Constitutive law very-elastic: material almost incompressible

Abstract:

One describes here the formulation adopted for a constitutive law very-elastic of Signorini. This model is a generalized version of the laws of Mooney-Rivlin often adopted for elastomers. The parameters characterizing the material are defined in DEFI_MATERIAU with the key word ELAS_HYPER.

This model is selected in command STAT_NON_LINE or DYNA_NON_LINE via the key word RELATION = ' ELAS_HYPER' under key words COMP_ELAS or COMP_INCR. This relation extends to great transformations; this functionality is selected via the key word DEFORMATION = "GROT_GDEP". It is available for the elements 3D, 3D_SI, C_PLAN and D_PLAN.

1 Kinematical potential of

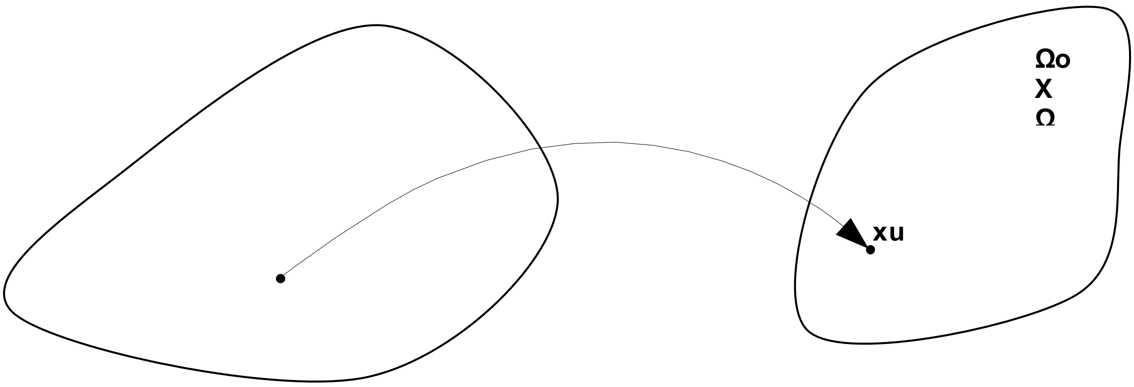
1.1 strain

One considers a solid Ω subjected to large deformations. That is to say $\mathbf{F}$ the tensor of the gradient of the transformation making pass the configuration initialeà Ω_0 deformed present configuration Ω_t . One notes $\mathbf{X}$ the position of a point in Ω_0 and $\mathbf{x}$ the position of this same point after strain in Ω_t . $\mathbf{u}$ is then displacement between the two configurations. One thus has:

$$\mathbf{x}=\mathbf{X}+\mathbf{u} \quad (1)$$

the tensor of the gradient of the transformation is written:

$$\mathbf{F}=\frac{\partial \mathbf{x}}{\partial \mathbf{X}}=\mathbf{I}+\nabla_{\mathbf{x}}\mathbf{u} \quad (2$$



1.1-1: Transformation of the initial configuration with finale

This tensor is not the best candidate to describe the structural deformation. In particular, it is not worth zero for rigid body motions and it describes all the transformation: change length of the infinitesimal elements but also their directional sense. However a rotation movement pure does not generate stresses and it is thus preferable to use a measurement of the strains which does not take into account this rigid rotation. Let us consider an element infinitesimal length noted $\mathrm{d}\mathbf{X}$ in the initial configuration and $\mathrm{d}\mathbf{x}$ the final configuration. If motion is a rigid rotation $\mathbf{R}$, one a:

$$\mathrm{d}\mathbf{x}=\mathbf{R}.\mathrm{d}\mathbf{X} \quad (3)$$

the norm of this vector after transformation is thus worth

$$d\,\mathbf{x}.d\,\mathbf{x}=\mathbf{R}.d\,\mathbf{X}.\mathbf{R}.d\,\mathbf{X}=\mathbf{R}^T.\mathbf{R}.d\,\mathbf{X}.d\,\mathbf{X} \quad (4)$$

