Resolution of the modal problem quadratic (QEP)

Summarized

the study of the dynamic stability of a damped and/or revolving structure leads to the resolution of a modal problem higher than the traditional standard modal problems (SEP) or generalized (GEP).

To apprehend them, ***Code_Aster*** **proposes a string of methods *via* operators** **MODE_ITER_INV** **and** **MODE_ITER_SIMULT**: powers opposite and method of Müller, Lanczos, WILL GO and QZ. They have each one their perimeter of use, their advantages/disadvantages and their history of development.

In the first part of the document we summarize the problems of resolution of a quadratic problem and its variation in the general architecture of a modal computation *Code_Aster* . Then we detail the numerical, data-processing and functional aspects of each approach available in the code. The various results, algorithms or parameters approached in this document are often based on the modal methods of orders lower (SEP and GEP) described in the document [R5.01.01]. The reading of this last is thus pre-necessary advised.

Contents

1 Introduction

a majority of study concerning **damped and/or the dynamic revolving structure behavior** is carried out by carrying out **a transient analysis on modal base**. To exhume these modes of vibrations, a string of algorithms have been developed for about sixty years. In order to cope with the continual increase in the size of the problems and with the degradation of conditionings of the discretized operators, only most effective and the more robust, in practice, were built-in the two modal operators of *Code_Aster*: MODE_ITER_SIMULT and MODE_ITER_INV.

The optimal perimeters of use of these operators can be dissociated. When it is a question **of determining some eigenvalues** (typically a half-dozen) or **of refining some estimates**, operator **MODE_ITER_INV** is completely indicated. He gathers heuristic algorithms and those of type powers (cf §3).

On the other hand, **to capture a significant part of the spectrum,** one has resorts to **MODE_ITER_SIMULT**. This last federates the methods known as of "subspace" (Lanczos, IRAM §4) which project the operator of work in order to obtain an approximated modal problem of more reduced size (treated then by a global method of the type QR).

This operator also allows to exhume in a robust way all the spectrum of the modal problem. For this making, one uses a total method of reference (method **QZ** cf §4) which calculates all the modes exhaustively. It is thus to hold for certain uses: problem of small (< 1000 degrees of freedom) or benchmark of algorithms.

The two operators can be complementary besides, because the methods put in work in MODE_ITER_INV are very powerful to optimize eigen modes already almost converged. In one or two iterations, they can thus improve the eigenvectors estimated beforehand *via* MODE_ITER_SIMULT. Projection on modal base will be only better!

In the first part of the document we summarize the problems of resolution of a quadratic problem and its variation in the general architecture of a modal computation *Code_Aster*. Then we detail the numerical, data-processing and functional aspects of each approach available in the code. One tries to give, for each method, his principal properties and his limitations by connecting these considerations, which can sometimes appear a little "ethereal", to a precise parameter setting of the operators *Aster*.

The various results, algorithms or parameters approached in this document are often based on the modal methods of order lower (SEP and GEP) described in the document [Boi09]. The reading of this last is thus pre-necessary advised!

Operator Perimeter of application	Algorithm	Key word	Advantages	Disadvantages Remarks
MODE_ITER_INV				
1st phase (heuristic)				Only symmetric reality
Computation of some Müller-Traub	modes	**"ADJUSTS"**		Cost computation
Improvement of some Initialization	estimates by the user	**"NEAR"**	Taken again eigenvalues estimated by another process. Cost computation of this phase almost zero	No capture of multiplicity
2nd phase (method of the powers itself)				Only symmetric reality
basic Method	Powers opposite	**"DIRECT"**	Very good construction of	Not very robust

Operator Perimeter of application	Algorithm	Key word	Advantages	Disadvantages Remarks
	(Jennings)		eigenvectors	
MODE_ITER_SIMULT				
	Lanczos	"TRI_DIAG"		Only symmetric reality
	IRAM (Sorensen)	**"SORENSEN"**	increased Robustness. Better complexities computation and memory. Quality control of the modes.	Method by default. Range into asymmetric and/or with **A** complexes.
Computation of all the spectrum then filtering of a QZ	part	"QZ"	Robustness. Method of reference.	Very expensive in CPU and memory. A to reserve for the small case (<103 degrees of freedom). Range into asymmetric and/or with **A** complexes.

Table 1-1. Summary of the modal methods to treat the QEP of Code_Aster.

Note :

- *The effective establishment and the maintenance of the modal solvers in Code_Aster are the fruit of a team work: D.Séligmann, B.Quinnez, G.Devesa, O.Boiteau, O.Nicolas, E.Boyère, I.Nistor…*
- *One tried constantly to bind different the items approached and to limit to the strict minimum the recourse to long mathematical demonstrations. In any event, the many references which enamel the text must make it possible to search the accurate information.*
- *The object of this document is not to detail all the aspects approached, of the complete works having already fulfilled this mission. One will find many references in the note [Boi09]. One will quote however the excellent synthesis on the QEP made by F.Tisseur and K.Meerbergen [TM01]. Our document exploits some of its illustrative elements and, in particular, some examples.*

2 Problematic

2.1 context

We consider the modal problem quadratic (QEP)

$$\text{To find } (\lambda, \boldsymbol{u}) \text{ such as } (\lambda^2 \mathbf{B} + \lambda \mathbf{C} + \mathbf{A})\mathbf{u} = \mathbf{0}, \mathbf{u} \neq 0 \qquad (2.1\text{-}1)$$

where $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ of the square matrixes of size n, with real or complex, symmetric coefficients or not. This kind of problem corresponds, in mechanics, in particular under investigation **of free vibrations of a damped and/or revolving structure**. For this structure, one searches the eigenvalues λ_i (and their associated eigenvectors $\mathbf{u}_i$) closest, in the complex plane, of a value of reference given (the "shift" σ) to know if an exiting force can create a resonance. In this standard case,

- the matrix $\mathbf{A}$ is the stiffness matrix, noted $\mathbf{K}$, symmetric real (possibly increased symmetric matrix complexes, noted $\mathbf{E}_{hyst}$, if the structure presents a hysteretic damping): $\mathbf{A} = \mathbf{K} + \mathbf{E}_{hyst}$. Thus $\mathbf{A}$ is symmetric real or complex.
- The matrix $\mathbf{B}$ is the mass matrix or of noted inertia $\mathbf{M}$ (symmetric real).
- The matrix $\mathbf{C}$ gathers, it, the possible gyroscopic effects and those of viscous damping *via* the combination:

$$\mathbf{C} := \mathbf{E}_{visq} + \xi \mathbf{G} \qquad (2.1\text{-}2)$$

with $\mathbf{E}_{visq}$, matrix (symmetric real) of damping induced by dissipative forces, $\mathbf{G}$ stamps gyroscopy (antisymmetric real) and ξ a real parameter representative rotational speed. Thus $\mathbf{C}$ is potentially asymmetric real.

This kind of problems are activated by the key word `TYPE_RESU= `DYNAMIQUE'`. In the general case of the QEP (2.1-1) and by exploiting only their property of symmetry and arithmetic of the matrixes, one can break up the perimeter of the modal operators *as follows* Code_Aster.

Nonrevolving structure with hysteretic damping.

Damped structure (viscous) nonrevolving .

Revolving and/or damped structure (viscous).

A / C	Real symmetri	Real asymmetri	Complex
Real symmetric	Case more running `SIMULT/INV` without restriction on the methods; Real or complex modes $(\lambda, \bar{\lambda})$.	`SIMULT` with "SORENSEN"/"QZ"; Real or complex modes $(\lambda, \bar{\lambda})$.	Case untreated
Real asymmetric	`SIMULT` with "SORENSEN"/"QZ"; Real or complex modes $(\lambda, \bar{\lambda})$.	`SIMULT` with "SORENSEN"/"QZ" real or complex Modes $(\lambda, \bar{\lambda})$.	Complex case
untreated symmetric	`SIMULT` with "SORENSEN"/"QZ" real Modes, unspecified complexes or $(\lambda, \bar{\lambda})$.	Case untreated	Case untreated
Other complexes (hermitian, asymmetric…)	Case untreated	Case untreated	Case untreated

Table 2.1-1. Perimeter of use of the operators Aster (MODE_ITER_SIMULT/INV) and of their méthodes d'analyze (key word METHODE) according to properties of the matrixes $\mathbf{A}$ and $\mathbf{C}$ the QEP (2.1-1). $\mathbf{B}$ is real unspecified.

In absence of hysteretic damping, **the standard QEP** to solve thus consists of **symmetric real matrixes**. These eigenvalues, either real, or complex are combined per pair $(\lambda, \bar{\lambda})$. The eigenvectors are then potentially with complex components.

In the presence of hysteretic damping, the QEP becomes complex and loses the property précédente1Pour[1]. These modes then are real, complex by pair or odd.

Note:

- *Contrary to the GEP [Boi09] treated by Code_Aster, key word TYPE_RESU= `FLAMBEMENT' is not licit in the frame as of QEP.*
- *By means of computer, the perimeter of operator MODE_ITER_SIMULT was extended to the real matrixes not symétriquesdu $\mathbf{A}, \mathbf{B}$ et $\mathbf{C}$ QEP. It is not yet the case for MODE_ITER_INV which is limited, for time, with symmetric real matrixes. This last cannot thus take into account gyroscopic effects.*
- *The modelization of damping [Lev96] in Code_Aster can break up into two classes: viscous damping proportional (known as of Rayleigh $\mathbf{E}_{visq} := \alpha \mathbf{K} + \beta \mathbf{M}$) or hysteretic damping ($\mathbf{E}_{hist} := (1 + i\eta)\mathbf{K}$). Each one is declined at the total level of structure or by means of localised depreciation (mesh groups, ad hoc discrete elements). The latter allowing to better represent the heterogeneity of structure compared to damping. Remain however the question delicate one of the identification of the coefficients $(\alpha, \beta \quad \eta)$ and their influences on result.*

2.2 Properties of the eigen modes

preceding table 2.2-1 does not take account of typical cases (pure gyroscopy, on-deadened structure…). For memory, one will state of them some which one often finds in the literature. But it is necessary well to keep in memory that the context of the modelizations *Code_Aster* (quasi-systematic use of Lagranges double) and the sorts of eigen modes programmed in the modal operators do not take account of these specificities. **Currently in QEP, one retains only the coupled modes $(\lambda, \bar{\lambda})$ and one preserves that at positive imaginary part**. According to the cases, the presence of eigenvalues of other natures (real or complex not combined) is announced by an ALARME or on a purely informative basis.

One thus has (by taking again the elements and the notations of the §3.3 of [Boi09] and paper [TM01]), the distribution of the following properties. Those are sometimes cumulative. Figures 2.2-1 and 2.2-2 illustrate, on canonical cases, certain cases.

