Model of Rousselier in large deformations

Summarized

One presents here an alternative of the model of Rousselier which makes it possible to describe the plastic growth of cavities in a steel. The behavior model is elastoplastic with isotropic hardening, allows the changes of plastic volume and is written in large deformations. These last lean on the theory suggested by Simo and Miehe, modified to facilitate the numerical integration of the constitutive law and to replace the model in the frame of the generalized standard materials.
This model is available in command STAT_NON_LINE via the key word RELATION: "ROUSSELIER" under factor key word the COMP_INCR and with the key word DEFORMATION: "SIMO_MIEHE".
This model is established for the three-dimensional modelizations (3D), axisymmetric (AXIS) and in plane strains (D_PLAN).

One presents the writing and the digital processing of this model.

Contents

1 Introduction

the mechanisms at the origin of the ductility fracture of steels are associated with the development of cavities within the material. Three phases are generally distinguished:

- germination: it is the initiation of the cavities, into cubes sites which correspond preferentially to the defaults of the material,
- growth: it is the phase which corresponds to the development itself of the cavities, controlled primarily by the yielding of the metal matrix which surrounds these cavities,
- coalescence: it is the phase which corresponds to the interaction of the cavities between them to create macroscopic cracks.

In what follows, we treat only the phases of growth and coalescence. Rousselier [bib1] proposed a capable constitutive law to give an account of these two phases. Compared to this formulation of origin, Lorentz and al. [bib2] introduced several modifications relating primarily to the processing of the large deformations (multiplicative decomposition), the evolution of porosity (function of the total deflection) and the form of the flow model at the singular point of surface threshold.

More precisely, the model is based on assumptions which introduce a microstructure made up of a cavity and a plastic rigid matrix thus isochoric. In this case, porosity f, definite like the relationship between the volume of the cavity V^C and the total volume V of representative ground volume, is directly connected to the macroscopic strain by:

$$J=\det\mathbf{F}=\frac{1-f_0}{1-f} \quad \text{avec} \quad f=\frac{V^c}{V}\Leftrightarrow \dot{f}=(1-f)\operatorname{tr}\mathbf{D} \qquad \text{éq 1-1}$$

where f_0 indicates initial porosity, $\mathbf{F}$ the tensor gradient of the transformation, J the variation of volume and $\mathbf{D}$ strain rate.

To build the model of growth of the cavities, Rousselier took as a starting point a phenomenologic analysis which leads it to the following ingredients:

- large deformations figure,
- irreversible changes of volume,
- isotropic hardening.

These considerations lead it to write the plasticity criterion F in the following form:

$$F(\boldsymbol{\tau},\ R)=\tau_{eq}+\sigma_1 D f \exp\left(\frac{\tau_H}{\sigma_1}\right)-R(p)-\sigma_y \qquad \text{éq 1-2}$$

where $\boldsymbol{\tau}$ is the stress of Kirchhoff, R isotropic hardening function of the cumulated plastic strain p and σ_1, D and σ_y of the parameters of the material. The presence in the plasticity criterion of the hydrostatic stress τ_H authorizes the changes of plastic volume. One also notices that this model does not comprise a specific variable of damage because only microstructural information selected is porosity, directly related to the macroscopic strain by the equation [éq 1-1].

As for the processing of the large deformations, one adopts the theory of Simo and Miehe but in a slightly modified formulation. The approximations brought make it possible to make easier the numerical integration of the constitutive law but also to replace the theory of Simo and Miehe in the frame of the generalized standard materials.

Thereafter, one briefly gives some notions of mechanics in large deformations, then one points out the theory of Simo and Miehe as well as the made modifications. One presents finally the behavior models of the model of Rousselier and his numerical integration.

2 Notations

One will note by:

$\mathbf{Id}$	stamp identity
$\mathrm{tr}\,A$	traces tensor **A**
$\mathbf{A}^{\mathbf{T}}$	transposed of the tensor **A**
$\det \mathbf{A}$	determinant of **A**
$\tilde{\mathbf{A}}$	deviatoric part of the tensor **A** defined by $\tilde{\mathbf{A}}=\mathbf{A}-(\frac{1}{3}\,\mathrm{tr}\,\mathbf{A})\,\mathbf{Id}$
A_H	hydrostatic part of the tensor **A** defined by $A_H=\frac{\mathrm{tr}\,\mathbf{A}}{3}$
$:$	doubly contracted product: $\mathbf{A}:\mathbf{B}=\sum_{i,j} A_{ij}B_{ij}=\mathrm{tr}(\mathbf{A}\mathbf{B}^{T})$
$\otimes$	tensor product: $(\mathbf{A}\otimes\mathbf{B})_{ijkl}=A_{ij}B_{kl}$
A_{eq}	equivalent value of Von Mises defined by $A_{eq}=\sqrt{\frac{3}{2}\tilde{A}:\tilde{A}}$
$\nabla_{\mathbf{X}}\mathbf{A}$	gradient : $\nabla_X \mathbf{A}=\frac{\partial \mathbf{A}}{\partial \mathbf{X}}$
λ,μ,E,ν,K	moduli of the isotropic elasticity
σ_y	elastic limit
α	thermal coefficient of thermal expansion
T	temperature
T_{ref}	reference temperature

In addition, in the frame of a discretization in time, all the quantities evaluated at previous time are subscripted by $^{-}$, the quantities evaluated at time $t+\Delta t$ are not subscripted and the increments are indicated par. Δ One has as follows:

$$\Delta Q=Q-Q^{-}$$

3 Theory of Simo and Miehe

3.1 Introduction

We point out here specificities of the formulation suggested by SIMO J.C and MIEHE C. [bib3] to treat the large deformations. This formulation was already used for models of thermo-élasto behavior (visco) - plastic with isotropic hardening and criterion of Von Mises, [R5.03.21] for a model without effect of the metallurgical transformations and [R4.04.03] for a model with effect of the metallurgical transformations.
The kinematical choices make it possible to treat large displacements and large deformations but also of large rotations in an exact way.
Specificities of these models are the following ones:

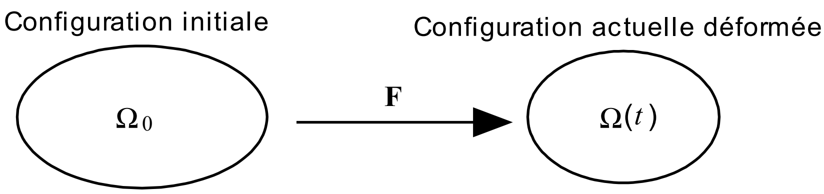
- just like in small strains, one supposes the existence of a slackened configuration, i.e. locally free of stress, which makes it possible to break up the total deflection into a thermo-elastic part and a plastic part,
- the decomposition of this thermo-elastic strain into cubes parts and plastic is not additive any more as in small strains (or for the model large deformations written in strain rate with for example a derivative of Jaumann) but multiplicative,
- the elastic strain are measured in the present configuration (deformed) while plastic strains are measured in the initial configuration,
- as in small strains, the stresses depend only on the thermoelastic strains,
- if the plasticity criterion does not depend that deviatoric stress, then plastic strains are done with constant volume. The variation of volume is then only due to the thermoelastic strains,
- this model during leads its numerical integration to a model incrémentalement objective (cf [§3.2.3]) what makes it possible to obtain the exact solution in the presence of large rotations.

Thereafter, one briefly points out some notions of mechanics in large deformations.

3.2 General information on the large deformations

3.2.1 Kinematics

Let us consider a solid subjected to large deformations. That is to say Ω_0 the field occupied by solid before strain and $\Omega(t)$ the field occupied at the moment T by deformed solid.



Appear 3.2.1-a: Representation of the configuration initial and deformed

In the initial configuration Ω_0 , the position of any particle of solid is indicated by $\mathbf{X}$ (Lagrangian description). After strain, the position at the time t of the particle which occupied the position $\mathbf{X}$ before strain is given by the variable $\mathbf{x}$ (eulerian description).
The total motion of solid is defined, with $\mathbf{u}$ displacement, by:

$$\mathbf{x}=\hat{\mathbf{x}}(\mathbf{X},t)=\mathbf{X}+\mathbf{u} \quad \text{éq 3.2.1-1}$$

to define the change of metric in the vicinity of a point, one introduces the tensor gradient of the transformation $\mathbf{F}$:

$$\mathbf{F}=\frac{\partial\hat{\mathbf{x}}}{\partial\mathbf{X}}=\mathbf{Id}+\nabla_{\mathbf{X}}\mathbf{u}$$ éq the 3.2.1-2

transformations of the volume element and the density are worth:

$$d\,\Omega=Jd\,\Omega_o \text{ with } J=\det F=\frac{\rho_o}{\rho}$$ éq 3.2.1-3

where ρ_o and ρ are respectively the density in the configurations initial and current.

Various strain tensors can be obtained by eliminating rotation in the local transformation. For example, by directly calculating the variations length and angle (variation of the scalar product), one obtains:

$$\mathbf{E}=\frac{1}{2}(\mathbf{C}-\mathbf{Id}) \text{ with } \mathbf{C}=\mathbf{F}^{\mathbf{T}}\mathbf{F}$$ éq 3.2.1-4

$$\mathbf{A}=\frac{1}{2}(\mathbf{Id}-\mathbf{b}^{-1}) \text{ with } \mathbf{b}=\mathbf{F}\mathbf{F}^{T}$$ éq 3.2.1-5

$\mathbf{E}$ and $\mathbf{A}$ are respectively the strain tensors of Green-Lagrange and Eulerian-Almansi and $\mathbf{C}$ $\mathbf{b}$, the tensors of right and left Cauchy-Green respectively.

In Lagrangian description, one will describe the strain by the tensors $\mathbf{C}$ or $\mathbf{E}$ because they are quantities defined on Ω_0 , and of eulerian description by the tensors $\mathbf{b}$ or $\mathbf{A}$ (definite on Ω).

