Elastoplastic integration of the behavior models of Summarized

Von Mises :

This document describes quantities calculated by the operator STAT_NON_LINE necessary to the implementation of the quasi static nonlinear algorithm describes in [R5.03.01] in the case as of elastoplastic behaviors. These quantities are calculated by the same subroutines in operator DYNA_NON_LINE in the case of a dynamic stress [R5.05.05].

This description is presented according to the various keywords which make it possible to the user to choose the desired behavior model. The behavior models treated here are:

- the behavior of Von Mises with isotropic hardening (linear or not linear)
- the behavior of Von Mises with linear kinematic hardening (models of Prager)

the integration method used is based on a direct implicit formulation. From the initial state, or from the time of preceding computation, one calculates the stress field resulting from an increment of strain. The tangent operator is also calculated.

Contents

1 Introduction

1.1 Behavior models described into this document

In operator `STAT_NON_LINE` [U4.51.03] (or `DYNA_NON_LINE` [U4.53.01]), two types of behaviors can be treated:

- the incremental behavior: key word factor `COMP_INCR`,
- the behavior in nonlinear elasticity: key word factor `COMP_ELAS`.

For each behavior one can choose:

- the behavior model: key word `RELATION`,
- mode of computation of the strains: key word `DEFORMATION`.

For more details, to consult the document [U4.51.03] user's manual, the behaviors described here not raising but of the key word factor `COMP_INCR`.

The relations treated in this document are:

`VMIS_ISOT_LINE:`	Von Mises with linear isotropic hardening,
`VMIS_ISOT_TRAC:`	Von Mises with isotropic hardening given by a curve of tension,
`VMIS_ISOT_PUIS:`	Von Mises with isotropic hardening given by an analytical curve,
`VMIS_JOHN_COOK:`	Von Mises with isotropic hardening of Johnson-Cook,
`VMIS_CINE_LINE:`	Von Mises with linear kinematic hardening.

1.2 Goal of integration

to solve the nonlinear total problem posed on the structure, the document [R5.03.01] described the algorithm used in *Code_Aster* for the nonlinear static (operator `STAT_NON_LINE)` and the document [R5.05.05] described the method used for the nonlinear dynamics (operator `DYNA_NON_LINE`).

These two algorithms lean on the computation of local quantities (in each point of integration of each finite element) which result from the integration of the behavior models.

A each iteration n of the method of Newton [R5.03.01 § 2.2.2.2] one must calculate the nodal forces $\mathbf{R}(\mathbf{u_i^n})=\mathbf{Q}^T\boldsymbol{\sigma}_i^n$ (options `RAPH_MECA` and `FULL_MECA`) the stresses σ_i^n being calculated in each point of integration of each element starting from displacements u_i^n via the behavior model. One must build also the tangent operator to compute: $\mathbf{K}_i^n$ (option `FULL_MECA`).

Before the first iteration, for the phase of prediction, one calculates $\mathbf{K}_{i-1}$ (option `RIGI_MECA_TANG`). The computation of $\mathbf{K}_{i-1}$, which is necessary to the phase of initialization [R5.03.01 § 2.2.2.2] corresponds to the computation of the tangent operator deduced from the problem of velocity.

This operator is not identical to that which is used to compute: $\mathbf{K}_i^n$ by option `FULL_MECA`, during iterations of Newton. Indeed, this last operator is tangent with the discretized problem in an implicit way.

One describes here for behavior models `VMIS_ISOT_LINE`, `VMIS_ISOT_TRAC`, `VMIS_ISOT_PUIS`, `VMIS_JOHN_COOK` and `VMIS_CINE_LINE`, the computation of the tangent matrix of the phase of prediction $\mathbf{K}_{i-1}$, then the computation of the stress field from an increment of strain, the computation of the nodal **forces** R and tangent matrix $\mathbf{K}_i^n$.

2 General notations and assumptions on the strains

All the quantities evaluated at previous time are subscripted par. $^-$

the quantities evaluated at time $t+\Delta t$ are not subscripted.

The increments are indicated par. Δ One has as follows:

$$\mathbf{Q}=\mathbf{Q}(t+\Delta t)=\mathbf{Q}(t)+\Delta\mathbf{Q}=\mathbf{Q}^{-}+\Delta\mathbf{Q}\,.$$

For the computation of derivatives, one will note: $\dot{\mathbf{Q}}$ derived from $\mathbf{Q}$ ratio with tensor

$\boldsymbol{\sigma}$	time from the stresses.
$\sim$	operator déviatoire: $\tilde{\boldsymbol{\sigma}}_{ij}=\boldsymbol{\sigma}-\frac{1}{3}\boldsymbol{\sigma}_{kk}\boldsymbol{\delta}_{ij}$.
$()_{eq}$	equivalent value of Von Mises: $\boldsymbol{\sigma}_{eq}=\sqrt{\frac{3}{2}\tilde{\boldsymbol{\sigma}}_{ij}\tilde{\boldsymbol{\sigma}}_{ij}}$
$\boldsymbol{\Delta}\,\varepsilon$	increment of strain.
$\mathbf{A}$	elasticity tensor.
λ,μ,E,ν,K	moduli of the isotropic elasticity, respectively: coefficients of Lamé, Young modulus, Poisson's ratio and modulus of compressibility.
$3\mathrm{K}=3\lambda+2\mu$	modulate compressibility
α	average thermal coefficient of thermal expansion.
t	time.
T	temperature.
$()_+$	positive part.

To compute: the tangent operators, one will adopt the convention of writing of the symmetric tensors of order 2 in the form of vectors with 6 components. Thus, for a tensor a :

$$\vec{a}={}^{t}\left[a_{xx}\quad a_{yy}\quad a_{zz}\,\sqrt{2}\,a_{xy}\quad\sqrt{2}\,a_{xz}\quad\sqrt{2}\,a_{yz}\right]$$

One introduces the hydrostatic vector $\vec{\mathbf{1}}$ and the matrix of deviatoric projection $\mathbf{P}$:

$$\vec{\mathbf{1}}={}^{t}\left[1\quad 1\quad 1\,0\quad 0\quad 0\right]$$
$$\mathbf{P}=\mathbf{Id}-\frac{1}{3}\vec{\mathbf{1}}\otimes\vec{\mathbf{1}}$$

2.1 Partition of the strains (small strains)

One writes for any time:

$$\boldsymbol{\varepsilon}(t)=\boldsymbol{\varepsilon}^{e}(t)+\boldsymbol{\varepsilon}^{th}(t)+\boldsymbol{\varepsilon}^{p}(t)$$

with

$$\boldsymbol{\varepsilon}^{e}(t)=\mathbf{A}^{-1}(T(t))\boldsymbol{\sigma}(t)$$

$$\boldsymbol{\varepsilon}^{th}(t)=\alpha(T(t))\left(T(t)-T_{ref}\right)\mathbf{Id}$$

or in a more general way:

$$\begin{aligned} \varepsilon^{th}(T) &= \alpha(T)\left(T-T_{def}\right)-\alpha\left(T_{ref}\right)\left(T_{ref}-T_{def}\right) \\ &= \hat{\alpha}(T)\left(T-T_{ref}\right) \\ \text{et} \quad \varepsilon^{th}\left(T_{ref}\right) &= 0 \end{aligned}$$

A depends on time t via the temperature. The thermal coefficient of thermal expansion $\alpha(T(t))$ is an average coefficient of thermal expansion which can depend on the temperature T. The temperature T_{ref} is the reference temperature, i.e. that for which thermal thermal expansion is supposed null if the average coefficient of thermal expansion is not known compared to T_{ref}, one can use a temperature of definition of the average coefficient of thermal expansion T_{def} (definite by the key word TEMP_DEF_ALPHA of DEFI_MATERIAU) different from the reference temperature [R4.08.01].

What leads to: $\dot{\boldsymbol{\varepsilon}}(t)=\overbrace{\mathbf{A}^{-1}(T(t))\boldsymbol{\sigma}(t)}^{\cdot}+\dot{\boldsymbol{\varepsilon}}^{th}(t)+\dot{\boldsymbol{\varepsilon}}^{p}(t)$

This choice is made by preoccupation with a coherence with elasticity: it is necessary to be able to find the same solution in elasticity (operator MECA_STATIQUE) and elastoplasticity (operator STAT_NON_LINE) when the characteristics of the material remain elastic. This choice leads to the discretization:

$$\Delta\varepsilon=\Delta\varepsilon^{p}+\Delta(A^{-1}\sigma)+\Delta\varepsilon^{th}$$

with:

$$\Delta(\mathbf{A}^{-1}\boldsymbol{\sigma})=\mathbf{A}^{-1}(t^{-}+\Delta t)(\boldsymbol{\sigma}^{-}+\Delta\boldsymbol{\sigma})-A^{-1}(t^{-})\boldsymbol{\sigma}^{-}$$

and

$$\Delta\boldsymbol{\varepsilon}^{\mathbf{th}}=(\alpha(t^{-}+\Delta t)(T-T_{ref})-\alpha(t^{-})(T^{-}-T_{ref}))\boldsymbol{Id}$$

2.2 Reactualization

In `STAT_NON_LINE`, under the key word factor `COMP_INCR`, several modes of computation of the strains are possible:

- `"PETIT"`
- `"SIMO_MIEHE"` [R5.03.21] (which carries out computation in large deformations for an isotropic hardening)
- `"GDEF_HYPO_ELAS"` [R5.03.24] which carries out the computation in large deformations, but with an hypo-elastic formulation, and which is usable for an unspecified hardening)
- `"GROT_GDEP"` [R5.03.22] (which carries out computation in large displacements and large rotations, but in small strains)
- `"PETIT_REAC"` (which is a substitute with computation in large deformations, valid for small increments of load, and for small rotations [bib2]).

This last possibility consists in reactualizing the geometry before you calculate $\Delta\varepsilon$:
One writes $x=x_0+u_{i-1}+\Delta u_i^n$, the computation of the gradients of Δu_i^n is thus made with the geometry x instead of the initial geometry x_0 .

