Elastoplastic behavior model with linear and isotropic kinematic hardening nonlinear. Modelizations 3D and plane stresses

Summarized:

This document describes elastoplastic constitutive law with hardening mixed, kinematical linear and isotropic nonlinear. The equations to solve integrate this behavior model numerically are specified, as well as the coherent tangent matrix.

This behavior is usable for the modelizations of continuums 3D, 2D (AXIS, C_PLAN, D_PLAN), and for the modelizations DKT, COQUE_3D and PIPE.

Contents

1 Introduction

When the way of loading is not monotonous any more, hardenings isotropic and kinematical are not equivalent any more. In particular, one can expect to have simultaneously a kinematical share and an isotropic share. If one seeks to precisely describe the effects of a cyclic loading, it is desirable to adopt modelizations sophisticated (but easy to use) such as the model of Taheri, for example, to see [R5.03.05]. On the other hand, for less complex ways of loading, one can wish to include only one linear kinematic hardening, all nonthe linearities of hardening being carried by the isotropic term. That makes it possible to follow a curve of tension precisely, while representing nevertheless phenomena such as the Bauschinger effect [bib1] (see for example it [Figure 5-a]).

The characteristics of hardening are then given by a curve of tension and a constant, called of Prager, for the linear term of kinematic hardening. They are introduced into command `DEFI_MATERIAU` :

Isotropic hardening linear	isotropic Hardening nonlinear

`DEFI_MATERIAU`
`ECRO_LINE`
`SY:` *elastic limit*
`D_SIGM_EPSI:` *slope of curve of tension*
`PRAGER: (C:` *constant of Prager* `)`
	`DEFI_MATERIAU (`
	`TENSION: (SIGM:` *curve of tension*
	`PRAGER: (C:` *constant of Prager* `)`

These characteristics can also depend on the temperature, on condition that employing then the key keys factors `ECMI_LINE_FO` and `ECMI_TRAC_FO` instead of `ECRO_LINE` and `TENSION`. The use of these constitutive laws is available in commands `STAT_NON_LINE` or `DYNA_NON_LINE` :

Isotropic hardening linear	isotropic Hardening nonlinear

`STAT_NON_LINE`
`COMP_INCR:`
`RELATION: "VMIS_ECMI_LINE"`
	`STAT_NON_LINE`
	`COMP_INCR:`
	`RELATION: "VMIS_ECMI_TRAC"`

In the continuation of this document, one precisely the model describes combined hardening. One presents then the detail of his numerical integration in restrain with the construction of the coherent tangent matrix. Lastly, a traction test uniaxial pressing illustrates the identification of the characteristics of the material.

2 Description of the model

At any moment, the state of the material is described by the strain ε , the temperature T , the plastic strain ε^{p} and the cumulated plastic strain p . The equations of state then define according to these variables of state the stress $\sigma=\sigma^{H}\mathbf{Id}+\tilde{\sigma}$ (broken up into hydrostatics parts and deviatoric), the isotropic share of hardening R and the kinematical share $\mathbf{X}$, so called forced recall:

$$\sigma^{H}=\frac{1}{3}tr(\sigma)=K\,tr(\varepsilon-\varepsilon^{\mathbf{th}})\;\text{avec}\;\varepsilon^{\mathbf{th}}=\alpha\left(T\text{-}T^{réf}\right)\mathbf{Id} \quad \text{éq 2-1}$$

$$\tilde{\sigma}=\sigma-\sigma^{H}\,\mathbf{Id}=2\mu\left(\tilde{\varepsilon}-\varepsilon^{p}\right)\;\text{où}\;\tilde{\varepsilon}=\varepsilon-\frac{1}{3}tr(\varepsilon)\,\mathbf{Id} \quad \text{éq 2-2}$$

$$R=\mathrm{R}(p) \quad \text{éq 2-3}$$

$$\mathbf{X}=C\,\varepsilon^{\mathrm{p}} \quad \text{éq 2-4}$$

where K,μ,α,R and C are characteristics of the material which can depend on the temperature. More precisely, they are respectively the moduli of compressibility and shears, the average thermal coefficient of thermal expansion (see [R4.08.01]), the isotropic function of hardening and the constant of Prager. As for $T^{réf}$, it is the reference temperature, for which the thermal strain is null.