As the transformation is purely rigid, one a:

$$\mathbf{R}^T.\mathbf{R}=\mathbf{I} \quad (5)$$

$\mathbf{R}$ is thus an orthogonal tensor. The tensor deformation gradient can be written like the product of an orthogonal tensor and a positive definite tensor (polar factorization):

$$\mathbf{F}=\mathbf{R}.\mathbf{U} \quad (6)$$

the tensor $\mathbf{U}$ (called tensor of lengthening) is thus the first measurement of large deformations. On the other hand, it requires the polar factorization of $\mathbf{F}$, which is expensive operation. One thus prefers to use the tensor of the strains of right Cauchy-Green:

$$\mathbf{C}=\mathbf{F}^{\mathrm{T}}.\mathbf{F}=\mathbf{U}^2 \quad (7)$$

This tensor is symmetric. The three invariants of the tensor of Cauchy-Green $\mathbf{C}$ are given par1On[1] :

1 note $\mathrm{tr}(\mathbf{C})=C_{11}+C_{22}+C_{33}$ the trace of the tensor $\mathbf{C}$.

$$I_c = \mathrm{tr}(\mathbf{C}) \quad (8)$$

$$II_c = \frac{1}{2}.\left((\mathrm{tr}(\mathbf{C}))^2 - \mathrm{tr}(\mathbf{C}^2)\right) \quad (9)$$

$$III_c = \det(\mathbf{C}) \quad (10)$$

the last invariant III_c describes the change of volume, one can also write it:

$$III_c = (\det \mathbf{F})^2 = J^2 \quad (11)$$

1.2 Potential of strain – compressible case

a model very-elastic supposes the existence of a potential density of internal energy Ψ , function scalar of the measurement of the strains. If an isotropic behavior is considered, the function Ψ owes the being too. It is shown that if the function Ψ depends only on the invariants of the tensor of the strains of right Cauchy-Green $\mathbf{C}$, then it describes an isotropic behavior. The potential is thus a function of these three invariants:

$$\Psi = \Psi(I_c, II_c, III_c) \quad (12)$$

the second tensor of the Piola-Kirchhoff stresses is obtained by derivative of this potential compared to the strains (see [R5.03.20]):

$$\mathbf{S} = 2.\frac{\partial \Psi}{\partial \mathbf{C}} \quad (13)$$

the most general form of a potential is polynomial. She is written according to the invariants of the tensor of right Cauchy-Green and materials parameters C_{pqr} according to Rivlin:

$$\Psi = \sum_{(p,q,r)=0}^{\infty} C_{pqr}.(I_c-3)^p.(II_c-3)^q.(III_c-1)^r \quad (14)$$

with $C_{000}=0$.
There exist particular forms of this potential which are used very frequently, with $r=0$:

	p		Q
	1	2	1
	C10	C20	C01
Signorini	YES	YES	YES
Mooney-Rivlin	YES	NON	YES
Néo-Hookéen	YES	NON	NON

1.3 Potential of strain – incompressible case

1.3.1 Principle

Most materials very-elastics (as elastomers) is incompressible, i.e.:

$$\det \mathbf{F} = 1 \quad (15)$$

And thus the third invariant III_c is thus worth:

$$III_c = 1 \quad (16)$$

the potential hyper elastic in incompressible mode is thus rewritten:

$$\Psi=\sum_{(p,q)=0}^{\infty} C_{pq}.\left(I_c-3\right)^p.\left(II_c-3\right)^q \quad (17)$$

Unfortunately, such a writing leads to severe numerical problems (except for the case of the plane stresses). We will thus propose a new writing which makes it possible to solve in a more effective way the case of the incompressibility and which has in more the good taste to be also valid in mode compressible, with a wise choice (and easy) of the coefficients.

