Nonlinear elastic behavior model in large displacements

Summarized:

One proposes to describe here a nonlinear elastic behavior model which coincides with the elastoplastic model of Hencky-Von Put (isotropic hardening) in the case of a loading which induces a radial and monotonous evolution stresses in any point of structure. This model is selected in command STAT_NON_LINE via key word RELATION=' ELAS_VMIS_LINE' or "ELAS_VMIS_TRAC" under factor key word the COMP_ELAS.

One extends then this behavior model to large displacements and large rotations, insofar as it derives from a potential (model hyper elastic); this functionality is selected via key word DEFORMATION=' GROT_GDEP'. It is available for all the isoparametric elements 2D and 3D.

nonlinear

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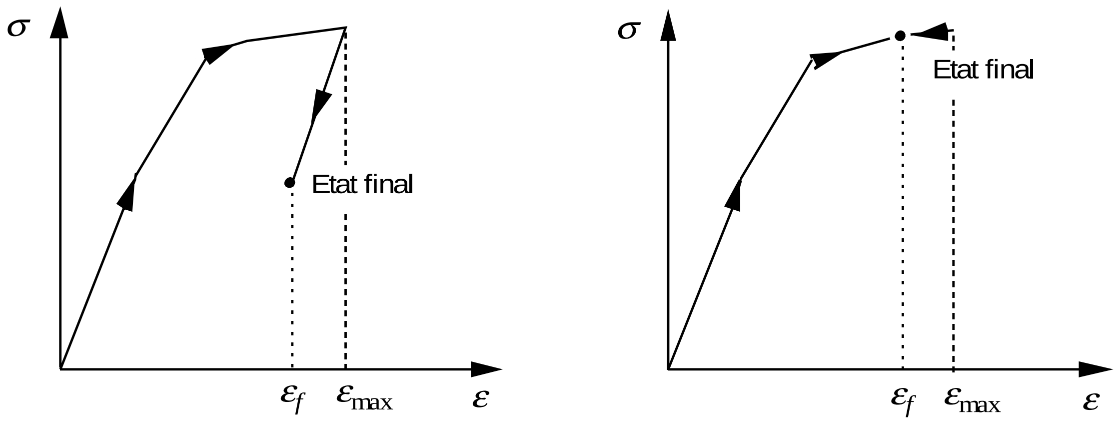
1.1 Purpose

In the frame of the global solution in fracture mechanics, one can give a meaning to rate of energy restitution only for behavior models hyper elastics, i.e. which derive from a potential, free energy. In order to be able nevertheless to deal with plastic problems élasto -, one proposes a nonlinear elastic behavior model which in the case of leads to results identical to those obtained by the plastic behavior model of Hencky-Von Put (isotropic hardening) an evolution of loading radial and monotonous in any point. The definition of the characteristics of the material (key word `DEFI_MATERIAU`) is identical to that of the isotropic plastic behavior. For further information on the model, one will be able to refer to [bib1]. To illustrate the common points and the differences between the models plastic and elastic, one presents Ci - below a curve of tension then compression obtained for a unidimensional bar.



Elasto-plasticité Elasticité non linéaire

1.2 Behavior model

After integration in time of the behavior model of Hencky-Von Put, formulated out of strainrates and of stresses in [R5.03.02] which one adopts the notations, the statement of the stresses according to the strains is:

$$\begin{cases} \boldsymbol{\sigma} = K(\mathrm{tr}\,\varepsilon)\,\mathbf{Id} + G(\varepsilon_{eq})\,\tilde{\varepsilon} & \text{éq } 1.2-1 \\ -\text{si} \quad \varepsilon_{eq} \le \dfrac{\sigma^y}{2\mu} & \\ \quad G = 2\mu \quad \text{et} \quad p = 0 & \\ -\text{si} \quad \varepsilon_{eq} > \dfrac{\sigma^y}{2\mu} & \\ \quad G = \dfrac{R(p)}{\varepsilon_{eq}} \quad \text{et} \quad p \text{ tel que : } \quad p + \dfrac{R(p)}{3\mu} = \dfrac{2\varepsilon_{eq}}{3} & \text{éq } 1.2-2 \end{cases}$$

In a way similar to plasticity, the function of hardening $R(p)$ is deduced from the data provided by a simple traction test (linear hardening with the key word `ELAS_VMIS_LINE` or defined well by points with the key word `ELAS_VMIS_TRAC`, cf [R5.03.02]).

