Algorithm of linear thermal transitory

Summarized:

One presents the linear algorithm of transient thermal established within command THER_LINEAIRE [U4.33.01]. The various computation options necessary were presented in the plane, axisymmetric and three-dimensional structural elements [U1.22.01], [U1.23.01] and [U1.24.01].

Contents

1 Statement of the equation of heat in linear thermal

1.1 Equation of heat

One is placed in open Ω of $\mathbb{R}^3$ regular border Γ.
In any point of Ω, the equation of heat can be written:

$$-\text{div}(\mathbf{q}(r,t))+s(r,t)=\rho C_p \frac{\partial T(r,t)}{\partial t}$$

with:

$\mathbf{q}$	vector heat flux (directed according to the decreasing temperatures),
s	heat per unit of volume dissipated by the internal sources,
ρC_p	voluminal heat with constant pressure,
T	temperature,
r	variable of space,
t	variable time.

This equation translates the phenomenon of change of the temperature (only through the phenomenon of diffusion, convection having been neglected) into any point of opened and at any moment. She admits in theory an infinity of solutions, but the data of the initial conditions and the variation of the boundary conditions in the course of time determines the evolution of the phenomenon perfectly.

1.2 Fourier analysis

In thermal conduction, the Fourier analysis provides an equation connecting heat flux to the gradient of the temperature (normal vector on the isothermal surface). This model reveals, in its most general form, a tensor of conductivity. In the case of an isotropic material, this tensor is reduced to a simple coefficient λ, the coefficient of thermal conductivity.

$$\mathbf{q}(r,t)=-\lambda \nabla T(r,t)$$

For the elements of anisotropic thermal one will refer to Establishment of the elements 2D and 2D-Axisymmetric in mechanics and thermal [R3.06.02].

1.3 Equation of the linear heat in the case of the model of thermal

By combining the two equations above, one obtains:

$$-\text{div}(-\lambda \nabla T(r,t))+s(r,t)=\rho C_p \frac{\partial T(r,t)}{\partial t}$$

2 Boundary conditions, loading and initial condition

One describes here only the thermal boundary conditions leading to linear equations in temperature, which excludes the conditions of type radiation.

2.1 Imposed temperatures

the conditions of the Dirichlet type, are usually treated by dualisation in *Code_Aster* (cf [R3.03.01]), but they can also be eliminated in certain cases (kinematical loads).

$$T(r,t)=T_1(r,t) \quad \text{sur } \Gamma_1$$

where $T_1(r,t)$ is a function of the variable of space and/or time.

2.2 Linear relations

It are of the conditions of the Dirichlet type, making it possible to define a linear relation between the values of the temperature:

- between two or several nodes: with an equation of the form

$$\sum_{i=1}^{n} \alpha_i T_i(r,t)=\beta(t)$$

- enters of the couples of nodes: with an equation of the form

$$\sum_{i=1}^{n_1} \alpha_{1i} T_{i/\Gamma_{12}}(r,t)+\sum_{i=1}^{n_2} \alpha_{2i} T_{i/\Gamma_{21}}(r,t)=\beta(t)$$

  where Γ_{12} and Γ_{21} are two under-parts of the border which one binds two to two the values of the temperature. This kind of boundary condition makes it possible to define conditions of periodicity.

2.3 Normal flux imposed

It is of the conditions of the Neumann type, defining flux entering the field.

$$-\mathbf{q}(r,t).\mathbf{n}=f(r,t) \quad \text{sur } \Gamma_2$$

where $f(r,t)$ is a function of the variable of space and/or time and $\mathbf{n}$ indicates the norm at the border Γ_2 .

2.4 Exchange

It are of the conditions of the Neumann type modelling the convective transfers on edges of the field.

$$-\mathbf{q}(r,t).\mathbf{n}=h(r,t)(T_{ext}(r,t)-T(r,t)) \quad \text{sur } \Gamma_3$$

where $T_{ext}(r,t)$ is a function of the variable of space and/or time representing the temperature of the external medium, and $h(r,t)$ is a function of the variable of space and/or time representing the convective coefficient of heat exchange on the border Γ_3 .

