Taking into account of the assumption of the plane stresses in the nonlinear behaviors

Abstract:

This document describes a general method of integration of the nonlinear models of behaviors (elastoplastic, viscoplastic, damaging,…) in plane stresses.

This is carried out by a method of static condensation due to R. of Borst.

This method makes it possible to use the modelization C_PLAN, or the modelizations COQUE_3D, DKT and PIPE for all the models of incremental behaviors of (STAT/DYNA) _NON_LINE available into axisymmetric or plane strains.

1 Introduction

One presents here a general method of integration of the nonlinear models of behaviors (plasticity, viscoplasticity, damage) in plane stresses. If the selected behavior is not integrated analytically in plane stresses, the method of R. De Borst is activated automatically for modelizations C_PLAN, DKT, COQUE3D and PIPE.

2 Difficulty of integration of the nonlinear behaviors in plane stresses

The modelization C_PLAN, (as well as modelizations COQUE_3D, DKT, PIPE) supposes that the local stress state is plane, i.e. that $\sigma_{zz}=0$, z representing the direction of the norm on the surface. The stress tensors and of strains thus take the following form (in C_PLAN):

$$\varepsilon=\begin{pmatrix}\varepsilon_{xx} & \varepsilon_{xy} & 0\\ \varepsilon_{xy} & \varepsilon_{yy} & 0\\ 0 & 0 & \varepsilon_{zz}\end{pmatrix}$$

$$\sigma=\begin{pmatrix}\sigma_{xx} & \sigma_{xy} & 0\\ \sigma_{xy} & \sigma_{yy} & 0\\ 0 & 0 & \sigma_{zz}\end{pmatrix} \qquad \text{eq.2-1}$$

Note::

For the shells, it is necessary to add terms due to the transverse shears ($\sigma_{xzn}\sigma_{yz}$), but those are treated elastically and do not intervene in the resolution of the local behavior.

This assumption implies that the corresponding strain is a priori undetermined (contrary to the other two-dimensional modelizations where one makes an assumption directly on ε_{zz}). It can be given only using the behavior model. However the condition $\sigma_{zz}=0$ is not alleviating for the integration of the behavior, where one calculates an increase in stress $\Delta\sigma$ according to the increase in strain $\Delta\varepsilon$ provided by the algorithm in Newton. In the case of linear elasticity, the taking into account of this condition is simple and makes it possible to find:

$$\varepsilon_{zz}=-\frac{\nu}{1-\nu}(\varepsilon_{xx}+\varepsilon_{yy})$$

But if the behavior is nonlinear, $\Delta\varepsilon_{zz}$ cannot be only calculated from Δu and does not result simply from the other components of the tensor of the strains. The taking into account of this assumption must then be made (when it is realizable) in a way specific to each behavior, and very often brings to additional difficulties of resolution: it is the case in particular for the behavior of Von Mises to isotropic hardening [R5.03.02]. So much of models of behavior are not available in plane stresses.

The method presented here has the large advantage of not requiring any particular development in the integration of the behavior to satisfy the assumption with the plane stresses. It is usable as soon as the model of behavior is available into axisymmetric or plane strains.

3 Principle of the processing of the plane stresses by the method De Borst

3.1 Recall of the method of Newton

In nonlinear static one is brought to solve the following equation ([cf R5.03.01]):

$$R(u)=f^{\text{int}}(u)-f^{ext}(t)\rightarrow 0 \ , \qquad \text{eq. 3.1-1}$$

where f^{ext} is the external force and f^{int} the internal force, definite like

$$f^{\int(u)}=\int_V B^T \sigma(\varepsilon(u))\,dV. \qquad \text{eq. 3.1-2}$$

By means of finite element method (MEF), the internal forces f^{int}, are obtained starting from the stress field, which is determined to him, by a constitutive law, starting from the strain field, the strain tensor discretizes ε, being defined like:

$$\varepsilon=Bu\ .$$

According to the method Newton one solves the system in [eq. 3.1-1] with the following iterative process:

1) $R^{(n)}=R(u^{(n)})$

2) $\Delta u^{(n+1)}=-\underbrace{\left(\dfrac{\partial R^{(n)}}{\partial u^{(n)}}\right)}_{K^{(n)^{-1}}}R^{(n)}=-K^{(n)^{-1}}R^{(n)}$

3) $u^{(n+1)}=u^{(n)}+\Delta u^{(n)}$, to repeat 1) until $R^{(n)}<tolérance$.