	Properties of the matrixes	**Properties of the eigenvalues**	Properties of the eigenvectors	**Example of restrains with the modelizations *Code_Aster***
1	$\mathbf{B}$ not singular	$2n$ finished λ eigenvalues	-	Structure not blocked or blocked by elimination
2	$\mathbf{B}$ singular	$2n$ finished λ or infinite eigenvalues	-	Structure blocked by dualisation
3	$\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ real	Real or per pair $(\lambda, \bar{\lambda})$	If $\mathbf{u}$ eigenvector on the right of λ, then $\bar{\mathbf{u}}$ is that of $\bar{\lambda}$	"classical" QEP (not of hysteretic damping)
4	$\mathbf{A}, \mathbf{B}$ hermitian $\mathbf{C}$ and	formula or per pair $(\lambda, \bar{\lambda})$	If $\mathbf{u}$ eigenvector on the right of λ, then it is also eigenvector	QEP without gyroscopy and with viscous damping

1 to recover it, one needs that the matrixes are, for example, complexes hermitian

	Properties of the matrixes	**Properties of the eigenvalues**	Properties of the eigenvectors	**Example of restrains with the modelizations *Code_Aster***
			on the left of $\bar{\lambda}$	
5	$\mathbf{A}, \mathbf{B}$ et $\mathbf{C}$ hermitian, $\mathbf{B}>0$ and $\mathbf{A}, \mathbf{C}\geq 0$	$\mathrm{Re}(\lambda)\leq 0$	-	Structure free or blocked by elimination, with a damping (dissipative) of Rayleigh and without gyroscopy
6	$\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ symmetric $\mathbf{B}, \mathbf{C}>0$ *and* $\mathbf{A}\geq 0$ + *Condition of on-damping* [2]	Negative real spectrum broken up into two packages of size n	n linearly independent eigenvectors are associated with n the greatest (or smaller) eigenvalues	Structure free or blocked by elimination, with an ad hoc *damping* and without gyroscopy
7	$\mathbf{A}$ and $\mathbf{B}$ hermitian, $\mathbf{C}=-\mathbf{C}^*$ (anti-hermitian) $\mathbf{B}>0$	Imaginary pure or per pair $(\lambda, -\bar{\lambda})$	If $\mathbf{u}$ eigenvector on the right of λ, then it is also eigenvector on the left of $-\bar{\lambda}$	Structure free or blocked by elimination, with gyroscopy
8	$\mathbf{A}$ et $\mathbf{B}$ symmetrical real >0, $\mathbf{C}=-\mathbf{C}^{\mathrm{T}}$ (antisymmetric)	Imaginary pure	-	-

Table 2.2-1. Properties of the modes of a standard QEP (of size n) according to those of its matrixes. Possible links with the modelizations Code_Aster.

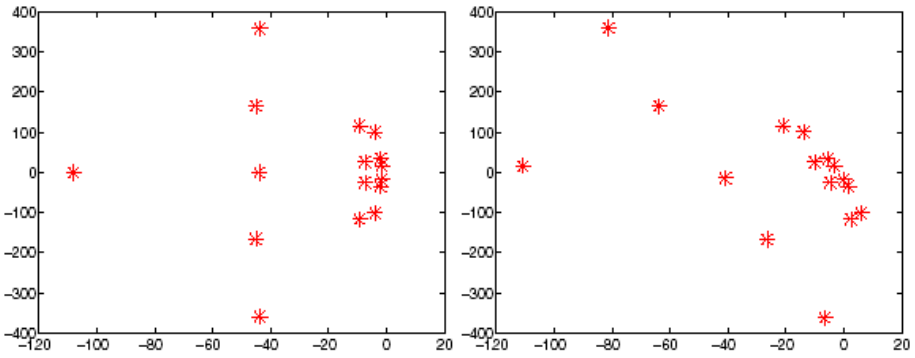

Figure 2.2-1. Examples of the spectrum of a QEP $(n=8)$ of a simplified model of nuclear enclosure [TM01]. Appear of left, one is in the case n°3 table 2.2-1. That of right, one thus adds $\mathbf{K}$ with hysteretic damping appear odd modes (general case n°1).

2 of on-damping: $\gamma(\mathbf{A},\mathbf{B},\mathbf{C}) := \min_{\|\mathbf{x}\|_2=1} \left[(\mathbf{x}^*\mathbf{C}\mathbf{x})^2 - 4(\mathbf{x}^*\mathbf{M}\mathbf{x})(\mathbf{x}^*\mathbf{K}\mathbf{x}) \right] > 0$.

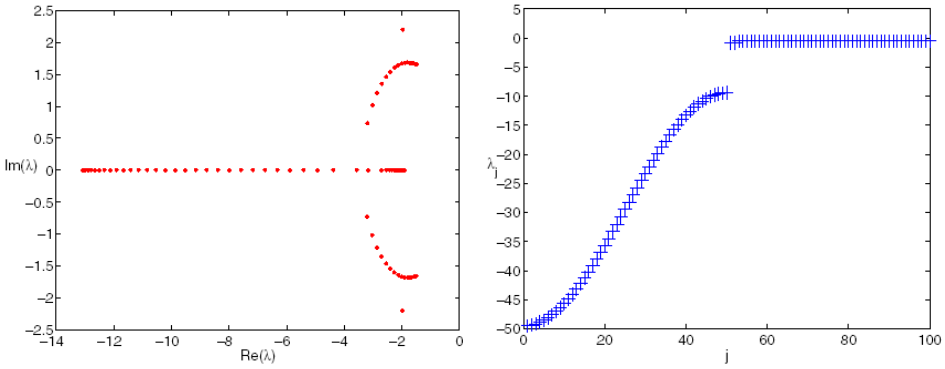

Figure 2.2-2. Examples of the spectrum of a QEP $(n=50)$ of a damped system mass-springs [TM01]. Appear of left, one is in the case n°5 table 2.2-1. That of right, one adds the stress of on-damping and the spectrum becomes real negative in two packages (case n°6).

2.3 Specificities of the QEP and elements of theory

a major algebraic specificity between the QEP and the other classes of modal problems of level lower (GEP and SEP) is **the existence of twice more eigen modes ($2n$) that size of the discretized problem**. The eigenvectors cannot be linearly independent any more, in addition, the eigenvalues can finished or infinite and all “this small world” generally lives in the complexe3C'[3]. The heuristic **robust ones should then be defined to filter the eigenvalues desired by the user** and to distinguish the real ones, of the complexes, the odd complexes of those coupled…

Another complication inherent in the QEP is of order algorithmic. **It does not exist of decomposition of Schur (resp. Schur generalized) as for the SEP (resp. GEP)** on which will be able to rest the algorithm of resolution. For example, for the SEP $\mathbf{A}\,\mathbf{u}=\lambda\,\mathbf{u}$, this decomposition the existence of a unit matrix ensures us (thus conditioned well and easily invertible) $\mathbf{U}$, allowing the rewriting of the matrix of work $\mathbf{A}$ in an easier form manipuler4Les[4]: the higher triangular matrix $\mathbf{T}$.

$$\mathbf{U}\,\mathbf{A}\,\mathbf{U}^{*}=\mathbf{T} \qquad (2.3\text{-}1)$$

the QEP thus constitute a very particular class and importante5Du[5] of the nonlinear problems. Their resolution is less routine than for the other classes of modal problems. In particular, few methods make it possible to solve the problem directly, it is often necessary to forward by a GEP (*via* an ad hoc *linearization*) then a SEP (spectral transformation) what induces a loss of spectral information and numerical instabilities (propagation of rounding errors).

In the theoretical literature, rather than in the QEP (2.1-1), one is interested in **the matric polynomial** (quadratic in λ)

$$\mathbf{Q}(\lambda):=(\lambda^{2}\mathbf{B}+\lambda\,\mathbf{C}+\mathbf{A}) \qquad (2.3\text{-}2)$$

One it call also λ-matrice **or stamps dynamic** and one seeks his noted spectrum $spectre(\mathbf{Q}):=\{\lambda\in C\,/\,det\,\mathbf{Q}(\lambda)=0\}$. When this determinant is null whatever λ , is λ-matrice tell to it singular. In the contrary case, it is known as regular (one places oneself in general in this frame!).

3 plane is already the case for certain classes of GEP: asymmetric or with complex matrix

4 eigenvalues of the problem are found on the principal diagonal of $\mathbf{T}$ and one can deduce easily from the vectors of Schur (the vectors columns of $\mathbf{Q}$), their associated eigenvectors.

5 makes in particular these increasing applications in industry: analyzes vibratory, vibro-accoustic, mechanics of the fluids with turbulence, electronics (VLSI)…

The characteristic polynomial of the problem is thus written in the form $det\,\mathbf{Q}(\lambda)=det(\mathbf{B})\lambda^{2n}+$ terms of a lower nature in λ. Thus, as soon as $\mathbf{B}=\mathbf{M}$ is singular, this polynomial admits $r<2n$ finished roots for which it is necessary to add $2n-r$ infinite. In addition, these distinct eigenvalues can divide the same eigenvectors. It is the "dual situation" of the multiple eigenvalues of the SEP where the same eigenvalue is attached to several eigenvectors. Thus, for the canonical QEP

$$\mathbf{A}=\mathbf{I}_3,\ \mathbf{B}=\begin{bmatrix}0&6&0\\0&6&0\\0&0&1\end{bmatrix}\text{ et }\boldsymbol{C}=\begin{bmatrix}1&-6&0\\2&-7&0\\0&0&0\end{bmatrix} \tag{2.3-3}$$

there is the following spectrum

$$\begin{array}{cccccccc} i & 1 & 2 & 3 & 4 & 5 & 6 \\ \lambda_i & 1/3 & 1/2 & 1 & i & -i & \infty \\ \mathbf{u}_i & \begin{bmatrix}1\\1\\0\end{bmatrix} & \begin{bmatrix}1\\1\\0\end{bmatrix} & \begin{bmatrix}0\\1\\0\end{bmatrix} & \begin{bmatrix}0\\0\\1\end{bmatrix} & \begin{bmatrix}0\\0\\1\end{bmatrix} & \begin{bmatrix}1\\0\\0\end{bmatrix} \end{array} \tag{2.3-4}$$

Of the singularity of $\mathbf{B}$ comes the infinite eigenvalue. One and the notes also the division of eigenvectors by the first two eigenvalues 4th/5th.

To more easily apprehend the infinite eigenvalues of $\mathbf{Q}(\lambda)$, one associates them with the null eigenvalues of **the polynomial reverses**

$$rev\,\mathbf{Q}(\lambda):=\lambda^2\mathbf{Q}\left(\frac{1}{\lambda}\right)=\left(\mathbf{B}+\lambda\mathbf{C}+\lambda^2\mathbf{A}\right) \tag{2.3-5}$$

These **eigenvalues finished of** $rev\,\mathbf{Q}(\lambda)$ **is easier to calculate** and shares the same spectral characteristics (multiplicities geometrical and algebraic, clean space…) that initial infinite values.