2.5 Exchange wall

It are of the conditions of the Neumann type bringing into play two pennies left the border in opposite. This kind of boundary condition models a thermal strength of interface.

$$\lambda\frac{\partial T_1}{\partial n_1}=h(r,t)(T_2(r,t)-T_1(r,t)) \text{ sur } \Gamma_{12} \qquad n_1 \text{ norm external with } \Gamma_{12}$$

$$\lambda\frac{\partial T_2}{\partial n_2}=h(r,t)(T_1(r,t)-T_2(r,t)) \text{ sur } \Gamma_{21} \qquad n_2 \text{ norm external with } \Gamma_{21}$$

($n_1=-n_2$ in general)

2.6 Volumic source

It is the function $s(r,t)$ term of the variable of space and/or time.

2.7 Initial condition

It is the statement of the field of temperature at initial time $t=0$:

$$T(r,0)=T_0(r)$$

where $T_0(r)$ is a function of the variable of space.

3 Variational formulation of the problem

We will restrict ourselves here to present the problem with only the boundary conditions of imposed temperature [§2.1], imposed normal flux [§2.3] or of exchange [§2.4]. The boundary conditions of exchange wall [§2.5] are treated with [the §4] and those with linear relations [§2.2] are brought back without difficulties to that of [§2.1].

That is to say Ω open of $\mathbb{R}^3$, border $\Gamma=\Gamma_1\cup\Gamma_2\cup\Gamma_3$.

The weak formulation of the equation of heat is:

$$\int_\Omega \rho C_p \frac{\partial T}{\partial t}.v\,d\Omega+\int_\Omega \lambda\nabla T.\nabla v\,d\Omega-\int_\Gamma \lambda\frac{\partial T}{\partial n}.v\,d\Gamma=\int_\Omega s.v\,d\Omega$$

where v is a sufficiently regular function cancelling itself uniformly on Γ_1. With the following boundary conditions:

$$\begin{cases} T=T_1(r,t) & \text{sur } \Gamma_1\\ \lambda\dfrac{\partial T}{\partial n}=q(r,t) & \text{sur } \Gamma_2\\ \lambda\dfrac{\partial T}{\partial n}=h(r,t)(T_{ext}(r,t)-T) & \text{sur } \Gamma_3 \end{cases}$$

The variational formulation of the problem is:

$$\int_\Omega \rho C_p \frac{\partial T}{\partial t}.v\,d\Omega+\int_\Omega \lambda\nabla T.\nabla v\,d\Omega+\int_{\Gamma_3} hT.v\,d\Gamma=\int_\Omega s.v\,d\Omega+\int_{\Gamma_2} q.v\,d\Gamma+\int_{\Gamma_3} hT_{ext}.v\,d\Gamma$$

4 Variational formulation of the problem with condition of exchange between two walls

One considers the "simplified" problem where does not appear any more source term and where the boundary conditions are only of the standard imposed temperature and wall exchanges.

That is to say Ω open of $\mathbb{R}^3$, border $\Gamma=\Gamma_1\cup\Gamma_{12}\cup\Gamma_{21}$.

The boundary conditions are in this case:

$$\begin{cases} T=T_1(r,t) & \text{sur } \Gamma_1\\ \lambda\dfrac{\partial T_1}{\partial n_1}=h(r,t)(T_2(r,t)-T_1(r,t)) & \text{sur } \Gamma_{12}\\ \lambda\dfrac{\partial T_2}{\partial n_2}=h(r,t)(T_1(r,t)-T_2(r,t)) & \text{sur } \Gamma_{21} \end{cases}$$

In substituent in the weak formulation of the equation of heat, one obtains:

$$\int_{\Omega} \rho C_p \frac{\partial T}{\partial t} . v \, d\Omega + \int_{\Omega} \lambda \nabla T . \nabla v \, d\Omega + \int_{\Gamma_{12}} h(T_{/\Gamma_{12}} - T_{/\Gamma_{21}}) . v \, d\Gamma_{12} + \int_{\Gamma_{21}} h(T_{/\Gamma_{21}} - T_{/\Gamma_{12}}) . v \, d\Gamma_{21} = 0$$

where v cancels itself uniformly on Γ_1 .

This kind of boundary conditions reveals new terms connecting degrees of freedom located on the two borders in relation.

5 Discretization in time of the differential equation

a classical way to discretize a first order differential equation consists in using one θ - method. Let us consider the following differential equation:

$$\begin{cases} \dfrac{\partial}{\partial t} y(t) = \varphi(t, y(t)) \\ y(0) = y_0 \end{cases}$$

$\theta -$ The method consists in discretizing the equation by a diagram with the finite differences

$$\frac{1}{\Delta t}(y_{n+1} - y_n) = \theta \varphi(t_{n+1}, y_{n+1}) + (1-\theta) \varphi(t_n, y_n)$$

where y_{n+1} is an approximation of $y(t_{n+1})$, y_n being supposed known and θ is the parameter of the method $\theta \in [0,1]$.