3.2.2 Stresses

the tensor of the stresses used in the theory of Simo and Miehe is the tensor of Kirchhoff $\boldsymbol{\tau}$ defined by:

$$J\boldsymbol{\sigma}=\boldsymbol{\tau}$$ éq 3.2.2-1

where $\boldsymbol{\sigma}$ is the eulerian tensor of Cauchy. The tensor $\boldsymbol{\tau}$ thus results from a “scaling” by the variation of volume of the tensor of Cauchy $\boldsymbol{\sigma}$.

3.2.3 Objectivity

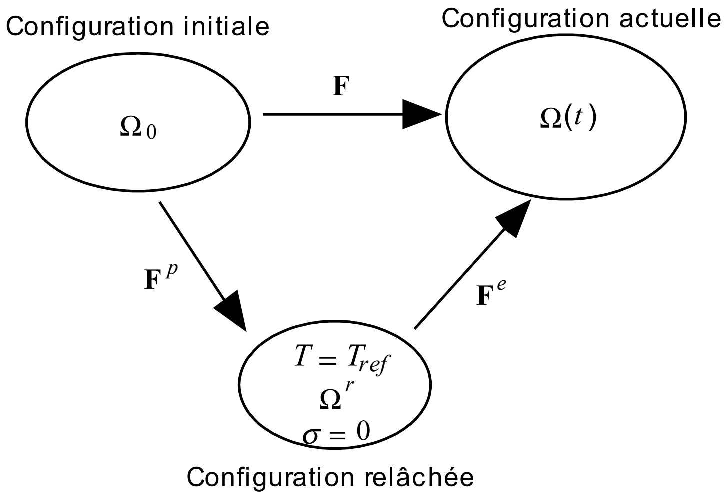
When a constitutive law in large deformations is written, one must check that this model is objective, i.e. invariant by any change of spatial reference frame of the form:

$$\mathbf{x}^{*}=\mathbf{c}(t)+\mathbf{Q}(t)\mathbf{x}$$ éq 3.2.3-1

where $\mathbf{Q}$ is an orthogonal tensor which represent the rotation of the reference frame and $\mathbf{c}$ a vector which represents the translation.
More concretely, if one carries out a traction test in the direction $\mathbf{e}_1$, for example, followed by a rotation of 90° around $\mathbf{e}_3$, which amounts carrying out a traction test according to $\mathbf{e}_2$, then the danger with a nonobjective constitutive law is not to find a uniaxial tensor of the stresses in the direction $\mathbf{e}_2$ (what is in particular the case with the kinematics `PETIT_REAC`).

3.3 Formulation of Simo and Miehe

Thereafter, one will note $\mathbf{F}$ the tensor gradient which makes pass from the initial configuration Ω_0 to the present configuration $\Omega(t)$, $\mathbf{F^p}$ the tensor gradient which makes pass from the configuration Ω_0 to the slackened configuration Ω^r , and $\mathbf{F^e}$ of the configuration Ω^r with $\Omega(t)$. The index p refers to the plastic part, the index e with the elastic part.



Appear 3.3-a: Decomposition of the tensor gradient $\mathbf{F}$ in an elastic and $\mathbf{F^e}$ plastic part $\mathbf{F^p}$

By composition of motions, one obtains following multiplicative decomposition:

$$\mathbf{F}=\mathbf{F^e}\,\mathbf{F^p} \qquad \text{éq the 3.3-1}$$

elastic strain are measured in the present configuration with the eulerian tensor of left Cauchy-Green $\mathbf{b^e}$ and the plastic strains in the initial configuration by the tensor $\mathbf{G^p}$ (Lagrangian description). These two tensors are defined by:

$$\mathbf{b^e}=\mathbf{F^e}\,\mathbf{F}^{eT}\ ,\ \mathbf{G^p}=(\mathbf{F}^{pT}\,\mathbf{F^p})^{-1}\ \text{ from where }\ \mathbf{b^e}=\mathbf{F}\mathbf{G}^p\,\mathbf{F^T} \qquad \text{éq 3.3-2}$$

However, one will employ alternatively another measurement of the elastic strain $\mathbf{e}$ which coincides with the opposite of the linearized strains when the elastic strain are small:

$$\mathbf{e}=\frac{1}{2}\left(\mathbf{Id}-\mathbf{b^e}\right) \qquad \text{éq 3.3-3}$$

In the case of an isotropic material, one can show that the potential free energy depends only on the left tensor of Cauchy-Green $\mathbf{b^e}$ (where in our case of the tensor $\mathbf{e}$) and in plasticity of the variable p related to isotropic hardening. Moreover, one supposes that the voluminal free energy breaks up, just like in small strains, in a hyper elastic part which depends only on the elastic strain and another related to the mechanism on hardening:

$$\Phi\left(\mathbf{e},\ p\right)=\Phi^{el}\left(\mathbf{e}\right)+\Phi^{bl}\left(\mathbf{p}\right) \qquad \text{éq 3.3-4}$$

So instead of using the stress of Cauchy $\boldsymbol{\sigma}$, one uses the stress of Kirchhoff $\boldsymbol{\tau}$, the inequality of Clausius-Duhem is written (one forgets the thermal part):

$$\boldsymbol{\tau}:\mathbf{D}-\dot{\Phi}\geq 0 \quad \text{éq 3.3-5}$$

statement in which $\mathbf{D}$ represents eulerian strain rate.

Under the preceding assumptions, dissipation is still written:

$$\left(\boldsymbol{\tau}+\frac{\partial\Phi}{\partial\mathbf{e}}\mathbf{b}^{\mathbf{e}}\right):\mathbf{D}+\frac{1}{2}\frac{\partial\Phi}{\partial\mathbf{e}}:\left(\mathbf{F}\dot{\mathbf{G}}^{p}\mathbf{F}^{\mathrm{T}}\right)-\frac{\partial\Phi}{\partial p}\dot{p}\geq 0 \quad \text{éq 3.3-6}$$

the second principle of the thermodynamics then requires the following statement for the relation stress-strain:

$$\boldsymbol{\tau}=-\frac{\partial\Phi}{\partial\mathbf{e}}\mathbf{b}^{\mathbf{e}} \quad \text{éq 3.3-7}$$

One defines finally the thermodynamic forces associated with the elastic strain and the plastic strain cumulated in accordance with the frame with the generalized standard materials:

$$\mathbf{s}=-\frac{\partial\Phi}{\partial\mathbf{e}} \quad \text{soit} \quad \boldsymbol{\tau}=\mathbf{s}\ \mathbf{b}^{\mathbf{e}} \quad \text{éq 3.3-8}$$

$$A=-\frac{\partial\Phi}{\partial p} \quad \text{éq 3.3-9}$$

where the thermodynamic force A is the opposite of the isotropic variable of hardening R.

It remains then for dissipation:

$$\boldsymbol{\tau}:(-\frac{1}{2}\mathbf{F}\dot{\mathbf{G}}^{\mathbf{p}}\mathbf{F}^{\mathrm{T}}\mathbf{b}^{\mathbf{e}-\mathbf{1}})+A\dot{p}\ =\ \mathbf{s}:(-\frac{1}{2}\mathbf{F}\dot{\mathbf{G}}^{p}\mathbf{F}^{\mathrm{T}})+A\dot{p}\geq 0 \quad \text{éq 3.3-10}$$

3.3.1 original Formulation

the principle of maximum dissipation applied starting from the threshold of elasticity F, function of the stress of Kirchhoff $\boldsymbol{\tau}$ and the thermodynamic force A makes it possible to deduce the laws of evolution from them from the plastic strain and of the cumulated plastic strain, that is to say:

$$-\frac{1}{2}\mathbf{F}\dot{\mathbf{G}}^{p}\mathbf{F}^{\mathrm{T}}\mathbf{b}^{\mathbf{e}\text{-}\mathbf{1}}=\dot{\lambda}\ \frac{\partial F}{\partial\boldsymbol{\tau}} \quad \text{éq 3.3.1-1}$$

$$\dot{p}=\dot{\lambda}\frac{\partial F}{\partial A} \quad \text{éq 3.3.1-2}$$

$$\dot{\lambda}\geq 0 \quad F\leq 0 \quad F\dot{\lambda}=0 \quad \text{éq 3.3.1-3}$$

3.3.2 modified Formulation

the approximation introduced here on the original formulation of Simo and Miehe relates to the form of the flow model, all the more reduced approximation as the elastic strain are small, since $\boldsymbol{\tau}=\mathbf{s}\ \mathbf{b}^{\mathbf{e}}$. Indeed, one henceforth expresses the threshold of elasticity like a function of the thermodynamic forces and either of the stresses $F(\mathbf{s},\ A)\leq 0$, and it is compared to these variables that one applies the principle of maximum dissipation, which leads to the following flow models:

$$-\frac{1}{2}\mathbf{F}\,\dot{\mathbf{G}}^p\,\mathbf{F}^{\mathbf{T}}=\dot{\lambda}\ \frac{\partial F}{\partial \mathbf{s}} \qquad \text{éq 3.3.2-1}$$

$$\dot{p}=\dot{\lambda}\ \frac{\partial F}{\partial A} \qquad \text{éq 3.3.2-2}$$

$$\dot{\lambda}\geq 0 \quad F\leq 0 \quad F\dot{\lambda}\ =0 \qquad \text{éq 3.3.2-3}$$

3.3.3 Consequences of the approximation

By replacing the stress $\boldsymbol{\tau}$ by the force thermodynamiqueassociée $\mathbf{s}$ with the elastic strain in the statement of the plasticity criterion, one introduces in fact a disturbance of the border of the field of reversibility of about size of $2\|\mathbf{e}\|$. Compared to the initial formulation, it results from it obviously an influence on the elastic limit observed but also on the direction from flow: in particular, the derivative compared to the time of the plastic variation of volume is written then:

$$\dot{J}^p=\dot{\lambda}\ J^p\,\mathbf{b}^{\mathbf{e}\text{-}\mathbf{1}}:\frac{\partial F}{\partial \mathbf{s}} \qquad \text{éq 3.3.3-1}$$

so that if the criterion F depends only on the deviator of the tensor of the stresses $\mathbf{s}$, one does not find $J^p=1$: the isochoric character of the plastic strain is not preserved perfectly any more. We will then be brought to introduce a correction of volume a posteriori.