For further information on these theoretical aspects one will be able to consult, for example, the works of P.Lancaster [GLR82] [LT85] and the thesis of D.S.Mackey [Mac06] (and its many bibliographical references). Strategy

2.4 of linearization Introduction

2.4.1 the standard

approach to solve a modal problem is to transform it to reveal canonical matrixes revealing the required modes. For example, the decomposition of Schur for the SEP or that generalized for the GEP. Unfortunately, this practice cannot spread with the modal problems of a higher nature. Two solutions are offered then: To solve

- **the nonlinear problem directly (root of** polynomial, factorization of, methods λ-matrice of the Newton type…). To linearize
- **in a GEP, then to treat** this last via the adapted methods [Boi09]. The first

strategy is declined in MODE_ITER _INV (method of Müller-Traub) to capture a first estimate of the eigenvalues. One then blends it **with the following strategy, because** to refine the estimates and to accelerate convergence, one calls on a method of the type powers opposite (alternative of Jennings) on the linearized QEP. Certain authors propose complete direct approaches. They are often based on alternatives of Newton. But their convergence is slow (a mode at the same time) and not always assured (cf Kublanovskaya 1970, G.Peters & J.H.Wilkinson 1979 and A.Ruhe 1973). The strategy of linearization
is thus often privileged. It doubles the size of the modal problem to treating, changes its nature (with the risk to lose information spectrale6On finds the solution[6] instabilities (sensitivities to the errors of arrondis7L' modal

6 of a "GEP approximate" but this one does not correspond inevitably to the exact solution of a "approximate QEP" because solvers GEP do not respect particular structure of the linearized problem (cf [TM01] §5).) and introduced numerical

application of[7] 8Les results can[8] the immense advantage of re-using all “numerical artillery” already deployed for GEP9On made in the same way for[9]

all the spectrum simultaneously (cf method QZ of MODE_ITER_SIMULT) or only it left centered on a zone interest (cf IRAM or Lanczos) by resorting to the spectral transformation idoine. Principle Now that

2.4.2 the context

was brushed, to consolidate itself with the terminology employed in the literature, one from now on will simplify the notations by considering the QEP associated with. This matrix λ-matrice $\mathbf{Q}(\lambda):=\left(\lambda^2\mathbf{M}+\lambda\mathbf{C}+\mathbf{K}\right)$ is also called “dynamic matrix” in the jargon of the dynamician of structures (and in the data-processing documents/sources of Code_Aster). It is *during* that described in the note dedicated to the GEP: . Definition 1 From $\mathbf{Q}(\lambda):=\left(\mathbf{K}-\lambda\,\mathbf{M}\right)$ a theoretical

point of view
, to linearize come down to be found another $\mathbf{Q}(\lambda)$ (complex square of size λ-matrice) of the type such as $2n$ there exist two $\mathbf{A}-\lambda\mathbf{B}$ different (formula and formula) λ-matrices with $\mathbf{E}(\lambda)$ $\mathbf{F}(\lambda)$ determinant, checking the relation (2.4-1) the matrixes

$$\begin{bmatrix}\mathbf{Q}(\lambda) & \mathbf{0}_n\\ \mathbf{0}_n & \mathbf{I}_n\end{bmatrix}=\mathbf{E}(\lambda)\left(\mathbf{A}-\lambda\,\mathbf{B}\right)\mathbf{F}(\lambda)$$ and

are called $\mathbf{A}$ the matrixes $\mathbf{B}$ “companion” of the linearization of the QEP. Clearly, the spectrums

of the QEP initialet that of its linearization $\left(\mathbf{Q}(\lambda)\right)$ coincide. However $\left(\mathbf{A}-\lambda\,\mathbf{B}\right)$ this decomposition N **“is not single and it is necessary to choose, if possible** , that which: Preserve the properties

- **of the initial matrixes** (arithmetic, symmetry…), Presents less
- sensitivity **to the errors D” rounded (“backward stable”:** balanced, well conditioned, invertible matrixes…), Handles canonical
- matrixes (identity **, triangular** …), Adapts to a broad
- perimeter of QEP **(viscous damping** , gyroscopy, on-damping…). A more intuitive way

to apprehend the techniques of linearization, consists in introducing a change of variable of the 10Comme type for the solutions $\mathbf{v}=\lambda\,\mathbf{u}$ [10] it in matric form: (2.4-2) This “

$$\left(\underbrace{\begin{bmatrix}\mathbf{0}_n & \mathbf{I}_n\\ -\mathbf{K} & -\mathbf{C}\end{bmatrix}}_{\mathbf{A}}-\lambda\underbrace{\begin{bmatrix}\mathbf{I}_n & \mathbf{0}_n\\ \mathbf{0}_n & \mathbf{M}\end{bmatrix}}_{\mathbf{B}}\right)\begin{bmatrix}\mathbf{u}\\ \mathbf{v}\end{bmatrix}=\mathbf{0}_{2n}$$ very

classical” linearization returns in the frame of definition 1 if they are used: (2.4-3) In fact, λ-matrices the linearization

$$\mathbf{E}(\lambda):=\begin{bmatrix}-(\mathbf{C}+\lambda\,\mathbf{M}) & -\mathbf{I}_n\\ \mathbf{I}_n & \mathbf{0}_n\end{bmatrix},\ \mathbf{F}(\lambda):=\begin{bmatrix}\mathbf{I}_n & \mathbf{0}_n\\ \lambda\,\mathbf{I}_n & \mathbf{I}_n\end{bmatrix}$$

of a QEP marries the general frame of that **of the matric polynomials. While posing (here formula** etc), one $\mathbf{Q}(\lambda):=\sum_{i=1}^{k}\lambda_i\mathbf{X}_i$ $k=2,\mathbf{X}_i=\mathbf{M}$ is always ensured to find two linearizations, known as “companions”, of the form and with (2.4-4) In practice $\mathbf{A}_1-\lambda\,\mathbf{B}_1$ $\boldsymbol{A}_2-\lambda\,\boldsymbol{B}_2$, one

7 solver of GEP to a linearized QEP is not “backward stable”. I.e. this technique has a propensity to transmit/amplify the rounding errors. and with the numerical parameters
8 be sensitive to the various parameters of the modal solvers used in fine, but more especially , with those of the two preprocessings: linearization and spectral transformation.). But she has
9 transition GEP/SEP ([Boi09] §3.7). One can thus capture
10 of differential equations of the second order of the lower school! in (2.1-1) and to rearrange

$$\mathbf{B}_1=\mathbf{B}_2:=\begin{bmatrix}-\mathbf{X}_k & \mathbf{0}_{(k-1)n}\\ \mathbf{0}_{(k-1)n} & -\mathbf{I}_{(k-1)n}\end{bmatrix},$$

$$\mathbf{A}_1:=\begin{bmatrix}\mathbf{X}_{k-1} & \mathbf{X}_{k-2} & \dots & \mathbf{X}_0\\ \mathbf{I}_n & \mathbf{0}_n & \dots & \mathbf{0}_n\\ \dots & \dots & \dots & \dots\\ \mathbf{0}_n & \dots & -\mathbf{I}_n & \mathbf{0}_n\end{bmatrix},\quad \mathbf{A}_2:=\begin{bmatrix}\mathbf{X}_{k-1} & -\mathbf{I}_n & \dots & \mathbf{0}_n\\ \mathbf{X}_{k-2} & \mathbf{0}_n & \dots & \dots\\ \dots & \dots & \dots & -\mathbf{I}_n\\ \mathbf{X}_0 & \mathbf{0}_n & \dots & \mathbf{0}_n\end{bmatrix}$$

prefers to them linearizations more adapted to the treated case (cf following paragraph). Some examples In

2.4.3 the literature,

it is often question following **linearizations (with a square regular** N matrix complexes of size): () (2.4-5) () (2.4 n - 6

) L_1 $\left(\underbrace{\begin{bmatrix}\mathbf{0}_n & \mathbf{N}\\ -\mathbf{K} & -\mathbf{C}\end{bmatrix}}_{\mathbf{A}}-\lambda\underbrace{\begin{bmatrix}\mathbf{N} & \mathbf{0}_n\\ \mathbf{0}_n & \mathbf{M}\end{bmatrix}}_{\mathbf{B}}\right)\begin{bmatrix}\mathbf{u}\\ \lambda\mathbf{u}\end{bmatrix}=\mathbf{0}_{2n}$ choices depends

L_2 $\left(\underbrace{\begin{bmatrix}-\mathbf{K} & \mathbf{0}_n\\ \mathbf{0}_n & \mathbf{N}\end{bmatrix}}_{\mathbf{A}}-\lambda\underbrace{\begin{bmatrix}\mathbf{C} & \mathbf{M}\\ \mathbf{N} & \mathbf{0}_n\end{bmatrix}}_{\mathbf{B}}\right)\begin{bmatrix}\mathbf{u}\\ \lambda\mathbf{u}\end{bmatrix}=\mathbf{0}_{2n}$ on

the potential singularity of the matriceset. One wishes to be able $\mathbf{K}$ to reverse $\mathbf{M}$ the diagonal terms, therefore if is singular (resp. $\mathbf{K}$), one uses (formula $\mathbf{M}$) (resp. (formula L_1 To balance L_2 the terms of the matrixes and to facilitate handling, one often takes with real factor of $\mathbf{N}=\alpha\mathbf{I}_n$ balancing α (cf [Boi09] appendix 1) built on the standard basis matric of the other terms. To work with $\alpha=\alpha(\|\mathbf{K}\|,\|\mathbf{M}\|,\|\mathbf{C}\|)$ a symmetric GEP (even definite or semi-definite positive), one can also initialize the auxiliary matrix. More effectively $\mathbf{N}$ à$\alpha\mathbf{K}$ ou$\beta\mathbf{M}(\alpha,\beta\in\Re)$ to solve

the QEP with gyroscopic effects **, one advises in** the literature hanging “the Hamiltoniens11Une stamps square[11]linearizations. “Companions are the matrixes” () (2.4-7) () (2.4-8

) L_3 Here $\left(\underbrace{\begin{bmatrix}\mathbf{K} & \mathbf{0}_n\\ \mathbf{C} & \mathbf{N}\end{bmatrix}}_{\mathbf{A}}-\lambda\underbrace{\begin{bmatrix}\mathbf{0}_n & \mathbf{K}\\ -\mathbf{M} & \mathbf{0}_n\end{bmatrix}}_{\mathbf{B}}\right)\begin{bmatrix}\mathbf{u}\\ \lambda\mathbf{u}\end{bmatrix}=\mathbf{0}_{2n}$ the purpose

L_4 $\left(\underbrace{\begin{bmatrix}\mathbf{0}_n & -\mathbf{K}\\ \mathbf{M} & \mathbf{N}\end{bmatrix}}_{\mathbf{A}}-\lambda\underbrace{\begin{bmatrix}\mathbf{M} & \mathbf{C}\\ \mathbf{0}_n & \mathbf{M}\end{bmatrix}}_{\mathbf{B}}\right)\begin{bmatrix}\mathbf{u}\\ \lambda\mathbf{u}\end{bmatrix}=\mathbf{0}_{2n}$ is not

to respect a property of symmetry but rather those “Hamiltoniennes12C' be-with-to say that[12] singular (resp. $\mathbf{K}$), one uses () (resp $\mathbf{M}$. ()). In the case of L_4 L_3 a QEP

