Thermal nonlinear out of Summarized

mobile coordinate system

One presents the formulation and the algorithm of the problem of convection-diffusion in steady nonlinear thermal introduced within command THER_NON_LINE_MO [U4.33.04].

The goal is to solve the equation of heat in a mobile reference frame related to a loading and moving in a given direction and at a velocity.

Nonthe linearities of the problems come as well from the characteristics of the material which depend on the temperature, as boundary conditions of type radiation.

The problems of this type can be treated with models using of the finite elements structure planes, axisymmetric and three-dimensional.

Contents

1 Presentation of the problem

the equation of heat present strong not linearities under certain conditions. It is the case when the material undergoes phase changes: those are accompanied by an abrupt variation of the characteristic quantities (heat capacity, enthalpy). This nonlinearity is all the more accentuated when with the problem of convection-diffusion is dealt, where appears the term of transport depend on the function enthalpy. The goal of this modelization is to deal with this last problem in permanent mode (steady case).

In all the cases, it is supposed that the velocity field is known a priori. The case of a mobile solid is rather frequent in practice. It relates to in particular the applications of welding or the surface treatment which bring into play a heat source moving in a given direction and at a velocity. The problem of thermal is then studied in a reference frame related to the source.

The problem with derivatives partial results from the equation of the total heat balance on any field Ω which is written:

$$\underbrace{\frac{d}{dt}\int_{\Omega}\rho\beta d\,\Omega}_{\text{accumulation}}=\underbrace{\int_{\Omega}Q d\,\Omega}_{\text{création}}+\underbrace{-\int_{\partial\Omega}\boldsymbol{q}\,.\boldsymbol{n}\,d\Gamma}_{\text{entrée-sortie}} \quad \text{éq 1-1}$$

In this equation, Ω represents a related, interior field with the studied system, which one follows in his motion, β represents the specific enthalpy of the material and ρ indicates his density. Q is a voluminal heat source, $\boldsymbol{q}$ is the heat flux through the border $\partial\Omega$ ($\boldsymbol{n}$ being the external norm), and d/dt is **the particulate derivative**.

The first term of [éq 1-1] is written (see for example [bib1]):

$$\frac{d}{dt}\int_{\Omega}\rho\beta d\,\Omega=\int_{\Omega}\left(\frac{\partial(\rho\beta)}{\partial t}+\operatorname{div}(\rho\beta\boldsymbol{V})\right)d\,\Omega \quad \text{éq 1-2}$$

or, taking into account the conservation of mass $\left(\frac{\partial\rho}{\partial t}+\operatorname{div}(\rho\boldsymbol{V})=0\right)$ [bib1]:

$$\frac{d}{dt}\int_{\Omega}\rho\,\beta\,d\,\Omega=\int_{\Omega}\left(\rho\frac{\partial\beta}{\partial t}+\rho\,\boldsymbol{V}\,.\mathbf{grad}\,\beta\right)d\,\Omega \quad \text{éq 1-3}$$

where V is the Flight Path Vector of displacement of the field $\boldsymbol{\Omega}$. V is well informed under key word simple CONVECTION of commands AFFE_CHAR_THER and AFFE_CHAR_THER_F.

The second term of the second member of [éq 1-1] is written, taking into account the theorem of the divergence and the Fourier analysis $(\boldsymbol{q}=-k(T)\mathbf{grad}\,T)$:

$$\int_{\partial\Omega} \boldsymbol{q}.\boldsymbol{n}\,d\Gamma=\int_{\Omega} div\,\boldsymbol{q}\,d\Omega=-\int_{\Omega} div(k(T)\mathbf{grad}\,T)\,d\Omega \qquad \text{éq 1-4}$$

where T is the temperature and $k(T)$ is the thermal conductivity of the material, function of the temperature.