Insofar as the elastic strain remain small, the results got with this modified model do not deviate significantly from those obtained with the old formulation (cf [bib4]), while numerical integration will be simplified by it. Indeed, one will see thereafter whom this model follows the same diagram of integration as that of the models written in small strains.

Note:

This new formulation of the large deformations makes it possible to replace the theory of Simo and Miehe in the frame of the generalized standard materials. From a numerical point of view, this results in to express the resolution of the constitutive law like a problem of minimization compared to the increments of local variables.

Indeed, it is pointed out that in the frame of the generalized standard materials, the data of the two potentials the free energy $\Phi(\boldsymbol{\varepsilon}, a)$ and the potential of dissipation $D(\dot{a})$, function of the tensor of déformationet $\boldsymbol{\varepsilon}$ of a certain number of local variables a , makes it possible to define the constitutive law completely (one places the materials in the case of independent of time).

$$\boldsymbol{\sigma}=\frac{\partial \Phi}{\partial \boldsymbol{\varepsilon}}\ ,\ A=-\frac{\partial \Phi}{\partial a}\in\partial D(\dot{a}) \qquad \text{éq 3.3.3-2}$$

where $\partial D(\dot{a})$ is under differential of the potential of dissipation D .

The constitutive laws of the standard type generalized which do not depend on time are characterized by a potential of dissipation positively homogeneous of degree 1, which results in the following property:

$$\forall \dot{a} \quad \forall \lambda > 0 \quad D(\lambda \dot{a}) = \lambda D(\dot{a}) \quad \Rightarrow \quad \partial D(\lambda \dot{a}) = \partial D(\dot{a}) \qquad \text{éq 3.3.3-3}$$

Now if one writes the problem [éq 3.3.3-2] in form discretized in time and if one uses the property of under differentials [éq 3.3.3-3], one obtains the following discretized problem:

$$\boldsymbol{\sigma} = \frac{\partial \Phi}{\partial \boldsymbol{\varepsilon}} \ , \ A = -\frac{\partial \Phi}{\partial a} \in \partial D(\Delta a) \qquad \text{éq 3.3.3-4}$$

One can show that the equation [éq 3.3.3-4] is equivalent (cf [bib5]) to solve the problem of minimization compared to the increments of local variables Δa according to:

$$-\frac{\partial \Phi}{\partial a} \in \partial D(\Delta a) \Leftrightarrow \Delta a = \mathrm{Arg}\underset{\Delta a^*}{\mathrm{Min}}\left[\Phi(a^- + \Delta a^*) + D(\Delta a^*)\right] \qquad \text{éq 3.3.3-5}$$

the application of the equation [éq 3.3.3-5] to the model of Rousselier in modified large deformations is written:

$$\underset{\text{énergie continue}}{\Phi(\mathbf{e}, p) \ \text{et} \ D(\mathbf{D}^\mathbf{p}, \dot{p})} \quad \underset{\text{discrétisation}}{=>} \quad \underset{\text{énergie discrétisée}}{\Phi(\mathbf{e}^{Tr} + \Delta \mathbf{e}, p^- + \Delta p) \ \text{et} \ D(\Delta \mathbf{e}, \Delta p)} \qquad \text{éq 3.3.3-6}$$

$$A = -\frac{\partial \Phi}{\partial a} = \begin{cases} \mathbf{s} = -\dfrac{\partial \Phi}{\partial \mathbf{e}} \\ -R = -\dfrac{\partial \Phi}{\partial p} \end{cases} \in \partial D(\Delta \mathbf{e}, \Delta p)$$
$$\Leftrightarrow \underset{\Delta \mathbf{e}, \Delta p}{\mathrm{Min}}\left[\Phi(\mathbf{e}^{Tr} + \Delta \mathbf{e}, p^- + \Delta p) + D(\Delta \mathbf{e}, \Delta p)\right] \qquad \text{éq. 3.3.3-7}$$

One will find in the paragraph [§4], the relation which binds plastic strain rate $\mathbf{D}^\mathbf{p}$ once discretized and the increment of elastic strain $\Delta \mathbf{e}$, as well as the definition of $\mathbf{e}^{Tr}$.

One sees well here whom if one takes the initial formulation of Simo and Miehe, one cannot write any more the problem of minimization [éq 3.3.3-7] with the stress of Kirchhoff $\boldsymbol{\tau}$ because of term in $\mathbf{b}^\mathbf{e}$ the statement:

$$\boldsymbol{\tau} = -\frac{\partial \Phi}{\partial e} \mathbf{b}^\mathbf{e} \qquad \text{éq 3.3.3-8}$$

4 Models of Rousselier

We now describe the application of the large deformations to the model of Rousselier presented in introduction.

4.1 Equations of the model

to describe a thermoelastoplastic model with isotropic hardening (the equivalent in small strains with the model with isotropic hardening and criterion of Von Mises), Simo and Miehe propose an elastic potential polyconvexe. By reason of simplicity, one chooses here the potential of Coming Saint who is written:

$$\Phi(\mathbf{e},\ p)=\frac{1}{2}\left[K(\mathrm{tr}\,\mathbf{e})^2+2\mu\,\tilde{\mathbf{e}}:\tilde{\mathbf{e}}+6\mathrm{K}\,\alpha\,\Delta T\,\mathrm{tr}\,\mathbf{e}\right]+\int_0^p R(u)\,du \qquad \text{éq 4.1-1}$$

In accordance with the equations [éq 3.3-8] and [éq 3.3-9], the state models which derive from the elastic potential above write then:

$$\mathbf{s}=-\left[K\,\mathrm{tr}\,\mathbf{e}\ \ \mathbf{Id}+2\mu\,\tilde{\mathbf{e}}+3\mathrm{K}\,\alpha\,\Delta T\,\mathbf{Id}\right] \qquad \text{éq 4.1-2}$$

$$A=-R(p) \qquad \text{éq 4.1-3}$$

the threshold of elasticity is given by:

$$F(\mathbf{s},R)=s_{eq}+\sigma_1\,Df\,\exp\left(\frac{s_H}{\sigma_1}\right)-R-\sigma_y \qquad \text{éq 4.1-4}$$

According to the equations [éq 3.3.2-1] and [éq 3.3.2-2], the flow models are defined by:

$$-\frac{1}{2}\mathbf{F}\,\dot{\mathbf{G}}^p\,\mathbf{F}^{\mathbf{T}}=\dot{\lambda}\left[\frac{3\,\tilde{\mathbf{s}}}{2\,s_{eq}}+\frac{Df}{3}\exp\left(\frac{s_H}{\sigma_1}\right)\mathbf{Id}\right] \qquad \text{éq 4.1-5}$$

$$\dot{p}=\dot{\lambda} \qquad \text{éq 4.1-6}$$

$$\dot{\lambda}\geq 0 \quad F\leq 0 \quad F\dot{\lambda}=0 \qquad \text{éq 4.1-7}$$

4.2 Processing of the singular points

In fact, the flow equation [éq 4.1-5] translated the belonging direction of flow to the normal cone on the surface of the field of elasticity. It is valid only at the regular points, characterized by:

$$s_{eq}\neq 0 \qquad \text{éq 4.2-1}$$

It thus remains to characterize the normal cone at the singular points, i.e. checking:

$$\tilde{\mathbf{s}}=0 \quad \text{et} \quad \sigma_1 D\,f\,\exp\left(\frac{s_H}{\sigma_1}\right)-R=\sigma_y \qquad \text{éq 4.2-2}$$

the normal cone with convex of elasticity in such a point is all the flow directions which carry out the problem of maximization according to:

$$\Delta^{*}(\mathbf{s},\mathrm{R})=\sup_{\mathbf{D^p},\,\dot p}\left[\mathbf{s}:\mathbf{D^p}-R\,\dot p-\Delta\left(\mathbf{D^p},\dot p\right)\right] \quad \text{éq 4.2-3}$$

where Δ^{*} is the indicating function of convex F and $\Delta(\mathbf{D^p},\dot p)$ potential of dissipation obtained by transform of Legendre-Fenchel of the indicating function of F :

$$\Delta(\mathbf{D^p},\dot p)=\underset{\substack{\mathbf{s},R\\ F(\mathbf{s},R)\le 0}}{\mathrm{Sup}}\left[\mathbf{s}:\mathbf{D^p}-R\,\dot p\right] \quad \text{éq 4.2-4}$$

After some computations, one obtains:

$$\Delta(\mathbf{D^p},\dot p)=\sigma_y\,\dot p+\sigma_1\,\mathrm{tr}\,\mathbf{D^p}\left(\ln\frac{\mathrm{tr}\,\mathbf{D^p}}{D\,f\,\dot p}-1\right)+I_{\mathbb{R}^+}(\mathrm{tr}\,\mathbf{D^p})+I_{\mathbb{R}^+}(\dot p-\frac{2}{3}D^p_{eq}) \quad \text{éq 4.2-5}$$

with

$$I_{\mathbb{R}^+}(x)=\begin{cases}0 & \text{si} \quad x\ge 0\\ +\infty & \text{sinon}\end{cases} \quad \text{éq 4.2-6}$$

For $\tilde{\mathbf{s}}=0$, Δ^{*} is worth:

$$\Delta^{*}(\mathbf{s},R)=\underset{\substack{\mathbf{D^p},\,\dot p\\ \mathrm{tr}\,\mathbf{D^p}\ge 0\\ \dot p-\frac{2}{3}D^p_{eq}\ge 0}}{\mathrm{Sup}}\left[\underbrace{s_H\,\mathrm{tr}\,\mathbf{D^p}-\sigma_1\,\mathrm{tr}\,\mathbf{D^p}\left(\ln\frac{\mathrm{tr}\,\mathbf{D^p}}{D\,f\,\dot p}-1\right)}_{G(\mathrm{tr}\,\mathbf{D^p})}-R\,\dot p-\sigma_y\,\dot p\right] \quad \text{éq 4.2-7}$$