11 order, is known as Hamiltonian $2n$, $\mathbf{A}$ (resp. Anti-Hamiltonian) if it checks the relation (resp. with the matrix $(\mathbf{AJ})^{\mathrm{T}}=\mathbf{AJ}$ anti $(\mathbf{AJ})^{\mathrm{T}}=-\mathbf{AJ}$ unit $\mathbf{J}$ - symmetric.” preceding $\mathbf{J}:=\begin{bmatrix}\mathbf{0}_n & \mathbf{I}_n\\ -\mathbf{I}_n & \mathbf{0}_n\end{bmatrix}$

12 the solutions are symmetric compared to the axes reality and imaginary: ” problem. If is $(\lambda,\bar{\lambda},-\lambda,-\bar{\lambda})$.

says "T-palindromique **" [Mac06] one prefers** the linearization $(\mathbf{K}=\mathbf{M}^T \text{ et } \mathbf{C}=\mathbf{C}^T)$ () (2.4-9) Strategies

$$L_5 \text{ retained } \left(\underbrace{\begin{bmatrix}\mathbf{K} & \mathbf{K}\\ \mathbf{C}-\mathbf{M} & \mathbf{K}\end{bmatrix}}_{\mathbf{A}}-\lambda\underbrace{\begin{bmatrix}-\mathbf{M} & \mathbf{K}-\mathbf{C}\\ -\mathbf{M} & -\mathbf{M}\end{bmatrix}}_{\mathbf{B}}\right)\begin{bmatrix}\mathbf{u}\\ \lambda\mathbf{u}\end{bmatrix}=\mathbf{0}_{2n} \quad \text{in}$$

2.4.4 Code_Aster One thus has a very large

variety of choice to adapt the linearization of the QEP to each situation. For time, Code_Aster does not propose to the user of parameter making it possible to modulate only the linearization. This one is fixed for a given modal solver: With MODE_ITER_INV (2nd

- ***phase) transition QEP /GEP*** is carried out via the companion matrixes of (L1), by posing the linearized system $\mathbf{N}=\mathbf{I}_n$. is not symmetric but it is not pre-necessary to apply a method **of the type powers opposite. (L1) "(2.4-10) With**

$$\text{MODE\_ITER\_SIMULT } \left(\underbrace{\begin{bmatrix}\mathbf{0}_n & \mathbf{I}_n\\ -\mathbf{K} & -\mathbf{C}\end{bmatrix}}_{\mathbf{A}}-\lambda\underbrace{\begin{bmatrix}\mathbf{I}_n & \mathbf{0}_n\\ \mathbf{0}_n & \mathbf{M}\end{bmatrix}}_{\mathbf{B}}\right)\begin{bmatrix}\mathbf{u}\\ \lambda\mathbf{u}\end{bmatrix}=\mathbf{0}_{2n}$$

- ***(METHODE =" TRI_DIAG' and "SORENSEN") transition QEP/GEP*** is carried out via the companion matrixes of (L2) while posing. If the stiffness matrixes $\mathbf{N}=\mathbf{M}$, of mass and damping are symmetric real, one falls down in the perimeter of the standard GEP (symmetric realities). The methods of Lanczos and Sorensen are then accessible. If becomes complex or if $\mathbf{K}$ one of them is asymmetric, only IRAM remains in race (with QZ cf item according to)! (L2) "(2.4-11) However

$$\text{, when } \left(\underbrace{\begin{bmatrix}-\mathbf{K} & \mathbf{0}_n\\ \mathbf{0}_n & \mathbf{M}\end{bmatrix}}_{\mathbf{A}}-\lambda\underbrace{\begin{bmatrix}\mathbf{C} & \mathbf{M}\\ \mathbf{M} & \mathbf{0}_n\end{bmatrix}}_{\mathbf{B}}\right)\begin{bmatrix}\mathbf{u}\\ \lambda\mathbf{u}\end{bmatrix}=\mathbf{0}_{2n} \quad \text{formula}$$

N" is not invertible $\mathbf{M}$ **(fact for example of** Lagranges of blocking), the inversion of the diagonal term of formula is not assured any more $\mathbf{A}$ To mitigate this problem, one introduces a "regularization" **of the matrix formulates, noted, in** $\mathbf{M}$ term $\boldsymbol{M}_R$. This one has the property to cancel , by multiplication, all the components of the core of. Thus, one can define $\mathbf{M}:\mathbf{M}_R\mathbf{u}=\mathbf{0}$ si $\mathbf{u}\in Ker(\mathbf{M})$ who formally checks $\mathbf{M}_R^{-1}$ the property on the components " $\mathbf{M}_R^{-1}\mathbf{M}\approx\mathbf{M}\mathbf{M}_R^{-1}\approx\mathbf{I}_n$ except core of" handled vectors $\mathbf{M}$. Because of particular structure of the dualized matrix (cf [Boi09] §3.2), $\mathbf{M}$ this handling amounts proscribing the components related on Lagranges and the blocked degrees of freedom (of the demonstration of $\mathbf{v}_i^{2,3}$ the property n°4 [Boi09]). What does not obstruct of anything the process since one is interested only in the physical modes and not in these artefacts of modélisations13Un proceeded identical[13]With

$$\text{MODE\_ITER\_SIMULT } \left(\underbrace{\begin{bmatrix}-\mathbf{K} & \mathbf{0}_n\\ \mathbf{0}_n & \mathbf{M}_R\end{bmatrix}}_{\mathbf{A}}-\lambda\underbrace{\begin{bmatrix}\mathbf{C} & \mathbf{M}\\ \mathbf{M} & \mathbf{0}_n\end{bmatrix}}_{\mathbf{B}}\right)\begin{bmatrix}\mathbf{u}\\ \lambda\mathbf{u}\end{bmatrix}=\mathbf{0}_{2n}$$

- ***(METHODE =" QZ"), one preferred to avoid*** these complications. The purpose being to privilege the numerical **robustness for treating small** cases (degrees of freedom). $<10^3$ The

13 is often used to solve the GEP. It consists in initializing the algorithm of resolution by a vector of the image group of (thus filtering the elements $\mathbf{M}$ of the cores). Instead of the random vector, one takes. (L2) " $\mathbf{u}_i$ " (2.4-12) $\mathbf{M}\mathbf{u}_i$.

scenario (L2) was empirically retained while posing, with, (L2) * (2.4-13 $\mathbf{N}=\alpha\mathbf{I}_n$) Establishment

$$\alpha := \frac{1}{3n}\left(\|\mathbf{K}\|+\|\mathbf{M}\|+\|\mathbf{C}\|\right)$$

$$\left(\underbrace{\begin{bmatrix}-\mathbf{K} & \mathbf{0}_n\\ \mathbf{0}_n & \alpha\mathbf{I}_n\end{bmatrix}}_{\mathbf{A}}-\lambda\underbrace{\begin{bmatrix}\mathbf{C} & \mathbf{M}\\ \alpha\mathbf{I}_n & \mathbf{0}_n\end{bmatrix}}_{\mathbf{B}}\right)\begin{bmatrix}\mathbf{u}\\ \lambda\mathbf{u}\end{bmatrix}=\mathbf{0}_{2n}$$

in Code_Aster

2.5 the course *of*

a quadratic modal computation is, by construction, very near to that of a generalized problem. One will thus be attached to the functional diagram of resolution of the GEP describes in [Boi09] (cf § 3.8). Its stages break up as follows: Preprocessings (linearization

2.5.1 , computation of the shift) Determination of

the linearized form and **the spectral transformation associated according to the adopted method** (power, Lanczos, IRAM and QZ) and the parameters "ADJUSTS"/"NEAR " (for MODE_ITER_INV) and "PLUS_PETITE "/"CENTER" (for MODE_ITER_SIMULT). It should be noted that the eigen modes

being in the complex plane, the test of Sturm type currently established is inoperative. Thus the heuristics of MODE_ITER_INV is not based on the modal positions and the option "BANDAGES" MODE_ITER_SIMULT is proscribed . In the same way for the post-checking based on the test of Sturm type [Boi12]. For the method of

subspace **(MODE_ITER_SIMULT+METHODE** = `TRI_DIAG' and "SORENSEN"), once the updated linearized problem, one applies (L_2)'' a spectral transformation of standard "shift-and-invert" (cf [Boi09] §3.6/ §5). That will make it possible to complete the transformation in SEP14Sur which apply[14] (S_1) of the spectrum in a zone of interest for the user and to better separate the eigenvalues. (2.5-1) the method of

$$(S_1)\quad \underbrace{(\mathbf{A}-\sigma\mathbf{B})^{-1}}_{\mathbf{A}_\sigma}\mathbf{B}\underbrace{\begin{bmatrix}\mathbf{u}\\ \lambda\mathbf{u}\end{bmatrix}}_{\mathbf{W}}=\underbrace{\frac{1}{\lambda-\sigma}}_{\mu}\underbrace{\begin{bmatrix}\mathbf{u}\\ \lambda\mathbf{u}\end{bmatrix}}_{\mathbf{W}}$$

the powers

opposite applies it a spectral transformation particular to the linearized problem. As for the method (L_1)'

QZ (MODE_ITER_ **SIMULT+METHODE=** `QZ'), this spectral transformation is superfluous. The algorithm deals with the linearized problem directly and calculates all its modes $(L_2)^*$. According to the wishes of user (OPTION, NMAX_FREQ…), it filters then the results. Note: As for

the techniques

- *of linearization, no parameter user makes it possible yet to change spectral transformation. Into quadratic, it would be interesting to use transformations more adapted to the complex plane than traditional "the shift-and-invert". For example, a transformation of the Cayley type to make converge simultaneously the couples of complex-conjugate eigenvalues (2.5-2) This strategy* $(\lambda,\bar{\lambda})$

$$\underbrace{(\mathbf{A}-\sigma\mathbf{B})\,\mathbf{B}^{-1}(\mathbf{A}-\bar{\sigma}\mathbf{B})}_{\mathbf{A}_\sigma}\,\mathbf{B}^{-1}\,\mathbf{w}=\underbrace{\frac{1}{(\lambda-\sigma)(\bar{\lambda}-\sigma)}}_{\mu}\mathbf{W}$$

of Cayley

- *spreads to treat the gyroscopy (cf V.Mehrman and D.Watkins 2001) and to make converge simultaneously the quadruplets Certain authors also* $(\lambda,\bar{\lambda},-\lambda,-\bar{\lambda})$

14 the classical modal solvers. , to direct the search

- *propose to invert the phase of linearization and that of spectral transformation [Mee07]. Factorization of dynamic*

2.5.2 matrixes As one already underlined

these factorizations of dynamic matrixes of the typene are not used to implement $\mathbf{Q}(\lambda)$ a test of Sturm type. This last being illicit for the QEP. However they remain an ingredient very present and very couteux15Complexity computation enavecla[15] cost

- of the heuristic part (method of Müller-Traub) with MODE_ITER_INV (OPTION=' AJUSTE'). When one must handle
- a matrix of work comprising of the opposite. To be more effective, one seeks a formulation then utilizing only (MODE_ITER_INV+' DIRECT' $\mathbf{Q}(\lambda)^{-1}$ and MODE_ITER_SIMULT+' TRI _DIAG'/"IRAM"). Comprehensive approach (