K,μ are connected to the Young modulus E and the Poisson's ratio by:

$$3\mathrm{K}=3\lambda+2\mu=\frac{E}{1-2\nu}$$

$$2\mu=\frac{E}{1+\nu}$$

Note:

Concerning the kinematical share of hardening [éq 2-4], one notes that it is linear in this model. In addition, it is necessary to take care of the fact that in certain references, one calls constant of Prager $2\mathrm{C}/3$ and not C . In the same way, for the isotropic function of hardening, the elastic limit is included there by $R(0)=\sigma^{y}$, certain references treating it except for.

The evolution of the local variables ε^{p} et p is controlled by a normal flow model associated with a plasticity criterion F :

$$\mathrm{F}(\sigma,R,\mathbf{X})=(\tilde{\sigma}-\mathbf{X})_{eq}-R \;\text{ with }\; \mathbf{A}_{eq}=\sqrt{\frac{3}{2}\tilde{\mathbf{A}}\cdot\tilde{\mathbf{A}}} \quad \text{éq 2-5}$$

$$\dot{\varepsilon}^{p}=\dot{\lambda}\frac{\partial\mathrm{F}}{\partial\sigma}=\frac{3}{2}\dot{\lambda}\frac{\tilde{\sigma}-\mathbf{X}}{(\tilde{\sigma}-\mathbf{X})_{eq}} \quad \text{éq 2-6}$$

$$\dot{p}=\dot{\lambda}=\sqrt{\frac{2}{3}\dot{\varepsilon}^{p}\cdot\dot{\varepsilon}^{\mathrm{p}}} \quad \text{éq 2-7}$$

As for the plastic multiplier $\dot{\lambda}$, it is obtained by the condition of following coherence:

$$\begin{cases}\text{si}\,\mathrm{F}<0\;\text{ou}\;\dot{\mathrm{F}}<0 & \dot{\lambda}=0\\ \text{si}\,\mathrm{F}=0\;\text{et}\;\dot{\mathrm{F}}=0 & \dot{\lambda}>=0\end{cases} \quad \text{éq 2-8}$$

3 Integration of the behavior model

to carry out the integration of the constitutive law numerically, one carries out a discretization in time and one adopts a diagram of implicit, famous Eulerian adapted for elastoplastic behavior models.

Henceforth, the following notations will be employed: A^{-}, A et ΔA represent respectively the values of a quantity A at the beginning and at the end of time step considered thus that its increment during the step. The problem is then the following: knowing the state at time t^{-} as well as the increments of strain $\Delta\varepsilon$ and temperature ΔT, to determine the state at time t as well as the stresses σ.

Initially, one compared to the takes into account the variations of the characteristics temperature by noticing that:

$$\sigma^{H}=\frac{K}{K^{-}}\sigma^{H^{-}}+K\,\mathrm{tr}\left(\Delta\varepsilon-\Delta\varepsilon^{\mathbf{th}}\right) \quad \text{éq 3-1}$$

$$\tilde{\sigma}=\frac{\mu}{\mu^{-}}\tilde{\sigma}^{-}+2\mu\left(\Delta\tilde{\varepsilon}-\Delta\varepsilon^{\mathrm{p}}\right) \quad \text{éq 3-2}$$

$$\mathbf{X}=\frac{C}{C^{-}}\mathbf{X}^{-}+C\,\Delta\varepsilon^{\mathrm{p}} \quad \text{éq 3-3}$$

Within sight of the equation [éq 3-1], one notes that the hydrostatic behavior is purely elastic. Only the processing of the deviatoric component is delicate. To reduce the writings to come, one introduces $\tilde{\mathbf{s}}^{e}$ the difference $\tilde{\sigma}-\mathbf{X}$ in the absence of increment of plastic strains, so that:

$$\tilde{\sigma}-\mathbf{X}=\underbrace{\frac{\mu}{\mu^{-}}\tilde{\sigma}^{-}-\frac{C}{C^{-}}\mathbf{X}^{-}+2\mu\,\Delta\tilde{e}}_{\tilde{s}^{e}}-\left(2\mu+C\right)\Delta\varepsilon^{p} \quad \text{éq 3-4}$$

the flow equations [éq 2-6] and [éq 2-7] and the condition of coherence [éq 2-8] are written once discretized and by noticing that $p=\lambda$:

$$\Delta\varepsilon^{p}=\frac{3}{2}\Delta p\frac{\tilde{\sigma}\text{-}\mathbf{X}}{\left(\tilde{\sigma}\text{-}\mathbf{X}\right)_{eq}} \quad \text{éq 3-5}$$

$$\mathrm{F}<=0 \quad \Delta p>=0 \quad \mathrm{F}\,\Delta p=0 \quad \text{éq the 3-6}$$

processing of the condition of coherence [éq 3-6] is classical. One starts with test elastic ($\Delta p=0$) which is well the solution if the plasticity criterion is not exceeded, i.e. if:

$$\mathrm{F}=s_{eq}^{e}\text{-}\mathrm{R}\left(p^{-}\right)<=0 \quad \text{éq 3-7}$$