By noticing that for $\mathrm{tr}\,\mathbf{D^p}\ge 0$, the function $G(\mathrm{tr}\,\mathbf{D^p})$ is concave, the suprémum compared to the trace of plastic strain rate $\mathbf{D^p}$ is obtained for:

$$G'(\mathrm{tr}\,\mathbf{D^p})=0 \qquad \text{d'où} \qquad \mathrm{tr}\,\mathbf{D^p}=Df\,\dot p\,\exp\left(\frac{s_H}{\sigma_1}\right) \quad \text{éq 4.2-8}$$

Note::

One finds well then for the indicating function of the threshold of elasticity F *.*

$$D^{*}(\mathbf{s},\mathrm{R})=\underset{\substack{\dot p\\ \dot p\ge\frac{2}{3}D^p_{eq}\ge 0}}{\mathrm{Sup}}\left[F\,\dot p\right]=\begin{cases}0 & \text{si}\,F\le 0\\ +\infty & \text{sinon}\end{cases} \quad \text{éq 4.2-9}$$

In a singular point, the normal cone, together of the acceptable flow directions, is thus characterized by:

$$\mathrm{tr}\,\mathbf{D^p}=D\;f\;\dot p\exp\left(\frac{s_H}{\sigma_1}\right) \quad \text{éq 4.2-10}$$

$$\dot p\ge\frac{2}{3}D^p_{\mathrm{eq}}\ge 0 \quad \text{éq 4.2-11}$$

4.3 Statement of porosity

One saw in introduction that the microscopic inspiration of the model of Rousselier is based on a microstructure made up of a cavity and a plastic rigid matrix, therefore isochoric. In this case, porosity is directly connected to the macroscopic strain by the relation eq. 1-1.

In this statement, the change of elastic volume of origin is neglected. Without special precaution, this approximation can prove penalizing in the presence of elastic compression, even reasonable, because it leads to a possibly negative porosity.

One thus prefers the following equivalent statement to him, together with an explicit decrease by initial porosity:

$$f=\max\left(f_{0,}1-\frac{1-f_0}{J}\right) \qquad \text{éq 4.3-1}$$

Rousselier proposes as for him to express porosity while being based on plastic strain rate $\mathbf{D^p}$. The relation is written in incremental form:

$$\dot{f}=(1-f)\,\mathrm{tr}\,\mathbf{D^p} \qquad \text{éq 4.3-2}$$

That means that variable porosity employed to parameterize the plasticity criterion F depends only on the plastic strain. In fact, plastic strain rate is a quantity evaluated in the slackened configuration. Its transport in the present configuration (as $\mathbf{D}$) is still expressed:

$$\mathbf{F^e}\,\mathbf{D^p}\,\mathbf{F^{e^T}}=-\frac{1}{2}\mathbf{F}\,\dot{\mathbf{G}}^p\,\mathbf{F^T} \qquad \text{éq 4.3-3}$$

In this case, the law of evolution of porosity is expressed:

$$\dot{f}=(1-f)\,\mathrm{tr}\left(-\frac{1}{2}\mathbf{F}\,\dot{\mathbf{G}}^p\,\mathbf{F^T}\right) \qquad \text{éq 4.3-4}$$

to avoid that the integration of porosity does not interfere with that of plasticity (since the two variables are coupled), it is necessary to separate integration from the constitutive law in two times: integration of plasticity with porosity built-in with its value at the beginning of time step, then integration of porosity by means of the equation 4.3-4 where the plastic evolution is that calculated with the preceding phase.

Notice important:

There are thus two possible versions of the model, according to whether one chooses the total deflection or the plastic strain in the evolution of porosity. Surprisingly, the choice seems to have an impact determining on the response at the level of structure, since it privileges or not the bifurcations of the zone where the strains are located. At the present time, version 4.3-1 is established in the code, version 4.3-4 being accessible to the developers by modification from parameter TYPORO in routine LCROLO. However, this strong sensitivity must lead the user to a greatest caution in the use of this model. Searches are in hand to understand this sensitivity and to discriminate the two alternatives for the evolution of porosity.

4.4 Relation "`ROUSSELIER` `

This behavior model is available via L" argument "`ROUSSELIER` `of key word `COMP_INCR` under L" operator `STAT_NON_LINE`, with argument "`SIMO_MIEHE` `of factor key word `the DEFORMATION`.

L" together of the parameters of the model is provided under the key keys factors "`ROUSSELIER` `or "`ROUSSELIER_FO` `and "`TENSION` `(to define curve of tension) of the command `DEFI_MATERIAU` ([U4.43.01]).

Note:

L" user must S" ensure well that "experimental" curve of tension used, either directly, or to deduce from it the slope D" hardening is well given in the plane forced rational $\sigma=F/S$ *- logarithmic strain* $\ln(1+\Delta l/l_0)$ where l_0 *is the initial length of the useful part of the test-tube,* Δl *the variation length after strain,* F *the applied force and* S *current surface*.

4.5 Stresses and local variables

the stresses are the stresses of Cauchy, $\boldsymbol{\sigma}$ thus calculated on the present configuration (six components in 3D, four in 2D).

The local variables produced in *the Code_Aster* are:

- V1, cumulated plastic strain p ,
- V2, porosity f ,
- V3 with V8, the elastic strain tensor $\mathbf{e}$,
- V9, the indicator of plasticity (0 if the last calculated increment is elastic, 1 if regular plastic solution, 2 if singular plastic solution).

Note:

If the user wants to possibly recover strains in postprocessing of his computation, it is necessary to trace the strains of Green-Lagrange $\mathbf{E}$ *, which represents a measurement of the strains in large deformations (option* EPSG_ELGA *or* EPSG_ELNO *of CALC_CHAMP). The classical linearized* $\boldsymbol{\varepsilon}$ *strains measure strains under the assumption of the small strains and do not have a meaning in large deformations.*

5 Numerical formulation

For the variational formulation, it acts of same as that given in the note [R5.03.21] and which refers to the constitutive law with isotropic hardening and criterion of Von Mises in large deformations. We point out only that it is about an eulerian formulation, with reactualization of the geometry to each increment and each iteration, and that one takes account of the stiffness of behavior and the geometrical stiffness.
We present the numerical integration of the constitutive law now and give the form of the tangent matrix (options FULL_MECA and RIGI_MECA_TANG).

5.1 Form of the discretized model

Knowing the stress $\boldsymbol{\sigma}^-$, the cumulated plastic strain p^- , the elastic strain $\mathbf{e}^-$, displacements $\mathbf{u}^-$ and $\Delta\mathbf{u}$, one seeks to determine $(\boldsymbol{\sigma}, p, \mathbf{e})$.
Displacements being known, the gradients of the transformation of Ω_0 with Ω^- , noted $\mathbf{F}^-$, and of Ω^- with Ω , noted $\Delta\mathbf{F}$, are known.

To integrate this model of behavior, one employs an implicit diagram of Eulerian, porosity being an explicit function of the strain via equation 4.3-1, therefore known during the integration of the behavior.

Once discretized, the following system then is obtained:

$$\mathbf{F}=\Delta\mathbf{F}\,\mathbf{F}^- \quad \text{éq 5.1-1}$$
$$J=\det\mathbf{F} \quad \text{éq 5.1-2}$$
$$J\boldsymbol{\sigma}=\boldsymbol{\tau} \quad \text{éq 5.1-3}$$

$\boldsymbol{\tau} = \mathbf{s}\ \mathbf{b}^e$ éq 5.1-4

$\mathbf{b}^e = \mathbf{Id} - 2\,\mathbf{e}$ éq 5.1-5

- Equations of state:

$\mathbf{s} = -\left[2\,\mu\,\tilde{\mathbf{e}} + K\ \mathrm{tr}\,\mathbf{e}\ \mathbf{Id} + 3\mathrm{K}\,\alpha\,\Delta T\ \mathbf{Id}\right]$ éq. 5.1-6

$A = -R(p)$ éq 5.1-7

Thereafter, one expresses the flow models and the plasticity criterion directly according to the tensor of the elastic strain $\mathbf{e}$.

- Flow models

$$\begin{aligned}\mathbf{D^p} &\simeq -\frac{1}{2}\mathbf{F}\dot{\mathbf{G}}^p\mathbf{F^T} = -\frac{1}{2\Delta t}\left[\underbrace{\mathbf{F}\mathbf{G}^p F^T}_{\mathbf{b^e}} - \Delta\mathbf{F}\underbrace{\mathbf{F^-}\mathbf{G}^{p-}\mathbf{F^{-T}}}_{\mathbf{b^{e-}}}\Delta\mathbf{F^T}\right]\\ &= -\frac{1}{2\Delta t}\left[\mathbf{Id} - 2\,\mathbf{e} - \Delta F\left(\mathbf{Id} - 2\,\mathbf{e^-}\right)\Delta\mathbf{F^T}\right]\\ &= (\mathbf{e} - \underbrace{\frac{1}{2}\left[\mathbf{Id} - \Delta\mathbf{F}\left(\mathbf{Id} - 2\,\mathbf{e^-}\right)\Delta\mathbf{F^T}\right]}_{\mathbf{e^{Tr}}})/\Delta t = (\mathbf{e} - \mathbf{e^{Tr}})/\Delta t\end{aligned} \quad \text{éq 5.1-8}$$