MODE_ITER_SIMULT+' QZ') is not concerned with this preliminary factorization of the dynamic matrix. Modal computation The computation

2.5.3 modal itself

is carried out: one solves standard problem SEP, one returns with the intermediate GEP, then with the initial QEP. A each one of these conversions, one filters and transcribes the results of preceding computation. Concerning the eigenvalues (cf **§2.1/2.2) one** does not retain that with positive imaginary part λ of the coupled modes your problem is strongly $(\lambda, \bar{\lambda})$.

```
damped. value (S) clean (S) real
 (S): 2 value (S) clean (                  S)  complex
 (S) with combined: 107 value (S) clean (S  ) complex
 (S) without combined: 0 Example 1. Printings
```

in the message file of alarms recapitulating the number of modes not selected (here 2 realities). Extract of the sdll123a benchmark. For the eigenvectors

, two **possibilities remain** open: To take directly

- the first components n of (part known as "high" w), To choose the last
- (left known as n "low") possibly divided by the same scalar. In Code_Aster, it

is *the second* option which was retained. This choice is not always anodin16La alternative of the method[16] the quality of the modes and their sensitivities to the rounding errors. In MODE_ITER_INV only one

method is available, an alternative of the method of the inverse iterations due to Jennings ("DIRECT `). It refines the eigenvalues detected beforehand by the method of Müller-Traub (" AJUSTE') or the estimates provided by the user ("NEAR `). As regards MODE_ITER_SIMULT, it allows the use of three distinct methods: method of Lanczos ("TRI_DIAG"), that of Sorensen ("SORENSEN") and QZ ("QZ"). Only the two last are available with real or complex matrixes asymmetric symmetric (for only). Remarks $\mathbf{K}$ Each one of these

methods

- *has internal tests of stops. Not counting the methods of projection employ auxiliary modal methods: for Lanczos and IRAM. They* $\mathbf{QR}/\mathbf{QL}$ *require also tests of stops. The user often has access to these parameters, although it is warmly recommended, at least initially, to preserve their values by default. Contrary to the GEP,*
- *it is not forced - orthonormalization of* $\mathbf{M}$ *the eigenvectors. It N" has any meaning in QEP. Moreover, by applying result proposal 2 of [Boi09] to the linearized problem, it is clear that the eigenvectors are and - orthogonal. But* $\mathbf{A}$ *because of* $\boldsymbol{B}$ *complexity of the linear reductions*

15 bandwidth $O(bn^2)$ eta b cuts problem. n for: To estimate the function

16 QZ suggested by F.Tisseur [TM01] is based on the values of "the opposite error" of each mode, to choose the high or low part eigenvectors. in particular with respect to

used, that does not lead (except typical case) to eigenvectors, or - orthogonal! $\mathbf{K}$ Post $\mathbf{M}$ - processing $\mathbf{C}$ of checking

2.5.4 This last part

gathers the only postprocessing which checks the good progress of computation (in the absence of test of Sturm type adapted to the complex case). One calculates the relative error on **the hermitian norme17Norme and[17] Algorithm 1. Test of**

$$\mathbf{u} \Leftarrow \frac{\mathbf{u}}{\left[\mathbf{u}\right]_\infty}$$

Si $|\lambda| >$ SEUIL_FREQ alors

$$\frac{\left\| \lambda^2 \mathbf{Mu} + \lambda \mathbf{Cu} + \mathbf{Ku} \right\|_2}{\left\| \mathbf{Ku} \right\|_2} ? < \text{SEUIL},$$

Sinon

$$\left\| \lambda^2 \mathbf{Mu} + \lambda \mathbf{Cu} + \mathbf{Ku} \right\|_2 ? < \text{SEUIL},$$

Fin si.

the norm of the residue. This sequence is parameterized

by key word SEUIL and SEUIL_FREQ, belonging respectively to the key word factors VERI_MODE and CALC_FREQ . In addition , this post - processing is activated by the initialization with “OUI' (default value) of STOP_ERREUR in factor key word the VERI_MODE. When this rule is activated and non-observance, computation stops, if not the error is just announced by an alarm. One could of course only too much recommend not to disable this parameter preferential treatment! Display in the message file

2.5.5 In the message file

are mentioned the relative information with the modes selected. For example, in the case of a QEP, one specifies the modal solver used and lists it eigenvalues retained by order ascending λ_i of imaginary part (column FREQUENCY). $\frac{\mathrm{Im}(\lambda_i)}{2\pi}$ Then one specifies his damping and his error norm $\xi_i = \frac{-\mathrm{Re}(\lambda_i)}{\lambda_i}$ (cf algorithm n°1). ----------

```
MODAL COMPUTATION: METHODE
      GLOBALE OF TYPE  QR ALGORITHME QZ_EQUI NUMERO
                    FREQUENCY (HZ)

   AMORTISSEMENT    Error norm    1 1.23915    E+02 1.55604
        E      08 9.93686E-       02 2 1.24546       E+02 -3.76772
        E      - 10 1.81854E      - 01 3 4.97033       E+02 -5.88927
        E      - 10 1.41043E      - 03 4 4.99575       E+02 1.41315
        E      12 2.14401E-       03 5 1.11531       E+03 4.75075
        E      12 1.17669E-       04 ..... Example       2. Printings
```

in the message file of the eigenvalues S selected during QEP. Extract of the sdll123a benchmark. Now that the context

of the QEP in Code_Aster was brushed , *we* will be interested more particularly in the methods of type power and their establishment in operator MODE_ITER_INV. Method of the powers

17 on the components of the physical degrees of freedom. residue of the initial QEP

3 opposite (MODE_ITER_INV) Introduction As for

3.1 the processing

of the GEP, in QEP this operator functions in two parts: Localization of the eigenvalues

- **. One determines an approximation** of each eigenvalue contained in an interval given by L" algorithm of Müller **- Traub [Mul56] [DB08] or one takes** an estimate provided by the user. Improvement of these
- **estimates and the computation of their** eigenvectors associated by a method of inverse iterations **(alternative of Jennings [** Jen77]). The search for a value

approached for each eigenvalue considered is selected in factor key word the CALC_FREQ via key word OPTION *: If* OPTION=' ADJUSTS ", in

- each interval of frequencies defined by the key word FREQ, an approximate value of each eigenvalue contained in this interval is calculated by means of the method of Müller-Traub (cf §3.2). Either one retains only
- the NMAX_FREQ the lowest frequencies contained in the maximum interval specified by the user, or one calculates all the values of this interval (if NMAX_FREQ=0). If OPTION = ' NEAR",
- frequencies given by the key word FREQ, are regarded as the approximate values of the sought eigenvalues. Note: Of course,

as one

- *already specified, this operator is to be used only to determine or refine some eigenvalues. For a wider search it is necessary to use operator MODE_ITER_SIMULT. With the "CLOSE" option*
- *one cannot calculate multiple modes. It is an expensive algorithm*
- *because he calls much on factorizations of a dynamic matrix of work Method of Müller-Traub* $\boldsymbol{Q}(\lambda)$.

3.2 As in QEP, the astute

test of Sturm type cannot apply more, the heuristics of localization of the eigenvalues must be based on another strategy. One "folds back oneself" then on the classic searches of **the zeros of the characteristic polynomial associated with the QEP in the complex plane** . $p(\lambda)=$dét $\mathbf{Q}(\lambda)$ As one then does not lay out

any more a relation of order, it is not possible to apply the dichotomy put in work for the GEP. A generalization at three points of the secant method makes it possible however to be left there. It is the method suggested by D.E.Muller [Mul56] [DB08] in 1956. It is an iterative method using as curve of interpolation a parabola with horizontal axis. Whereas the secant method leans on a linear interpolation between the two reiterated last the method known as of "Müller-Traub $\left\{(\lambda_{k-1}, p(\lambda_{k-1})),(\lambda_k, p(\lambda_k))\right\}$, " seeks to build the interpolated quadratic one passing by the three last By writing that the interpolated

$$\left\{(\lambda_{k-2}, p(\lambda_{k-2})),(\lambda_{k-1}, p(\lambda_{k-1})),(\lambda_k, p(\lambda_k))\right\}.$$

quadratic one of Lagrange of the characteristic polynomial (3.2-1) must pass by the last three

$$p(\lambda)\approx a_0\lambda^2+a_1\lambda+a_2$$

quoted points, one builds a system of three equations to three unknowns. From his analytical resolution, one deduces the root nearest of the reiterated last (3.2-2) To, one deduces

$$\rho_{k+1} = \frac{-2p(\lambda_k)\delta_k}{g_k \pm \sqrt{g_k^2 - 4p(\lambda_k)\delta_k\rho_k\left[p(\lambda_{k-2})\rho_k - p(\lambda_{k-1})\delta_k + p(\lambda_k)\right]}}$$

avec $g_k := p(\lambda_{k-2})\rho_k^2 - p(\lambda_{k-1})\delta_i^2 + p(\lambda_k)(\rho_k + \delta_k)$ the new one

$$\rho_k := \frac{\lambda_k - \lambda_{k-1}}{\lambda_{k-1} - \lambda_{k-2}}$$

easily ρ_k then reiterated. The sign of the denominator is λ_{k+1} selected so as to minimize thus one privileges the "conservative $|\rho_{k+1}|$ " choice of the zero nearest to reiterated preceding. It is considered that the method λ_k converged, when the following relative criterion was reached (avantNMAX_ITER_AJUSTE iterations) (3.2-3 $k+1<$) `This method` is relatively

$$\frac{|\lambda_{k+1} - \lambda_k|}{|\lambda_k|} < \varepsilon_{\text{PREC\_AJUSTE}}$$

easy to implement but it lends itself badly in the searches of zeros of real functions with real roots, because it plunges the interpolation in the complex plane and this even on the basis of actual values. Its interest is related to the class **of this method called "by curves of interpolation", namely: The safety of the method** of bisection

•, since the search is carried out in a ball of approximésse reducing gradually, Only $(\lambda_k)_k$ the computation of the function is

•necessary (computation step of derived such as for example in the method of Newton-Raphson). Convergence is connected with

•a quadratic convergence (q~1.89). In practice, the evaluating

is done not directly on the determinant of the characteristic polynomial but on its " $p(\lambda) := \det \mathbf{Q}(\lambda)$ déflaté" to take account **of already exhumed spectral information (if not one reconverge always towards** the same mode). In addition, after having calculated eigenvalues, one déflate k also by taking account $(\lambda_i)_{i=1}^k$ **of their conjugué18C' is as for this reason**[18] 4) the evaluating of the denominator

$$p(\lambda) := \frac{\det \mathbf{Q}(\lambda)}{\prod_{i=1}^{k} (\lambda - \lambda_i)(\lambda - \bar{\lambda}_i)}$$

is commonplace. That of the numerator, is carried out in factorisant19On $\det \mathbf{Q}(\lambda)$ expects to be able to factorize[19] and by carrying out LDL^T the product of $\mathbf{Q}(\lambda)$ the diagonal terms of the diagonal matrix. This factorization of dynamic $\mathbf{D}$ matrix

is very expensive with a computation complexity in where is the bandwidth and $O(bn^2)$ the size b of the problem. It is n the part rebukes it in capacity for treatment and core memory of the algorithm. It is to be reproduced time with the number of desired modes $p(i+2)$ and p the average nombre of iterations i with convergence. The algorithm is thus rather **to reserve for the QEP small (<104 degrees of freedom) or for the benchmarks of methods. Note: Müller [Mul56] proposes**

a particular

- *and effective procedure to initialize the process if the function to be interpolated is a polynomial of order whose $p(\lambda)$ coefficients are known n . The case which interests us does not return unfortunately in this class of problem. The initialization of the algorithm thus requires 2 preliminary evaluatings of This method does not calculate p.*