2.3 Initial conditions

They are taken into account via $\boldsymbol{\sigma}^-$, p^- , $\mathbf{u}^-$.

In the event of poursuite or resumption of a preceding computation, there is directly the initial state $\boldsymbol{\sigma}^-$, p^- , $\mathbf{u}^-$ on the basis of $\boldsymbol{\sigma}, p, \mathbf{u}$ preceding computation at specified time.

3 Relation from Von Mises with isotropic hardening

3.1 Form of the behavior models

These relations are obtained by key keys `VMIS_ISOT_LINE`, `VMIS_ISOT_TRAC` and `VMIS_ISOT_PUIS`.

One describes here these relations into small strain (`DEFORMATION=' PETIT')`:

$$\left|\begin{array}{l}\dot{\boldsymbol{\varepsilon}}^p=\frac{3}{2}\dot{p}\frac{\tilde{\boldsymbol{\sigma}}}{\sigma_{eq}}=\dot{\boldsymbol{\varepsilon}}-\overbrace{A^{-1}\boldsymbol{\sigma}}^{\cdot}-\dot{\boldsymbol{\varepsilon}}^{th}\\ \sigma_{eq}-R(p)\le 0\\ \left(\begin{array}{l}\dot{p}=0 \text{ si } \sigma_{eq}-R(p)<0\\ \dot{p}\ge 0 \text{ si } \sigma_{eq}-R(p)=0\end{array}\right.\end{array}\right.$$

$\dot{\boldsymbol{\varepsilon}}^p$: vitesse de déformation plastique,
p: déformation plastique cumulée,
$\boldsymbol{\varepsilon}^{th}$: déformation d'origine thermique: $\boldsymbol{\varepsilon}^{th}=\alpha\left(T-T_{ref}\right)\mathbf{Id}$.

The function of hardening $R(p)$ is deduced from a monotonous simple traction test and isotherm the user can choose a linear hardening (relation VMIS_ISOT_LINE) or a given curve of tension either points by points (relation VMIS_ISOT_TRAC), or by an analytical statement (relation VMIS_ISOT_PUIS).

Constitutive law VMIS_JOHN_COOK differs from the preceding ones in the meaning where the function of hardening depends on the velocity of the cumulated plastic strain and the temperature.

One describes here these relations into small strain (DEFORMATION=' PETIT'):

$$\begin{cases} \dot{\boldsymbol{\varepsilon}}^{p} = \frac{3}{2}\dot{p}\frac{\tilde{\boldsymbol{\sigma}}}{\sigma_{eq}} = \dot{\boldsymbol{\varepsilon}} - \dot{\overbrace{A^{-1}\boldsymbol{\sigma}}} - \dot{\boldsymbol{\varepsilon}}^{th} \\ \sigma_{eq} - R(p, \dot{p}, T) \leq 0 \\ \begin{cases} \dot{p} = 0 \text{ si } \sigma_{eq} - R(p, \dot{p}, T) < 0 \\ \dot{p} \geq 0 \text{ si } \sigma_{eq} - R(p, \dot{p}, T) = 0 \end{cases} \end{cases}$$

$\dot{\boldsymbol{\varepsilon}}^{p}$: vitesse de déformation plastique,
p: déformation plastique cumulée,
$\dot{p}$: vitesse de déformation plastique cumulée,
$\boldsymbol{\varepsilon}^{th}$: déformation d'origine thermique: $\boldsymbol{\varepsilon}^{th} = \alpha\left(T - T_{ref}\right)\mathbf{Id}$.

The function of hardening $R(p, \dot{p}, T)$ is deduced from a series of traction tests simple monotonous at different strainrate and different temperature.

3.1.1 Relation VMIS_ISOT_LINE

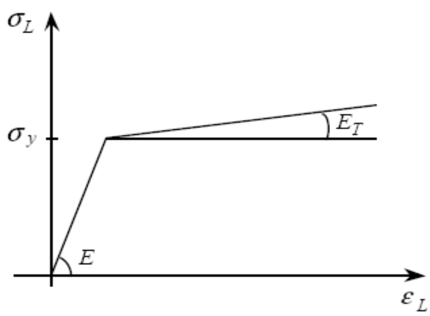
the data of the characteristics of materials are those provided under the key word factor ECRO_LINE or ECRO_LINE_FO of operator DEFI_MATERIAU [U4.43.01].

```
/ECRO_LINE =_F (D_SIGM_EPSI = E_T , SY = σ_y )
/ECRO_LINE_FO =_F (D_SIGM_EPSI = E_T , SY = σ_y )
```

ECRO_LINE_FO corresponds if E_T and σ_y depends on the temperature and is then calculated for the temperature of the current Gauss point.

The Young modulus E and the Poisson's ratio ν are those provided under the key keys factors ELAS or ELAS_FO.

In this case curve of tension is the following one:



I.e.:

$$\begin{cases} \sigma_L = E\,\varepsilon_L & \text{si}\,\varepsilon_L \le \dfrac{\sigma_y}{E} \\ \sigma_L = \sigma_y + E_T(\varepsilon_L - \dfrac{\sigma_y}{E}) & \text{si}\,\varepsilon_L \ge \dfrac{\sigma_y}{E} \end{cases}$$

Note:

σ_y is the elastic limit (the choice of σ_y falls to the user: it can correspond at the end of linearity of real curve of tension, either to a lawful or conventional elastic limit. At all events, one uses here the single value defined under `ECRO_LINE`).

When the criterion is reached one a: $\sigma_{eq} - R(p) = 0$. To identify $R(p)$, one uses the properties of the uniaxial stress state:

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_L & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{.donc} \quad \begin{array}{l} \sigma_{eq} = \sigma_L \\ p = \varepsilon_L^P = \varepsilon_L - \dfrac{\sigma_L}{E} \end{array} \quad \text{and the criterion is written: } \sigma_L - R(p) = 0$$

$$\sigma_L - \sigma_y = E_T(\varepsilon_L - \frac{\sigma_y}{E}) = E_T(\frac{\sigma_L}{E} + p - \frac{\sigma_y}{E})$$

thus

$$(\sigma_L - \sigma_y)\left(1 - \frac{E_T}{E}\right) = E_T\, p \ \text{ is } \ (\sigma_L - \sigma_y) = \frac{E_T . E}{(E - E_T)}\, p$$

from where the linear function of hardening: $R(p) = \dfrac{E_T\, E}{E - E_T}\, p + \sigma_y$

3.1.2 Relation VMIS_ISOT_TRAC

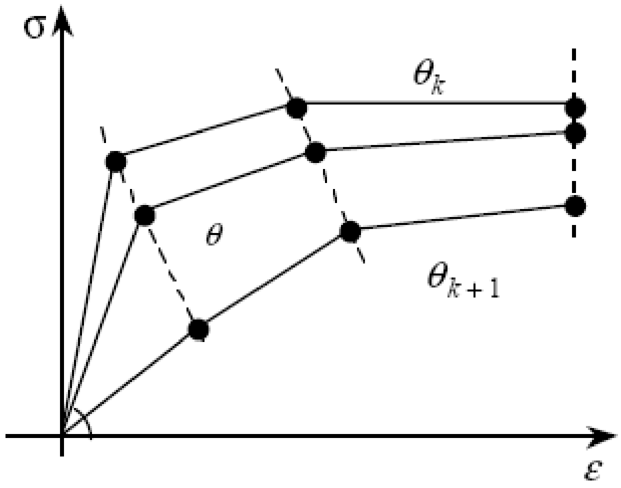
the data of the material are those provided under the key word factor `TENSION =_F (SIGM = F)`, of operator `DEFI_MATERIAU`.

`F` is a function with one or two variables representing curves of tension simple. The first variable is obligatorily the strain, the second if it exists is the temperature (parameter of a three-dimensions function). For each temperature, curve of tension must be such as:

- the X-coordinates (strains) are strictly increasing,
- the slope between 2 successive points is lower than the elastic slope between 0 and the first point of the curve.

The interpolation compared to the temperature is carried out in the following way:

That is to say θ temperature considered, if there exists k such as $\theta \in [\theta_k, \theta_k + 1]$ where k the index of the curves of tension contained in the three-dimensions function indicates, one point by point builds curve of tension with the temperature θ while interpolating compared to θ the X-coordinates and the ordered of the points of two extreme curves of tension.



If θ is apart from the intervals of definition of curves of tension, one extrapolates in accordance with the prolongations specified by the user in `DEFI_NAPPE` [U4.31.03] and according to the preceding principle.

Note:

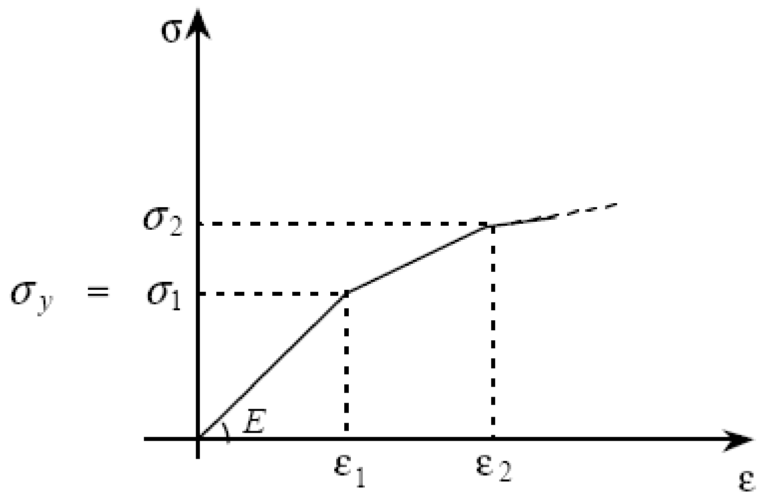
It is disadvised and dangerous to extrapolate curves of tension for values of temperature very far away from the extreme temperatures to which the curves are defined. It is always preferable to provide curves of tension for values of temperature framing the temperatures of computation.

If the numbers of points of discretization of curve of tension with θ_k and θ_{k+1} are different, one interpolates between the last point of the poorest curve with all the remaining points of the richest curve. Consequently, it is preferable to have a number of points of homogeneous enough discretization for the various temperatures.