In the contrary case, the solution is plastic ($\Delta p>0$) and the condition of coherence [éq 3-6] is reduced to $\mathrm{F}=0$. To solve it, one starts by showing that one can bring back oneself to a scalar problem while eliminating $\Delta\varepsilon^{p}$. Indeed, by taking account of [éq 3-4] and [éq 3-5], one notes that $\Delta\varepsilon^{p}$ is collinear to $\tilde{\mathbf{s}}^{e}$ because:

$$\Delta\varepsilon^{\mathrm{p}}=\frac{3}{2}\frac{\Delta p}{\left(\tilde{\sigma}-\mathbf{X}\right)_{eq}}\left[\tilde{s}^{e}\text{-}\left(2\mu+C\right)\Delta\varepsilon^{p}\right] \quad \text{éq 3-8}$$

In addition, according to [éq 3-5], the norm of $\Delta\epsilon^{p}$ is worth:

$$\left(\Delta\varepsilon^{\mathrm{p}}\right)_{eq}=\frac{3}{2}\Delta p \quad \text{éq 3-9}$$

One from of thus deduced immediately the statement from $\Delta\varepsilon^{\mathrm{p}}$ according to Δp :

$$\Delta\varepsilon^{p}=\frac{3}{2}\Delta p\frac{\tilde{\mathbf{s}}^{e}}{s_{eq}^{e}} \quad \text{éq 3-10}$$

It now only remains to be replaced $\Delta\varepsilon^{\mathbf{p}}$ by its statement [éq 3-10] in the equation [éq 3-4] one obtains:

$$\tilde{\sigma}-\mathbf{X}=\tilde{s}^e\left[1-\frac{\frac{3}{2}(2\mu+C)\Delta p}{s^e_{eq}}\right]$$

while deferring $\tilde{\sigma}-\mathbf{X}$ in the equation $\mathrm{F}=0$, one brings back oneself to a scalar equation in Δp to solve, namely:

$$|s^e_{eq}-\frac{3}{2}(2\mu+C)\Delta p|-\mathrm{R}(p^-+\Delta p)=0 \quad \text{éq 3-11}$$

Insofar as the function R is positive, which one will admit henceforth, there exists a solution Δp with this equation, characterized by:

$$\frac{3}{2}(2\mu+C)\Delta p+\mathrm{R}(p^-+\Delta p)=s^e_{eq} \quad \text{where } 0<\Delta p<\frac{2}{3}\frac{s^e_{eq}}{2\mu+C} \quad \text{éq 3-12}$$

Let us note that in the interval specified in [éq 3-12], the solution is single. For details as for the solution of this equation, one will refer to [R5.03.02].

The typical case of the plane stresses is studied with [§6].

4 Computation of the tangent stiffness

In order to allow a resolution of the total problem (balance equations) by a method of Newton, it is necessary to determine the coherent tangent matrix of the incremental problem. For that, one once more adopts the convention of writing of the symmetric tensors of order 2 in the form of vectors with 6 components. Thus, for a tensor $\mathbf{a}$:

$$\mathbf{a}={}^t\left[a_{xx} \quad a_{yy} \quad a_{zz}\sqrt{2}a_{xy} \quad \sqrt{2}a_{xz} \quad \sqrt{2}a_{yz}\right] \quad \text{éq 4-1}$$

If one introduces moreover the hydrostatic vector $\mathbf{1}$ and the matrix of deviatoric projection $\mathbf{P}$:

$$\mathbf{1}={}^t[1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0] \quad \text{éq 4-2}$$

$$\mathbf{P}=\mathbf{Id}-\frac{1}{3}\mathbf{1}\otimes\mathbf{1} \quad \text{éq the 4-3}$$