18 the perimeter of use of MODE_ITER_INV is restricted to the standard `QEP (symmetric` reality). One expects to find only complex-conjugate eigenvalues per pair. not to recompute them (3.2 -

19 without encumbers (without resorting to the swivelling) and to handle one factorized symmetric. It is another reason which limits the perimeter of MODE_ITER_INV to the real symmetric `QEP. in form` the dynamic matrix

- *a modal position, it thus manages badly the multiplicities. Contrary to the dichotomy put in work for the GEP. The notion of threshold in on this side which two numerical values are supposed to be multiple is related to the parameter One retains in the formula (3.2* $\varepsilon_{\text{PREC\_AJUSTE}}$
- *- 2) only the complex with positive imaginary part. Method of the powers opposite*

3.3 (alternative of Jennings) the second part consists in refining

the estimates of the eigenvalues calculated by the method of Müller-Traub and with calculating their associated eigenvectors. With this intention, one linearizes the QEP according to **the strategy (L1) "(cf §2.4) and one applies**
a particular spectral transformation due to Jennings [Jen77] (1977). Its purpose is to reveal , like reverse of matrix, only that of the dynamic matrix. L" algorithm set up in $\mathbf{Q}(\lambda)^{-1}$ the code cuts out as follows: Initialization of the eigenvalue

•from the estimate of the first phase: . Construction of the dynamic matrix λ_0
•and its factorization. Initialization $\mathbf{Q}(\lambda_0)$ of the iterative process
•by the vectors: To make Standardize To solve $\mathbf{x}_0$ aléatoire, $\mathbf{y}_0=\lambda_0\mathbf{x}_0$.
•To calculate $k=0,$ NMAX_ITER-1 To evaluate

- (3.3 $\mathbf{x}_k \leftarrow \frac{\mathbf{x}_k}{(\bar{\mathbf{x}}_k)^t\mathbf{M}\mathbf{x}_k}$, $\mathbf{y}_k \leftarrow \frac{\mathbf{y}_k}{(\bar{\mathbf{y}}_k)^t\mathbf{M}\mathbf{y}_k}$
- - 1) If and $\mathbf{Q}(\lambda_0)\mathbf{x}_{k+1}=\mathbf{C}\mathbf{x}_k+\lambda_0\mathbf{M}\mathbf{x}_{k-1}+\mathbf{M}\mathbf{y}_{k-1}$
- then ($\mathbf{y}_{k+1}=-\lambda_0\mathbf{x}_{k+1}+\mathbf{x}_k$
- 3.3-2) $\lambda_k=\frac{(\bar{\mathbf{x}}_k)^T\mathbf{C}\mathbf{x}_k+\sqrt{\left((\bar{\mathbf{x}}_k)^T\mathbf{C}\mathbf{x}_k\right)^2-4\left((\bar{\mathbf{x}}_k)^T\mathbf{M}\mathbf{x}_k\right)\left((\bar{\mathbf{x}}_k)^T\mathbf{K}\mathbf{x}_k\right)}}{2\left((\bar{\mathbf{x}}_k)^T\mathbf{M}\mathbf{x}_k\right)}$ the solution
- $\frac{|\lambda_{k+1}-\lambda_k|}{|\lambda_k|}<$ PREC eigen mode $\max(|\text{Re}(\lambda_k-\lambda_0)|,|\text{Im}(\lambda_k-\lambda_0)|)<$ PREC is, exit
- . Algorithm 2. Method of $(\lambda_k,\mathbf{x}_k)$ the powers

opposite (alternative of Jennings). The stopping criteria PREC and

the maximum number of authorized iterations NMAX_ITER are arguments of `factor key word` the CALC_MODE. Note: The stopping criteria

are based

- *on the extension of the quotient of Rayleigh (cf [Boi09] §4.3) to the QEP. Knowing an eigenvector a good estimate of its eigenvalue* $\mathbf{x}_k$ *is provided by the formula (3.3 -* λ_k *1). The algorithm is stopped as soon as relative variations (in complex) and absolute (on the parts real and imaginary separately), compared to estimated initial, are lower than a certain value. Thus jointly, one refines estimated initial of the eigenvalue and one determines his associated eigenvector. In spite of the linearization of the initial*
- *problem, the "astute" formulation of Jennings handles only entities of size. Actually, algorithm 2* n *can be schematized in the vectorial form (3.3-3) Perimeter of use*
- $\begin{bmatrix}\mathbf{Q}(\lambda_0) & \mathbf{0}_n\\ -\lambda_0\mathbf{I}_n & \mathbf{I}_n\end{bmatrix}\begin{Bmatrix}\mathbf{x}_{k+1}\\ \mathbf{y}_{k+1}\end{Bmatrix}=\begin{bmatrix}-\boldsymbol{C}-\lambda_0\mathbf{M} & -\mathbf{M}\\ -\mathbf{I}_n & \mathbf{0}_n\end{bmatrix}\begin{Bmatrix}\mathbf{x}_k\\ \mathbf{y}_k\end{Bmatrix}$ *QEP*

3.4 to symmetric real matrixes

. The user can specify

only TYPE_RESU=' DYNAMIQUE' like classifies membership of his computation (not of buckling). One then informs possibly vector FREQ (and not CHAR_CRIT). Display in the message file

3.5 In the message file, the results

are displayed in the form of table ----------

```
LE NUMBER OF D.O.F. TOTAL IS: 56
OF LAGRANGE IS:
   32 LE NUMBER OF D.O.F.
   ACTIVE IS:                 8
-------------------------------------------------------------------------
MODAL COMPUTATION: OPPOSITE ITERATION METHOD
       MULLER  INVERSE NUMERO FREQUENCY
                                        (HZ) AMORTISSEMENT
NB_ITER accuracy NB_ITER accuracy Error norm  1 5.52915  E+00 1.52090   E-02 5
9.49522    E-07 1     4.47353E-17      2.17748   E-15 2 1.08959       E+01 2.87575     E-02
3
1.17993    E-05 1     3.24203E-17      4.83597   E-15 3 1.59270       E+01 3.95645     E-02
5
2.53170    E-05 1     8.86844E-18      7.12414   E-16…
-------------------------------------------------------------------------

VERIFICATION A POSTERIORI OF THE MODES
          ------------------------------------------------------------------------
Example 3. Trace MODE_ITER_INV
```

(QEP) in the message file . Extract of the sdld27b benchmark. For each eigenvalue (represented

in form FREQUENCE= and AMORTISSEMENT=cf §2.5), one $\frac{\mathrm{Im}(\lambda_i)}{2\pi}$ traces the nombre of iterations $\frac{-\mathrm{Re}(\lambda_i)}{|\lambda_i|}$ and the accuracy obtenue20Précision of the method of Müller[20] operator. With the "CLOSE" option, the columns concerning the method of Müller do not appear obviously. The last column, Error norm, take again the error norm of the determined residue following the algorithm n°1 (§2.5). Summary of the parameter setting Let us recapitulate

3.6 now the parameter setting

of operator MODE_ITER_INV. Operand Key word Default value

References		*TYPE_RESU= ' DYNAMIQUE*	*""DYNAMIQUE*
	" §2.1 CALC_FREQ FREQ	(so NEAR	) §3.1
OPTION	**= `ADJUSTS** " "ADJUSTS		" §
	3.1 " CLOSE RELATION" §3.1	NMAX_FREQ	0 §
	3.1 NMAX_ITER		_AJUSTE
	15 §3.2		PREC_AJUSTE
	1.E-04 §3.2	CALC_MODE	
	OPTION= `	DIRECT	""DIRECT
" §3.3	"RAYLEIGH" (even	processing	

20 within the meaning of the formula (3.2-3), that of the method of Jennings takes again the formula (3.3-2). of the two phases of the modal

References		TYPE_RESU=' DYNAMIQUE	""DYNAMIQUE
	) §3.3 PREC 1. E-05 §3.3 NMAX_ITER		30 §
	3.3	VERI_MODE	STOP
	_ERREUR="	OUI'	"OUI'
[Boi12]	§3.2 "NON" SEUIL	1.E-02	[Boi12] §3.2
	Table		
	3.6	- 1. Summary	of

the parameter setting of MODE_ITER_INV (QEP). Note: One finds all

the "tripaille"

- *of parameters related to postprocessings of checking (SEUIL_FREQ, VERI_MODE). At the time as of first transitions, it*
- *is strongly advised to modify only the principal parameters noted in fat. The others relate to more the mysteries of L" algorithm and they were initialized empirically with values standards. Method of subspace (METHODE*

4 = ' TRI_DIAG'/"SORENSEN") Of the QEP with the SEP Transition QEP/GEP: linearization

4.1

4.1.1 As one explained to the §2.4

, the transition of the QEP with the GEP is carried out via the following linearization (*L2) "* "(4.1-1) the GEP obtained

$$\text{is structurally } \left(\underbrace{\begin{bmatrix} -\mathbf{K} & \mathbf{0}_n \\ \mathbf{0}_n & \mathbf{M}_R \end{bmatrix}}_{\mathbf{A}} - \lambda \underbrace{\begin{bmatrix} -\mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0}_n \end{bmatrix}}_{\mathbf{B}} \right) \begin{bmatrix} \mathbf{u} \\ \lambda \mathbf{u} \end{bmatrix} = \mathbf{0}_{2n}$$

symmetric reality and inherits the properties of the matrixes, and. As soon as L" D" it is $\mathbf{K}$ $\mathbf{M}$ $\mathbf{C}$ asymmetric or complex (possible only with), the GEP becomes asymmetric $\mathbf{K}$ or complex. The symmetric aspect or not of the GEP is not in fact not blocking because each method will get busy **to find the couple "operator of work/scalar product" who allows him to function. Moreover, only Lanczos needs** a symmetric couple. Method IRAM can it function very well into asymmetric and with often more robustesse21Au price however of a surplus[21]

4.1.2 transformation Then a spectral transformation

(cf [Boi09] §3.7/§5) makes it possible to complete the phase of preprocessing of the QEP by transforming the preceding GEP into a SEP. When (L_2)'' complex modes are handled, two cases are possible: To work completely in arithmetic