1.3.2 Tensor of Cauchy-Green modified right

One starts by defining a new tensor of right Cauchy-Green $\mathbf{C}^*$ (pure or isochoric deviatoric thermal expansions) such as:

$$\mathbf{C}^*=J^{-\frac{2}{3}}.\mathbf{C} \quad (18)$$

Indeed, $J^{\frac{1}{3}}.\mathbf{I}$ indicates the pure voluminal strain. This tensor remains symmetric. Its invariants are:

$$I_c^*=J^{-\frac{2}{3}}.I_c \quad (19)$$

$$II_c^*=J^{-\frac{4}{3}}.II_c \quad (20)$$

$$III_c^*=1 \quad (21)$$

Knowing that:

$$J=\left(C_{11}.C_{22}.C_{33}-C_{12}^2.C_{33}-C_{23}^2.C_{11}+2.C_{12}.C_{13}.C_{23}-C_{13}^2.C_{22}\right)^{\frac{1}{2}} \quad (22)$$

With:

$$I_c^*=\frac{C_{11}+C_{22}+C_{33}}{J^{2/3}} \quad (23)$$

And:

$$II_c^*=\frac{C_{22}.C_{33}+C_{11}.C_{33}+C_{11}.C_{22}}{J^{4/3}} \quad (24)$$

These invariants are also called reduced **invariants** of $\mathbf{C}$.

1.3.3 Potential of strain modified

If one expresses the potential of strain using the reduced invariants of $\mathbf{C}$, one can break up the potential into two parts:

$$\Psi=\Psi^{iso}+\Psi^{vol} \quad (25)$$

There is the partiequi Ψ^{iso} corresponds to the isochoric strains ($J=1$):

$$\Psi^{\text{iso}}=\sum_{p,q=0}^{\infty} C_{pq}.\left(I_c^*-3\right)^p.\left(II_c^*-3\right)^q \quad (26)$$

And the part Ψ^{vol} which corresponds to the voluminal strains ($J\neq 1$):

$$\Psi^{\text{vol}}=\frac{K}{2}.\left(J-1\right)^2 \quad (27)$$

K is the coefficient of compressibility. This formulation makes it possible to keep account of the effects incompressible and compressible:

1. In the incompressible case, and the frame of a formulation by finite elements the parameter K plays the part of a coefficient of penalization of the condition of incompressibility;

2. In the compressible case this same coefficient translates a material property: hydrostatic compressibility.

So the model characterizing the material is of Mooney-Rivlin type ($p=q=1$), K can be given by:

$$K=\frac{4(C_{01}+C_{10})(1+\nu)}{3(1-2\nu)} \tag{28}$$

In the case of the small strains, $E=4(C_{01}+C_{10})(1+\nu)$ represents the Young's modulus while $G=2(C_{01}+C_{10})$ the shear modulus.

1.3.4 Tensor of Piola-Kirchhoff stresses 2

the stress tensor of Piola-Kirchhoff 2, representing the stresses measured in the initial configuration, is written:

$$\mathbf{S}=2.\frac{\partial\Psi}{\partial\mathbf{C}} \tag{29}$$

factor 2 makes it possible to find the usual statement in small strains. One can separate it in two parts:

$$\mathbf{S}=\mathbf{S}^{\text{iso}}+\mathbf{S}^{\text{vol}} \tag{30}$$

With:

$$\mathbf{S}^{\text{iso}}=2.\frac{\partial\Psi^{\text{iso}}}{\partial\mathbf{C}} \tag{31}$$

And:

$$\mathbf{S}^{\text{vol}}=2.\frac{\partial\Psi^{\text{vol}}}{\partial\mathbf{C}} \tag{32}$$

$\mathbf{S}^{\text{iso}}$ is the tensor of the isochoric stresses and $\mathbf{S}^{\text{vol}}$ is that of the voluminal or hydrostatic stresses.

1.3.5 Lagrangian elasticity tensor

the elastic tensor of stiffness (“tangent” matrix for the problem of Newton) is given by double derivative of the potential:

$$\mathbf{K}=4.\frac{\partial^2\Psi}{\partial\mathbf{C}.\partial\mathbf{C}}=4.\frac{\partial^2\Psi^{\text{iso}}}{\partial\mathbf{C}.\partial\mathbf{C}}+4.\frac{\partial^2\Psi^{\text{vol}}}{\partial\mathbf{C}.\partial\mathbf{C}} \tag{33}$$