Then coherent tangent stiffness matrix is written for an elastic behavior:

$$\frac{\partial s}{\partial\Delta\varepsilon}=K\,\mathbf{1}\otimes\mathbf{1}+2\mu\,\mathbf{P} \quad \text{éq 4-4}$$

and for a plastic behavior:

$$\frac{\partial s}{\partial\Delta\varepsilon}=K\,\mathbf{1}\otimes\mathbf{1}+2\mu\left(1-\frac{3\mu\Delta p}{s^e_{eq}}\right)\mathbf{P}+9\mu^2\left(\frac{\Delta p}{s^e_{eq}}-\frac{1}{\mathrm{R}'(p)+\frac{3}{2}(2\mu+C)}\right)\left(\frac{\tilde{\mathbf{s}}^e}{s^e_{eq}}\otimes\frac{\tilde{\mathbf{s}}^e}{s^e_{eq}}\right) \quad \text{éq the 4-5}$$

initial tangent matrix, used by option RIGI_MECA_TANG is obtained by adopting the behavior of the preceding step (elastic or plastic, meant by local variable x being worth 0 or 1) and while taking $\Delta p=0$ in the equation [éq 4-5].

Note:

*RIGI_MECA_TANG is the operator linearized compared to **time** (cf [R5.03.01], [R5.03.05]) and corresponds to what is called the problem of velocity; in this case, the linearization compared to Δu , in $\Delta u=0$, provides the same statement.*

One now proposes to show the statement [éq 4-5]. By differentiating them [éq 2-1] and [éq 2-2] with fixed temperature, one obtains immediately:

$$\delta\sigma=\left[K\,\mathbf{1}\otimes\mathbf{1}+2\mu\,\mathbf{P}\right]\delta\varepsilon-2\mu\,\delta\varepsilon^{p} \quad \text{éq 4-6}$$

If the mode of behavior is plastic, the incremental flow model [éq 3-10] provides then:

$$\delta\varepsilon^{p}=\frac{3}{2}\delta p\frac{\tilde{\mathbf{s}}^{\mathbf{e}}}{s_{eq}^{e}}+\frac{3}{2}\Delta p\,\delta\left(\frac{\tilde{\mathbf{s}}^{\mathbf{e}}}{s_{eq}^{e}}\right) \quad \text{éq 4-7}$$

As for dp, it is obtained by differentiating the implicit equation [éq 3-12]:

$$\left[\frac{3}{2}(2\mu+C)+\mathrm{R}'(p)\right]\delta p=\delta s_{eq}^{e} \quad \text{éq 4-8}$$

Lastly, it any more but does not remain to provide the variations of $\tilde{\mathbf{s}}^{\mathbf{e}}$:

$$\delta\tilde{\mathbf{s}}^{\mathbf{e}}=2\mu\,\delta\tilde{\varepsilon}\,ds_{eq}^{e}=3\mu\frac{\tilde{\mathbf{s}}^{\mathbf{e}}}{s_{eq}^{e}}\cdot\delta\tilde{\varepsilon}\,\delta\left(\frac{\tilde{\mathbf{s}}^{\mathbf{e}}}{s_{eq}^{e}}\right)=\frac{1}{s_{eq}^{e}}\left(2\mu-3\mu\frac{\tilde{\mathbf{s}}^{\mathbf{e}}}{s_{eq}^{e}}\otimes\frac{\tilde{\mathbf{s}}^{\mathbf{e}}}{s_{eq}^{e}}\right)\cdot\delta\tilde{\varepsilon} \quad \text{éq 4-9}$$

While replacing then [éq 4-7], [éq 4-8] and [éq 4-9] in [éq 4-6], one obtains well the statement [éq 4-5].

This statement is formally identical to that defined in R5.03.02: [éq 4-3] and is written:

$$\frac{\partial\sigma}{\partial\Delta\varepsilon}=K\,\mathbf{1}\otimes\mathbf{1}+2\mu\left(1-\frac{3\mu\,\xi\,\Delta p}{s_{eq}^{e}}\right)\left(\mathbf{Id}-\frac{1}{3}\mathbf{1}\otimes\mathbf{1}\right)+9\mu^{2}\xi\left(\frac{\Delta p}{s_{eq}^{e}}-\frac{1}{\mathrm{R}'+\frac{3}{2}(2\mu+C)}\right)\left(\frac{\tilde{\mathbf{s}}^{\mathbf{e}}}{s_{eq}^{e}}\otimes\frac{\tilde{\mathbf{s}}^{\mathbf{e}}}{s_{eq}^{e}}\right)$$

with $\xi=1$ if $\Delta\varepsilon$ led to a plasticization, and $\xi=0$ if not.