The equation [éq 1-1] in front of being satisfied for any field Ω , it comes then:

$$\rho\frac{\partial\beta}{\partial t}+\rho\boldsymbol{V}.\mathbf{grad}\,\beta-div(k(T)\mathbf{grad}\,T)=Q \text{ in } \Omega \qquad \text{1-5}$$

Note::

Let us note that the classical case with, $k(T)=k$ (constant) and $V=0$, and where the specific enthalpy is a linear function of the temperature, $\beta(T)=CT$ gives again the well-known classical equation:

$$\rho C\frac{\partial T}{\partial t}-kDT=Q \text{ in } \Omega$$

where Δ is the Laplacian and C (constant) the specific heat represents.

The problem with partial derivatives treaty by the command THER_NON_LINE_MO [U4.33.04], consists in solving the equation [éq 1-5] in the steady case (directly at the permanent state) with boundary conditions on the border $\partial\Omega$.

This problem is formally written in the following form:

$$\begin{array}{ll} \boldsymbol{V}.\mathbf{grad}\,u(T)-div(k(T)\mathbf{grad}\,T)=Q, & \text{dans}\,\Omega, \\ +\text{conditions aux limites} & \text{sur}\,\partial\Omega \end{array} \qquad \text{éq 1-6}$$

where we adopted the notation, valid for all the continuation, $u(T)=\rho\beta(T)$ where ρ is constant, defining the voluminal enthalpy.

2 Boundary conditions. Problem of reference to be solved

One will refer, for example, with [R5.02.01] for more information on the thermal boundary conditions of type Dirichlet, Neumann and linear Fourier, and with [R5.02.02] for the nonlinear normal flux boundary conditions type (nonlinear Fourier).

Of enthalpic formulation, the steady problem of **thermal** thus consists in solving in a field Ω of border on $\partial\Omega$.

$$\boldsymbol{V}\,.\,\mathbf{grad}\,u(T) - div\left(k(T)\,\mathbf{grad}\,T\right) = Q \quad \textit{in} \quad \Omega \qquad \text{2-1}$$

$$\textit{with} \quad k(T)\frac{\partial T}{\partial \boldsymbol{n}} = \gamma\left(T_{ext} - T\right) \quad \textit{out of} \quad \partial_1\Omega \qquad \text{2-2}$$

$$k(T)\frac{\partial T}{\partial \boldsymbol{n}} = q_0 \quad \textit{on} \quad \partial_2\Omega \qquad \text{2-3}$$

$$k(T)\frac{\partial T}{\partial \boldsymbol{n}} = \alpha(T) \quad \textit{on} \quad \partial_3\Omega \qquad \text{2-4}$$

$$T = T_0 \quad \textit{on} \quad \partial_4\Omega \qquad \text{2-5}$$

where:

- T_0 : is the temperature imposed on $\partial_4\Omega$;
- q_0 : is the normal flux imposed on $\partial_2\Omega$;
- γ : is the thermal coefficient of heat exchange ;
- T_{ext} : is the outside temperature ;
- $\alpha(T)$: is the normal flux of nonlinear type Fourier (radiation).

The equations [éq 2-2], [éq 2-5] the boundary conditions of the types represent, respectively: Linear Fourier, Neumann, nonlinear Fourier and Dirichlet.

The problem of reference [éq 2-1], [éq 2-5] is strongly nonlinear because of nonthe linearities on $k(T), u(T)$ (phase change) and $\alpha(T)$ (radiation).

3 Variational formulation of the problem

Is Ω open of $\boldsymbol{R}^3$, border $\partial\Omega=\partial_1\Omega\cup\partial_2\Omega\cup\partial_3\Omega\cup\partial_4\Omega$ such as,

for $i\neq j$ and $i,j=1,\dots,4$, one a: $\partial_i\Omega\cap\partial_j\Omega=\emptyset$.

That is to say still ψ a sufficiently regular function which is cancelled on $\partial_4\Omega$: $\psi\in V=\left\{y \text{ régulière et } \psi|_{\partial_4\Omega}=0\right\}$.