2 Analytical statements

2.1 Case of the stresses

We will detail the analytical statement of the Piola-Kirchhoff stresses for the potential very-elastic of Signorini ($p=2$ and $q=1$) in the incompressible case. There is thus the stress tensor of Piola-Kirchhoff 2, representing the stresses measured in the initial configuration which is written:

$$\mathbf{S}=2.\frac{\partial\Psi^{\text{iso}}}{\partial\mathbf{C}}+2.\frac{\partial\Psi^{\text{vol}}}{\partial\mathbf{C}} \quad (34)$$

With the two potentials:

$$\begin{gathered}\Psi^{\text{iso}}=C_{10}.\left(I_c^*-3\right)+C_{01}.\left(II_c^*-3\right)+C_{20}.\left(I_c^*-3\right)^2\\ \Psi^{\text{vol}}=\frac{K}{2}.\left(J-1\right)^2\end{gathered} \quad (35)$$

to obtain the stresses, the potential should be derived:

$$\begin{gathered}\mathbf{S}_{ij}^{iso}=2\frac{\partial\Psi^{iso}}{\partial I_c^*}.\frac{\partial I_c^*}{\partial\mathbf{C}_{ij}}+2\frac{\partial\Psi^{iso}}{\partial II_c^*}.\frac{\partial II_c^*}{\partial\mathbf{C}_{ij}}\\ \mathbf{S}_{ij}^{vol}=2\frac{\partial\Psi^{vol}}{\partial J}.\frac{\partial J}{\partial\mathbf{C}_{ij}}\end{gathered} \quad (36)$$

With:

$$\frac{\partial\Psi^{iso}}{\partial I_c^*}=C_{10}+2.\mathrm{C}_{20}.\left(I_c^*-3\right) \qquad \frac{\partial\Psi^{iso}}{\partial II_c^*}=C_{01} \qquad \frac{\partial\Psi^{vol}}{\partial J}=K.\left(J-1\right) \quad (37)$$

As well as derivatives of the reduced invariants (*cf* page 26 of [5] for derivatives of the invariants of a tensor):

$$\frac{\partial I_c^*}{\partial\mathbf{C}_{ij}}=III_c^{-\frac{1}{3}}.\left(\delta_{ij}-\frac{1}{3}.\mathbf{C}_{ij}^{-1}.I_c\right) \quad (38)$$

$$\frac{\partial II_c^*}{\partial\mathbf{C}_{ij}}=III_c^{-\frac{2}{3}}.\left(I_c.\ \delta_{ij}-\mathbf{C}_{ij}-\frac{2}{3}.\mathbf{C}_{ij}^{-1}.II_c\right) \quad (39)$$

$$\frac{\partial J}{\partial\mathbf{C}_{ij}}=\frac{1}{2}.III_c^{\frac{1}{2}}.\mathbf{C}_{ij}^{-1} \quad (40)$$

Here thus the analytical statement of the voluminal stresses:

$$\mathbf{S}_{ij}^{vol}=K.\left(J-1\right).J.\mathbf{C}_{ij}^{-1} \quad (41)$$

And of the isochoric stresses:

$$\mathbf{S}_{ij}^{iso}=2\left(\left(C_{10}+2.\mathrm{C}_{20}.\left(I_c^*-3\right)\right).J^{-\frac{2}{3}}.\left(\delta_{ij}-\frac{1}{3}.\mathbf{C}_{ij}^{-1}.I_c\right)+C_{01}.J^{-\frac{4}{3}}.\left(I_c.\ \delta_{ij}-\mathbf{C}_{ij}-\frac{2}{3}.\mathbf{C}_{ij}^{-1}.II_c\right)\right) \quad (42)$$

2.2 Case of the elastic matrix

We will detail the analytical statement of the elastic matrix for the potential very-elastic of Signorini ($p=2$ and $q=1$) in the incompressible case. One thus has:

$$\mathbf{K}=4.\frac{\partial^2\Psi^{\mathrm{iso}}}{\partial\mathbf{C}.\partial\mathbf{C}}+4.\frac{\partial^2\Psi^{\mathrm{vol}}}{\partial\mathbf{C}.\partial\mathbf{C}} \tag{43}$$