By means of [éq 3-12], one finds:

$$\frac{\partial s}{\partial\Delta\varepsilon}=\lambda^{*}\,\vec{\mathbf{1}}\otimes\vec{\mathbf{1}}+2\mu^{*}\,\mathbf{Id}-\xi\frac{9\mu^{2}}{H(p)}\left(1-\frac{\mathrm{R}'.\Delta p}{\mathrm{R}(p)}\right)\frac{1}{\mathrm{R}'+\frac{3}{2}(2\mu+C)}\left(\frac{\sigma^{\mathbf{dev}}}{R(p)}\otimes\frac{\sigma^{\mathbf{dev}}}{R(p)}\right)$$

with $\lambda^{*}=K-\frac{2\mu}{3}\frac{G(\Delta p)}{H(\Delta p)}\,2\mu^{*}=2\mu\frac{G(\Delta p)}{H(\Delta p)}$

for the option FULL_MECA : $\boldsymbol{\sigma}^{\mathbf{dev}}=\tilde{\boldsymbol{\sigma}}-\mathbf{X}$

for the option RIGI_MECA_TANG : $\boldsymbol{\sigma}^{\mathbf{dev}}=\tilde{\boldsymbol{\sigma}}^{-}-\mathbf{X}^{-}$

with $H(\Delta p)=1+\dfrac{\frac{3}{2}(2\mu+C)\,\xi\,\Delta p}{R(p)}$

and $G(\Delta p)=1+\frac{3}{2}C\,\xi\,\frac{\Delta p}{R(p)}$

5 Identification of the characteristics of the material

Let us consider a traction test uniaxial pressing, [Figure 5-a]. One proposes to show how it makes it possible to identify the constant of Prager and the isotropic function of hardening. In such a test, the various tensors are with fixed directions, i.e.:

$$\tilde{\sigma}=\sigma\,\Delta\mathbf{X}=\mathbf{X}\,\Delta\varepsilon^{p}=\frac{3}{2}\varepsilon^{p}\,\mathbf{D}\ \text{ with }\ \mathbf{D}=\begin{bmatrix}2/3 & & \\ & -1/3 & \\ & & -1/3\end{bmatrix} \qquad \text{éq. 5-1}$$

As long as the loading is monotonous, therefore in phase of tension, one obtains the following relations immediately:

$$p=\varepsilon^{p}\quad X=\frac{3}{2}C\,\varepsilon^{p}\quad s^{t}=\frac{3}{2}C\,\varepsilon^{p}+\mathrm{R}\left(\varepsilon^{p}\right) \qquad \text{éq. 5-2}$$

the constant of Prager is determined by the first plasticization in compression, since one a:

$$\begin{cases}\sigma^{t}_{A}=\frac{3}{2}C\,\varepsilon^{p}_{A}+R\left(\varepsilon^{p}_{A}\right)\\ \sigma^{c}_{A}=\frac{3}{2}C\,\varepsilon^{p}_{A}\text{-}\,R\left(\varepsilon^{p}_{A}\right)\end{cases}\Rightarrow C=\frac{\sigma^{t}_{A}+\sigma^{c}_{A}}{3\,\varepsilon^{p}_{A}} \qquad \text{éq 5-3}$$

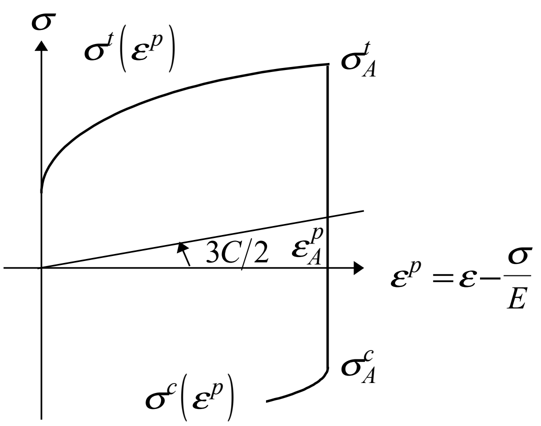

Figure 5-a: Traction test uniaxial pressing

the curve of hardening $\sigma^{t}=F\left(\varepsilon^{p}\right)$ is deduced from the curve of tension $\sigma^{t}=F\left(\varepsilon\right)$ provided by the user under key keys `ECRO_LINE` ((`SY` and `D_SIGM_EPSI` (linear hardening)) or `TENSION` (unspecified hardening). It finally makes it possible to obtain the isotropic function of hardening by [éq 5-2]:

$$\mathrm{R}\left(\varepsilon^{p}\right)=s^{t}\left(\varepsilon^{p}\right)-\frac{3}{2}C\,\varepsilon^{p}\ .$$