Let us multiply by ψ the two members of the equation [éq 2-1], then integrate on Ω . An integration by parts gives then:

$$\begin{aligned}\int_\Omega Q y d\Omega &= \int_\Omega V\mathbf{.grad}\,u(T)\,y d\Omega - \int_\Omega div\left(k(T)\mathbf{grad}\,T\right) y d\Omega \\ &= \int_\Omega V\mathbf{.grad}\,u(T)\,y d\Omega + \int_\Omega k(T)\mathbf{grad}\,T.\mathbf{grad}\,y d\Omega - \int_{\partial\Omega-\partial_4\Omega}\left(k(T)\frac{\partial T}{\partial n}y\right)dG\end{aligned}$$

éq 3-1

since ψ is null on $\partial_4\Omega$.

From where, by taking account of the boundary conditions [éq 2-2], [éq 2-3] and [éq 2-4], the variational formulation of the problem of reference which is given by the following equation:

$\forall\,\psi\in V$

$$\begin{aligned}&\int_\Omega k(T)\mathbf{grad}\,T.\mathbf{grad}\,y d\Omega + \int_\Omega V.\mathrm{grad}\,u(T)\,y d\Omega + \int_{\partial_1\Omega} gTydG - \int_{\partial_3\Omega}\alpha(T)\,ydG \\ &= \int_\Omega Q y d\Omega + \int_{\partial_1\Omega}\gamma\,T_{ext}\,ydG + \int_{\partial_2\Omega} q_0\,y d\Omega,\end{aligned}$$

éq 3-2

4 Processing of nonthe linearities

For the numerical resolution of the nonlinear problem that we consider, it is necessary to treat all nonthe linearities.

In our case, let us quote the strong not linearity related to the function enthalpy $u(T)$ which takes into account the solid-liquid phase change, as well as nonthe linearity related to the possible presence of a boundary condition of normal flux nonlinear (radiation).

Let us recall that in the classical case of the nonlinear problems of transient thermal without convection, that is to say $V=0$, several methods of resolution are proposed in the literature. There as well exist methods using of the enthalpic formulations as methods using of the formulations in temperature, all having for goal as well as possible treating to it not linearity related to the enthalpy (phase change).

We return the reader to the reference [bib5] for abstract of the principal methods met in the literature. However, let us note that because of the difficulty related to the presence of the term of transport $\boldsymbol{V}\,.\,\boldsymbol{grad}\,u(T)$ in the problem, none of these methods will be employed in the continuation.

As in any iterative process, the goal of the numerical diagram in sight is to find a field of temperature T^{n+1} to the iteration $n+1$, starting from the field of temperature T^n , solution of the preceding iteration.

4.1 Processing of nonthe linearity related to the enthalpy

In order to treat this nonlinearity, the strategy employed in this study was inspired by a technique of resolution of the free problems of borders [bib3], which, in the beginning was proposed in [bib4].

Let us regard the function enthalpy $u(T)$ as being given in a reciprocal form: Temperature function of the enthalpy (opposite of the function $u(T)$). In other terms one will have to treat the relation following Temperature-enthalpy:

$$T=\tau(u) \quad \text{éq 4.1-1}$$

the reason of this choice will be clearer in what follows. Indeed we will have to deal with a problem with two fields: a field of temperature and a field enthalpy. The discretization of the opposite function [éq 4.1-1] makes it possible to as follows increment the field enthalpy according to the current field of temperature (and not the reverse):

The development with the first order of the function $\tau(u)$ is the following,

$$T^{n+1}=\tau(u^n)+\tau'(u^n)(u^{n+1}-u^n), \quad \text{éq 4.1-2}$$

where τ' is the derivative of the function defined by [éq 4.1-1] compared to its argument.
In order to take into account this nonlinearity, and from [éq 4.1-2], one replaces u^{n+1} by an approximation according to the unknown field of temperature T^{n+1} in the following way:

$$u^{n+1}-u^n=\omega(T^{n+1}-\tau(u^n)), \quad \text{éq 4.1-3}$$

where ω is a parameter of relaxation, constant on all the field and during all the iterative process, representing the term $\frac{1}{\tau'(u^n)}$.

