^d(e)\right)\right)} \qquad \text{. 5.2 5.2.3-5}$$

the preceding statement is rewritten in order to reveal a function whose restricted development is known: éq

$$\begin{aligned}\sigma_{vm}(u) &= \sqrt{\frac{3}{2}\left(\sigma^d(u^h):\sigma^d(u^h)+2\,\sigma^d(u^h):\sigma^d(e)+\sigma^d(e):\sigma^d(e)\right)}\\ &= \sigma_{vm}(u^h)\sqrt{1+\frac{3}{2}\frac{2\,\sigma^d(u^h):\sigma^d(e)+\sigma^d(e):\sigma^d(e)}{\sigma^2_{vm}(u^h)}}\end{aligned} \qquad \text{. 5.2 5.2.3-6}$$

the development limited to order 1 when tends e towards zero makes it possible to obtain the following approximation: éq

$$\begin{aligned}\sigma_{vm}(u) &\approx \sigma_{vm}(u^h)\left(1+\frac{3}{2}\frac{\sigma^d(u^h):\sigma^d(e)}{\sigma^2_{vm}(u^h)}\right)\\ &\approx \sigma_{vm}(u^h)+\frac{3}{2}\frac{\sigma^d(u^h):\sigma^d(e)}{\sigma_{vm}(u^h)}\end{aligned} \qquad \text{. 5.2 5.2.3-7}$$

, an approximation of the error on the von Mises stress is given: éq

$$\sigma_{vm}(u)-\sigma_{vm}(u^h)\approx\frac{3}{2}\frac{\sigma^d(u^h):\sigma^d(e)}{\sigma_{vm}(u^h)} \qquad \text{. 5.2 5.2.3-8}$$

, to estimate the error on the average of the von Mises stress in a field: éq

$$\varepsilon^Q=\frac{1}{|\omega|}\int_\omega\left(\sigma_{vm}(u)-\sigma_{vm}(u^h)\right)\,d\,\Omega \qquad \text{. 5.2 5.2.3-9}$$

the following functional calculus perhaps used: éq

$$Q^{vm}(v)=\frac{3}{2|\omega|}\int_\omega\frac{\sigma^d(u^h):\sigma^d(v)}{\sigma_{vm}(u^h)}\,d\,\Omega \qquad \text{. 5.2 5.2.3-10}$$

for the component of the stresses on a field, to determine, $Q^{vm}(v)$ it is enough to determine: éq

$$-a\left(u^Q,v\right)=\int_\Omega\Sigma(u^Q):\sigma(v)\,d\,\Omega \qquad \text{. 5.2 5.2.3-11}$$

corresponds to the typical case where and $f^Q=0$ $F^Q=0$ where is $\Sigma\left(u^Q\right)$ a constant operator symmetric by element on and ω no one elsewhere. The fact that in the equation 5.2 5.2.3-10 and $\sigma^d(v)$ not requires $\sigma(v)$ to rewrite the functional calculus $Q^{vm}(v)$ to reveal: $\sigma(v)$ éq

$$Q^{vm}(v)=\frac{3}{2|\omega|}\int_{\omega}\frac{\sigma^{d}(u^{h}):\left(\sigma(v)-\sigma^{s}(v)\right)}{\sigma_{vm}(u^{h})}\,d\Omega$$ **. 5.2 5.2.3-12**

$$Q^{vm}(v)=\frac{3}{2|\omega|}\int_{\omega}\frac{\sigma^{d}(u^{h}):\sigma(v)-\sigma^{d}(u^{h}):\sigma^{s}(v)}{\sigma_{vm}(u^{h})}\,d\Omega$$ **. 5.2 5.2.3-13**

the double contraction enters a tensor deviatoric and a spherical tensor being null, the functional calculus is written: éq

$$Q^{vm}(v)=\frac{1}{|\omega|}\int_{\omega}\left(\frac{3}{2\,\sigma_{vm}(u^{h})}\sigma^{d}(u^{h})\right):\sigma(v)\,d\Omega$$ **. 5.2 5.2.3-14**