For the effective computation of R (p), according to the R5.03.02 document, one titrates party of the linearity (ECMI_LINE) or the linearity per pieces of curve of tension (ECMI_TRAC):

ECMI_LINE :

$$\sigma^t=F\left(\varepsilon^p\right)=\sigma_y+\frac{E.E_T}{E-E_T}p$$
$$R(p)=\sigma_y+\left(\frac{E.E_T}{E-E_T}-\frac{3}{2}C\right)p=\sigma_y+R'.p$$ éq the 5-4

equation [éq 3-12] becomes then:

$$\frac{3}{2}(2\mu+C)\Delta p+\sigma_y+R'.(p+\Delta p)=s^e_{eq}$$ éq 5-5

ECMI_TRAC:

$$\sigma^t=F\left(\varepsilon^p\right)=\sigma_i+\frac{\sigma_{i+1}-s_i}{p_{i-1}-p_i}(p-p_i),\text{ pour } p_i<=p<=p_{i+1}$$
$$R(p)=\sigma_i+\frac{\sigma_{i+1}-\sigma_i}{p_{i-1}-p_i}(p-p_i)-\frac{3}{2}Cp=\sigma_i-\frac{\sigma_{i+1}-\sigma_i}{p_{i-1}-p_i}p_i+R'.p$$ éq 5-6

Note::

For the use: the correspondence enters the model of behavior VMIS_CINE_LINE and behavior VMIS_ECMI_LINE is the following one:
With VMIS_CINE_LINE , it is necessary to introduce into DEFI_MATERIAU a linear hardening of slope And by:
D_SIGM_EPSI : And
For VMIS_ECMI_LINE , to reproduce same behavior with linear kinematic hardening, it is necessary to give in DEFI_MATERIAU .

- *a linear hardening of slope E_T : D_SIGM_EPSI : And*
- *the constant of Prager C : PRAGER : C*

C is determined by: $C=\frac{2}{3}\frac{EE_T}{E-E_T}$

It should well be noticed that the identification of C and of $R\left(\varepsilon^p\right)$ have meaning only in one restricted field of strains (small strains). In particular, so $\sigma^t\left(\varepsilon^p\right)$ present an asymptote $\sigma^t_{\max}$ for ε^p sufficiently large, then the kinematical contribution of hardening does not have any more meaning. It is thus advised to restrict itself with the field where hardening is strictly positive.

6 Typical case of the plane stresses: computation of Δ p

It is necessary to add to the equations [éq 3-1] with [éq 3-4] the constraint plane $s_{33}=0$, which adds an unknown (corresponding strain):

$$\sigma^H=\frac{K}{K^-}s^{H^-}+K\,\mathrm{tr}\left(\Delta\varepsilon-\Delta\varepsilon^{\mathbf{th}}\right)$$ éq 6-1

$$\tilde{\sigma}=\frac{\mu}{\mu^-}\tilde{\sigma}^-+2\mu\left(\Delta\tilde{\varepsilon}-\Delta\varepsilon^p\right)$$ éq 6-2

$$\mathbf{X}=\frac{C}{C^-}\mathbf{X}^- + C\Delta\varepsilon^p \quad \text{éq 6-3}$$

$$\sigma_{33}=0 \quad \text{éq 6-4}$$

Then, the equation [éq 3-4] becomes:

$$\tilde{\sigma}-X=\frac{m}{m^-}\tilde{\sigma}^- - \frac{C}{C^-}X^- + 2\mathrm{mD}\tilde{\varepsilon}^c - (2\mu+C)\Delta\varepsilon^p + 2\mu\Delta\tilde{e}^y = \tilde{\sigma}^e - (2\mu+C)\Delta\varepsilon^p + 2\mu\Delta\tilde{\varepsilon}^y \quad \text{éq 6-5}$$

where $\Delta\tilde{e}^c$ is entirely determined by the elastic behavior:

$$\Delta\tilde{\varepsilon}^c_{33}=\frac{-\nu}{1-\nu}\left(\Delta\tilde{\varepsilon}^c_{11}+\Delta\tilde{\varepsilon}^c_{22}\right), \Delta\tilde{\varepsilon}^c_{11}=\Delta\tilde{\varepsilon}_{11}, \Delta\tilde{\varepsilon}^c_{22}=\Delta\tilde{\varepsilon}_{22}$$

and $\Delta\tilde{\varepsilon}^y=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \Delta Y\end{bmatrix}$ is unknown. It is also supposed that $\sigma_{13}=\sigma_{23}=\varepsilon_{13}=\varepsilon_{23}=0$.