- complex, To generally remain **into arithmetic**
- real by isolating **the contributions real** and imaginary from the classical spectral transformation. The first strategy classically

uses the spectral transformation known as of "shift-and-invert" and metamorphose in (4.1-2) It is implemented (L_2)''

$$(S_1) \qquad \underbrace{(\mathbf{A}-\sigma\mathbf{B})^{-1}}_{\mathbf{A}_\sigma}\mathbf{B}\,\mathbf{w} = \mu\,\mathbf{w}$$
$$\text{avec } \mu := \frac{1}{\lambda-\sigma},\ \mathbf{w} := \begin{bmatrix} \mathbf{u} \\ \lambda\,\mathbf{u} \end{bmatrix}$$

by activating

the complex approach of IRAM ("SORENSEN" +APPROCHE=' COMPLEXE'). It is the only alternative when one handles a matrix complexe22Dès which is complex $\mathbf{K}$ symmetric[22] symmetric or not. $\mathbf{K}, \mathbf{M}$ $\mathbf{C}$ strategy rests on

the spectral transformation with "double shifts adds" introduced in 1987 by B.N.Parlett & Y.Saad [PS87], which results in handling two types of SEP (4.1-3) (4.1-4) These two alternatives are implemented

$$(S_2) \qquad \mathrm{Re}\underbrace{\left((\mathbf{A}-\sigma\mathbf{B})^{-1}\mathbf{B}\right)}_{\mathbf{A}_\sigma^+}\mathbf{w} = \mu^+\,\mathbf{w}$$
$$\text{avec } \mu^+ := \frac{1}{2}\left(\frac{1}{\lambda-\sigma}+\frac{1}{\lambda-\bar{\sigma}}\right),\ \mathbf{w} := \begin{bmatrix} \mathbf{u} \\ \lambda\,\mathbf{u} \end{bmatrix}$$

21 of computations and storages. Transition GEP/SEP: spectral

22 and which $\mathbf{K}$ one chose IRAM, some is the approach chosen in the command file, it is the approach "COMPLEXE" which is taken by default. It is usable with and formulates

$$\underbrace{\mathrm{Im}\left(\left(\mathbf{A}-\sigma\,\mathbf{B}\right)^{-1}\mathbf{B}\right)}_{\mathbf{A}_{\sigma}^{-}}\mathbf{w}=\mu^{-}\mathbf{w}$$

(S_3) with

$$\text{avec } \mu^{-}:=\frac{1}{2i}\left(\frac{1}{\lambda-\sigma}+\frac{1}{\lambda-\bar{\sigma}}\right),\ \mathbf{w}:=\begin{bmatrix}\mathbf{u}\\ \lambda\,\mathbf{u}\end{bmatrix}$$

Lanczos and IRAM via the values, respectively, "REEL" *and* "IMAG" of key word APPROCHE. `With` Lanczos, the perimeter of `use` is restricted with the classical QEP (symmetric realities), while IRAM is less restricted and allows also the asymmetric one. Note: The origin of the denomination

"doubles shifts

- *adds" spectral transformation above appears more clearly besides if one reformulates the operators as follows (4.1-5) the terminology "sum" in opposition to*

$$\mathbf{A}_{\sigma}^{+}:=\frac{1}{2}\left[\left(\mathbf{A}-\sigma\,\mathbf{B}\right)^{-1}+\left(\mathbf{A}-\bar{\sigma}\,\mathbf{B}\right)^{-1}\right]\mathbf{B}$$

$$\mathbf{A}_{\sigma}^{-}:=\frac{1}{2i}\left[\left(\mathbf{A}-\sigma\,\mathbf{B}\right)^{-1}+\left(\mathbf{A}-\bar{\sigma}\,\mathbf{B}\right)^{-1}\right]\mathbf{B}$$

- *the "double shifts produced" is initially proposed by J.C.Francis and is used, in writing pad, by the method "implicit" QZ and the restartings of IRAM (cf [Boi09] Appendix 1): (4.1-6) This one has the major drawback to produce*

$$\mathbf{A}_{\sigma}:=\left[\left(\mathbf{A}-\sigma\,\mathbf{B}\right)+\left(\mathbf{A}-\bar{\sigma}\,\mathbf{B}\right)\right]^{-1}\mathbf{B}$$

compared to the matrixes of work very filled (thus more expensive to handle and store) strategy of Parlett & Saad. Once determined the operator of work, it remains

to choose its associated scalar product. For IRAM, one can be satisfied with a couple asymmetric "operator/scalar product" (cf [Boi09] §6), but for Lanczos, this one must obligatorily be symmetric (cf [Boi09] §7). Several choices of (pseudonym) matric produce-scalars allow this symmetry: (4.1-7) the first functions with the two formulations

$$\begin{aligned}\left(\mathbf{w}\,,\mathbf{z}\right)_{\mathbf{B}}&:=\mathbf{z}^{T}\mathbf{B}\mathbf{w}\\ \left(\mathbf{w}\,,\mathbf{z}\right)_{+}&:=\mathbf{z}^{T}\,\mathrm{Re}\left[\left(\mathbf{A}-\sigma\,\mathbf{B}\right)^{-1}\right]^{-1}\mathbf{w}\\ \left(\mathbf{w}\,,\mathbf{z}\right)_{-}&:=\mathbf{z}^{T}\,\mathrm{Im}\left[\left(\mathbf{A}-\sigma\,\mathbf{B}\right)^{-1}\right]^{-1}\mathbf{w}\end{aligned}\qquad (S2)$$

and (S3), while the second (resp. the third) makes symmetric only (resp.). In the effective establishment in Code_Aster $\mathbf{A}_{\sigma}^{+}$, these $\mathbf{A}_{\sigma}^{-}$ are the two last which are retained *for* the approaches real and imaginary of Lanczos. Note: These scalar products are the "natural

"

- *extensions of (pseudonym) the matric scalar product used by the alternative of Newmann-Pipano in GEP: (4.1-8) Establishment in Code_Aster Choice of the spectral*

$$\left(\mathbf{w}\,,\mathbf{z}\right):=\mathbf{z}^{T}\left(\mathbf{A}-\sigma\,\mathbf{B}\right)\mathbf{w}$$

shift

4.2 In GEP it there are *four ways*

4.2.1 of choosing this shift (cf

[Boi09] §5.4). In QEP, the option "BANDAGES" being proscribed because of absence of test `of` adapted Sturm type, this figure is tiny room to three: one seeks the smallest eigenvalues of the starting

- $\sigma=0$, problem. This corresponds to OPTION=' PLUS_PETITE' under factor key word the CALC_FREQ . `one seeks the frequencies` close to the frequency FREQ
- $\sigma=\sigma_0$ avec $\sigma_0=(2\pi f_0)^2$, = (OPTION=' CENTER'). one seeks the frequencies close to f_0 `the frequency` FREQ

• $\sigma=\sigma_0$ avec $\sigma_0=(2\pi f_0)^2$ et $\sigma_1=\eta(2\pi f_0)^2/2$ = (OPTION=' CENTER') and to reduced damping f_0 AMOR_REDUIT= (OPTION=' CENTER'). The number of frequencies to calculating η is given in general

by the user using NMAX_FREQ under factor key word the CALC_FREQ. Computation of the operator and the scalar product of work

4.2.2 One recapitulates in the table according to all

the couples (operator of work, scalar product) possible, according to the options chosen by the user. APPROCHE/METHODE “TRI_DIAG” “SORENSEN” (by default

) “COMPLEXE	” Unavailable	“REEL” (by default)
“IMAG” Table	4.2-1. Together	$\left(A_\sigma, L^2\right)$
couples (operator	$\left(A_\sigma^+, (\,)_+\right)$	$\left(A_\sigma^+, L^2\right)$
of work	$\left(A_\sigma^-, (\,)_-\right)$	$\left(A_\sigma^-, L^2\right)$

, scalar product) possible according to the options chosen by the user. The computation operators of work can bring back themselves

to simple products matrix-vector $\mathbf{A}_\sigma^\pm$ and with an inversion of the dynamic matrix (factorized beforehand) via the formulation (astute $\mathbf{Q}(\lambda)$) following (4.2-1) In addition , this formulation has the “good taste

$$\left(\mathbf{A}-\sigma\mathbf{B}\right)^{-1}\mathbf{B}=\begin{bmatrix}\mathbf{0}_n & \mathbf{0}_n\\ \mathbf{M}_R^{-1}\mathbf{M} & \mathbf{0}_n\end{bmatrix}-\begin{bmatrix}\mathbf{Q}(\sigma)^{-1} & \mathbf{0}_n\\ \mathbf{0}_n & \sigma\mathbf{M}_R^{-1}\mathbf{M}\mathbf{Q}(\sigma)^{-1}\end{bmatrix}\begin{bmatrix}\mathbf{C}+\sigma\mathbf{M} & \mathbf{M}\\ \mathbf{C}+\sigma\mathbf{M} & \mathbf{M}\end{bmatrix}$$ „

to allow processing into pure arithmetic real and to preserve hollow structure of the matrixes. One can thus build the following procedure: Preparation dans23Car the shift (thus queavec IRAM) can

•be complex C [23] , cf § 2.5): To form Factorisersous formula forms LDLT. Computation of (several $\mathbf{Q}(\sigma):=\left(\sigma^2\mathbf{M}+\sigma\mathbf{C}+\mathbf{K}\right)$ by iteration $\mathbf{Q}(\sigma)$ of Lanczos or

•IRAM) $\mathbf{A}_{\sigma^\pm}\underbrace{\begin{bmatrix}\mathbf{u}\\ \mathbf{v}\end{bmatrix}}_{\mathbf{w}}$: To form in Calculating in Following the choice of the approach

Algorithm R $\mathbf{u}_1:=\mathbf{C}\mathbf{u}$, $\mathbf{u}_2:=\mathbf{M}\mathbf{u}$, $\mathbf{u}_3:=\mathbf{M}\mathbf{v}$,

3. Procedure C $\mathbf{u}_4:=\mathbf{Q}(\sigma)^{-1}+\sigma\,\mathbf{u}_2+\mathbf{u}_3$

•to compute: operators

$$\mathbf{A}_\sigma^+\mathbf{w}=\begin{bmatrix}-\mathrm{Re}(\mathbf{u}_4)\\ \mathbf{M}_R^{-1}\mathbf{M}(\mathbf{u}-\mathrm{Re}(\sigma\,\mathbf{u}_4))\end{bmatrix}$$

$$\mathbf{A}_\sigma^-\mathbf{w}=\begin{bmatrix}-\mathrm{Im}(\mathbf{u}_4)\\ -\mathbf{M}_R^{-1}\mathbf{M}(\mathbf{u}-\mathrm{Im}(\sigma\,\mathbf{u}_4))\end{bmatrix}$$