The index (n) means that the variable concerned corresponds to n - ième iteration, known as *total,* since it relates to the computation of the field of displacement, contrary to the process says *local*, which makes it possible to calculate the component ε_{zz} so that the stresses is plane and which will be detailed in the continuation.

The stiffness matrix $K^{(n)}$, is calculated like:

$$K^{(n)}(u)=\int_V B^T \underbrace{\frac{\partial\sigma^{(n)}}{\partial u^{(n)}}}_{D^{(n)}} B\,dV=\int_V B^T D^{(n)} B\,dV \ ,$$

where $D^{(n)}$ is the tangent matrix corresponding to the constitutive law used. To extend the method of Newton to the use of unspecified constitutive laws, in [bib1] one proposes to condense the value of ε_{zz} so that $\sigma_{zz}<tolérance$ in a converged state. Here two alternatives of this approach are presented: in the original approach, the variables $u^{(k+1)}$ and $\varepsilon_{zz}^{(k+1)}$ are corrected simultaneously for each total iteration, while in the modified approach a local iterative process is added, so that the condition $\sigma_{zz}<tolérance$ is satisfied for each value with $u^{(k+1)}$ and NON-PAS only for the converged state. It is interesting to use the approach modified in particular when convergence on the total equilibrium is faster than convergence for the satisfaction of the plane stresses. In certain cases, but not always, the modified approach can reduce the nombre of iterations total and thus accelerate computations. In term of use, the original approach corresponds to the value of the parameter `ITER_CPLAN_MAXI` = 1 and modified with ITER_CPLAN_MAXI approaches `it` > 1.

Besides the computation of ε_{zz} at the local level which results in modifying the local stresses, the approach known as of DEBORST especially consists in intervening on the level of the stiffness matrix

$$\hat{K}^{(n)}(u)=\int_V B^T \hat{D}^{(n)} B\, dV ,$$

where D par. The computation $\hat{D}$ of as well $\hat{D}$ was replaced as of ε_{zz} are detailed in the continuation.

3.2 Approaches origin

the idea of the method due to R. of Borst [bib1] consists in not treating the constraint plane with the level of the constitutive law but with the level of the equilibrium. One obtains thus during iterations of the algorithm of total resolution of STAT_NON_LINE of the stress fields which tend towards a stress field plane as iterations:

$$\sigma_{zz}^{(n)} \rightarrow 0$$

where n the number of iteration of Newton indicates.
One thus obtains the constraint planes not exactly, but in an approached way, with convergence of the iterations of Newton, for each calculated increment. One checks, as specified thereafter, that the component above is lower than a given tolerance.

The method consists in breaking up the fields (strains or stresses) into a purely plane part (specified by a "hat") and a component according to z . One then reveals explicitly the components " zz " in the form of the strain tensors and stresses:

$$\varepsilon=\begin{pmatrix}\hat{\varepsilon}\\ \varepsilon_{zz}\end{pmatrix} \quad \text{and} \quad \sigma=\begin{pmatrix}\hat{\sigma}\\ \sigma_{zz}\end{pmatrix}.$$

like in the statement of the tangent operator:

$$D^{(n)}=\frac{\partial \sigma^{(n)}}{\partial \varepsilon^{(n)}}=\begin{pmatrix} D_{11}^{(n)} & D_{12}^{(n)}\\ D_{21}^{(n)} & D_{22}^{(n)}\end{pmatrix} \quad \text{.eq} \quad 3.2\text{-}1$$

Lastly, with each iteration Newton one corrects the values of strain,

$$\hat{\varepsilon}^{(n+1)}=\hat{\varepsilon}^{(n)}+\Delta\hat{\varepsilon}^{(n)} \quad \text{and} \quad \varepsilon_{zz}^{(n+1)}=\varepsilon_{zz}^{(n)}+\Delta\varepsilon_{zz}^{(n)} .$$

By means of [eq. 3.2-1] one can write

$$\begin{pmatrix}\Delta\hat{\sigma}\\ \Delta\sigma_{zz}\end{pmatrix}=D^{(n)}\begin{pmatrix}\Delta\hat{\varepsilon}\\ \Delta\varepsilon_{zz}\end{pmatrix} \text{, with } \Delta\sigma_{zz}=\sigma_{zz}^{(n+1)}-\sigma_{zz}^{(n)} ,$$

and one obtains:

$$\sigma_{zz}^{(n+1)}\approx\sigma_{zz}^{(n)}+D_{21}^{(n)}\Delta\hat{\varepsilon}^{(n)}+D_{22}^{(n)}\Delta\varepsilon_{zz}^{(n)}\rightarrow 0 , \quad \text{eq } 3.2\text{-}2$$

From [eq. 3.2-2] one can condense $\Delta\varepsilon_{zz}^{(n)}$ like,

$$\Delta\varepsilon_{zz}^{(n)}=-\frac{\sigma_{zz}^{(n)}+D_{21}^{(n)}\Delta\hat{\varepsilon}^{(n)}}{D_{22}^{(n)}} \quad \text{.eq} \quad 3.2\text{-}3$$

It is exactly the condensation in [eq 3.2-3] which enables us 2D to use the frame for the resolution with the MEF. With final, one seeks to correct the components 2D (noted by index [1]), as well for the stresses as for the tangent operator, so that the constraint plane is satisfied.