In all the cases, curve of tension considered is a linear function per pieces:

$$\sigma=\sigma_i+\frac{\sigma_{i+1}-\sigma_i}{\varepsilon_{i+1}-\varepsilon_i}(\varepsilon-\varepsilon_i) \text{ for } \varepsilon\in[\varepsilon_i,\varepsilon_{i+1}] \text{, for } i+1\leq n$$

n being the number of points of interpolation with a linear extrapolation, constant or excluded according to the choice carried out in `DEFI_FONCTION` by the user (cf [U4.31.02] for more precise details on extrapolation considered).



The first point makes it possible to define:

$$\sigma_y = \sigma_1$$

$$E = \frac{\sigma_1}{\varepsilon_1}$$

It is this Young modulus who is used in the integration of the behavior model.

One thus has for all i :

$$p_i = \varepsilon_i - \frac{\sigma_i}{E}.$$

The function of hardening is then:

$$R(p) = \sigma(i) + \frac{\sigma_{i+1} - \sigma_i}{p_{i+1} - p_i}(p - p_i)\,\text{pour}\; p \in [p_i, p_{i+1}].$$

The user must also give the Poisson's ratio ν and a fictitious modulus Young yg (which serves only to compute: the elastic stiffness matrix if key word `NEWTON: (MATRICE: "ELASTIC")` is present in `STAT_NON_LINE`) by the key keys:

```
/ELAS    =_F  (NU = ν ,    E = E )
/ELAS_FO =_F  (NU = ν ,    E = E )
```

3.1.3 Relation VMIS_ISOT_PUIS

the data of the material are those provided under the key word factor `ECRO_PUIS` or `ECRO_PUIS_FO` of operator `DEFI_MATERIAU` [U4.43.01].

```
ECRO_PUIS=_F (SY= σy ,  A_PUIS =a,  N_PUIS =n)
```

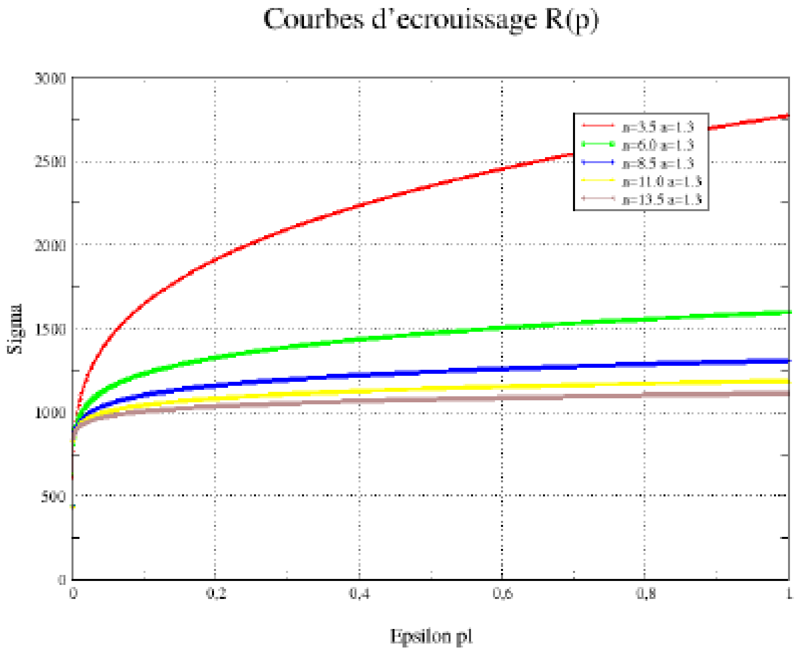
the curve of hardening is deduced from the uniaxial curve connecting the strains to the stresses, whose statement is:

$$\varepsilon = \frac{\sigma}{E} + a\frac{\sigma_y}{E}\left(\frac{\sigma - \sigma_y}{\sigma_y}\right)^n \quad \text{for} \quad \sigma > \sigma_y$$

what gives the curve of hardening:

$$R(p) = \sigma_y + \sigma_y\left(\frac{E}{a\sigma_y}p\right)^{\frac{1}{n}}$$

The curve representative of such a function takes the following form, for various values of N:



3.1.4 Relation `VMIS_JOHN_COOK`

the data of the material are those provided under the key word factor `ECRO_COOK` or `ECRO_COOK_FO` of operator `DEFI_MATERIAU` [U4.43.01].

```
ECRO_COOK =_F (A=A, B=B, C=C, N_PUIS=n, M_PUIS=m, EPSP0=epsp0, TROOM=troom,
TMELT=tmelt,)
```

the curve of hardening is written in the following way:

$$R(p,\dot{p},T)=(A+B\,p^n)\left(1+C\ln\left(\frac{\dot{p}}{\dot{p}_0}\right)\right)\left(1-\left(\frac{T-T_{room}}{T_{melt}-T_{room}}\right)^m\right)$$

or in a more concise way:

$$R(p,\dot{p},T)=(A+B\,p^n)(1+C\dot{p}^*)(1-T^{*m})$$

with $\dot{p}^*=\begin{cases}\dfrac{\dot{p}}{\dot{p}_0} & si \quad \dot{p}\geq\dot{p}_0\\ 1 & si \quad \dot{p}\leq\dot{p}_0\end{cases}$ and $T^*=\begin{cases}\dfrac{T-T_{room}}{T_{melt}-T_{room}} & si \quad T\geq T_{room}\\ 0 & si \quad T\leq T_{room}\end{cases}$

3.2 tangent Operator. Option `RIGI_MECA_TANG`

the goal of this paragraph is to calculate the tangent operator K_{i-1} (computation option `RIGI_MECA_TANG` called with the first iteration of a new increment of load) starting from the results known at previous time t_{i-1}.

For that, if the tensor of the stresses with t_{i-1} is on the border of the field of elasticity, the condition is written:

$$\dot{f}=0$$

who must be checked (for the continuous problem in time) jointly with the condition:

$$f=0$$

with:

$$f(\sigma, p)=\sigma_{eq}-R(p)$$

So on the other hand the tensor of the stresses with t_{i-1} is inside the field $f<0$, then the tangent operator is the operator of elasticity.

The quantities intervening in this statement are calculated at previous time t_{i-1}, which are the only known ones at the time of the phase of prediction. One thus obtains:

$$\dot{f}=\frac{\partial f}{\partial \boldsymbol{\sigma}}\dot{\boldsymbol{\sigma}}+\frac{\partial f}{\partial p}\dot{p}=\frac{\partial f}{\partial \boldsymbol{\sigma}}\dot{\tilde{\boldsymbol{\sigma}}}+\frac{\partial f}{\partial p}\dot{p}=\frac{\partial f}{\partial \boldsymbol{\sigma}}\left(2\mu\dot{\tilde{\boldsymbol{\varepsilon}}}-2\mu\dot{\boldsymbol{\varepsilon}}^P\right)+\frac{\partial f}{\partial p}\dot{p}$$
$$=\frac{\partial f}{\partial \boldsymbol{\sigma}}\left(2\mu\dot{\boldsymbol{\varepsilon}}-2\mu\dot{\boldsymbol{\varepsilon}}^P\right)+\frac{\partial f}{\partial p}\dot{p},$$

because of being $\frac{\partial f}{\partial \boldsymbol{\sigma}}$ deviative.

With

$$\boldsymbol{\sigma}=\boldsymbol{\sigma}^-=\boldsymbol{\sigma}\left(t_{i\text{-}1}\right)\quad \boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}^-=\boldsymbol{\varepsilon}\left(t_{i-1}\right),\ \boldsymbol{\varepsilon}^{\mathbf{p}}=\boldsymbol{\varepsilon}^{\mathbf{p}^-}=\boldsymbol{\varepsilon}^{\mathbf{p}}\left(t_{i\text{-}1}\right)\ \text{and}\ p=p^-=p\left(t_{i\text{-}1}\right)$$

Note:

One does not take account in this statement of the variation of the elastic coefficients with the temperature. It is an approximation, without important consequence, since this operator is only used to initialize the iterations of Newton. On the other hand, the dependence of the tangent operator compared to the thermal strains is well taken into account on the level of the total algorithm [R5.03.01].

One has then: $\frac{3}{2}\frac{\tilde{\boldsymbol{\sigma}}}{\sigma_{eq}}\left(2\mu\dot{\boldsymbol{\varepsilon}}-2\mu\dot{p}\frac{3}{2}\frac{\tilde{\boldsymbol{\sigma}}}{\sigma_{eq}}\right)-R'(p)\dot{p}=0$

what leads to: $\dot{p}=\frac{(3\mu)}{\sigma_{eq}}\frac{(\tilde{\boldsymbol{\sigma}}.\dot{\tilde{\boldsymbol{\varepsilon}}})}{3\mu+R'(p)}$ thus

$$\begin{cases}\dot{\boldsymbol{\varepsilon}}^p=\begin{cases}\frac{9\mu}{2}\frac{(\tilde{\boldsymbol{\sigma}}.\dot{\tilde{\boldsymbol{\varepsilon}}})}{3\mu+R'(p)}\frac{\tilde{\boldsymbol{\sigma}}}{\sigma_{eq}^2}, & \text{si } f(\sigma,p)=\sigma_{eq}-R(p)=0\\ 0, & \text{si } \sigma_{eq}-R(p)<0\end{cases}\\ \dot{\boldsymbol{\sigma}}_{ij}=K\dot{\boldsymbol{\varepsilon}}_{kk}\boldsymbol{\delta}_{ij}+2\mu\left(\dot{\tilde{\boldsymbol{\varepsilon}}}_{ij}-\dot{\boldsymbol{\varepsilon}}_{ij}^p\right)\end{cases}$$

Note::

The information $\boldsymbol{\sigma}_{eq}^- - R(p^-)=0$ is stored in the form of a local variable ξ which is worth 1 in this case and 0 if $\boldsymbol{\sigma}_{eq}^- < R(p^-)$.