By taking the parts traces and deviatoric of the equation [éq 4.1-5], one obtains:

$$\mathrm{tr}\,\mathbf{e} - \mathrm{tr}\,\mathbf{e}^{Tr} = \Delta p\, D\, f \exp\left(\frac{-3K\alpha\Delta T}{\sigma_1}\right)\exp\left(\frac{-K\,\mathrm{tr}\,\mathbf{e}}{\sigma_1}\right) \quad \text{éq 5.1-9}$$

$$\tilde{\mathbf{e}} = \begin{cases}\tilde{\mathbf{e}}^{Tr} - \dfrac{3}{2}\Delta p\dfrac{\tilde{\mathbf{e}}}{e_{eq}} & \text{si solution régulière}\\ 0 \quad \text{et} \quad \Delta p \geq \dfrac{2}{3}(\tilde{\mathbf{e}} - \tilde{\mathbf{e}}^{Tr})_{eq} & \text{si solution singulière}\end{cases} \quad \text{éq 5.1-10}$$

- Conditions of coherence

$$F = \begin{cases}2\mu e_{eq} + \sigma_1 D f \exp\left(\dfrac{-3K\alpha\Delta T}{\sigma_1}\right)\exp\left(\dfrac{-K\,tr\,\mathbf{e}}{\sigma_1}\right) - R - \sigma_y & \text{si solution régulière}\\ \sigma_1 D f \exp\left(\dfrac{-3K\alpha\Delta T}{\sigma_1}\right)\exp\left(\dfrac{-K\,tr\,\mathbf{e}}{\sigma_1}\right) - R - \sigma_y & \text{si solution singulière}\end{cases} \quad \text{éq 5.1-11}$$

$$\text{avec}\, F \leq 0 \qquad \Delta p \geq 0 \qquad F\,\Delta p = 0$$

5.2 Resolution of the nonlinear system

the integration of the constitutive law is thus summarized to solve the system [éq 5.1-9], [éq 5.1-10] and [éq 5.1-11]. We will see that this resolution is brought back to that of only one scalar equation, whose unknown x is the increment of the trace of the elastic strain:

$$x = \mathrm{tr}\,\mathbf{e} - \mathrm{tr}\,\mathbf{e}^{Tr} \qquad \text{éq 5.2-1}$$

Thanks to this choice, that the solution is elastic or plastic, attack in a singular point or not, the equation [éq 5.1-9] bearing on the trail of the elastic increment is always valid and makes it possible to express the increment of cumulated plastic strain:

$$\mathrm{tr}\,\mathbf{e} - \mathrm{tr}\,\mathbf{e}^{Tr} = \underbrace{\Delta p\, D\, f \exp\left(\frac{-K\,\mathrm{tr}\,\mathbf{e}^{Tr}}{\sigma_1}\right)\exp\left(\frac{-3\,K\,\alpha\,\Delta T}{\sigma_1}\right)}_{G}\exp\left(\frac{-K\left(\mathrm{tr}\,\mathbf{e} - \mathrm{tr}\,\mathbf{e}^{Tr}\right)}{\sigma_1}\right)$$

$$\Rightarrow \Delta p(x) = \frac{1}{G}\, x \exp\frac{K\,x}{\sigma_1} \qquad \text{éq 5.2-2}$$

This equation shows us that one can seek $x \geq 0$ to guarantee a positive cumulated plastic strain and that the elastic solution is obtained for $x = 0$. It is also noticed that the increment of cumulated plastic strain is a function continuous and strictly increasing of x . With the help of these remarks, if one notes by S the term [éq 5.2-3] in the plasticity criterion, it acts then, there too, of a continuous and strictly increasing function of x :

$$F = 2\mu\, e_{eq} - S(x) \quad \text{avec} \quad S(x) = -\sigma_1\, \mathrm{G} \exp\left(-\frac{Kx}{\sigma_1}\right) + R(p(x)) + \sigma_y \qquad \text{éq 5.2-3}$$

A this stage, the approach of resolution breaks up into two times.

5.2.1 Examination of the singular points

Such a singular point is characterized by [éq 5.1-10] (low) and [éq 5.1-11] (low), therefore in particular par. $S(x) = 0$ Because of the properties of S , this equation admits with more the one positive solution, say x^S which exists if and only if $S(0) \leq 0$. The knowledge of x^S makes it possible to deduce from it the strain tensor elastics $\mathbf{e}$, the plastic strain cumulated p as well as the thermodynamic forces $\mathbf{s}$ and R .

Finally, this singular point will be solution if the inequality in [éq 5.1-11] (low) is checked, i.e. if:

$$\Delta p^s \geq \frac{2}{3}\left(\tilde{\mathbf{e}}^s - \tilde{\mathbf{e}}^{\mathrm{Tr}}\right)_{\mathrm{eq}} \qquad \text{éq 5.2.1-1}$$

5.2.2 regular Solution

the flow equation [éq 5.1-10] (high) which determines the deviatoric part of the elastic strain tensor makes it possible to deduce a scalar equation from it function from the increment of cumulated plastic strain:

$$\tilde{\mathbf{e}}-\tilde{\mathbf{e}}^{\mathrm{Tr}}=-\frac{3}{2}\Delta p\frac{\tilde{\mathbf{e}}}{e_{\mathrm{eq}}} \quad \Rightarrow \quad \begin{cases} e_{\mathrm{eq}}=e_{\mathrm{eq}}^{\mathrm{Tr}}-\frac{3}{2}\Delta p \\ \tilde{\mathbf{e}}=\frac{e_{\mathrm{eq}}}{e_{\mathrm{eq}}^{\mathrm{Tr}}}\tilde{\mathbf{e}}^{\mathrm{Tr}} \end{cases} \qquad \text{éq 5.2.2-1}$$

One notes that because of positivity of e_{eq}, the value sells by auction of Δp is limited:

$$\Delta p\leq\frac{2}{3}e_{\mathrm{eq}}^{\mathrm{Tr}} \qquad \text{éq 5.2.2-2}$$

the condition of coherence determines now x :

$$F=2\mu e_{eq}^{Tr}-S(x)-3\mu\Delta p\ \leq 0 \qquad \text{éq 5.2.2-3}$$

being given this statement, the increase of the licit value of Δp is reduced to the only condition $S(x)\geq 0$ or, in an equivalent way, with $x\geq x^S$.

The elastic solution is obtained for x = 0. It is the solution of the problem if and only if:

$$F(0)=2\mu\ e_{eq}^{Tr}-S(0)<0 \qquad \text{éq 5.2.2-4}$$

In the contrary case, one must then solve:

$$F(x)=2\mu e_{eq}^{Tr}-S(x)-\frac{3\mu}{G}x\exp\left(\frac{Kx}{\sigma_1}\right)=0 \quad \text{avec} \quad \begin{cases} x>x^s & \text{si } x^s \text{existe} \\ x>0 & \text{sinon} \end{cases} \qquad \text{éq 5.2.2-5}$$

This function is continuous and strictly decreasing and tends towards $-\infty$ with x . She thus admits with more the one solution. The demonstration of the existence of this solution is immediate. Indeed, it is enough to prove that F is positive on the lower limit of the interval of search.
When x^S does not exist, $F(0)>0$ since the solution is not elastic.

When x^S exists, the function is worth:

$$F(x^s)=2\mu e_{\mathrm{eq}}^{\mathrm{Tr}}-3\mu\Delta p^s>0\Leftrightarrow\Delta p^s<\frac{2}{3}e_{\mathrm{eq}}^{\mathrm{Tr}} \qquad \text{éq 5.2.2-6}$$

This condition is checked since one rejected the solution singular.

5.3 Course of computation

the approach to solve all the equations of the model is the following one:

1) One searches if the solution is elastic
   - computation of $F(0)$
   - if $F(0)<0$, the solution of the problem is the elastic solution $x^{Sol}=0$
   - if not one passes in 2)
2) If $S(0)>0$, the solution is plastic and regular
   - one passes in 4)
3) If $S(0)<0$, one seeks if the solution is singular
   - one solves $S(x^s)=0$
   - if x^s the inequality checks $\Delta p^s \geq \frac{2}{3}(\tilde{\mathbf{e}}^s - \tilde{\mathbf{e}}^{\mathrm{Tr}})_{\mathrm{eq}}$, then the solution is singular $x^{Sol}=x^s$
   - if not, x^s is a lower limit to solve $F(x)=0$, one passes in 4)
4) The solution is plastic and regular
   - one solves $F(x)=0$

5.4 Resolution

to solve the two equations $S(x)=0$ and $F(x)=0$, one employs a method of Newton with controlled limits coupled to dichothomy when Newton gives a solution apart from the interval of the two limits. One now presents the determination of the limits for each preceding case (points 2) 3) and 4) preceding paragraph).

5.4.1 Hight delimiters and lower if the function S is strictly positive in the beginning

One solves:

$$\begin{cases} F(x)=0 \\ F(0)>0 \end{cases} \Leftrightarrow \begin{cases} \underbrace{2\mu e_{eq}^{Tr} - S(x)}_{f_1} = \underbrace{\frac{3\mu}{G} x \exp\left(\frac{Kx}{\sigma_1}\right)}_{3\mu\,\Delta p} \\ f_1(0)>0 \end{cases} \qquad \text{éq 5.4.1-1}$$

where the function $\Delta p(x)$ is continuous, strictly increasing and null in the beginning and the function $f_1(x)$ is continuous, strictly decreasing and strictly positive at the origin (see [Figure 5.4.1-a]).
One poses:

$$f_1 = \underbrace{2\mu e_{eq}^{Tr} - R(x) - \sigma_y}_{f_2} + \sigma_1 \mathrm{G}\exp\left(-\frac{Kx}{\sigma_1}\right) \quad \text{alors} \quad f_2(x) < f_1(x) \quad \forall x \geq 0 \qquad \text{éq 5.4.1-2}$$

where the function $f_2(x)$ is continuous, strictly decreasing. In this case, the resolution of the equations:

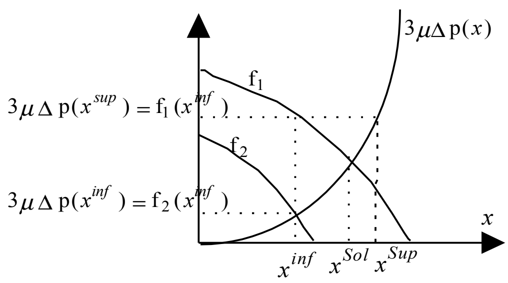
$$f_2(\Delta p^{inf})=3\mu\Delta p^{inf} \text{ and } x^{inf}\exp(\frac{Kx^{inf}}{\sigma_1})=G\Delta p^{inf} \quad \text{éq 5.4.1-3}$$

to deduce some successively Δp then x gives a lower limit x^{Inf} which corresponds to the solution of the model to isotropic hardening and criterion of Von Mises. If $f_2(0)<0$, the lower limit is taken equalizes to zero: $x^{inf}=0$.