One always has:

$$\Delta\varepsilon^{\mathrm{p}}=\frac{3}{2}\Delta p\frac{\tilde{\sigma}-\mathbf{X}}{\left(\tilde{\sigma}-\mathbf{X}\right)_{eq}} \quad \text{éq 6-6}$$

$$\mathrm{F}=\sigma_{eq}-R(p) <= 0 \; \Delta p³0 \quad F\Delta p=0 \quad \text{éq 6-7}$$

elastic Test:

- If

$$\mathrm{F}=s^e_{eq}-R\left(p^-\right)<=0 \quad \text{éq 6-8}$$

then

$$\tilde{\sigma}=\tilde{\mathbf{s}}_e \quad \Delta p=0\,,\ \Delta Y=0 \quad \text{éq 6-9}$$

$$\sigma^H=\frac{K}{K^-}\sigma^{H^-}+K\,\mathrm{tr}\left(\Delta\varepsilon^c-\Delta\varepsilon^{\mathbf{th}}\right) \quad \text{éq 6-10}$$

- If not, the solution is plastic: $\Delta p>0 \quad \Delta Y\neq 0$. One can still bring back oneself to a scalar problem in Δp .

By taking account of [éq 6-5] and [éq 6-6], one notes that $\tilde{\sigma}-X$ is collinear to $\tilde{\mathbf{s}}^{\mathrm{e}}+2\mu\Delta\tilde{\varepsilon}^y$ because:

$$\left(\tilde{\sigma}-\mathbf{X}\right)\left(1+\frac{\frac{3}{2}(2\mu+C)\Delta p}{R(p)}\right)=\left(\tilde{\sigma}-\mathbf{X}\right)H\left(\Delta p\right)=\left[\tilde{\mathbf{s}}^e+2\mu\Delta\tilde{\varepsilon}^y\right] \quad \text{éq 6-11}$$

Thus:

$$\left(\tilde{\sigma}_{33}-X_{33}\right)H\left(\Delta p\right)=\left[\tilde{s}^e_{33}+\frac{4}{3}\mu\Delta Y\right] \quad \text{éq 6-12}$$

We will express the equation [éq 6-12] according to Δp only. According to [éq 6-4]:

$$\sigma_{33}=0=\tilde{\sigma}_{33}+\sigma^{H}=\tilde{\sigma}_{33}+\sigma^H_e+K.\Delta Y \text{, with } \sigma^H_e=\frac{K}{K^-}\sigma^{H^-}+K\,\mathrm{tr}\left(\Delta\varepsilon^c-\Delta\varepsilon^{\mathbf{th}}\right) \quad \text{éq 6-13}$$

By means of [éq 6-5], [éq 6-6] and the incompressibility of plastic strains, one can show that:

$$\tilde{s}^e_{33}+\sigma^H_e=-\frac{C}{C^-}X^-_{33}$$ éq 6-14

Then:

$$\tilde{\sigma}_{33}=\tilde{s}^e_{33}-K.\Delta Y+\frac{C}{C^-}X^-_{33}$$ éq 6-15

As according to [éq 6-3]:

$$X_{33}=\frac{C}{C^-}X^-_{33}+C.\Delta\varepsilon^p_{33}=\frac{C}{C^-}X^-_{33}+C.\frac{3}{2}\Delta p\frac{\tilde{\sigma}_{33}-X_{33}}{R(p)}$$ éq 6-16

$$X_{33}.G(\Delta p)=\frac{C}{C^-}X^-_{33}+\frac{3}{2}C\Delta p\frac{\tilde{\sigma}_{33}}{R(p)}\text{ , with }G(\Delta p)=1+\frac{3}{2}C\frac{\Delta p}{R(p)}$$ éq 6-17