23 (only once, in phase $\mathbf{K}$ of initialization of the algorithm

of work. In complex approach, computation is done directly $\mathbf{A}_\sigma^{\pm}$ *via*

the formula (4.1-2) into arithmetic complex. Concerning the matric scalar products, their handling in Code_Aster leans on the following formulations *: (4.2-2) Note:* : The computation matric scalar products

$$(\mathbf{w},\mathbf{z})_+ := \mathbf{z}^T\left[(\mathbf{A}-\mathrm{Re}(\sigma)\mathbf{B})+\mathrm{Im}^2(\sigma)\mathbf{B}(\mathbf{A}-\mathrm{Re}(\sigma)\mathbf{B})^{-1}\mathbf{B}\right]\mathbf{w}$$

$$(\mathbf{w},\mathbf{z})_- := \mathbf{z}^T\left[\frac{1}{\mathrm{Im}(\sigma)}(\mathbf{A}-\mathrm{Re}(\sigma)\mathbf{B})\mathbf{B}^{-1}(\mathbf{A}-\mathrm{Re}(\sigma)\mathbf{B})+\mathrm{Im}^2(\sigma)\mathbf{B}\right]\mathbf{w}$$

can be carried out

- *only into arithmetic real. For the first scalar product one needs an additional dynamic matrix and for its factorized (from where a overcost computation and memory* $\mathbf{Q}(\mathrm{Re}(\sigma))$ *). Perimeter of use QEP to real symmetric*

4.3 matrixes for Lanczos, possibly

asymmetric (and can be symmetric **complex) for IRAM. The user** $\mathbf{K}$ can specify only TYPE_RESU=' DYNAMIQUE

"like classifies membership `of his computation (not of buckling`). One then informs possibly vector FREQ (and not CHAR_CRIT). Display in the message file In `the message file`

4.4 , the results are displayed

in the form of table ----------

```
LE NUMBER OF D.O.F. TOTAL IS: 130 OF LAGRANGE IS: 16
LE NUMBER OF D.O.F.
   ACTIVE IS:                        106
   ------------------------------------------------------------------------
--

MODAL COMPUTATION: METHODE Of ITERATION SIMULTANEOUS METHODE
OF SORENSEN NUMERO  FREQUENCY (HZ) AMORTISSEMENT
      Error norm

    1 1.24163    E+02 -1.89229    E-02 1.16550E-    09 2 1.24164E+
         02      1.89229E-02      1.04882E-09…       6 1.07321
         E+      03 1.63495E-       01 1.36092E-       13 AVERAGE

         ERROR NORM      : 0.36947       E-09 Example       4. Trace
 MODE_ITER_SIMULT with  METHODE= `
```

SORENSEN" (QEP) in `the message file . Extract of the benchmark sdll` 123b. For each eigenvalue (represented in form

FREQUENCE= and AMORTISSEMENT= cf §2.5), one traces the error norm $\frac{\mathrm{Im}(\lambda_i)}{2\pi}$ `of the determined residue` $\frac{-\mathrm{Re}(\lambda_i)}{|\lambda_i|}$ following the algorithm n°1 (§2.5). Summary of the parameter setting Let us recapitulate now

4.5 the parameter setting of operator

MODE_ITER_SIMULT for the methods of Lanczos and Sorensen . Operand Key word Default value References TYPE

_RESU=' DYNAMIQUE	“	“DYNAMIQUE” §2.1 METHODE	= `TRI_DIAG
	” “SORENSEN”	[Boi09] §	6 “SORENSEN
	” [Boi09] §7	OPTION= `MODE_	RIGIDE' Non
	available CALC_FREQ		FREQ
	(if CENTER) §4.2 AMOR_REDUIT	(if CENTER	
) §4.2 APPROCHE	= “REEL”		“
	REEL” §4.1 “ IMAG”		§4.1
	(if SORENSEN) “COMPLEXE	”	§4.1
	OPTION = `		PLUS_PETIT E
	““PLUS_PETITE ” §4.2		“CENTER
	” §4.2 NMAX_FREQ 10	§4.2 DIM_SOUS_ESPA CE	
	COEF_DIM_ESPACE		Calculated
	[Boi	09]	§6/7
	…. Usual parameter setting of “TRI_DIAG	”	/” SORENSEN'
	[Boi		
	09] §6/7 Table 4.5 - 1. Summary of the parameter setting		of MODE_ITER_ SIMULT

(QEP) with METHODE= `TRI_DIAG'/“ SORENSEN”. Note: During the first transitions, it

is strongly advised

- *to modify only the principal parameters noted in fat. The others relate to more the mysteries of the algorithm and they were initialized empirically with values standards. In particular, to improve quality of a mode*
- *, the fundamental parameter is the dimension of the subspace of projection of the SEP, DIM_SOUS_ESPACE. Total method QZ (METHODE=' QZ') Of the QEP with the GEP Contrary*

5 to the methods seen previously

5.1 , the method

QZ is able to manage a GEP directly. One thus has that only one transformation to be applied to the initial QEP **, it is the linearization. That empirically retained and detailed with the § 2.4** is (L2) * (5.1-1) the GEP obtained is not structurally

$$\text{symmetric} \left(\underbrace{\begin{bmatrix} -\mathbf{K} & \mathbf{0}_n \\ \mathbf{0}_n & \alpha \mathbf{I}_n \end{bmatrix}}_{\mathbf{A}} - \lambda \underbrace{\begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \alpha \mathbf{I}_n & \mathbf{0}_n \end{bmatrix}}_{\mathbf{B}} \right) \begin{bmatrix} \mathbf{u} \\ \lambda \mathbf{u} \end{bmatrix} = \mathbf{0}_{2n}$$

(or hermitian) but it is not prejudicial with computation because the drivers of LAPACK adapted work only into asymmetric. For more robustness and to simplify the structuring **of the code, one formulates and solves the GEP (L2) * in the complex plane**. As for the GEP with complex modes one **uses** the routines LAPACK standard (ZGGEV) or expert (ZGGEVX) according to parameter TYPE_QZ='QZ_SIMPLE'/"QZ_EQUI". To resulting from modal computation , QZ returns to Code_Aster

the calculated modes and the latter will be *checked* , filtered and ordered according to the same recommendations as for the complex modes of the GEP (cf [Boi09] §9.4). Perimeter of use QEP to symmetric matrixes

5.2 or not (and can be symmetric

complex). The user can $\mathbf{K}$ specify only TYPE_RESU=' DYNAMIQUE
"like classifies membership of his computation (not of buckling). One then informs possibly vector FREQ (and not CHAR_CRIT). Printings in the message file an extract with accompanying notes

5.3 of the sdll113a benchmark is taken again

with the §2.5. Summary of the parameter setting Let us recapitulate now

5.4 the parameter setting of operator

MODE_ITER_SIMULT for the method QZ. Operand Key word Default value References TYPE

_RESU=' DYNAMIQUE	"	"DYNAMIQUE" §2.1 METHODE	= `QZ
	""SORENSEN" [Boi	09] §9 OPTION	= `
	MODE_RIGIDE " Non	available CALC_FREQ	FREQ
	(if CENTER) [Boi09] §9.4	AMOR_REDUIT (	
if CENTER)	**[Boi** 09] §9.4 APPROCHE		=… Without
	object OPTION = "PLUS_PETITE		" "PLUS_PETITE
	"	[Boi09] §	
	9.4 "CENTER " "TOUT" [Boi	09] §9.4 NMAX_FREQ	10 [Boi09] §
	9.4…. Usual		parameter setting of

_RESU=' DYNAMIQUE	"	"DYNAMIQUE" §2.1 METHODE	= `QZ
	"QZ"	[Boi	09] §9 Table
	5.4		
	- 1. Summary of the parameter setting		of MODE_ITER_ SIMULT

(QEP) with METHODE= `QZ'. Note: At the time as of first transitions , it

is strongly advised

- *to modify only the principal parameters noted in fat. The others relate to more the details of the algorithm and they were initialized empirically with values standards. Bibliography Books/articles/proceedings/theses… [*

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6.3 of the versions

of the document Version Aster Author (S) Organization (S) Description of the modifications

7 3d.SELIGMANN-R&D/MMN initial

Text …	**… 5 O.BOITEAU EDF-R&**	**D/MMN V8.4 O.BOITEAU EDF-R&D/SINETICS**
3	O.NICOLAS EDF	R & D /AMA 9.4 O.
BOITEAU		
5	- R&D/SINETICS Recasting	
of the document	, and extension of the perimeter of the solver	
QZ to	asymmetric v10.4 O.BOITEAU	EDF-R&D/SINETICS large work of re-arrangement (formula , figure, legend,
notice	, character blanks, cast iron	…). Update concerning parameters (in particular NPREC, COEF_DIM_ESPACE, TOUT), the aspects linear solvers and parallelism. V11.2 O.BOITEAU EDF-R&D/SINETICS Work of re-arrangement, correction of formulas in agreement with
	Docs. R 5.01.01/04	. V11.2.13 O.BOITEAU EDF-R&D/SINETICS Still some fitness + taken into account EL16710
. Appendix	1. Interpretation of the complex	eigenvalues In the case of a symmetric damping

8 and in internal absence of damping, relations

of orthogonalities **and owing to the fact that the clean elements appear per combined** pairs, one has the following relations: If one notes, one can then define One can write the complex eigenvalue in the following

$$\frac{\boldsymbol{\Phi}_i^{*T}\mathbf{C}\boldsymbol{\Phi}_i}{\boldsymbol{\Phi}_i^{*T}\mathbf{M}\boldsymbol{\Phi}_i}=\frac{c_i}{m_i}=2\,\mathrm{Re}(\lambda_i)$$

$$\frac{\boldsymbol{\Phi}_i^{*T}\mathbf{K}\boldsymbol{\Phi}_i}{\boldsymbol{\Phi}_i^{*T}\mathbf{M}\boldsymbol{\Phi}_i}=\frac{k_i}{m_i}=|\lambda_i|^2$$

form This $\lambda_i=\alpha_i\pm i\,\beta_i$ formulation induces

$$\omega_i=|\lambda_i|=\sqrt{\alpha_i^2+\beta_i^2}=\frac{\beta_i}{\xi_i}$$

$$\xi_i=\mathrm{Re}\frac{(\lambda_i)}{\omega_i}=\frac{\alpha_i}{\sqrt{\alpha_i^2+\beta_i^2}}\,\omega_i$$

following physical interpretation: The real term represents

$$\lambda_i=-\xi_i\omega_i\pm i\,\omega_i\sqrt{1-\xi_i^2}$$

the dissipative character of the system**. The imaginary part** represents

- the oscillatory part of the solution. is the pulsation of
- the ème mode. is the damping of the ème mode, is the reduced damping
- ω_i of the ème mode. As for i
- ξ_i the physical interpretation i of the eigenvectors
- $\omega_{d_i}=\omega_i\sqrt{1-\xi_i^2}$: The physical meaning i of

the existence **of a complex eigenvector, lies in the fact**

- that if the structure vibrates on an eigen mode, its various degrees of freedom do not vibrate with the same phase the ones compared to the others. The bellies and the modal nodes do not correspond of the steady points, but move
- during motion. Note: One finds the classical formulation of the deadened systems with 1 degree of freedom

. and are

- *real. They are many intrinsic quantities with a mode (modal quantities)*
- k_i, m_i *thus* c_i *dependant on its standardization. We point out that the modes of the GEP (just like complex modes of the GEP) do not diagonalisent*
- *the matrixes If the real part of the eigenvalue is negative, then the eigen mode is a deadened* $\mathbf{M}$, $\mathbf{K}$ et $\mathbf{C}$.
- *periodic motion of pulsation. If on the contrary, it is positive, then the eigen mode is a periodic motion of amplitude* ω *increasing and thus unstable.*