The upper limit x^{Sup} is such as:

$$x^{\mathrm{Sup}}\exp(\frac{\mathrm{Kx}^{\mathrm{Sup}}}{\sigma_1})=\frac{G}{3\mu}f_1(x^{\mathrm{Inf}}) \quad \text{éq 5.4.1-4}$$

the equation of the type $x\exp(\frac{Kx}{\sigma_1})=constante$ is solved by a method of Newton.



Appear 5.4.1-a: chart of the hight delimiters and lower

5.4.2 Hight delimiters and lower if the function S is negative or null at the origin

the system to be solved is the following:

$$\begin{cases} S(x)=0 \\ S(0)<0 \end{cases} \Leftrightarrow \begin{cases} R\left(p^- + \frac{x}{G}\exp(\frac{Kx}{\sigma_1})\right)+\sigma_y=\sigma_1 \mathrm{G}\exp(-\frac{Kx}{\sigma_1}) \\ R\left(p^-\right)+\sigma_y<\sigma_1 G \end{cases} \quad \text{éq 5.4.2-1}$$

the part of left is a function continuous, strictly increasing of x and strictly positive in the beginning, the part of right is a function continuous, strictly decreasing of x and strictly positive at the origin. Using the properties of these two functions, a chart (cf [Figure 5.4.2-a]) of these functions shows that the higher limit x^{Sup} is such as:

$$\sigma_1 \mathrm{G}\ \exp(-\frac{Kx^{Sup}}{\sigma_1})=R\left(p^-\right)+\sigma_y \quad \Leftrightarrow \quad x^{Sup}=\frac{\sigma_1}{K}\log\left(\frac{\sigma_1 G}{R\left(p^-\right)+\sigma_y}\right) \quad \text{éq 5.4.2-2}$$

the lower limit x^{Inf} is such as:

$$\begin{aligned} &\sigma_1 \mathrm{G}\exp(-\frac{Kx^{Inf}}{\sigma_1})=R\left(p^-+\frac{x^{Sup}}{G}\exp(\frac{Kx^{Sup}}{\sigma_1})\right)+\sigma_y \\ &\Leftrightarrow x^{Inf}=\left\langle \frac{\sigma_1}{K}\log\left(\frac{\sigma_1 G}{R\left(p^-+\frac{x^{Sup}}{G}\exp(\frac{Kx^{Sup}}{\sigma_1})\right)+\sigma_y}\right)\right\rangle^+ \end{aligned} \quad \text{éq 5.4.2-3}$$

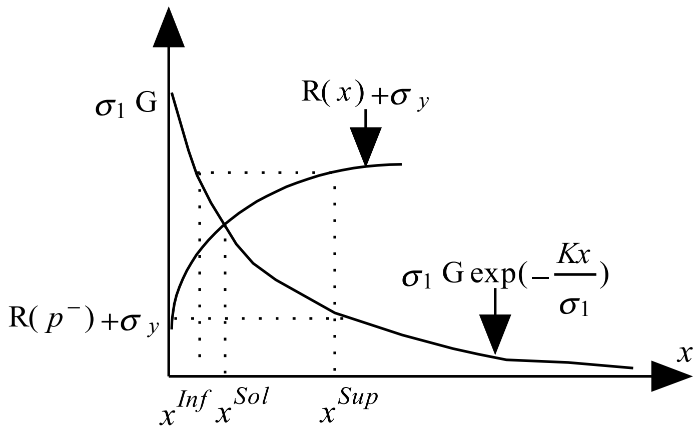


Figure 5.4.2-a: chart of the hight delimiters and lower

5.4.3 Hight delimiters and lower if the function S is strictly negative in the beginning and x^s not solution

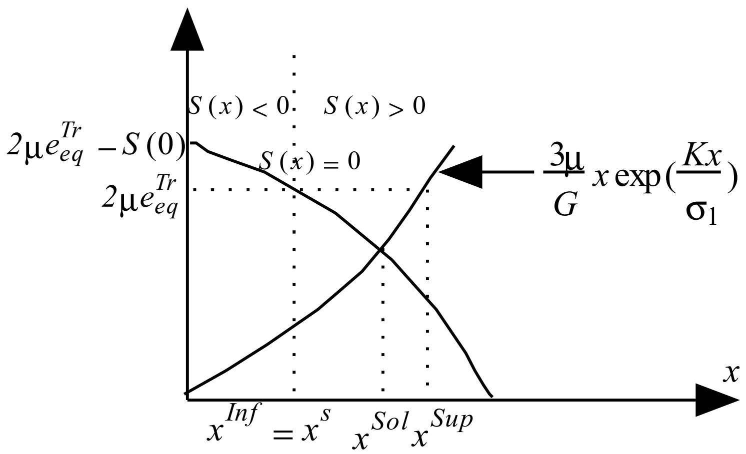
One solves the following system:

$$\begin{cases} F(x)=0 \\ S(0)<0 \\ S(x^s)=0 \end{cases} \Leftrightarrow \begin{cases} \underbrace{2\mu e_{eq}^{Tr}-S(x)}_{f_1}=\underbrace{\frac{3\mu}{G}x\exp(\frac{Kx}{\sigma_1})}_{3\mu\Delta p} \\ f_1(0)>0 \\ 2\mu e_{eq}^{Tr}=\frac{3\mu}{G}x^s\exp(\frac{Kx^s}{\sigma_1}) \end{cases} \quad \text{éq 5.4.3-1}$$

the solution x^{Sol} is such as $S(x^{Sol})>0$.
For the lower limit, one takes $x^{Inf}=x^s$. Being given the properties of the functions f_1 (strictly decreasing) and $3\mu\Delta p(x)$ (strictly increasing), the higher limit x^{Sup} is such as (cf [Figure 5.4.3-a]):

$$x^{Sup}\exp(\frac{Kx^{Sup}}{\sigma_1})=\frac{2G}{3}e_{eq}^{Tr} \quad \text{éq 5.4.3-2}$$

This equation is solved by a method of Newton.



Appear 5.4.3-a: chart of the hight delimiters and lower

5.5 Correction of volume a posteriori

In the case of the model of Rousselier, the change of volume plays a crucial role, so that the mistake made by the approximationdans $\mathbf{s}=\boldsymbol{\tau}$ the equations of flow can lead to an evolution can precise of porosity, cf [Bib2]. While following the proposal for this reference, one will correct a posteriori (i.e after having calculated all the quantities) the trace of the elastic strain. Indeed

, in the absence of approximation, the hydrostatic part of the flow equation leads to: (éq

$$\frac{d}{dt}\left(\ln J^p\right)=\dot{p}\,D\,f\exp\left(\frac{\mathrm{tr}\,\boldsymbol{\tau}}{3\,\sigma_1}\right)$$. 5.5-1) After

discretization in time, one then obtains a proposal corrected for the changes of plastic and elastic volume: (éq

$$J^p_{corr}=J^{p-}\exp\left(\mathrm{tr}\,\mathbf{e}-\mathrm{tr}\,\mathbf{e}^{Tr}\right)\quad;\quad J^e_{corr}=\frac{J}{J^p_{corr}}$$. 5.5-2) One

then will seek a new trace of elastic strain such as the elastic change of volume corresponds to the value corrected above: (éq

$$\mathbf{e}_{\mathbf{corr}}=\tilde{\mathbf{e}}+t\,\mathbf{Id}\qquad \text{où } t \text{ tel que : } \det\left(\mathbf{Id}-2\,\mathbf{e}_{\mathbf{corr}}\right)={J^e_{corr}}^2$$. 5.5-3) That

led to a polynomial of degree 3 in, t which one will choose the solution nearest to. $\mathbf{e}$ Form

5.6 of the tangent matrix of the behavior One

gives the form of the tangent matrix here (option FULL `_MECA during` the iterations of Newton, option RIGI `_MECA_TANG for` the first iteration). For the option
FULL `_MECA, this one` is obtained by linearizing the system of equations which governs the constitutive law. We give hereafter the broad outlines of this linearization. For the option
RIGI `_MECA_TANG, it` acts of the same statements as those given for FULL `_MECA but` with. $\Delta\,p=0$ In particular, one will have. $\Delta\,\mathbf{F}=\mathbf{Id}$