The algorithm of plane stresses entirely local, is applied in the frame of a total architecture finite elements corresponding to a modelization 2D equivalent to that of the plane strain. One positions in a Gauss point, where one knows the value of the strain to the iteration (n+1)

$\hat{\varepsilon}^{(n+1)}=\hat{\varepsilon}^{(n)}+\Delta\hat{\varepsilon}^{(n)}$, and one must calculate the values of stress $\hat{\sigma}^{(n+1)}$ and the tangent operator $\hat{D}^{(n+1)}$.

Algorithm 1:

1) To bring up to date $\varepsilon_{zz}^{(n)}$,

$$\varepsilon_{zz}^{(n+1)}=\varepsilon_{zz}^{(n)}-\frac{\sigma_{zz}^{(n)}+D_{\mathbf{21}}^{(n)}\Delta\hat{\varepsilon}^{(n)}}{D_{22}^{(n)}}$$

2) Calculate the stresses and the tangent intermediaries, $\sigma^{(n+1)}$ by $D^{(n+1)}$ the constitutive law 3D, imposing operator $\varepsilon_{xz}=\varepsilon_{yz}=0$,

$$\sigma^{(n+1)}=\sigma(\hat{\varepsilon}^{(n+1)},\varepsilon_{zz}^{(n+1)}) \quad \text{and} \quad D^{(n+1)}=D(\hat{\varepsilon}^{(n+1)},\varepsilon_{zz}^{(n+1)})$$

3) To calculate the final stress by means of intermediate correction of $\tilde{\varepsilon}_{zz}^{(n+1)}$,

$$\Delta\tilde{\varepsilon}_{zz}^{(n+1)}=-\frac{\sigma_{zz}^{(n+1)}}{D_{22}^{(n+1)}} \qquad \text{eq 3.2-4}$$

which enables us to write the final stress $\hat{\sigma}^{(n+1)}$ like:

$$\hat{\sigma}^{(n+1)}=\sigma^{(n+1)}+D_{12}^{(n+1)}\Delta\tilde{\varepsilon}_{zz}^{(n+1)}=\sigma^{(n+1)}-\frac{D_{12}^{(n+1)}\sigma_{zz}^{(n+1)}}{D_{22}^{(n+1)}}$$

4) To calculate the final tangent operator:

$$\hat{D}^{(n+1)}=D_{11}^{(n+1)}-\frac{D_{12}^{(n+1)}D_{\mathbf{21}}^{(n+1)}}{D_{22}^{(n+1)}} .$$

Notice 1 *: In 2) the stress σ is calculated as a tensor in 3D*

$\sigma=(\sigma_{xx} \quad \sigma_{yy} \quad \sigma_{zz} \quad \sigma_{xy} \quad \sigma_{xz} \quad \sigma_{yz})^T$.

On the other hand, one only uses $\sigma=(\sigma_{xx} \quad \sigma_{yy} \quad \sigma_{zz} \quad \sigma_{xy})^T$, the components σ_{xz} and σ_{yz} not needing to be extracted. Thus, D is of size 4×4 .

Notice 2 *: In 3) the statement of $\Delta\tilde{\varepsilon}_{zz}^{(n+1)}$ is obtained by supposing that the value extrapolated of $\sigma_{zz}^{n+2}\approx0$*

$$\sigma_{zz}^{(n+2)}\approx\sigma_{zz}^{(n+1)}+D_{22}^{(n+1)}\Delta\tilde{\varepsilon}_{zz}^{(n+1)}=0 .$$