The tangent operator binds the vector of virtual strains $\boldsymbol{\varepsilon}^*$ to a stress vector virtual $\boldsymbol{\sigma}^*$.

The tangent stiffness matrix is written for an elastic behavior:

$$\boldsymbol{\sigma}^*=\left(K\vec{\mathbf{1}}\otimes\vec{\mathbf{1}}+2\mu\mathbf{P}\right)\boldsymbol{\varepsilon}^*$$

and for a plastic behavior:

$$\boldsymbol{\sigma}^*=\left(K\vec{\mathbf{1}}\otimes\vec{\mathbf{1}}+2\mu\mathbf{P}-C_p\mathbf{s}\otimes\mathbf{s}\right)\boldsymbol{\varepsilon}^*$$

with $\mathbf{s}$ the vector of the deviatoric stresses associated with $\boldsymbol{\sigma}^-$ defined by:

$$\mathbf{s}^T=\left(\tilde{\sigma}_{11}^-,\tilde{\sigma}_{22}^-,\tilde{\sigma}_{33}^-,\sqrt{2}\,\tilde{\sigma}_{12}^-,\sqrt{2}\,\tilde{\sigma}_{23}^-,\sqrt{2}\,\tilde{\sigma}_{31}^-\right)$$

and:

$$C_p=\xi\frac{\left(3\mu\right)^2}{\left(\sigma_{eq}\right)^2}\frac{1}{3\mu+R'}$$

$$\xi=\begin{cases}1 & \text{si}\,\sigma_{eq}^- - R\left(p^-\right)=0\\ 0 & \text{sinon}\end{cases}$$

In the case of the first increment of loading, therefore if the state at previous time corresponds in a nonconstrained initial state, the tangent operator is identical to the operator of elasticity.

3.3 Computation of the stresses and the local variables

the decomposition of the strains makes it possible to write:

$$\Delta\boldsymbol{\varepsilon}=\Delta\boldsymbol{\varepsilon}^p+\Delta\left(\mathbf{A}^{-1}\boldsymbol{\sigma}\right)+\Delta\boldsymbol{\varepsilon}^{th}$$

Maybe, by taking the spherical and deviatoric parts:

$$\Delta\tilde{\boldsymbol{\varepsilon}}=\Delta\boldsymbol{\varepsilon}^p+\Delta\left(\frac{\tilde{\boldsymbol{\sigma}}}{2\mu}\right)\ \text{because}\ \Delta\tilde{\boldsymbol{\varepsilon}}^{th}=0.$$

$$tr\,\Delta\boldsymbol{\varepsilon}=\Delta\left(\frac{\text{tr}\,\boldsymbol{\sigma}}{3\text{K}}\right)+tr\,\Delta\boldsymbol{\varepsilon}^{th}\quad\text{tr}\,\Delta\boldsymbol{\varepsilon}^p=0.$$

By direct implicit discretization of the behavior models for isotropic hardening, one obtains then:

$$2\mu\Delta\tilde{\boldsymbol{\varepsilon}}-(\tilde{\boldsymbol{\sigma}}^-+\Delta\tilde{\boldsymbol{\sigma}})=\frac{3}{2}2\mu\Delta p\frac{\tilde{\boldsymbol{\sigma}}^-+\Delta\tilde{\boldsymbol{\sigma}}}{(\tilde{\boldsymbol{\sigma}}^-+\Delta\tilde{\boldsymbol{\sigma}})_{eq}}-2\frac{\mu}{2\mu^-}\tilde{\boldsymbol{\sigma}}^-$$

$$tr\,\boldsymbol{\sigma}=\frac{3\text{K}}{3\text{K}^-}tr\,\boldsymbol{\sigma}^-+3K\,tr\,\boldsymbol{\Delta}\,\boldsymbol{\varepsilon}-3K\,tr\,\boldsymbol{\Delta}\,\boldsymbol{\varepsilon}^{th}$$

$$\left(\boldsymbol{\sigma}^-+\boldsymbol{\Delta}\boldsymbol{\sigma}\right)_{eq}-R\left(p^-+\Delta p\right)\leq 0$$

$$\Delta p=0\,\text{si}\left(\boldsymbol{\sigma}^-+\boldsymbol{\Delta}\boldsymbol{\sigma}\right)_{eq}<R\left(p^-+\Delta p\right)$$

$$\Delta p\geq 0\,\text{si}\left(\boldsymbol{\sigma}^-+\boldsymbol{\Delta}\boldsymbol{\sigma}\right)_{eq}=R\left(p^-+\Delta p\right)$$

One defines, to simplify the notations, the tensor σ^e such as:

$$\tilde{\boldsymbol{\sigma}}^e = \frac{2\mu}{2\mu^-}\tilde{\boldsymbol{\sigma}}^- + 2\mu\boldsymbol{\Delta}\tilde{\boldsymbol{\varepsilon}} \text{ and } \operatorname{tr}\sigma^e = \operatorname{tr}\sigma .$$

Two cases arise:

- $(\boldsymbol{\sigma}^- + \boldsymbol{\Delta\sigma})_{eq} < R(p^- + \Delta p)$

  in this case: $\Delta p = 0 \,\text{soit}\, \tilde{\boldsymbol{\sigma}} = \tilde{\boldsymbol{\sigma}}^- + \Delta\tilde{\boldsymbol{\sigma}} = \tilde{\boldsymbol{\sigma}}^e$, therefore: $(\tilde{\boldsymbol{\sigma}}^e)_{eq} < R(p^-)$

- $(\boldsymbol{\sigma}^- + \boldsymbol{\Delta\sigma})_{eq} = R(p^- + \Delta p)$

  in this case: $\Delta p \geq 0$ thus: $(\tilde{\boldsymbol{\sigma}}^e)_{eq} \geq R(p^-)$

One from of deduced the algorithm from resolution:

- so $\tilde{\sigma}^e_{eq} \leq R(p^-)$ then $\Delta p = 0 \,\text{soit}\, \tilde{\boldsymbol{\sigma}} = \tilde{\boldsymbol{\sigma}}^- + \Delta\tilde{\boldsymbol{\sigma}} = \tilde{\boldsymbol{\sigma}}^e$
- so $\tilde{\sigma}^e_{eq} > R(p^-)$

  then it is necessary to solve: $\tilde{\boldsymbol{\sigma}}^e = \tilde{\boldsymbol{\sigma}}^- + \Delta\tilde{\boldsymbol{\sigma}} + \frac{3}{2}2\mu\Delta p \frac{\tilde{\boldsymbol{\sigma}}^- + \Delta\tilde{\boldsymbol{\sigma}}}{(\boldsymbol{\sigma}^- + \Delta\boldsymbol{\sigma})_{eq}}$

thus while factorizing $\tilde{\boldsymbol{\sigma}}^- + \Delta\tilde{\boldsymbol{\sigma}}$ and by taking the equivalent value of Von Mises:

$$\sigma^e_{eq} = \left(1 + \frac{3}{2}\frac{2\mu\Delta p}{(\boldsymbol{\sigma}^- + \Delta\boldsymbol{\sigma})_{\text{eq}}}\right)(\boldsymbol{\sigma}^- + \Delta\boldsymbol{\sigma})_{\text{eq}}$$

that is to say:

$$\sigma^e_{eq} = R(p^- + \Delta p) + 3\mu\Delta p$$

because: $\sigma_{eq} = (\boldsymbol{\sigma}^- + \boldsymbol{\Delta\sigma})_{eq} = R(\overline{p} + \Delta p)$

It is a scalar equation in Δp , linear or not according to $R(p)$. Δp is calculated in the following way:

- if hardening is linear (relation `VMIS_ISOT_LINE`), one obtains directly:

$$\Delta p = \frac{\sigma^e_{eq} - \sigma_y - R' p^-}{R' + 3\mu} \quad R' = \frac{E E_T}{E - E_T} \text{ with}$$

- if hardening is given by a curve of tension closely connected per pieces, (relation `VMIS_ISOT_TRAC)`, one benefits from the linearity per pieces to determine exactly Δp to see [§An1];
- in the case of a hardening defined by a model in power (relation `VMIS_ISOT_PUIS`), Δp is solution of the nonlinear equation: $R(p^- + \Delta p) + 3\mu\Delta p - \sigma^e_{eq} = 0$. This equation is solved by an iterative method (algorithm of the secant type). In the vicinity of the origin, one linearizes $R(p)$, because the derivative $R' = \frac{E}{an}\left(\frac{E}{a\sigma_y}p\right)^{\frac{1}{n}-1}$ is infinite in $p = 0$. Thus if $p < p_0$, one replaces $R(p)$ by $R^{lin}(p) = \sigma_y + \frac{p}{p_0}(R(p_0) - \sigma_y)$, which avoids the search for a solution numerically almost null. In practice, one chooses $p_0 = 10^{-10}$.

Once Δp determined, one calculates σ by:

$$\tilde{\boldsymbol{\sigma}}^{-}+\Delta\tilde{\boldsymbol{\sigma}}=\frac{\sigma_{eq}^{e}-3\mu\Delta p}{\sigma_{eq}^{e}}.\tilde{\boldsymbol{\sigma}}^{e}$$

and

$$\mathrm{tr}\left(\boldsymbol{\sigma}^{-}+\Delta\boldsymbol{\sigma}\right)=\mathrm{tr}\,\boldsymbol{\sigma}^{\mathbf{e}}\ .$$

Options RAPH_MECA and FULL_MECA carry out both the preceding computation, which clarifies the computation of $\mathbf{R}\left(\mathbf{u}_i^n\right)$. It is noticed that actually, $\mathbf{R}\left(\mathbf{u}_\mathbf{i}^\mathbf{n}\right)=\mathbf{Q}^T\boldsymbol{\sigma}_i^n$ where $\boldsymbol{\sigma}_i^n$ is calculated not according to $\mathbf{u}_i^n$, but $\boldsymbol{\sigma}_{i-1}$ et $\boldsymbol{\Delta}\,\mathbf{u}_i^n$.

Note:

Typical case of the plane stresses.