Because of the nonconvexity of the function $\tau(u)$, this parameter of relaxation must necessarily check the following condition [bib2], [bib3]:

$$\omega \le \frac{1}{\max_n \tau'(u^n)}$$ éq 4.1-4

In practice one takes $\omega = \frac{1}{\max_n \tau'(u^n)}$.

In taking into account the approximation [éq 4.1-3], the discretization of the second term of the equation [éq 3-2] is expressed in the following way:

$$\int_\Omega \mathbf{V} . \mathbf{grad}\, u^{n+1} \psi \, d\Omega = \int_\Omega \mathbf{V} . \mathbf{grad}\, u^n \psi \, d\Omega + \int_\Omega \omega \mathbf{V} . \mathbf{grad}\, T^{n+1} \psi \, d\Omega - \int_\Omega \omega \mathbf{V} . \mathbf{grad}\, \tau(u^n) \psi \, d\Omega ,$$

éq 4.1-5

4.2 Processing of nonthe linearities related on the nonlinear condition of Fourier and thermal conductivity

It not linearity related to the condition of normal flux nonlinear is treated by considering the development with the first order of the function (supposed sufficiently regular) $\alpha(T)$ which is given by:

$$\alpha(T^{n+1}) = \alpha(T^n) + \alpha'(T^n)(T^{n+1} - T^n) ,$$ éq 4.2-1

where $(.)'$ the derivative of the function compared to $(.)$ its argument indicates.

It appeared necessary to decide on a strategy of discretization of the term $k(T)\, \mathbf{grad}\, T$ in the equation [éq 3-2] in order to be able to treat this nonlinearity for the steady problem which we consider. For that, we adopted the following approximation:

$$k(T^{n+1})\, \mathbf{grad}\, T^{n+1} = k(T^n)\, \mathbf{grad}\, T^{n+1} - [k(T^n) - k(T^{n-1})]\, \mathbf{grad}\, T^n$$ éq 4.2-2

This discretization is in fact a simplification of the development to the first order of the term $k(T)\, \mathbf{grad}\, T$. It is effective being in particular because of the low not linearity of the function $k(T)$ in practice.

Note:

Also let us note that the following purely explicit approximation:

$$k(T^{n+1})\, \mathbf{grad}\, T^{n+1} \approx k(T^n)\, \mathbf{grad}\, T^{n+1} ,$$

also give satisfactory results. This observation was checked from several numerical experiments.

5 Algorithm established in *Code_Aster*

the numerical diagram employed for the resolution of the problem of reference [éq 2-1], [éq 2-5] is deduced from the variational formulation [éq 3-2] and from the processing of the various not linearities, [éq 4.1-5], [éq 4.2-1], [éq 4.2-2], discussed in the preceding section.

The algorithm of resolution is consisted the sequence of two successive operations to each iteration of computation.

Knowing the fields solutions with the iteration n : T^n with the nodes and u^n Gauss points, one seeks the solutions T^{n+1} et u^{n+1} with the iteration $n+1$ as follows:

$$\forall \psi \in V\,,$$

$$\begin{aligned}
&\int_\Omega k\left(T^n\right)\mathbf{grad}\,T^{n+1}.\,\mathbf{grad}\,\psi\, d\,\Omega \quad +\int_\Omega \omega\, \boldsymbol{V}\,.\,\mathbf{grad}\,T^{n+1}\psi\, d\,\Omega \\
&\qquad +\int_{\partial_1\Omega} \gamma\, T^{n+1}\psi\, d\,\Gamma - \int_{\partial_3\Omega} \alpha'\left(T^n\right)T^{n+1}\psi\, d\,\Gamma \\
&=\int_\Omega Q\,\psi\, d\,\Omega + \int_{\partial_1\Omega} \gamma\, T_{ext}\,\psi\, d\,\Gamma + \int_{\partial_2\Omega} q_0\,\psi\, d\,\Omega \\
&+\int_{\partial_3\Omega} \left(\alpha\left(T^n\right)-\alpha'\left(T^n\right)T^n\right)\psi\, d\,\Gamma + \int_\Omega \left[k\left(T^n\right)-k\left(T^{n-1}\right)\right]\mathbf{grad}\,T^n.\,\mathbf{grad}\,\psi\, d\,\Omega \\
&+\int_\Omega \omega\, \boldsymbol{V}\,.\,\mathbf{grad}\,t\left(u^n\right)\psi\, d\,\Omega - \int_\Omega \boldsymbol{V}\,.\,\mathbf{grad}\,u^n\psi\, d\,\Omega\,,
\end{aligned} \qquad \text{éq 5-1}$$