As for the variable p , it deserves a few times of attention. In the model plastic, its meaning is clear. It is the cumulated plastic strain, always increasing; it is a local variable of the model. On the other hand, in the elastic case, it does not have any more the statute of local variable, since there is no dissipation. Moreover, it decrease discharges during. In fact, its value coincides with that obtained in plasticity as long as the evolution of the loading is radial and monotonous.

In addition to the behavior model itself, it is necessary to know the value of the free energy for a state given for computations of rate of energy restitution. Without demonstration, this potential of which derives the behavior model is worth:

- If $\varepsilon_{eq} \leq \frac{\sigma^y}{2\mu}$ $\qquad \psi(\varepsilon)=\frac{1}{2}K(\mathrm{tr}\,\varepsilon)^2+\frac{2\mu}{3}\varepsilon_{eq^2}$
- If $\varepsilon_{eq} > \frac{\sigma^y}{2\mu}$ $\qquad \psi(\varepsilon)=\frac{1}{2}K(\mathrm{tr}\,\varepsilon)^2+\frac{R(p(\varepsilon_{eq}))^2}{6\mu}+\int_0^{p(e_{eq})} R(s)\,ds$

éq 1.2-3

1.3 Resolution of the equation in p

One could note in the preceding paragraph that the statement of the stresses requires the solution of an equation relating to the variable p . Insofar as the function of hardening R is increasing, this equation can be written by gathering the terms where appear p in the first member (who is then increasing with p):

More precisely, the first member is linear per pieces in p . To solve the equation, it is then enough sequentially to traverse each interval until finding that in which the solution is. An equation closely connected provides the value then of p .

1.4 Computation of the behavior model and tangent stiffness

The computation of the stresses and the tangent stiffness, i.e. the variation of the stresses compared to the strains, is carried out according to the algorithm presented below. By adopting the convention of *Code_Aster*, the stresses and the strains are arranged in a vector with six components, while the tangent stiffness is a matrix 6×6.

$$\{\boldsymbol{\varepsilon}\}=\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \sqrt{2}\,\varepsilon_{xy}\\ \sqrt{2}\,\varepsilon_{xz}\\ \sqrt{2}\,\varepsilon_{yz}\end{Bmatrix}\quad\{\boldsymbol{\sigma}\}=\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sqrt{2}\,\sigma_{xy}\\ \sqrt{2}\,\sigma_{xz}\\ \sqrt{2}\,\sigma_{yz}\end{Bmatrix}\quad\{\mathbf{1}\}=\begin{Bmatrix}1\\1\\1\\0\\0\\0\end{Bmatrix}$$

Behavior model:

$$\{\boldsymbol{\sigma}\}=K(\operatorname{tr}\boldsymbol{\varepsilon})\{\mathbf{1}\}+G\{\tilde{\boldsymbol{\varepsilon}}\}$$