The constitutive law can be put in the following general form: éq

$$\boldsymbol{\sigma}=\boldsymbol{\sigma}\left(\boldsymbol{\tau}\,,\Delta\,\mathbf{F}\right)$$ 5.6-1 éq

$$\boldsymbol{\tau}=\boldsymbol{\tau}\left(\mathbf{e}\right)$$ 5.6-2 éq

$$\mathbf{e}=\mathbf{e}\left(\mathbf{e}^{\mathbf{Tr}},f\right)$$ 5.6-3 éq

$$\mathbf{e}^{\mathbf{Tr}}=\mathbf{e}^{\mathbf{Tr}}(\Delta\,\mathbf{F})$$ 5.6-4

the linearization of this system gives: éq

$$\delta\,\boldsymbol{\sigma}=\left(\frac{\partial\boldsymbol{\sigma}}{\partial\boldsymbol{\tau}}:\frac{\partial\boldsymbol{\tau}}{\partial\mathbf{e}}:\left(\frac{\partial\,\mathbf{e}}{\partial\,\mathbf{e}^{\mathbf{Tr}}}:\frac{\partial\,\mathbf{e}^{\mathbf{Tr}}}{\partial\boldsymbol{\Delta}\,\mathbf{F}}+\frac{\partial\,\mathbf{e}}{\partial\,f}\frac{\partial\,f}{\partial\,J}\otimes\frac{\partial\,J}{\partial\boldsymbol{\Delta}\,\mathbf{F}}\right)+\frac{\partial\boldsymbol{\sigma}}{\partial\boldsymbol{\Delta}\,\mathbf{F}}\right):\delta\,\Delta\,\mathbf{F}=\mathbf{H}:\delta\,\Delta\,\mathbf{F}$$ 5.6-5 where

is $\mathbf{H}$ the tangent matrix. Thereafter, one separately determines the five terms of the preceding equation. In
the linearization of the system, one will often use the tensor defined $\mathbf{C}$ below and the two following equations: éq

$$\delta a_{ij}=\frac{1}{2}\left(\delta_{ik}\delta_{jl}+\delta_{jk}\delta_{il}\right)\delta a_{kl}$$ 5.6-6 éq

$$\delta\ a_{pp}=\delta_{kl}\delta\ a_{kl}$$ 5.5-7 éq

$$C_{ijkl}=\frac{1}{2}\left(\delta_{ik}\delta_{jl}+\delta_{jk}\delta_{il}\right)$$ 5.6-8 Computation

- of and $\frac{\partial\boldsymbol{\sigma}}{\partial\boldsymbol{\tau}}$ $\frac{\partial\boldsymbol{\sigma}}{\partial\Delta\mathbf{F}}$

  the linearization of the relation which binds the stress of Cauchy and $\boldsymbol{\sigma}$ the stress of Kirchhoff gives $\boldsymbol{\tau}$: éq

$$J\boldsymbol{\sigma}=\boldsymbol{\tau}\quad\Leftrightarrow\quad\delta\boldsymbol{\sigma}=\frac{1}{J}\delta\boldsymbol{\tau}-\left(\frac{\boldsymbol{\sigma}}{J}\otimes\frac{\partial J}{\partial\Delta F}\right):\delta\ \Delta\mathbf{F}$$ 5.6-9 By means of

the relation [éq 5.6-6], one obtains for: $\frac{\partial\boldsymbol{\sigma}}{\partial\boldsymbol{\tau}}$ éq

$$\frac{\partial\boldsymbol{\sigma}}{\partial\boldsymbol{\tau}}=C$$ 5.6-10 and

for: $\frac{\partial\boldsymbol{\sigma}}{\partial\Delta\mathbf{F}}$ éq

$$\frac{\partial\boldsymbol{\sigma}}{\partial\Delta\mathbf{F}}=-\frac{\boldsymbol{\sigma}}{J}\otimes\frac{\partial J}{\partial\Delta\mathbf{F}}$$ 5.6-11 with

éq

$$\frac{\partial J}{\partial F_{11}}=\Delta F_{22}\Delta F_{33}-\Delta F_{23}\Delta F_{32}$$
$$\frac{\partial J}{\partial F_{22}}=\Delta F_{11}\Delta F_{33}-\Delta F_{13}\Delta F_{31}$$
$$\frac{\partial J}{\partial F_{33}}=\Delta F_{11}\Delta F_{22}-\Delta F_{12}\Delta F_{21}$$
$$\frac{\partial J}{\partial F_{12}}=\Delta F_{31}\Delta F_{23}-\Delta F_{33}\Delta F_{21}\qquad\frac{\partial J}{\partial F_{21}}=\Delta F_{13}\Delta F_{32}-\Delta F_{33}\Delta F_{12}$$
$$\frac{\partial J}{\partial F_{13}}=\Delta F_{21}\Delta F_{32}-\Delta F_{22}\Delta F_{31}\qquad\frac{\partial J}{\partial F_{31}}=\Delta F_{12}\Delta F_{23}-\Delta F_{22}\Delta F_{13}$$
$$\frac{\partial J}{\partial F_{23}}=\Delta F_{31}\Delta F_{12}-\Delta F_{11}\Delta F_{32}\qquad\frac{\partial J}{\partial F_{32}}=\Delta F_{13}\Delta F_{21}-\Delta F_{11}\Delta F_{23}$$

5.6-12 Computation

- of $\dfrac{\partial \boldsymbol{\tau}}{\partial \mathbf{e}}$

the relation which binds the stress of Kirchhoff and $\boldsymbol{\tau}$ the elastic strain tensor $\mathbf{e}$ is given by: éq

$$\boldsymbol{\tau}=\mathbf{s}\,\mathbf{b}^{e}=-2\mu\ \mathbf{e}-\lambda\ \mathrm{Tr}\,\mathbf{e}\,\mathbf{Id}+4\mu\ \mathbf{e}\,\mathbf{e}+2\lambda\ (\mathrm{tr}\,\mathbf{e})e \\ -\ 3\mathrm{K}\,\alpha\,\Delta T\,\mathbf{Id}+6\mathrm{K}\,\alpha\,\Delta T\,\mathbf{e}$$ 5.6-13 One

obtains after linearization: éq

$$\delta\boldsymbol{\tau}=2(\lambda\ \mathrm{tr}\,\mathbf{e}-\mu+3\mathrm{K}\,\alpha\,\Delta T)\delta\mathbf{e}+\lambda(2e-\mathbf{Id})\mathrm{Tr}\,\delta\,\mathbf{e}\ +4\mu(\delta\,\mathbf{e}\,\mathbf{e}+\mathbf{e}\,\delta\,\mathbf{e})$$ 5.6-14 from where

éq

$$\frac{\partial\tau_{ij}}{\partial e_{kl}}=2(\lambda\,tr\,e-\mu+3\mathrm{K}\,\alpha\,\Delta T)C_{ijkl}+\lambda(2e_{ij}-\delta_{ij})\delta_{kl}+2\mu(\delta_{ik}e_{lj}+\delta_{il}e_{kj}+e_{il}\delta_{kj}+e_{ik}\delta_{jl})$$ 5.6-15 Computation

- of $\dfrac{\partial \mathbf{e}^{Tr}}{\partial \Delta \mathbf{F}}$

the relation between the elastic strain tensor and $\mathbf{e}^{\mathrm{Tr}}$ the increment of the gradient of the transformation $\Delta\mathbf{F}$ is written: éq

$$\mathbf{e}^{\mathrm{Tr}}=\frac{1}{2}(\mathbf{Id}-\boldsymbol{\Delta}\,\mathbf{F}\,\mathbf{b}^{e-}\,\boldsymbol{\Delta}\,\mathbf{F}^{\mathrm{T}})$$ 5.6-16 Its

linearization gives: éq

$$\frac{\partial e_{ij}^{\mathrm{Tr}}}{\partial\Delta F_{kl}}=-\frac{1}{2}(\delta_{ik}\,\Delta F_{jp}\,b_{pl}^{e-}+\Delta F_{ip}\,b_{pl}^{e-}\,\delta_{jk})$$ 5.6-17 Computation

- of elastic $\dfrac{\partial \mathbf{e}}{\partial \mathbf{e}^{\mathrm{Tr}}}$

Case In

the elastic case, the computation of is $\dfrac{\partial \mathbf{e}}{\partial \mathbf{e}^{\mathrm{Tr}}}$ immediate since from where $\delta\,\mathbf{e}=\delta\,\mathbf{e}^{\mathrm{Tr}}$ éq

$$\frac{\partial\mathbf{e}}{\partial\mathbf{e}^{\mathrm{Tr}}}=C$$ 5.6-18 Cases

plastic – regular Solution

to determine, $\frac{\partial \mathbf{e}}{\partial \mathbf{e}^{\mathbf{Tr}}}$ one operates in two stages. By the flow model discretized, one calculates in first according to $\delta\,\mathbf{e}$ and $\delta\,\mathbf{e}^{\mathbf{Tr}}$. $\delta\,\Delta\,p$ Then the condition of coherence makes it possible to deduce some according to $\delta\,\Delta\,p$. $\delta\,\mathbf{e}^{\mathbf{Tr}}$ These two stages are thereafter detailed.
The deviatoric part of the flow model discretized is written: éq

$$\tilde{\mathbf{e}}-\tilde{\mathbf{e}}^{\mathrm{Tr}}=-\frac{3}{2}\Delta\,p\frac{\tilde{\mathbf{e}}}{\mathbf{e}_{eq}}$$ 5.6-19 One

obtains after linearization: éq

$$\underbrace{\left(1+\frac{3}{2}\frac{\Delta\,p}{e_{\mathrm{eq}}}\right)}_{1/\alpha}\delta\,\tilde{\mathbf{e}}=\delta\,\tilde{\mathbf{e}}^{\mathrm{Tr}}-\frac{3}{2}\frac{\tilde{\mathbf{e}}}{e_{\mathrm{eq}}}\delta\,\Delta\,p+\frac{9}{4}\Delta\,p\frac{\tilde{\mathbf{e}}}{e_{\mathrm{eq}}^{3}}\left(\tilde{\mathbf{e}}:\delta\,\tilde{\mathbf{e}}\right)$$ 5.6-20

to determine, $\tilde{\mathbf{e}}:\delta\,\tilde{\mathbf{e}}$ one contracts the equation [éq 5.6-20] with what $\tilde{\mathbf{e}}$ gives: éq

$$\tilde{\mathbf{e}}:\delta\,\tilde{\mathbf{e}}=\tilde{\mathbf{e}}:\delta\,\tilde{\mathbf{e}}^{\mathrm{Tr}}-e_{\mathrm{eq}}\delta\,\Delta\,p$$ 5.6-21 from where