and $\Sigma\left(u^{Q}\right)=\frac{3}{2\,\sigma_{vm}(u^{Q})}\sigma^{d}(u^{Q})$. $u^{Q}=u^{h}$ In

a way similar to the case of the average of the stresses, the local dual problem is written: éq

$$\begin{cases}\left(\sigma_{ij}(\omega)-a_{ijkl}\Sigma_{kl}\right)_{,j} &= 0 \quad dans \quad \Omega\\ u_{i}(x) &= 0 \quad sur \quad \Gamma_{U}\\ \left(\sigma_{ij}(\omega)-a_{ijkl}\Sigma_{kl}\right)n_{j}(x) &= 0 \quad sur \quad \Gamma_{F}\end{cases}$$ **. 5.2 5.2.3-15**

as $Q^{vm}(v)$ loading of the dual problem amounts imposing a strain field initial. Quantities

5.3 related to the fracture mechanics

5.3.1 the factors of intensity of the stresses

a méthode de calcul of, K_{I} and K_{II} in K_{III} 2D (plane and axisymmetric) and 3D is pointed out; it is based on the extrapolation of the jumps of displacements on the lips of crack. This method is applicable only to the case of plane cracks, in homogeneous and isotropic materials; it is used in command POST _K1_K2_K3.

The stress intensity factors are identified starting from the jump of displacements by $[U]$ a method of the least squares on a segment length. r_{max} This is expressed by the problem of minimization according to: éq

$$K \quad \text{minimise} \quad F(k)=\frac{1}{2}\int_{0}^{r_{max}}\left(C[U(r)]-k\sqrt{r}\right)^{2}dr$$ **. 5.3 5.3.1-1**

to solve this problem, a value of k is sought such as derivative of F is cancelled: éq

$$F'(K)=\int_{0}^{r_{max}}\sqrt{r}\left(K\sqrt{r}-C[U(r)]\right)dr=0$$ **. 5.3 5.3.1-2**

the separation of the integral gives: éq

$$\int_{0}^{r_{max}}r\,K\,dr=\int_{0}^{r_{max}}\sqrt{r}\,C[U(r)]\,dr$$ **. 5.3 5.3.1-3**

$$K\left[\frac{r^2}{2}\right]_0^{r_{max}} = \int_0^{r_{max}} \sqrt{r}\, C\left[U(r)\right] dr \qquad \text{. 5.3 5.3.1-4}$$

the formula clarifies to compute: K results directly: éq

$$K = \frac{2\mathrm{C}}{r_{max}^2} \int_0^{r_{max}} \sqrt{r}\left[U(r)\right] dr \qquad \text{. 5.3 5.3.1-5}$$

carrying out integration by a method of the trapezoids, the preceding relation becomes: éq

$$K = \frac{C}{r_{max}^2} \sum_{i=1}^{N} \left(\left[U\right]_{i-1} \sqrt{r_{i-1}} + \left[U\right]_i \sqrt{r_i} \right) \left(r_i - r_{i-1} \right) \qquad \text{. 5.3 5.3.1-6}$$

is N the number of nodes on the segment. $\left[0, r_{max}\right]$ One notes that is K the linear shape of the field of displacement for fixed r_{max} , typically that corresponds to 4 or 5 elements.

To express the factors of intensity of the stresses in term of loading for the dual problem, we will use the equation 5.3 5.3.1-5

revealing the components of displacement on the upper lip and the $U_i^{sup}(r)$ component of displacement on the lower lip, $U_i^{inf}(r)$ one obtains: éq

$$K_i = \frac{2\,C}{r_{max}^2} \int_\Gamma \left(U_i^{sup}(r) - U_i^{inf}(r) \right) \sqrt{r}\, dr \qquad \text{. 5.3 5.3.1-7}$$

contour Γ is made up by the two lips of crack; is Γ^{sup} the upper lip and is Γ^{inf} the lower lip. This makes it possible to separate the preceding integral in two parts: éq