The model of Von Mises with isotropic hardening (VMIS_ISOT_LINE, VMIS_ISOT_PUIS or VMIS_ISOT_TRAC) is also available in plane stresses, i.e. for modelizations C_PLAN, DKT, COQUE_3D, COQUE_AXIS, COQUE_D_PLAN, COQUE_C_PLAN, PIPE, TUYAU_6M.

In this case, the system to be solved comprises an additional equation. This computation is detailed in appendix 2.

3.4 Tangent operator. Option FULL_MECA

option FULL_MECA makes it possible to calculate the tangent matrix $\boldsymbol{K}_i^n$ with each iteration. The tangent operator who is used for building it is calculated directly on the preceding discretized system (one notes to simplify: $\tilde{\boldsymbol{\sigma}}=\tilde{\boldsymbol{\sigma}}^{-}+\boldsymbol{\Delta}\,\tilde{\boldsymbol{\sigma}}$, $p=p^{-}+\Delta p$) and one writes the statements only in the isothermal case.

- If the tensor of the stresses is on the border of the field, $f=0$ then one has, by differentiating the form of the model of normality in $\tilde{\boldsymbol{\sigma}}=\tilde{\boldsymbol{\sigma}}^{-}+\boldsymbol{\Delta}\,\tilde{\boldsymbol{\sigma}}$:

$$2\mu\,\boldsymbol{\delta}\boldsymbol{\epsilon}^{p}=2\mu\,\boldsymbol{\delta}\,\tilde{\boldsymbol{\epsilon}}-\boldsymbol{\delta}\,\tilde{\boldsymbol{\sigma}}=\frac{3}{2}2\mu\left(\delta p\frac{\tilde{\boldsymbol{\sigma}}}{\sigma_{eq}}+\Delta p\frac{\boldsymbol{\delta}\,\tilde{\boldsymbol{\sigma}}}{\sigma_{eq}}-\frac{3}{2}\Delta p\frac{\tilde{\sigma}:d\,\tilde{\sigma}}{\sigma_{eq^3}}.\tilde{\boldsymbol{\sigma}}\right)$$

where $\boldsymbol{\delta}\,\boldsymbol{\varepsilon}^{\mathbf{p}}\,\boldsymbol{\delta}\,\tilde{\boldsymbol{\varepsilon}}\,\boldsymbol{\delta}\,\tilde{\boldsymbol{\sigma}}$ represent infinitesimal increases around the solution in the incremental elastoplastic problem obtained previously.

Like:

$$\frac{3}{2}\frac{\tilde{\boldsymbol{\sigma}}:d\,\tilde{\boldsymbol{\sigma}}}{\sigma_{eq}}=R'(p)\,dp$$

by carrying out the tensor product of the first equation by $\tilde{\boldsymbol{\sigma}}$ one a:

$$2\mu\,\tilde{\boldsymbol{\sigma}}:\boldsymbol{\delta}\,\tilde{\boldsymbol{\varepsilon}}-\tilde{\boldsymbol{\sigma}}:\delta\,\tilde{\boldsymbol{\sigma}}=2\mu\,\delta p.\sigma_{eq}\ ,$$

while eliminating δp from the two last equations:

$$\tilde{\boldsymbol{\sigma}}:\delta\tilde{\boldsymbol{\sigma}}=\frac{2\mu\tilde{\boldsymbol{\sigma}}:\delta\tilde{\boldsymbol{\varepsilon}}}{1+\frac{3\mu}{R'(p)}}\ .$$

- So on the other hand if the tensor of the stresses is inside the field $f<0$, then the tangent operator is the operator of elasticity.

While expressing $\delta\ p$ and $\tilde{\boldsymbol{\sigma}}:\delta\ \tilde{\boldsymbol{\sigma}}$ in the first equation, one obtains:

$$2\mu\boldsymbol{\delta}\ \tilde{\boldsymbol{\varepsilon}}-\boldsymbol{\delta}\ \tilde{\boldsymbol{\sigma}}=\frac{3\mu\Delta p}{\sigma_{eq}}\boldsymbol{\delta}\ \tilde{\boldsymbol{\sigma}}+C_p.(\tilde{\boldsymbol{\sigma}}:\boldsymbol{\delta}\ \tilde{\boldsymbol{\varepsilon}})_+\tilde{\boldsymbol{\sigma}},$$

with:

$$C_p=\frac{9\mu^2}{\sigma_{eq^2}}\left(1-\frac{R'(p)\Delta p}{\sigma_{eq}}\right)\frac{1}{R'(p)+3\mu}$$

The positive part $(\tilde{\boldsymbol{\sigma}}:\boldsymbol{\delta}\ \tilde{\boldsymbol{\varepsilon}})_+$ makes it possible to gather in only one equation the two conditions:

- either $f<0$, which implies $\Delta p=0$
- or $f=0$

One obtains then:

$$\boldsymbol{\delta}\ \tilde{\boldsymbol{\sigma}}=\frac{2\mu}{a}\boldsymbol{\delta}\ \tilde{\boldsymbol{\varepsilon}}-\frac{C_p}{a}(\tilde{\boldsymbol{\sigma}}:\boldsymbol{\delta}\ \tilde{\boldsymbol{\varepsilon}})_+\tilde{\boldsymbol{\sigma}}$$

while posing:

$$a=1+\frac{3\mu\Delta p}{R(p+\Delta p)}$$

The tangent operator binds the vector of virtual strains $\boldsymbol{\varepsilon}^*$ to a stress vector virtual $\boldsymbol{\sigma}^*$.

The tangent stiffness matrix is written for an elastic behavior:

$$\boldsymbol{\sigma}^*=(K\vec{\mathbf{1}}\otimes\vec{\mathbf{1}}+2\mu\mathbf{P})\boldsymbol{\varepsilon}^*$$

and for a plastic behavior:

$$\boldsymbol{\sigma}^*=\left(K\vec{1}\otimes\vec{1}+\frac{2\mu}{a}\mathbf{P}-\frac{C_p}{a}\mathbf{s}\otimes\mathbf{s}\right)\boldsymbol{\varepsilon}^*$$

with $\mathbf{s}$ the vector of the deviatoric stresses associated with $\boldsymbol{\sigma}^-$ defined by:

$$\mathbf{s}^{\mathbf{T}}=\left(\tilde{\sigma}_{11}^-,\tilde{\sigma}_{22}^-,\tilde{\sigma}_{33}^-,\sqrt{2}\,\tilde{\sigma}_{12}^-,\sqrt{2}\,\tilde{\sigma}_{23}^-,\sqrt{2}\,\tilde{\sigma}_{31}^-\right)$$

and:

$$\xi=\begin{cases}1 \text{ si } \Delta\varepsilon \text{ conduit à une plastification} \quad \text{et } \tilde{\boldsymbol{\sigma}}\,.\,\Delta\tilde{\boldsymbol{\varepsilon}}\geq 0\\ 0 \text{ sinon}\end{cases}$$

It is noted that the tangent operator with the system resulting from the implicit discretization differs from the tangent operator to the problem of velocity (RIGI_MECA_TANG). One finds it while making: $\Delta p=0$ in the statements of C_p and a .

3.5 Local variables of behaviors VMIS_ISOT_LINE, VMIS_ISOT_PUIS, VMIS_ISOT_TRAC and VMIS_JOHN_COOK

behavior models VMIS_ISOT_LINE, VMIS_ISOT_PUIS and VMIS_ISOT_TRAC produce two local variables:

- p cumulated equivalent plastic strain,
- and χ indicator of plasticity at time considered (useful for the computation of the tangent operator).

VMIS_JOHN_COOK uses two local variables besides the two preceding ones:

- $\dot{p}^-$ plastic strainrate equivalent cumulated to time less,
- and Δt^- the increment of time step at time less.

4 Relation from Von Mises with linear kinematic hardening

4.1 Form of the behavior model

This relation is obtained by key word `VMIS_CINE_LINE` of the key word factor `COMP_INCR`.

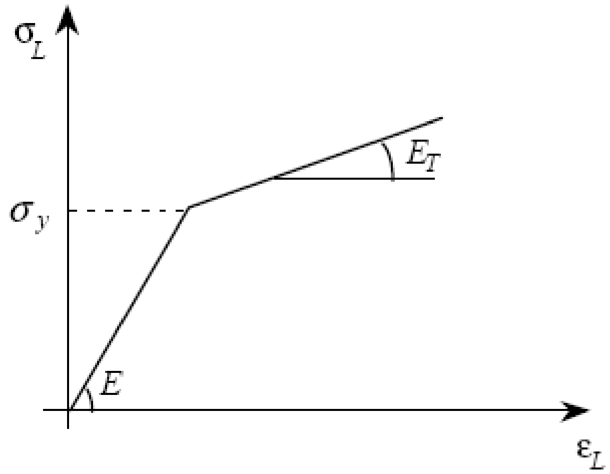
She is written (always in small strains):

$$\begin{cases} \dot{\boldsymbol{\varepsilon}}^P = \dfrac{3}{2}\dot{p}\dfrac{\tilde{\boldsymbol{\sigma}}-\tilde{\mathbf{X}}}{(\sigma-X)_{eq}} = \dfrac{3}{2}\dot{p}\dfrac{\tilde{\boldsymbol{\sigma}}-\mathbf{X}}{(\sigma-X)_{eq}} = \dot{\boldsymbol{\varepsilon}} - \dot{\overbrace{\mathbf{A}^{-1}}}\boldsymbol{\sigma} - \dot{\boldsymbol{\varepsilon}}^{th} \\ \mathbf{X} = C\,\boldsymbol{\varepsilon}^{\mathbf{p}}\boldsymbol{\varepsilon}^{th} = \alpha(T-T_{ref})\,\mathbf{Id} \\ (\boldsymbol{\sigma}-\mathbf{X})_{eq} - \sigma_y \le 0 \\ \begin{cases} \dot{p} = 0 \text{ si } (\boldsymbol{\sigma}-\mathbf{X})_{eq} - \sigma_y \le 0 \\ \dot{p} \ge 0 \text{ si } (s-X)_{eq} - s_y = 0 \end{cases} \end{cases} \qquad \text{éq 4.1-1}$$

σ_y is the elastic limit (the choice of σ_y falls to the user: it can correspond at the end of linearity of real curve of tension, either to a lawful or conventional elastic limit… At all events, one uses here the single value defined under `ECRO_LINE`).
C is the coefficient of hardening deduced from the data by a simple traction test.