Notice 3 *: In 4) one calculates the tangent operator like:*

$$\hat{D}^{(n+1)}=\frac{\delta\sigma^{(n+1)}}{\delta\hat{\varepsilon}^{(n+1)}}=\frac{\partial\sigma^{(n+1)}}{\partial\hat{\varepsilon}^{(n+1)}}+\frac{\partial\sigma^{(n+1)}}{\partial\varepsilon_{zz}^{(n+1)}}\frac{\partial\varepsilon_{zz}^{(n+1)}}{\partial\hat{\varepsilon}^{(n+1)}}=D_{11}^{(n+1)}-\frac{D_{12}^{(n+1)}D_{\mathbf{21}}^{(n+1)}}{D_{22}^{(n+1)}} ,$$

where δ means the total derivative contrary to ∂ which means partial derivative. In general, this tangent operator is not coherent compared to $\hat{\sigma}^{(n+1)}$

$$\hat{D}^{(n+1)} \neq \frac{\delta \hat{\sigma}^{(n+1)}}{\delta \hat{\varepsilon}^{(n+1)}} \quad .$$

The coherent tangent operator can be obtained in the following way:

$$\frac{\delta \hat{\sigma}^{(n+1)}}{\delta \hat{\varepsilon}^{(n+1)}} = \frac{\delta \sigma^{(n+1)}}{\delta \hat{\varepsilon}^{(n+1)}} - \frac{D_{12}^{(n+1)}}{D_{22}^{(n+1)}} \frac{\delta \sigma_{zz}^{(n+1)}}{\delta \hat{\varepsilon}^{(n+1)}} \quad ,$$

where

$$\frac{\delta \sigma^{(n+1)}}{\delta \hat{\varepsilon}^{(n+1)}} = \frac{\partial \sigma^{(n+1)}}{\partial \hat{\varepsilon}^{(n+1)}} + \frac{\partial \delta^{(n+1)}}{\partial \varepsilon_{zz}^{(n+1)}} \frac{\partial \varepsilon_{zz}^{(n+1)}}{\partial \hat{\varepsilon}^{(n+1)}} = D_{11}^{(n+1)} - \frac{D_{12}^{(n+1)} D_{\mathbf{21}}^{(n)}}{D_{22}^{(n)}}$$

$$\frac{\delta \sigma_{zz}^{(n+1)}}{\delta \hat{\varepsilon}^{(n+1)}} = \frac{\partial \sigma_{zz}^{(n+1)}}{\partial \hat{\varepsilon}^{(n+1)}} + \frac{\partial \sigma_{zz}^{(n+1)}}{\partial \varepsilon_{zz}^{(n+1)}} \frac{\partial \varepsilon_{zz}^{(n+1)}}{\partial \hat{\varepsilon}^{(n+1)}} = D_{\mathbf{21}}^{(n+1)} - \frac{D_{22}^{(n+1)} D_{\mathbf{21}}^{(n)}}{D_{22}^{(n)}} \quad .$$

With final, one obtains:

$$\frac{\delta \hat{\sigma}^{(n+1)}}{\delta \hat{\varepsilon}^{(n+1)}} = D_{11}^{(n+1)} - \frac{D_{12}^{(n+1)} D_{\mathbf{21}}^{(n)}}{D_{22}^{(n)}} - \frac{D_{12}^{(n+1)}}{D_{22}^{(n+1)}} \left(D_{\mathbf{21}}^{(n+1)} - \frac{D_{22}^{(n+1)} D_{\mathbf{21}}^{(n)}}{D_{22}^{(n)}} \right) \quad .$$

The operator used $\hat{D}^{(n+1)}$, tends towards the coherent operator tangent at the time of the convergence of the criterion of the plane stresses, since $D^{(n+1)} \rightarrow D^{(n)}$, when $\sigma_{zz}^{(n)} \rightarrow 0$, which thus carries out to:

$$\hat{D}^{(n+1)} \overset{\sigma_{zz}^{(n+1)} \rightarrow 0}{\longrightarrow} \frac{\delta \hat{\sigma}^{(n+1)}}{\delta \hat{\varepsilon}^{(n+1)}} \quad .$$

3.3 Modified In

the modified algorithm one approaches proposes to introduce an additional loop compared to the process describes above for better satisfying the plane stresses for each total iteration with Newton, (n). This new loop includes the points 2) and 3) algorithm presented in 3.2. Thus the new algorithm is written like:

Algorithm 2:

1) To bring up to date $\varepsilon_{zz}^{(n)}$,

$$\varepsilon_{zz}^{(n+1)} = \varepsilon_{zz}^{(n)} - \frac{\sigma_{zz}^{(n)} + D_{\mathbf{21}}^{(n)} \Delta \hat{\varepsilon}^{(n)}}{D_{22}^{(n)}}$$