It is thus necessary to derive (twice) the potential:

$$\mathbf{K}_{ijkl}^{iso}=\frac{\partial^2\Psi^{iso}}{\partial^2 I_c^*}.\frac{\partial^2 I_c^*}{\partial\mathbf{C}_{ij}.\partial\mathbf{C}_{kl}}+\frac{\partial^2\Psi^{iso}}{\partial^2 II_c^*}.\frac{\partial^2 II_c^*}{\partial\mathbf{C}_{ij}.\partial\mathbf{C}_{kl}}$$
$$\mathbf{K}_{ijkl}^{vol}=\frac{\partial^2\Psi^{vol}}{\partial^2 J}.\frac{\partial^2 J}{\partial\mathbf{C}_{ij}.\partial\mathbf{C}_{kl}} \tag{44}$$

constant the materials are supposed to be constant. One thus has:

$$\frac{\partial^2\Psi^{iso}}{\partial^2 I_c^*}=2.\mathrm{C}_{20}\qquad\frac{\partial^2\Psi^{iso}}{\partial^2 II_c^*}=0\qquad\frac{\partial^2\Psi^{vol}}{\partial^2 J}=K \tag{45}$$

It is seen that the coefficient K is well a coefficient of penalization and that its choice impacts the conditioning of the matrix. Derivatives of the reduced invariants:

$$\frac{\partial^2 I_c^*}{\partial\mathbf{C}_{ij}.\partial\mathbf{C}_{kl}}=III_c^{-\frac{1}{3}}.\left(\mathbf{C}_{ki}^{-1}.\mathbf{C}_{lj}^{-1}.I_c-\mathbf{C}_{ij}^{-1}.\delta_{kl}-\mathbf{C}_{kl}^{-1}.\delta_{ij}+\frac{1}{3}.\mathbf{C}_{kl}^{-1}.\mathbf{C}_{ij}^{-1}.I_c\right) \tag{46}$$

$$\frac{\partial^2 II_c^*}{\partial\mathbf{C}_{ij}.\partial\mathbf{C}_{kl}}=\begin{array}{c}-\frac{2}{3}.III_c^{-\frac{2}{3}}.\mathbf{C}_{kl}^{-1}.\left(I_c.\delta_{ij}-\mathbf{C}_{ij}-\frac{2}{3}.\mathbf{C}_{ij}^{-1}.II_c\right)+\\ III_c^{-\frac{2}{3}}.\left(\delta_{kl}.\delta_{ij}-\delta_{ik}.\delta_{jl}+\frac{2}{3}.\mathbf{C}_{ki}^{-1}.\mathbf{C}_{lj}^{-1}.II_c-\frac{2}{3}.\mathbf{C}_{ij}^{-1}.\left(I_c.\delta_{kl}-\mathbf{C}_{kl}\right)\right)\end{array} \tag{47}$$

$$\frac{\partial^2 J}{\partial\mathbf{C}_{ij}.\partial\mathbf{C}_{kl}}=\frac{1}{4}.III_c^{\frac{1}{4}}.\left(\mathbf{C}_{kl}^{-1}.\mathbf{C}_{ij}^{-1}-2.\mathbf{C}_{ki}^{-1}.\mathbf{C}_{lj}^{-1}\right) \tag{48}$$

3 bibliographical References

[1] G.A. Holzapfel – "Nonlinear solid mechanics" – Wiley – 2001.

[2] J. Bonnet – " Nonlinear continuum mechanics for finite element analysis " – Cambridge University Close – 1997.

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[4] A. Delalleau – "Analysis of the structural mechanics behavior of the skin *in vivo* " – Doctorate University Jean Monnet of Saint Étienne – 2007.

[5] C. Truesdell, W. Nool - The Non-Linear Field Theories of Mechanics, vol. 3, Springer, 2004.

4 Description of the versions of the document

Version Aster	Author (S) Organization (S)	Description of modifications
8.4	Mr. Abbas, T. Baranger EDF-R&D/AMA, UCBL	text inital
8.5	M.Abbas, EDF-R&D/AMA	Correction page 2, cf drives REX 11026
10.1	J.M.Proix EDF-R&D/ AMA Replacement	of GREEN by GROT_DEP 10.2 Mr.
Abbas	, EDF-R&D/AMA complete	Rewriting, analytical statements of the stresses and the tangent matrix