Note:

if $\theta = 0$ *the diagram is known as explicit,*
if $\theta \geq 0$ *the diagram is known as implicit.*

5.1.1 Accuracy of the method

Let us suppose y sufficiently regular (at least 3 times differentiable), by a development of Taylor at the point t_n one obtains:

$$y(t_{n+1}) - y(t_n) = \Delta t \, y'(t_n) + \frac{\Delta t^2}{2} y''(t_n) + O(\Delta t^2)$$

and

$$\begin{aligned} \theta \varphi(t_{n+1}, y(t_{n+1})) + (1-\theta) \varphi(t_n, y(t_n)) &= \theta y'(t_{n+1}) + (1-\theta) y'(t_n) \\ &= y'(t_{n+1}) + \theta (y'(t_{n+1}) - y'(t_n)) \\ &= y'(t_n) + \theta \Delta t \, y''(t_n) + O(\Delta t^2) \end{aligned}$$

the solution thus checks roughly:

$$\frac{1}{\Delta t}\left(y(t_{n+1})-y(t_n)\right)=\theta\varphi(t_{n+1},y(t_{n+1}))+(1-\theta)\varphi(t_n,y(t_n))+(\frac{1}{2}-\theta)\Delta t\, y''(t_n)+O(\Delta t^2)$$

The diagram is of order 1 in time if $\theta\neq\frac{1}{2}$, and of order 2 if $\theta=\frac{1}{2}$ (diagram of Crank - Nicolson).

5.1.2 Stability of the method

Let us consider the following differential equation:

$$\begin{cases} y'=-\lambda y \quad t\geq 0 \quad \lambda\in R \\ y(0)=y_0 \end{cases}$$

By means of it θ - method in this differential equation one obtains:

$$y_{n+1}=\frac{1-(1-\theta)\lambda\Delta t}{1+\theta\lambda\Delta t}y_n \quad 0\leq n\leq N-1$$

That is to say still:

$$y_{n+1}=r^n(\lambda\Delta t)y_0 \quad \text{avec} \quad r(x)=\frac{1-(1-\theta)x}{1+\theta x}$$

The approximate solution y_n must be limited (the exact solution of the initial problem being it), which imposes the following condition:

$$|r(\lambda\Delta t)|\leq 1$$

By studying the variations of the function $r(x)$, one notes easily that:

- if $\theta\geq 1/2$ the condition is checked whatever Δt , the diagram is unconditionally stable;
- if $\theta< 1/2$ the condition is checked only if $\Delta t\leq\frac{2}{\lambda(1-2\theta)}$, the diagram is conditionally stable.

In command `THER_LINEAIRE` [U4.33.01], the parameter θ is a data being able to be provided by the user, the value by default is fixed at 0.57. This value has the reputation to be preferable with the value of Crank - Nicolson (0,5) and “optimal” for the quadratic interpolations, but we did not find trace of the justifications.

5.1.3 Application to the equation of heat

Let us use it θ - method in the variational formulation of the equation of heat, where one posed:

$$\begin{array}{llll} T^+=T(r,t+\Delta t) & T^-=T(r,t) & h^+=h(r,t+\Delta t) & h^-=h(r,t) \\ f^+=f(r,t+\Delta t) & f^-=f(r,t) & T^+_{ext}=T_{ext}(r,t+\Delta t) & T^-_{ext}=T_{ext}(r,t) \\ s^+=s(r,t+\Delta t) & s^-=s(r,t) & T^+_1=T_1(r,t+\Delta t) & T^-_1=T_1(r,t) \end{array}$$

Let us introduce following spaces of functions:

$$\begin{aligned} V_{t^+} &= \left\{ v\in H^1(\Omega) \quad v_{/\Gamma_1}=T_1(r,t^+) \right\} \\ V_{t^-} &= \left\{ v\in H^1(\Omega) \quad v_{/\Gamma_1}=T_1(r,t^-) \right\} \\ V_0 &= \left\{ v\in H^1(\Omega) \quad v_{/\Gamma_1}=0 \right\} \end{aligned}$$