2) To initialize the loop

$$\tilde{\varepsilon}_{zz}^{(k=0,n+1)} = \varepsilon_{zz}^{(n+1)}$$

<u>Beginning buckles</u> $k=0, K_{\max}$

3) Compute the stresses and the tangent operator intermediaries $\sigma^{(k,n+1)}$, $D^{(k,n+1)}$ by the constitutive law 3D,

$$\sigma^{k,(n+1)} = \sigma(\hat{\varepsilon}^{(n+1)}, \tilde{\varepsilon}_{zz}^{(k,n+1)}) \quad \text{and} \quad D^{(k,n+1)} = D(\hat{\varepsilon}^{(n+1)}, \tilde{\varepsilon}_{zz}^{(k,n)})$$

4) Compute the intermediate correction of $\tilde{\varepsilon}_{zz}^{(k,n+1)}$,

$$\Delta \tilde{\varepsilon}_{zz}^{(k,n+1)} = -\frac{\sigma_{zz}^{(k,n+1)}}{D_{22}^{(k,n+1)}} \quad \text{and} \quad \tilde{\varepsilon}_{zz}^{(k+1,n+1)} = \tilde{\varepsilon}_{zz}^{(k,n+1)} + \Delta \tilde{\varepsilon}_{zz}^{(k,n+1)} \quad .$$

To finish the loop if $|\sigma_{zz}^{(k,n+1)}|<\sigma_{tol}$ or if $k=K_{\max}$.
(With the parameter σ_{tol} one defines the tolerance on the value of σ_{zz} .)

5) To allot to the tensor stresses 2D converged values:

$$\hat{\sigma}^{(n+1)}=\sigma^{(k,n+1)}+D_{12}^{(k,n+1)}\Delta\tilde{\varepsilon}_{zz}^{(k,n+1)}=\sigma^{(k,n+1)}-\frac{D_{12}^{(k,n+1)}\sigma_{zz}^{(k,n+1)}}{D_{22}^{(k,n+1)}}$$

In theory, the second term of the equation above can be omitted, if the parameter of the convergence criterion σ_{tol} , is selected sufficiently small.

6) To calculate the final tangent operator:

$$\hat{D}^{(n+1)}=D_{11}^{(n+1)}-\frac{D_{12}^{(n+1)}D_{21}^{(n+1)}}{D_{22}^{(n+1)}}\ .$$

***Notice 4** : In the modified version, the tangent operator $\hat{D}^{(n+1)}$, is coherent compared to $\hat{\sigma}^{(n+1)}$, contrary with the operator of the version of origin (see remark 3), since $|\sigma_{zz}^{(n+1)}|<\sigma_{tol}$.*

4 Practical aspects of use

This method is used automatically as soon as the selected behavior is not available in plane stresses, for modelizations `C_PLAN` or of standard shell: `COQUE_3D, DKT, PIPE`. In practice, that increases (automatically) by 4 the number of local variables of the behavior.

For converging well, it is advised to reactualize the tangent matrix if possible (, with all the iterations: `REAC_ITER = 1,` or all n iterations, with n small).

This method thus allows a great flexibility in use compared to the behaviors: it is enough that a behavior is available in axisymetry or plane strain so that it is also usable in plane stresses.

As for all integrations of models of behaviors nonlinear, it is highly advised to give a small convergence criterion (to leave the value by default with $10^{-}6$.).

The advantage of the modified approach is a better satisfaction of the constraint plane in each Gauss point ($|\sigma_{zz}^{\mathrm{mod}if,(n)}|<<|\sigma_{zz}^{orig,(n)}|$ for each n). In certain cases, it is essential to make converge a computation, in particular for the lenitive constitutive laws.

On the other hand, because of an additional loop the modified procedure is more expensive, especially because the loop 3D includes the call to the modulus "constitutive law". Nevertheless, the overcost because of heavier local computations can be compensated by a gain on the level of the nombre of iterations total of Newton, which is generally less low for the modified algorithm. This gain of nombre of iterations total is not guaranteed, which has as a consequence that the additional iterative loop is not activated by default (ITER_CPLAN_MAXI=1). It was also observed that moment when one chooses `ITER_CPLAN_MAXI` > 1, it is preferable to use `ITER_CPLAN_MAXI >` 5.

5 Bibliography

1 R of Borst "The zero normal stress condition in plane stress and Shell elastoplasticity" Communications in applied numerical methods, Flight 7,29-33 (1991)

6 Historical of the versions of the document

Version Aster	Author (S) or contributor (S), organization	Description of the modifications
5.4	J.M. PROIX, E. LORENTZ	initial Version.
9.2	D. MARKOVIC	Addition of the iterative loop interns to improve convergence.
11.2	J.M.PROIX card-indexes 18398	