In this case (uniaxial stress tensor, strain tensor plastics isochoric and orthotropic):

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_L & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{X} = \begin{pmatrix} X_L & 0 & 0 \\ 0 & -\dfrac{X_L}{2} & 0 \\ 0 & 0 & -\dfrac{X_L}{2} \end{pmatrix}$$



$$(\boldsymbol{\sigma}-\mathbf{X})_{eq} = \sigma_L - \frac{3}{2}X_L \quad \text{and the} \quad X_L = C\,\varepsilon_L^p = C\left(\varepsilon_L - \frac{\sigma_L}{E}\right)$$

material characteristics are those provided under the key word factor ECRO_LINE or ECRO_LINE_FO of operator DEFI_MATERIAU :

/ECRO_LINE =_F (D_SIGM_EPSI = E_T , SY = σ_y)
/ECRO_LINE_FO =_F (D_SIGM_EPSI = E_T , SY = σ_y)

ECRO_LINE_FO corresponds if E_T and σ_y depends on the temperature and is then calculated for the temperature of the current Gauss point.

The Young modulus E and the Poisson's ratio are those provided under the key keys factors ELAS or ELAS_FO.

For $\varepsilon_L > \frac{\sigma_y}{E}$ $\sigma_L = \sigma_y + E_T\left(\varepsilon_L - \frac{\sigma_y}{E}\right)$,

but one also has:

$$\begin{cases} \sigma_L - \frac{3}{2} X_L = \sigma_y \\ X_L = C\left(\varepsilon_L - \frac{\sigma_L}{E}\right) \end{cases}$$

from where, while eliminating X_L and while identifying:

$$C = \frac{2}{3}\frac{E E_T}{E - E_T}.$$

4.2 Tangent operator. Option RIGI_MECA_TANG

the goal of this paragraph is to calculate the tangent operator K_{i-1} (computation option RIGI_MECA_TANG called with the first iteration of a new increment of load) starting from the results known at previous time t_{i-1}.

For that, if the tensor of the stresses with t_{i-1} is on the border of the field of elasticity, the condition is written:

$$\dot{f} = 0$$

who must be checked (for the continuous problem in time) jointly with the condition:

$$f = 0$$

with

$$f = f\left(\boldsymbol{\sigma}^-, \mathbf{X}^-\right) = \left(\boldsymbol{\sigma}^- - \mathbf{X}^-\right)_{eq} - \sigma_y$$

So on the other hand the tensor of the stresses with t_{i-1} is inside the field $f < 0$, then the tangent operator is the operator of elasticity.

One poses:

$$\boldsymbol{\sigma}^{dev}=\tilde{\boldsymbol{\sigma}}^{-}-\mathbf{X}^{-} \text{ et } \gamma=\begin{cases}1 & \text{si } \left(\boldsymbol{\sigma}^{-}-\mathbf{X}^{-}\right)_{eq}-\sigma_y=0 \quad (\text{variable interne } \chi) \\ 0 & \text{sinon}\end{cases}$$

The problem of velocity is written in this case:

$$\begin{cases}\dot{\boldsymbol{\varepsilon}}^{p}=\begin{cases}\dfrac{1}{2\mu}\dfrac{3}{2}\left(\dfrac{2\mu}{\sigma_y}\right)^2\dfrac{\left(\left(\tilde{\boldsymbol{\sigma}}-\mathbf{X}\right).\dot{\tilde{\boldsymbol{\varepsilon}}}\right)\left(\tilde{\boldsymbol{\sigma}}-\mathbf{X}\right)}{C+2\mu}\text{si}\left(\boldsymbol{\sigma}-\mathbf{X}\right)-\sigma_y=0 \\ 0\,\text{si}\left(s-X\right)_{\text{eq}}-s_y<0\end{cases} \\ \dot{\sigma}_{ij}=K\,\dot{\varepsilon}_{kk}\,\delta_{ij}+2\mu\left(\dot{\tilde{\varepsilon}}_{ij}-\dot{\varepsilon}_{ij}^{p}\right)\end{cases}$$

The tangent operator binds the vector of virtual strains $\boldsymbol{\varepsilon}^{*}$ to a stress vector virtual $\boldsymbol{\sigma}^{*}$.
The tangent stiffness matrix is written for an elastic behavior:

$$\boldsymbol{\sigma}^{*}=\left(K\,\vec{\mathbf{1}}\otimes\vec{\mathbf{1}}+2\mu\,\mathbf{P}\right)\boldsymbol{\varepsilon}^{*}$$

and for a plastic behavior:

$$\boldsymbol{\sigma}^{*}=\left(K\,\vec{\mathbf{1}}\otimes\vec{\mathbf{1}}+2\mu\,\mathbf{P}-C_p\,\mathbf{s}\otimes\mathbf{s}\right)\boldsymbol{\varepsilon}^{*}$$

with $\mathbf{s}$ the vector of the deviatoric stresses associated with $\boldsymbol{\sigma}^{dev}$ defined by:

$$\mathbf{s}^{\mathbf{T}}=\left(\sigma_{11}^{dev},\sigma_{22}^{dev},\sigma_{33}^{dev},\sqrt{2}\,\sigma_{12}^{dev},\sqrt{2}\,\sigma_{23}^{dev},\sqrt{2}\,\sigma_{31}^{dev}\right)$$

and:

$$C_p=\gamma\frac{3}{2}\left(\frac{2\mu}{\sigma_y}\right)^2\frac{1}{2\mu+C}$$

In the case of the first increment of loading, therefore if the state at previous time corresponds in a nonconstrained initial state, the tangent operator is identical to the operator of elasticity.

4.3 Computation of the stresses and local variables

the direct implicit discretization of the continuous relations results in solving:

$$\begin{cases}2\mu\,\boldsymbol{\Delta}\,\boldsymbol{\varepsilon}^{p}=2\mu\left(\boldsymbol{\Delta}\,\tilde{\boldsymbol{\varepsilon}}+\dfrac{\tilde{\boldsymbol{\sigma}}^{-}}{2\mu^{-}}-\dfrac{\tilde{\boldsymbol{\sigma}}}{2\mu}\right)=\dfrac{3}{2}2\mu\,\Delta\,p\,\dfrac{\tilde{\boldsymbol{\sigma}}-\mathbf{X}}{\sigma_y} \\ \mathbf{X}=\dfrac{C}{C^{-}}\mathbf{X}^{-}+C\,\boldsymbol{\Delta}\,\boldsymbol{\varepsilon}^{p} \\ \left(\boldsymbol{\sigma}-\mathbf{X}\right)_{eq}\leq\sigma_y \\ \Delta\,p=0\,\text{si}\left(\boldsymbol{\sigma}-\mathbf{X}\right)_{eq}<\sigma_y \\ \Delta\,p\geq0\,\text{sinon} \\ \text{tr}\left(\boldsymbol{\sigma}^{-}+\boldsymbol{\Delta}\,\boldsymbol{\sigma}\right)=\dfrac{3\text{K}}{3\text{K}^{-}}\text{tr}\,\boldsymbol{\sigma}^{-}+3\text{K}\,\text{tr}\,\boldsymbol{\Delta}\,\boldsymbol{\varepsilon}-3\text{K}\,\text{tr}\,\boldsymbol{\Delta}\,\boldsymbol{\varepsilon}^{th}\end{cases}$$

One still poses:

$$\tilde{\boldsymbol{\sigma}}^e=\frac{2\mu}{2\mu^-}\tilde{\boldsymbol{\sigma}}^-+2\mu\boldsymbol{\Delta}\tilde{\boldsymbol{\varepsilon}}-\frac{C}{C^-}\mathbf{X}^- \ .$$

The first equation is also written:

$$\left(2\mu\boldsymbol{\Delta}\tilde{\boldsymbol{\varepsilon}}+\frac{2\mu}{2\mu^-}\tilde{\boldsymbol{\sigma}}^-\right)=\tilde{\boldsymbol{\sigma}}+\frac{3}{2}2\mu\Delta p\frac{\tilde{\boldsymbol{\sigma}}-\mathbf{X}}{\sigma_y}$$

while cutting off $\mathbf{X}=\frac{C}{C^-}\mathbf{X}^- +C\boldsymbol{\Delta}\boldsymbol{\varepsilon}^p$ has each term, one obtains:

$$2\mu\boldsymbol{\Delta}\tilde{\boldsymbol{\varepsilon}}+\frac{2\mu}{2\mu^-}\tilde{\boldsymbol{\sigma}}^--\frac{C}{C^-}\mathbf{X}^-=\tilde{\boldsymbol{\sigma}}-\mathbf{X}+\frac{3}{2}2\mu\Delta p\frac{\tilde{\boldsymbol{\sigma}}-\mathbf{X}}{\sigma_y}+C\boldsymbol{\Delta}\boldsymbol{\varepsilon}^{\mathbf{P}}$$

or, by means of the flow model:

$$\tilde{\boldsymbol{\sigma}}^e=\left(\tilde{\boldsymbol{\sigma}}-\mathbf{X}\right)\left(1+\frac{3}{2}\left(2\mu+C\right)\frac{\Delta p}{\sigma_y}\right)$$

One still obtains a scalar equation by Δp taking some the equivalent values of Von Mises:

$$\sigma_{eq}^e=\sigma_y+\frac{3}{2}\left(2\mu+C\right)\Delta p$$

what gives directly:

$$\Delta p=\frac{\sigma_{eq}^e-\sigma_y}{\frac{3}{2}\left(2\mu+C\right)}$$

And σ is obtained by: $\tilde{\boldsymbol{\sigma}}=\frac{2\mu}{2\mu^-}\tilde{\boldsymbol{\sigma}}^- +2\mu\boldsymbol{\Delta}\tilde{\boldsymbol{\varepsilon}}+2\mu\boldsymbol{\Delta}\boldsymbol{\epsilon}^p$