$$K_i = \frac{2\mathrm{C}}{r_{max}^2} \left[\int_{\Gamma^{sup}} U_i(r) \sqrt{r}\, dr - \int_{\Gamma^{inf}} U_i(r)\, \sqrt{r}\, dr \right] \qquad \text{. 5.3 5.3.1-8}$$

reveal the function test, v the components of displacements are written like the scalar product of two vectors: éq

$$K_i = \frac{2\mathrm{C}}{r_{max}^2} \left[\int_{\Gamma^{sup}} v \cdot g_i(r) \sqrt{r}\, dr - \int_{\Gamma^{inf}} v \cdot g_i(r) \sqrt{r}\, dr \right] \qquad \text{. 5.3 5.3.1-9}$$

; $i = I, II, III$ g_I the normal vector with the plane of crack, g_{II} the normal vector with the crack tip and g_{III} the tangent vector with the crack tip. This thus means that the statement of the quantity of interest is valid in dimension 3 and thus allows the estimate of error of the factors of intensity of the stresses for the problems in dimension 3. Finally

by gathering all the terms, the following statement is obtained: éq

$$Q(v) = K_i(v) = \int_{\Gamma^{sup}} f^{sup} \cdot v\ dr + \int_{\Gamma^{inf}} f^{inf} \cdot v\ dr \qquad \text{. 5.3 5.3.1-10}$$

: and

$$f^{sup} = \frac{2\mathrm{C}}{r_{max}^2} g_i(r) \sqrt{r} \quad \text{éq} \quad f^{inf} = -\frac{2\mathrm{C}}{r_{max}^2} g_i(r) \sqrt{r} \qquad \text{. 5.3 5.3.1-11}$$

statement corresponds to a linear loading in dimension 2 (surface in dimension 3) to impose on a partition of the lips of crack on the basis of the point and length. r_{max} The choice of the radius must r_{max} be made so that it is in the singular zone and contain the possible elements of Barsoum. Implemented

6 and use in Code_Aster

the estimator of error in quantity of interest is implemented in Code_Aster *in* mechanics for all the elements, modelizations C_PLAN , D_PLAN , AXIS and 3D .

To estimate the error in quantity of interest, we saw that it was necessary to solve an assistant problem whose loading in force is the quantity of interest. The following table summarizes and gathers the loadings to be imposed in Code_Aster *, by the means* of the command AFFE_CHAR_MECA , for each quantity of interest. The table is given for 2D but 3D results easily. For the component quantities of the stresses, the value to be indicated is 1.0 for the diagonal terms and 0.5 for the others. Key word

	$Q(v)$	of AFFE_CHAR_MECA Average	Component
of a component of displacement on a contour FORCE_CONTOUR	$\frac{1}{\lvert\gamma\rvert}\int_\gamma v_x \, d\Gamma$	FX	=1. FORCE_CONTOUR
	$\frac{1}{\lvert\gamma\rvert}\int_\gamma v_y \, d\Gamma$	FY	=1. Average
of a component of displacement on under - field FORCE_INTERNE	$\frac{1}{\lvert\omega\rvert}\int_\omega v_x \, d\Omega$	FX	=1. FORCE_INTERNE
	$\frac{1}{\lvert\omega\rvert}\int_\omega v_y \, d\Omega$	FY	=1. Average
of normal displacement on a contour PRES_REP	$\frac{1}{\lvert\gamma\rvert}\int_\gamma v_n \, d\Gamma$	NEAR	=1. Average
of a component of the stresses on under - field PRE	$\frac{1}{\lvert\omega\rvert}\int_\omega \sigma_{xx} \, d\Omega$	_EPSI EPXX	=1. PRE
	$\frac{1}{\lvert\omega\rvert}\int_\omega \sigma_{yy} \, d\Omega$	_EPSI EPYY	=1. PRE
	$\frac{1}{\lvert\omega\rvert}\int_\omega \sigma_{xy} \, d\Omega$	_EPSI EPXY	=0.5 When

the quantities of interest are not linear or when their statement does not make it possible to easily express them in term of loading a particular processing is necessary. For the nonlinear quantities, the theory does not apply; this is why they should be linearized. The von Mises stress is the typical example. The linearization brings us to the following formula: éq