From [éq 6-12], [éq 6-15], [éq 6-17], one obtains an equation flexible Δp and ΔY :

$$\Delta Y.\left(\frac{4}{3}\mu+K\frac{H(\Delta p)}{G(\Delta p)}\right)=\left[\tilde{\mathbf{s}}^{\mathbf{e}}_{33}\left(\frac{H(\Delta p)}{G(\Delta p)}-1\right)\right]$$ éq the 6-18

equation [éq 6-11] makes it possible to obtain the scalar equation in Δp to solve, namely:

$$(\tilde{\sigma}-\mathbf{X})_{eq}H(\Delta p)=R(p^-+\Delta p)H(\Delta p)=\left[\tilde{\mathbf{s}}^{\mathbf{e}}+2\mu\Delta\tilde{\varepsilon}^y\right]_{eq}$$ éq 6-19

Equation in which ΔY is function from Δp the equation [éq 6-18].

As in the case of isotropic hardening, one obtains a scalar equation in Δp , always nonlinear, which is solved by a method of secant.

Once Δp known, the computation of $\tilde{\sigma}$ and X is carried out from the statement of ΔY, therefore of $\Delta\varepsilon$ entirely known, by a approach similar to that of the equation [éq 3-10].

$$\Delta\varepsilon^{\mathrm{p}}=\frac{3}{2}\Delta p\frac{\tilde{\mathbf{s}}^{\mathbf{e}}+2\mu\Delta\tilde{\varepsilon}^y}{(\tilde{\mathbf{s}}^{\mathbf{e}}+2\mu\Delta\tilde{\varepsilon}^y)_{eq}}=\frac{3}{2}\Delta p\frac{\tilde{\sigma}-\mathbf{X}}{H(\Delta p)(\tilde{\mathbf{s}}^{\mathbf{e}}+2\mu\Delta\tilde{\varepsilon}^y)_{eq}}$$ éq 6-20

$$\tilde{\sigma}=\frac{\mu}{\mu^-}\tilde{\sigma}^-+2\mu(\Delta\tilde{\varepsilon}-\Delta\varepsilon^p)$$ éq 6-21

One obtains while eliminating $\Delta\varepsilon^p$ from [éq 6-6], [éq 6-3] and [éq 6-2]:

$$\tilde{\sigma}=\left(\frac{\mu}{\mu^-}\tilde{\sigma}^-+2\mu\Delta\tilde{\varepsilon}\right)\frac{G(\Delta p)}{H(\Delta p)}+\frac{3}{2}2\mu\frac{\Delta p}{R(p)H(\Delta p)}\frac{C}{C^-}\mathbf{X}^-$$ éq 6-22

$$\mathbf{X}=\frac{3}{2}C\frac{\Delta p}{R(p)H(\Delta p)}\left(\frac{\mu}{\mu^-}\tilde{\sigma}^-+2\mu\Delta\tilde{e}\right)+\left(1-\frac{3}{2}C\frac{\Delta p}{R(p)H(\Delta p)}\right)\frac{C}{C^-}\mathbf{X}^-$$ éq 6-23

7 Meaning of the local variables

the local variables of the model to Gauss points (`VARI_ELGA`) are for all the modelizations:

- `VARI_1` = p : cumulated plastic strain (positive or null)
- `VARI_2` = ξ : being worth 1 if the Gauss point plasticized during the increment or 0 if not.

$\mathbf{X}$: tensor of recall:

For the modelization 3D :

- VARI_3 = X_{11}
- VARI_4 = X_{22}
- VARI_5 = X_{33}
- VARI_6 = X_{12}
- VARI_7 = X_{13}
- VARI_8 = X_{23}

For modelizations D_PLAN, C_PLAN, AXIS

- VARI_3 = X_{11}
- VARI_4 = X_{22}
- VARI_5 = X_{33}
- VARI_6 = X_{12}

For the modelizations of shells (DKT, COQUE_3D), in local coordinate system and in each point of integration of each layer:

- VARI_3 = X_{11}
- VARI_4 = X_{22}
- VARI_5 = X_{33}
- VARI_6 = X_{12}

8 Bibliography

1) J. LEMAITRE, J.L. CHABOCHE: "Mechanical of the solid materials". Dunod 1992

Description of the versions of the document

Version Aster	Author (S) Organization (S)	Description of modifications
5.	J.M Proix, E.Lorentz EDF-R&D/AMA	initial Version
8.5	J.M.Proix, EDF-R&D/ AMA Change	of notation of the modulus of compressibility, cf drives REX 10218