The field $T^-\in V_{t^-}$ being supposed known, one seeks $T^+\in V_{t^+}$:

$$\begin{aligned} &\int_\Omega \rho C_p \frac{T^+-T^-}{\Delta t} v\, d\Omega + \int_\Omega (\theta \lambda \nabla T^+ . \nabla v + (1-\theta)\lambda \nabla T^- . \nabla v)\, d\Omega \\ &-\int_{\Gamma_2} (\theta f^+ + (1-\theta) f^-) v\, d\Gamma_2 - \int_{\Gamma_3} (\theta h^+ T^+_{ext} + (1-\theta) h^- T^-_{ext} - \theta h^+ T^+ - (1-\theta) h^- T^-) v\, d\Gamma_3 \\ &= \int_\Omega (\theta s^+ + (1-\theta) s^-) v\, d\Omega \\ &\forall v\in V_0 \end{aligned}$$

While posing:

$$\begin{aligned} (hT_{ext})^\theta &= \theta h^+ T^+_{ext} + (1-\theta) h^- T^-_{ext} \\ f^\theta &= \theta f^+ + (1-\theta) f^- \end{aligned}$$

one obtains finally:

$$\begin{aligned} &\int_\Omega \frac{\rho C_p}{\Delta t} T^+ v\, d\Omega + \int_\Omega \theta \lambda \nabla T^+ . \nabla v\, d\Omega + \int_{\Gamma_3} \theta h^+ T^+ v\, d\Gamma_3 \\ &= \int_\Omega \frac{\rho C_p}{\Delta t} T^- v\, d\Omega - \int_\Omega (1-\theta) \lambda \nabla T^- . \nabla v\, d\Omega + \int_{\Gamma_2} f^\theta v\, d\Gamma_2 \\ &+ \int_{\Gamma_3} ((hT_{ext})^\theta - (1-\theta) h^- T^-) v\, d\Gamma_3 + \int_\Omega (\theta s^+ + (1-\theta) s^-) v\, d\Omega \\ &\forall v\in V_0 \end{aligned}$$

6 Spatial discretization

Is P_h a space division Ω, indicate by N the number of nodes of the mesh, p_i the shape function associated with the node i. One indicates by J all the nodes belonging to the border Γ_1.

Are:

$$V^h_{t^+}=\{ v=\sum_{i=1,N} v_i p_i(x) \quad ; \quad v_j=T_1(x_j,t^+) \quad j\in J \}$$
$$V^h_{t^-}=\{ v=\sum_{i=1,N} v_i p_i(x) \quad ; \quad v_j=T_1(x_j,t^-) \quad j\in J \}$$
$$V^h_0=\{ v=\sum_{i=1,N} v_i p_i(x) \quad ; \quad v_j=0 \quad j\in J \}$$

Let us pose:

$$K_{ij}T_i=\int_{\Omega_h}\frac{\rho C_p}{\Delta t}T_i p_i p_j d\,\Omega_h+\int_{\Omega_h}\theta\lambda T_i \nabla p_i . \nabla p_j d\,\Omega_h+\int_{\Gamma_{h3}}\theta h^+ T_i p_i d\,\Gamma_{h3}$$

$$\begin{aligned}L_j= & \int_{\Omega_h}\frac{\rho C_p}{\Delta t}T^- p_j d\,\Omega_h-\int_{\Omega_h}(1-\theta)\lambda\nabla T^- . \nabla p_j d\,\Omega_h+\int_{\Gamma_{h2}} f^\theta p_j d\,\Gamma_{h2}\\ & +\int_{\Gamma_{h3}}((hT_{ext})^\theta-(1-\theta)h^- T^-)p_j d\,\Gamma_{h3}+\int_{\Omega_h}(\theta s^+ +(1-\theta)s^-)p_j d\,\Omega_h\end{aligned}$$

By dualisant the boundary conditions in imposed temperature ([R3.03.01]), one reveals the operator B defined by:

$$(Bv)_j=\begin{cases}0 & \text{si} \quad j\notin J\\ v_j & \text{si} \quad j\notin J\end{cases}$$

One obtains finally the following system:

$$\begin{cases}\sum_{i=1}^N K_{ij}T_i & + & ({}^t B\lambda)_j & = & L_j & \forall j\\ & & (BT)_j & = & T_1(x_j,t) & j\in J\end{cases}$$