$$Q^{vm}(v)=\frac{1}{\lvert\omega\rvert}\int_\omega \left(\frac{3}{2\,\sigma_{vm}(u^h)}\sigma^d(u^h)\right):\sigma(v)\;d\Omega \qquad \textbf{. 6 - 6-1}$$

one approaches the quantity of interest Von Mises by another linear quantity. The loading to be imposed in Code_Aster *is* a made up stress field: éq

$$\frac{3\mathrm{K}}{2}\frac{\sigma^d(u^h)}{\sigma_{vm}(u^h)} \qquad \textbf{. 6 - 6-2}$$

the definition of the quantity is not adapted to the statement of a loading, it is necessary, there too, to use a processing which will be specific to each quantity of interest. For our applications, the factors of intensity of stresses (FIC) are interesting quantities. By means of a method based on extrapolation of the jumps of displacements on the lips of crack, one can express the FIC in the following way, with in $C=1$ plane stress and $C=1-\nu^2$ plane strain: éq

$$K=\frac{2C}{r_{max}^2}\int_0^{r_{max}}\sqrt{r}\left[U(r)\right]dr \qquad \textbf{. 6 - 6-3}$$

makes it possible to obtain the following statement: éq

$$Q(v)=K_i(v)=\int_{\Gamma^{sup}} f^{sup}\cdot v\ dr+\int_{\Gamma^{inf}} f^{inf}\cdot v\ dr \qquad \textbf{. 6 - 6-4}$$

: and

$$f^{sup}=\frac{2C}{r_{max}^2}g(r)\sqrt{r}\ \text{ éq }\ f^{inf}=-\frac{2C}{r_{max}^2}g(r)\sqrt{r} \qquad \textbf{. 6 - 6-5}$$

vector $g(r)$ which directs crack. In

Code_Aster , *this* vector is known under the name of VECT_K1 . One sees thus that it is necessary to impose a force on each lip of crack (and). f^{inf} The following table shows these two results. Key word

	$Q(v)$	of AFFE_CHAR_MECA_F Average	Component
of von Mises stress PRE	$\frac{1}{\lvert\omega\rvert}\int_\omega \sigma_{vm}\,d\,\Omega$	_SIGM made up	Field Stress intensity factor
FORCE_CONTOUR	$K_I\ \ K_{II}\ \ K_{III}$	F+	on lip sup. F on lip inf. The computation

from the error in quantity of interest passes by the resolution of an elastic design (corresponding to the dual problem) whose loading is on the one hand, the same one as that of the problem principal (or primal) on the conditions of Dirichlet (thus in imposed displacement), and on the other hand, that representing the selected quantity of interest. It is then necessary to provide this result in CALC _ERREUR via key word RESU_DUAL . Lastly,

the various possible options in CALC _ERREUR are : “QIZ1_ELEM
•″ (respectively “QIZ2_ELEM ″): estimator of error in quantity of interest based on the method of Zhu-Zienkiewicz. It is necessary to calculate beforehand option “ERZ1_ELEM ″ or “ERZ2_ELEM ″. “QIRE_ELEM
•″ (by element) or “QIRE_ELNO ″ (by element with the nodes): estimator of error in quantity of interest based on the residues in mechanics. It is necessary to calculate option “ERME_ELEM beforehand ″.

Examples of application on industrial structures could be found in the thesis of J. DELMAS [bib26]. List

tests implementing the estimators of error in quantity of interest: Test

of the estimator of error in quantity of interest sslv	113c test
of the estimator of error in quantity of interest sslv	113d Error indicator
meshes voluminal and functions sslv	153a Detection
of the singularities in a fissured plate (TRIA3) ssnp	130a Estimators
of error in quantities of interest for the computation of the fic zzzz	257a Estimators
of error in quantities of interest for the computation of the fic zzzz	257b Estimators

of error in quantities of interest for the computation of the fic zzzz	257c Estimators
of error in quantities of interest for the computation of the fic zzzz	257d Bibliography

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