$$u^{n+1}=u^n+\omega\left(T^{n+1}-\tau\left(u^n\right)\right) \qquad \text{éq 5-2}$$

A each iteration, a linear problem of convection-diffusion is solved to obtain the field at nodes T^{n+1} [éq 5-1], and then a simple on-the-spot correction is carried out to obtain the field with Gauss points u^{n+1} [éq 5-2].

The stopping criteria adopted in *Code_Aster* utilize at the same time the two fields solutions: the field of temperature, and the field enthalpy.

The algorithm continues the iterations as long as at least one of the relative variations of reiterated is higher than the corresponding tolerance given by the user:

$$\frac{\left(\sum_{i=1,\dots,nddl}\left(T_i^{n+1}-T_i^n\right)^2\right)^{1/2}}{\left(\sum_{i=1,\dots,nddl}\left(T_i^{n+1}\right)^2\right)^{1/2}} > \text{tole } 1$$

$$\frac{\left(\sum_{i=1,\dots,npg}\left(u_i^{n+1}-T_i^n\right)^2\right)^{1/2}}{\left(\sum_{i=1,\dots,npg}\left(u_i^{n+1}\right)^2\right)^{1/2}} > \text{tole } 2$$

where $nddl$ is the nombre total of the degrees of freedom to the nodes, and npg is the nombre total of Gauss points.

$tole\,1$ is indicated under the key word `crit_temp_rela` key word factor convergence of the operator `ther_non_line_mo`.

$tole\,2$ is indicated under the key word `crit_enth_rela` key word factor convergence of the operator `ther_non_line_mo`.

6 Principal computation options in *Code_Aster*

One presents below the principal options of *Code_Aster* specific to the unfolding of the algorithm [éq 5-1], [éq 5-2] above. On the other hand, we will not mention the nonspecific options of *Code_Aster* and which are used in computation:

- Boundary conditions:

Linear Fourier

`RIGI_THER_COET_R`

`RIGI_THER_COET_F` $\int_{\partial_1 \Omega} \gamma T^{n+1} \psi \, d\Gamma$

Nonlinear Fourier

`RIGI_THER_FLUTNL` $\int_{\partial_3 \Omega} \alpha' (T^n) T^{n+1} \psi \, d\Gamma$

`CHAR_THER_FLUTNL` $\int_{\partial_3 \Omega} \left(\alpha(T^n) - \alpha'(T^n) T^n\right) \psi \, d\Gamma$

- elementary Matrixes and second member:

`RIGI_THER_TRANS` $\int_{\Omega} k(T^n) \, \mathbf{grad} \, T^{n+1} . \mathbf{grad} \, \psi \, d\Omega$

`RIGI_THER_CONV_T` $\int_{\Omega} \omega V . \mathbf{grad} \, T^{n+1} \psi \, d\Omega$

`CHAR_THER_TNL` $\int_{\Omega} \left[k(T^n) - k(T^{n-1})\right] \mathbf{grad} \, T^n . \mathbf{grad} \, \psi \, d\Omega$

$$+ \int_{\Omega} \omega \mathbf{V} . \mathbf{grad} \, \tau(u^n) \psi \, d\Omega - \int_{\Omega} V . \mathbf{grad} \, u^n \psi \, d\Omega$$

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8 Description of the versions of the document

Version Aster	Author (S) Organization (S)	Description of the modifications
04/01/00	*F. WAECKEL, B. NEDJAR (EDF/IMA/MMN, ENPC)*	initial Text