By noticing that: $\boldsymbol{\Delta}\boldsymbol{\varepsilon}^p=\frac{3}{2}\Delta p\frac{\tilde{\boldsymbol{\sigma}}-\mathbf{X}}{\sigma_y}=\frac{3}{2}\Delta p\frac{\tilde{\boldsymbol{\sigma}}^e}{\sigma_{eq}^e}$ because: $\frac{\tilde{\boldsymbol{\sigma}}-\mathbf{X}}{\sigma_y}=\frac{\tilde{\boldsymbol{\sigma}}^e}{\sigma_{eq}^e}$

one thus has:

$$\tilde{\boldsymbol{\sigma}}=\frac{2\mu}{2\mu^-}\tilde{\boldsymbol{\sigma}}^-+2\mu\boldsymbol{\Delta}\tilde{\boldsymbol{\varepsilon}}-\frac{2\mu}{2\mu+C}\frac{\left(s_{eq}^e-\sigma_y\right)_+}{\sigma_{eq}^e}.\tilde{\boldsymbol{\sigma}}^e$$

The local variables $\mathbf{X}$ are calculated by:

$$\mathbf{X}=\frac{C}{C^-}\mathbf{X}^-+C\boldsymbol{\Delta}\boldsymbol{\varepsilon}^{\mathbf{p}}=\frac{C}{C^-}\mathbf{X}^-+\frac{3}{2}C\Delta p\frac{\tilde{\boldsymbol{\sigma}}^e}{\sigma_{eq}^e}$$

Note: *Typical case of the plane stresses.*

The direct taking into account of the assumption of the plane stresses in the integration of the model of Von Mises with linear kinematic hardening was not made in *Code_Aster*. To take into account this assumption, i.e. to use an elastoplastic behavior of Von Mises with a linear kinematic hardening (model of Prager) with modelizations `C_PLAN`, `DKT`, `COQUE_3D`, `COQUE_AXIS`, `COQUE_D_PLAN`, `COQUE_C_PLAN`, `PIPE`, `TUYAU_6M`, one can:

- that is to say to use the method of condensation static (due to R. of Borst [R5.03.03]) which makes it possible to obtain a plane state of stresses with convergence of the total iterations of the algorithm of Newton;
- that is to say to use behavior `VMIS_ECMI_LINE` (cf [R5.03.16]).

4.4 Tangent operator. Option FULL_MECA

option `FULL_MECA` makes it possible to calculate the tangent matrix $\mathbf{K}_i^n$ with each iteration. The tangent operator who is used for building it is calculated directly on the preceding discretized system (one notes to simplify: $\tilde{\boldsymbol{\sigma}}=\boldsymbol{\sigma}^- +\boldsymbol{\Delta}\,\tilde{\boldsymbol{\sigma}}$, $p=p^- +\Delta\, p$) and one writes the statements only in the isothermal case.

One poses $\boldsymbol{\sigma}^{dev}=\tilde{\boldsymbol{\sigma}}-\mathbf{X}$ and $\gamma=\begin{cases}1 & \text{si} \quad \Delta\, p>0\,\text{et}\left(\tilde{\boldsymbol{\sigma}}-\mathbf{X}\right).\boldsymbol{\Delta}\tilde{\boldsymbol{\varepsilon}}\geq 0\\ 0 & \text{sinon}\end{cases}$

the tangent operator binds the vector of virtual strains $\boldsymbol{\varepsilon}^*$ to the stress vector virtual $\boldsymbol{\sigma}^*$. Then the tangent stiffness matrix is written:

$$\boldsymbol{\sigma}^*=\left(K\,\vec{\mathbf{1}}\otimes\vec{\mathbf{1}}+2\,\mu\, a_2 P-C_p\mathbf{s}\otimes\mathbf{s}\right)\boldsymbol{\varepsilon}^*$$

with $\mathbf{s}$ the stress vector associated with $\boldsymbol{\sigma}^{dev}$ by:

$$\mathbf{s}^{\mathbf{T}}=\left(\sigma_{11}^{dev},\sigma_{22}^{dev},\sigma_{33}^{dev},\sqrt{2}\,\sigma_{12}^{dev},\sqrt{2}\,\sigma_{23}^{dev},\sqrt{2}\,\sigma_{31}^{dev}\right)$$

and:

$$C_p=\gamma\frac{3}{2}\left(\frac{2\,\mu}{\sigma_y}\right)^2\frac{1}{2\,\mu+C}a_1 \text{ with } a_1=\frac{1}{1}+\frac{3}{2}\frac{(2\,\mu+C)\,\Delta\, p}{\sigma_y} \text{ and } a_2=a_1\left(1+\frac{3}{2}C\frac{\Delta\, p}{\sigma_y}\right)$$

4.5 Local variables of model VMIS_CINE_LINE

the local variables are 7:

- the tensor $\mathbf{X}$ stored on 6 components,
- the scalar variable χ.

5 Bibliography

1) P. MIALON, Elements of analysis and numerical resolution of the relations of elastoplasticity. EDF - Bulletin of the Management of the Studies and Searches - Series C - N° 3 1986, p. 57 - 89.

2) E.LORENTZ, J.M.PROIX, I.VAUTIER, F.VOLDOIRE, F.WAECKEL “Initiation with the thermo - plasticity in the Code_Aster *”, EDF/DER/HI* - 74/96/013 Description of

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VMIS_JOHN_COOK	Relation	VMIS_ISOT_TRAC : `complements`

Annexe 1 on `integration` the implicit discretization of

the behavior model results in solving an equation in [§5]. One solves the equation exactly Δp while

$\sigma_{eq}^{e}-3\mu\Delta p-R\left(p^{-}+\Delta p\right)=0$

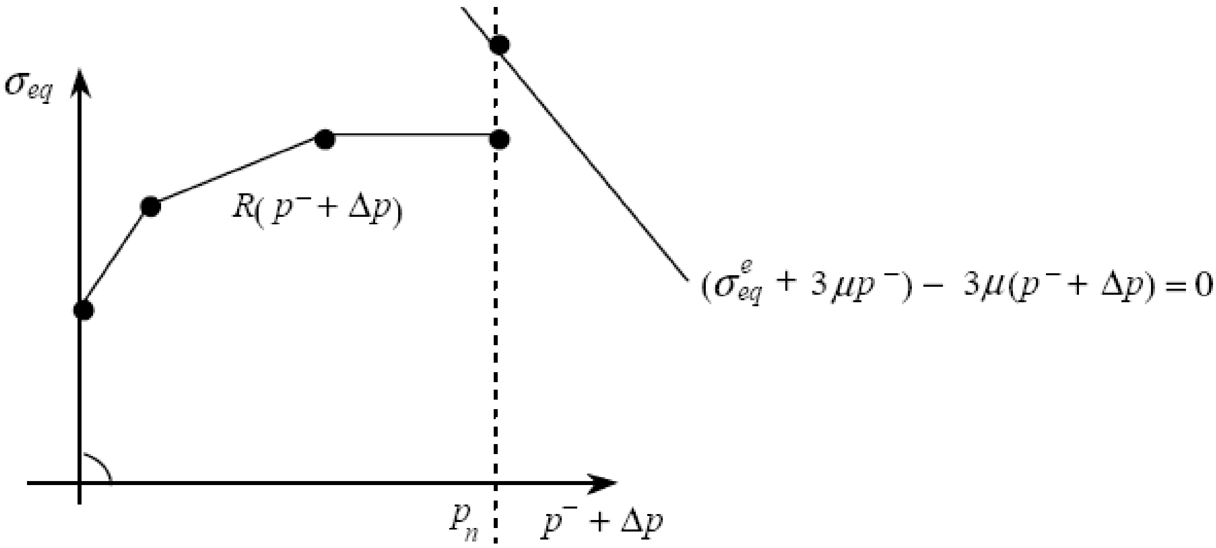
drawing left the linearity per pieces. One examines initially if the solution could

be apart from the limits of the points of discretization of the curve, i.e., if is a possible $R(p)$ solution. For that $p\geq p_n$: so then one is in

the following

- situation $\sigma_{eq}^{e}+3\mu\left(p^{-}-p_n\right)-\sigma_n\geq 0$

  : if the prolongation on the right is linear



- then: that is to say then: if the prolongation is

  constant

$$\alpha_{n-1}=\frac{\sigma_n-\sigma_{n-1}}{p_n-p_{n-1}} \qquad H_{n-1}=\sigma_{n-1}+\alpha_{n-1}\left(p^{-}-p_{n-1}\right)$$

:

$$\Delta p=\frac{\sigma_{eq}^{e}-H_{n-1}}{\alpha_{n-1}+3\mu}$$

- if not an error message is transmitted

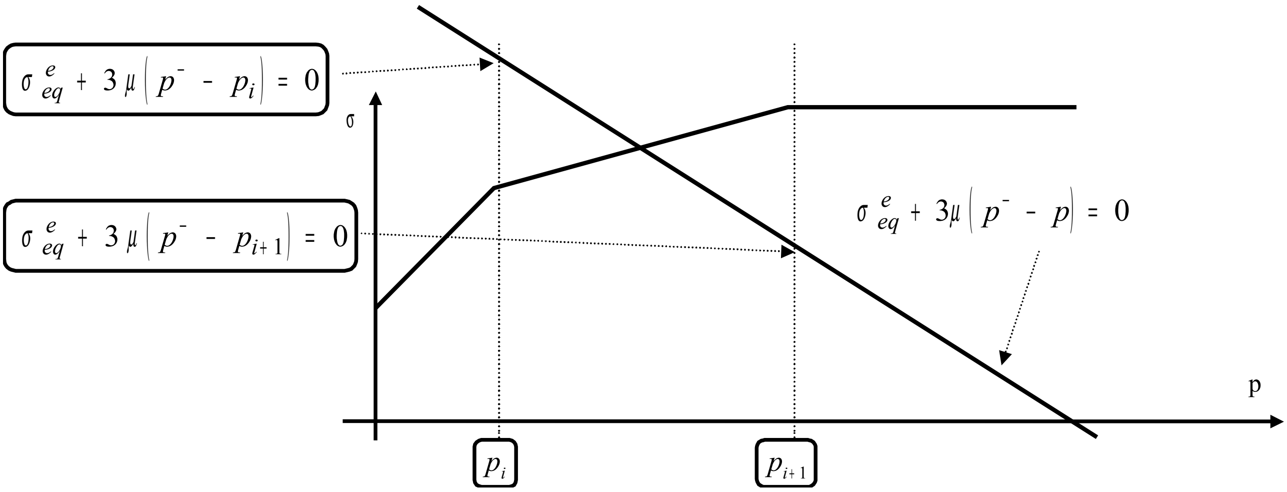
$$\Delta p=\frac{\sigma_{eq}^{e}-\sigma_{n}}{3\mu}$$