éq

$$\delta\,\tilde{\mathbf{e}}=\underbrace{\alpha\left[\frac{9\,\Delta\,p}{4\,e_{\mathrm{eq}}^{3}}\tilde{\mathbf{e}}\otimes\tilde{\mathbf{e}}+\mathbf{C}\right]}_{\mathbf{A}_1}:\delta\,\tilde{\mathbf{e}}^{\mathrm{Tr}}\underbrace{-\frac{3}{2}\frac{\tilde{\mathbf{e}}}{e_{\mathrm{eq}}}}_{\mathbf{A}_2}\delta\Delta p$$ 5.6-22 For

the part traces flow model discretized, one a: éq

$$\mathrm{Tr}\,\mathbf{e}-\mathrm{Tr}\,\mathbf{e}^{Tr}=D\,f\,\Delta\,p\exp\left(\frac{3\,K\,\alpha\,\Delta\,T}{\sigma_1}\right)\exp\left(-\frac{K}{\sigma_1}\mathrm{Tr}\,\mathbf{e}\right)$$ 5.6-23 what

gives, while posing: $k_1=1+\frac{D\,f\,K\,\Delta\,p}{\sigma_1}\exp\left(\frac{3\,K\,\alpha\,\Delta\,T}{\sigma_1}\right)\exp\left(-\frac{K}{\sigma_1}\mathrm{Tr}\,\mathbf{e}\right)$ éq

$$\mathrm{tr}\,\delta\,\mathbf{e}=\underbrace{\frac{1}{k_1}}_{\alpha_1}\mathrm{tr}\,\delta\,\mathbf{e}^{Tr}+\underbrace{\frac{D\,f\exp\left(\frac{-K}{\sigma_1}\mathrm{tr}\,\mathbf{e}\right)\exp\left(\frac{3\,K\,\alpha\,\Delta\,T}{\sigma_1}\right)}{k_1}}_{\alpha_2}\delta\,\Delta\,p+\underbrace{\frac{\Delta\,p\exp\left(\frac{-K}{\sigma_1}\mathrm{tr}\,\mathbf{e}\right)\exp\left(\frac{3\,K\,\alpha\,\Delta\,T}{\sigma_1}\right)}{k_1}}_{\beta_1}D\,\delta\,f$$ 5.6-24 In

the plastic case, the condition of coherence is worth: éq

$$2\mu e_{eq}+D\,f\,\sigma_1\exp\left(-\frac{3K\alpha\Delta T}{\sigma_1}\right)\exp\left(-\frac{K}{\sigma_1}\mathrm{Tr}\,\mathbf{e}\right)-R-\sigma_y=0$$ 5.6-25 from where

éq

$$\frac{3\mu}{e_{eq}}\left(\tilde{\mathbf{e}}:\delta\tilde{\mathbf{e}}\right)+D\sigma_1\exp\left(\frac{-3K\alpha\Delta T}{\sigma_1}\right)\exp\left(\frac{-K\,\mathrm{tr}\,\mathbf{e}}{\sigma_1}\right)\delta f$$
$$-D\,f\,K\exp\left(\frac{-3K\alpha\Delta T}{\sigma_1}\right)\exp\left(\frac{-K\,\mathrm{tr}\,\mathbf{e}}{\sigma_1}\right)\mathrm{tr}\,\delta\mathbf{e}-h\,\delta\Delta p=0$$ 5.6-26 By

injecting the relation [éq 5.6-21] in the equation above, one obtains then, while posing:

$$k_2=3\mu+h+\alpha_2 D\,f\,K\exp\left(\frac{-3K\alpha\Delta T}{\sigma_1}\right)\exp\left(-\frac{K}{\sigma_1}\mathrm{Tr}\,\mathbf{e}\right)$$ éq

$$\delta\Delta p=\begin{array}{l}\underbrace{\frac{3\mu}{e_{eq}}\frac{1}{k_2}}_{\alpha_3}\tilde{\mathbf{e}}:\delta\tilde{\mathbf{e}}^{Tr}-\underbrace{\frac{\alpha_1 D\,f\,K\exp\left(\frac{-3K\alpha\Delta T}{\sigma_1}\right)\exp\left(-\frac{K}{\sigma_1}\mathrm{Tr}\,\mathbf{e}\right)}{k_2}}_{\alpha_4}\mathrm{tr}\,\delta\mathbf{e}^{Tr}\\ +\underbrace{\frac{\exp\left(\frac{-3K\alpha\Delta T}{\sigma_1}\right)\exp\left(\frac{-K}{\sigma_1}\mathrm{Tr}\,\mathbf{e}\right)\left(\sigma_1-\beta_1 D\,f\,K\right)}{k_2}}_{\beta_2}D\,\delta f\end{array}$$ 5.6-27 While

replacing by $\delta\Delta p$ its value obtained above in the equations [éq 5.6-22] and [éq 5.6-24], one obtains: éq

$$\delta\mathbf{e}=\underbrace{\left[\mathbf{A_1}+\left(\mathbf{A_2}+\frac{1}{3}\alpha_2\,\mathbf{Id}\right)\otimes\left(\frac{3\mu\alpha_3}{e_{eq}}\tilde{\mathbf{e}}\right)\right]}_{\mathbf{ddvetr}}:\delta\tilde{\mathbf{e}}^{Tr}+\underbrace{\left[\frac{1}{3}\alpha_1\,\mathbf{Id}+\alpha_4\left(\mathbf{A_2}+\frac{1}{3}\alpha_2\,\mathbf{Id}\right)\right]}_{\mathbf{dtretr}}\mathrm{tr}\,\delta\mathbf{e}^{Tr}$$
$$+\underbrace{\left[\frac{1}{3}\beta_1\,\mathbf{Id}+\beta_2\left(\mathbf{A_2}+\frac{1}{3}\alpha_2\,\mathbf{Id}\right)\right]}_{\mathbf{dedf}}D\,\delta f$$ 5.6-28 from

where

éq

$$\frac{\partial\mathbf{e}}{\partial\mathbf{e}^{Tr}}=\mathbf{ddvetr}+\left(\mathbf{dtretr}-\frac{1}{3}\mathbf{ddvetr}:\mathbf{Id}\right)\otimes\mathbf{Id}$$ 5.6-29 plastic

Cases – singular Solution

the approach is identical to that used previously. One obtains for the flow model discretized: éq

$$\tilde{\mathbf{e}}=0 \quad \Leftrightarrow \quad \delta\,\tilde{\mathbf{e}}=0$$ 5.6-30 for

the deviatoric part and the part traces, the relation is identical to that found for the regular solution. éq

$$\mathrm{tr}\,\delta\,\mathbf{e}=\alpha_1\,\mathrm{tr}\,\delta\,\mathbf{e}^{\mathbf{Tr}}+\alpha_2\delta\,\Delta\,p+\beta_1\,D\,\delta\,f$$ 5.6-31 where

, α_1 and α_2 has β_1 the same definitions as in the preceding paragraph.

The condition of coherence then makes it possible to find according to $\delta\,\Delta\,p$. $\partial\,\mathbf{e}^{\mathbf{Tr}}$ éq

$$D\,f\,\sigma_1\exp\left(\frac{3\mathrm{K}\,\alpha\,\Delta\,T}{\sigma_1}\right)\exp\left(-\frac{K}{\sigma_1}\mathrm{Tr}\,e\right)-R-\sigma_y=0$$ 5.6-32 is

after linearization: éq

$$\delta\,\Delta\,p=\alpha_4\,\mathrm{tr}\,\delta\,\mathbf{e}^{\mathbf{Tr}}+\beta_2 D\,\delta\,f$$ 5.6-33 is

finally: éq

$$\delta\,\mathbf{e}=\underbrace{\frac{1}{3}\left[\alpha_1+\alpha_4\,\alpha_2\right]\mathbf{Id}\;\mathrm{tr}\,\delta\,\mathbf{e}^{\mathbf{Tr}}}_{\mathbf{dtretr}}+\underbrace{\frac{1}{3}\left[\beta_1+\beta_2\,\alpha_2\right]\mathbf{Id}\;D\,\delta\,f}_{\mathbf{dedf}}$$ 5.6-34 from where

éq

$$\frac{\partial\,\mathbf{e}}{\partial\,\mathbf{e}^{Tr}}=\mathbf{dtretr}\otimes\mathbf{Id}$$ 5.6-35 Computation

- of Taking into account $\frac{\partial f}{\partial J}$

relation 4.3-1, the derivative is written simply: éq

$$\begin{cases}\dfrac{\partial f}{\partial J}=D\dfrac{1-f_0}{J^2} & \mathrm{si}\ f>f_0\\ \dfrac{\partial f}{\partial J}=0 & \mathrm{si}\ f=f_0\end{cases}$$ 5.6-36 Note:

:

The tangent matrix is not deteriorated by the correction of volume because this one is carried out a posteriori, i.e. after the computation of the stresses. Bibliography

6 ROUSSELIER

1) G.: “Finite constitutive strain relations including ductile fracture ramming, In three dimensional constitutive relations and ductile fracture”, ED. Nemat - Nasser, North Holland publishing company, pp. 331-355. LORENTZ

2) E., BESSON J., CANO V., “Numerical simulation of ductile fracture: efficient year and robust constitutive implementation of the Rousselier law”, comp. Meth. Appl. Mech. Engrg. 197, pp. 1965-1982, 2008. SIMO

3) J.C., MIEHE C., “Associative coupled thermoplasticity At finite strains: Formulation, numerical analysis and implementation”, comp. Meth . Appl. Mech. Eng., 98, pp. 41-104, North Holland, 1992.SIDOROFF F., “Course on the large deformations”, Ratio Greco n51, 1982. LORENTZ

4) E., CANO V.: “A minimization principle for finite strain plasticity: incremental objectivity and immediate implementation”, Com. Num. Meth. Eng. 18, pp. 851-859, 2002. MIALON

5) P.: “Elements of analysis and resolution numerical of the relations of the élasto - plasticity “, EDF, bulletin of DER, series C mathematical data processing, 3, pp. 57-89, 1986 Description

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