7 Implementation in *Code_Aster*

7.1 introduced Equations

command THER_LINEAIRE [U4.33.01] makes it possible to treat the equation in the transitory case such as it is described above, but it also makes it possible to solve the steady problem which is reduced to the following equation:

$$-\mathrm{div}(\lambda \nabla T)=s \text{ dans } \Omega$$

and the following boundary conditions:

$$\begin{cases} T=T_1(r,t_s) & \text{sur}\,\Gamma_1 \\ \lambda \dfrac{\partial T}{\partial n}=q(r,t_s) & \text{sur}\,\Gamma_2 \\ \lambda \dfrac{\partial T}{\partial n}=h(r,t)(T_{ext}(r,t_s)-T) & \text{sur}\,\Gamma_3 \end{cases}$$

t_s being time taken to evaluate the boundary conditions of the equation.

In the transitory case, it is necessary to provide an initial state, this initial state (field of temperature) can be selected among the following:

- a field which can be uniform or unspecified created by the command CREA_CHAMP,
- a field result of a steady problem describe by the equations above, the time of computation is taken at the first time defined in the list of realities describing the temporal discretization defined by the user,
- a field extracted result from a transitory problem.

The discretization in time (value of Δt) must be provided in the shape of one or more lists of times. These lists are created by user by the command DEFI_LIST_REEL [U4.21.04].

A thermal transient can be calculated by carrying out several calls to command THER_LINEAIRE [U4.33.01] by enriching the same concept of the evol_ther type while providing starting from the second call initial time by recovery of computation (to obtain T^-) and possibly final moment.

The fields of temperatures resulting from a computation contain at the same time the value with the nodes of the mesh and the nodes of Lagrange. During a resumption of computation, it is possible to vary the type of the boundary conditions, the field used to initiate new in-house computation is then tiny room to the only nodes of the mesh. The result concept of the evol_ther type will contain fields at nodes then leaning on different classifications. The operators of *Code_Aster* interpolate then only with the nodes of the mesh when classification differs.

7.2 Principal thermal options calculated in *Code_Aster*

7.2.1 Boundary conditions and loadings

TEMP_IMPO	DDLI_R DDLI_F	$\int_{\Gamma_1} T^+ \Phi^* d\Gamma_1 + \int_{\Gamma_1} \Phi^\theta vdG_1$
	DDLI_R DDLI_F	$\int_{\Gamma_1} \Phi^* T_1 d\Gamma_1$
FLUX_REP	CHAR_THER_FLUN_R CHAR_THER_FLUN_F	$\int_{\Gamma_2} q^\theta vdG_2$
ECHANGE	CHAR_THER_COEF_R CHAR_THER_COEF_F	$\int_{\Gamma_3} \theta h^+ T^+ v\, d\Gamma_3$
	CHAR_THER_TEXT_R CHAR_THER_TEXT_F	$\int_{\Gamma_3} ((hT_{ext})^\theta - (1-\theta) h^- T^-) v\, d\Gamma_3$
ECHANGE_PAROI	RIGI_THER_PARO_R RIGI_THER_PARO_F	$\int_{\Gamma_{12}} \theta h^+ (T^+_{/\Gamma_{12}} - T^+_{/\Gamma_{21}}) v_1 d\Gamma_{12}$
	CHAR_THER_PARO_R CHAR_THER_PARO_F	$\int_{\Gamma_{12}} (1-\theta) h^- (T^-_{/\Gamma_{21}} - T^-_{/\Gamma_{12}}) v_1 d\Gamma_{12}$
SOURCE	CHAR_THER_SOUR_R CHAR_THER_SOUR_F	$\int_{\Omega} (\theta s^+ + (1-\theta) s^-) vd\Omega$

7.2.2 Computation of the elementary matrixes and transitory term

RIGI_THER	$\int_{\Omega} \theta \lambda \nabla T^+ . \nabla vd\Omega$
MASS_THER	$\int_{\Omega} \frac{\rho C_p}{\Delta t} T^+ vd\Omega$
CHAR_THER_EVOL	$\int_{\Omega} \frac{\rho C_p}{\Delta t} T^- vd\Omega - \int_{\Omega} (1-\theta) \lambda \nabla T^- . \nabla vd\Omega$

8 Description of the versions of the document

Version Aster	Author (S) Organization (S)	Description of the modifications
3	*J.P. LEFEBVRE (EDF/IMA/MMN)*	initial Text