- if not, the solution is to be sought

- in L" interval p such as: and then, is such that $\left[p_{i},p_{i+1}\right]$: Isotropic

$$\sigma_{i+1}>\sigma_{eq}^{e}+3\mu\left(p^{-}-p_{i+1}\right)$$

$$\sigma_{i}\le\sigma_{eq}^{e}+3\mu\left(p^{-}-p_{i}\right)$$



$$\alpha_{i}=\frac{\sigma_{i+1}-\sigma_{i}}{p_{i+1}-p_{i}} \qquad H_{i}=\sigma_{i}+\alpha_{i}\left(p^{-}-p_{i}\right)\text{pour}\,i=1\,\grave{a}\,n-1$$

hardening Δp in plane stresses

$$\Delta p=\frac{\sigma_{eq}^{e}-H_{i}}{\alpha_{i}+3\mu}\text{ et }p^{-}+\Delta p\in\left[p_{i},p_{i+1}\right]$$

Annexe 2 In this case, the system to be solved

comprises an equation moreover: . The following system then is obtained $\Delta\sigma_{33}=0$: With this assumption, is not entirely

$$\begin{aligned}
&2\mu\,\boldsymbol{\Delta}\tilde{\boldsymbol{\varepsilon}}-\boldsymbol{\Delta}\tilde{\boldsymbol{\sigma}}=\frac{3}{2}2\mu\,\Delta p\,\frac{\tilde{\boldsymbol{\sigma}}^{-}+\boldsymbol{\Delta}\tilde{\boldsymbol{\sigma}}}{\left(\boldsymbol{\sigma}^{-}+\boldsymbol{\Delta}\boldsymbol{\sigma}\right)_{eq}}\\
&\mathrm{tr}\,\boldsymbol{\Delta}\boldsymbol{\sigma}=3\mathrm{K}\,\mathrm{tr}\,\boldsymbol{\Delta}\boldsymbol{\varepsilon}\\
&\left(\boldsymbol{\sigma}^{-}+\boldsymbol{\Delta}\boldsymbol{\sigma}\right)_{eq}-R\left(p^{-}+\Delta p\right)\leq 0\\
&\Delta p=0\,\mathrm{si}\left(\boldsymbol{\sigma}^{-}+\boldsymbol{\Delta}\boldsymbol{\sigma}\right)_{eq}<R\left(p^{-}+\Delta p\right)\\
&\Delta p\geq 0\,\mathrm{si}\left(\boldsymbol{\sigma}^{-}+\boldsymbol{\Delta}\boldsymbol{\sigma}\right)_{eq}=R\left(p^{-}+\Delta p\right)\\
&\Delta\sigma_{33}=0
\end{aligned}$$

known: $\boldsymbol{\Delta}\boldsymbol{\varepsilon}$ cannot be only calculated from $\Delta\varepsilon_{33}$. Note: In the case as of modelizations $\boldsymbol{\Delta}\,\mathbf{u}_i^n$

other than

C_PLAN, therefore for example for the modelizations of shells (DKT, COQUE_3D), the assumptions on the transverse terms of shears and are defined by these modelizations $\Delta\sigma_{13}$ $\Delta\sigma_{23}$ (in general, the behavior related to the transverse shears linear, elastic and is uncoupled from the equations above). These terms thus do not enter on account here. One poses with entirely known from

$\boldsymbol{\Delta}\boldsymbol{\varepsilon}=\boldsymbol{\Delta}\boldsymbol{\varepsilon}^{q}+\boldsymbol{\Delta}\boldsymbol{\varepsilon}^{y}$ and $\boldsymbol{\Delta}\boldsymbol{\varepsilon}^{q}$ of elasticity, therefore is unknown $\boldsymbol{\Delta}\,\mathbf{u}_i^n$. Compared to the preceding

$\Delta\varepsilon_{33}^{q}=-\dfrac{\nu}{1-\nu}\left(\Delta\varepsilon_{11}^{q}+\Delta\varepsilon_{22}^{q}\right)\,\mathrm{et}\,\boldsymbol{\Delta}\boldsymbol{\varepsilon}^{\mathrm{y}}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&\Delta y\end{pmatrix}$ system

, there is an additional unknown. If thus i.e. If not, Δy

- the technique $\left(\tilde{\boldsymbol{\sigma}}^{-}+\boldsymbol{\Delta}\tilde{\boldsymbol{\sigma}}\right)_{eq}<R\left(p^{-}+\Delta p\right)\mathrm{alors}\,\Delta p=0\quad 2\mu\,\boldsymbol{\Delta}\tilde{\boldsymbol{\varepsilon}}=\boldsymbol{\Delta}\tilde{\boldsymbol{\sigma}}$, of resolution $\Delta y=0$.
- consists in expressing according to. One then obtains Δy a nonlinear scalar Δp equation in. One poses: ". In the same way that Δp

for L" integration $\tilde{\boldsymbol{\sigma}}^{e}=\dfrac{2\mu}{2\mu^{-}}\tilde{\boldsymbol{\sigma}}^{-}+2\mu\,\boldsymbol{\Delta}\tilde{\boldsymbol{\varepsilon}}^{q}$ except plane stresses, one obtains: . But this statement utilizes

$$\tilde{\boldsymbol{\sigma}}_{e}+2\mu\,\boldsymbol{\Delta}\tilde{\boldsymbol{\varepsilon}}^{y}=\left(\tilde{\boldsymbol{\sigma}}^{-}+\boldsymbol{\Delta}\tilde{\boldsymbol{\sigma}}\right)\left(1+\frac{3\mu\,\Delta p}{R\left(p+\Delta p\right)}\right)$$

an additional unknown: In particular: however and Like: Δy One obtains an equation

$$\tilde{\sigma}_{33}+2\mu\Delta\tilde{\varepsilon}^{y}_{33}=\left(\tilde{\sigma}^{-}_{33}+\Delta\tilde{\sigma}_{33}\right)\left(1+\frac{3\mu\Delta p}{R(p+\Delta p)}\right)$$

$$\Delta\tilde{\varepsilon}^{y}_{33}=\frac{2}{3}\Delta y$$

flexible $tr(\boldsymbol{\sigma}^{-}+\boldsymbol{\Delta\sigma})=3K\,tr\,\boldsymbol{\Delta\varepsilon}^{q}+3K^{+}\Delta y+\frac{3K^{+}}{3K^{-}}tr\,\boldsymbol{\sigma}^{-}-3K^{+}\Delta\varepsilon^{th}$

and

$$\tilde{\sigma}_{e33}+\Delta\tilde{\sigma}_{33}=\sigma_{e33}+\Delta\sigma_{33}=\frac{tr(\boldsymbol{\sigma}^{-}+\boldsymbol{\Delta\sigma})}{3}=0-\frac{tr(\boldsymbol{\sigma}^{-}+\boldsymbol{\Delta\sigma})}{3}$$

: with Is: by noticing that Δy : Δp and

$$\tilde{\sigma}_{33}+2\frac{2}{3}\Delta y=\left(1+\frac{3\mu\Delta p}{R(p+\Delta p)}\right)\left(\frac{-tr\,\sigma_{e}-3K\Delta y}{3}\right)$$

while

$$tr\,\boldsymbol{\sigma}_{\mathbf{e}}=\frac{3K}{3K^{-}}tr\,\boldsymbol{\sigma}^{-}+3K\,tr\,\boldsymbol{\Delta\varepsilon}^{q}+-3K\Delta\varepsilon^{th}$$

clarifying

$$\Delta y\left(\frac{4\mu}{3}+K\left(1+\frac{3\mu\Delta p}{R\left(p^{-}+\Delta p\right)}\right)\right)=-\tilde{\sigma}^{e}_{33}-\frac{\mathrm{tr}\,\sigma_{e}}{3}\left(1+\frac{3\mu\Delta p}{R\left(p^{-}+\Delta p\right)}\right)$$

, one obtains:

$$\tilde{\sigma}^{e}_{33}=\sigma^{e}_{33}-\frac{\mathrm{tr}\,\sigma_{e}}{3}=0-\frac{\mathrm{tr}\,\sigma_{e}}{3}$$

to defer in μ, K the equation in (identical

$$\Delta y=\frac{3\left(1-2\nu\right)\Delta p}{E\Delta p+2\left(1-\nu\right)R\left(p+\Delta p\right)}\tilde{\sigma}^{e}_{33}$$

to the preceding cases) where Δp expresses itself according to since

$$(\tilde{\boldsymbol{\sigma}}^{\mathbf{e}}+\mathbf{2}\boldsymbol{\mu}\,\boldsymbol{\Delta}\,\tilde{\boldsymbol{\varepsilon}}^{\mathbf{y}})_{\mathbf{eq}}-3\mu\Delta p-R(p^{-}+\Delta p)$$

Δy : The scalar equation in Δp thus obtained $\boldsymbol{\Delta}\,\tilde{\boldsymbol{\varepsilon}}^{y}=\frac{\Delta y}{3}\begin{pmatrix}-1 & & \\ & -1 & \\ & & 2\end{pmatrix}$

is always nonlinear Δp . This equation is solved by a research method of zeros of functions, based on an algorithm of secant. Once the known solution one calculates then. Δp Δy σ