Tangent stiffness:

$$\frac{d\{\boldsymbol{\sigma}\}}{d\{\boldsymbol{\varepsilon}\}}=\mathbf{K}=[\mathbf{K_1}]+[\mathbf{K_2}]$$

$$[\mathbf{K_1}]=\frac{3\mathrm{K}-G}{3}\{\mathbf{1}\}\otimes\{\mathbf{1}\}+G[\mathbf{Id}]$$

$$[\mathbf{K_2}]=\begin{cases}[\mathbf{0}] & \text{si}\ \ \boldsymbol{\varepsilon}_{eq}\leq\dfrac{\sigma^y}{2\mu}\\[2ex] \dfrac{3}{2\varepsilon_{eq}^2}\left[\dfrac{2\mu R'(p)}{R'(p)+3\mu}-G\right]\{\tilde{\boldsymbol{\varepsilon}}\}\otimes\{\tilde{\boldsymbol{\varepsilon}}\} & \text{si}\ \ \varepsilon_{eq}>\dfrac{\sigma^y}{2\mu}\end{cases}$$

1.5 Taken into account of strains of thermal origin

In a way identical to plasticity, one divides the total deflection into a mechanical part which checks the preceding behavior model [éq 1.2-1], [éq 1.2-2] and a thermal part, function of the temperature. Let us note moreover that the various characteristics of the material can also depend on the temperature.

$$\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}^m+\boldsymbol{\varepsilon}^{th}$$

$$\text{avec}\ \begin{cases}\boldsymbol{\sigma}=K(\operatorname{tr}\boldsymbol{\varepsilon}^m)\mathbf{Id}+G(\varepsilon_{eq})\tilde{\boldsymbol{\varepsilon}}\\ \boldsymbol{\varepsilon}^{th}=\alpha(T-T_{réf})\mathbf{Id}\end{cases}\qquad \text{éq 1.5.1}$$

α : thermal coefficient of thermal expansion

T_{ref} : reference temperature

It remains to supplement the potential free energy [éq 1.2-3] to include the temperature there. Several choices are possible, depend on the way in which one wishes to define the entropy (derivative of the free energy compared to the temperature). In our case, the adopted potential is:

- If $\varepsilon_{eq} \leq \dfrac{\sigma^y}{2\mu}$ $\qquad \psi(\boldsymbol{\varepsilon}, T) = \dfrac{1}{2} K \left(\mathrm{Tr}\left(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{th}\right)\right)^2 + \dfrac{2\mu}{3} \varepsilon_{eq}^2$
- If $\varepsilon_{eq} > \dfrac{\sigma^y}{2\mu}$ $\qquad \psi(\boldsymbol{\varepsilon}, T) = \dfrac{1}{2} K \left(\mathrm{Tr}\left(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{th}\right)\right)^2 + \dfrac{R\left(p\left(\varepsilon_{eq}\right)\right)^2}{6\mu} + \int_0^{p(\varepsilon_{eq})} R(s)\,ds$

1.6 particular Processing of the plane stresses

Usually, one seeks to determine the stresses knowing the strains and the temperature. However, it is not completely any more the case under the assumption of the plane stresses insofar as three of the components of the tensor of the strains are henceforth unknown, the dual quantities being built-in:

ε_{xz} , ε_{yz} and unknowns
ε_{zz}
$\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0$

It is thus necessary to start by determining these unknown components. The adopted method is exposed in [bib1] and [R5.03.02]. One can however point out here that the components xz and yz do not pose a problem, being given the form of the behavior model [éq 1.2-1]:

$$\varepsilon_{xz} = \varepsilon_{yz} = 0$$

On the other hand the determination of the component zz requires the solution (numerical) of a nonlinear scalar equation.

Lastly, a last warning is essential. Unlike the plane strains, the solutions which one obtains under the assumption of the plane stresses are generally not exact insofar as they do not check the compatibility conditions geometrical (integrability of the strain field). They are only approximate solutions.

2 Elasticity in great transformations

2.1 Purpose

Henceforth, one proposes to take into account large displacements and large rotations, functionality accessible by the key word DEFORMATION=' GROT_GDEP' in command STAT_NON_LINE. Let us specify as of now that one of the finite elements restricts oneself with isoparametric (D_PLAN, C_PLAN, AXIS and 3D) for whom the discretization of the continuous problem does not raise particular difficulties, cf [R3.01.00].

To this end, it is admitted that the second tensor of the Piola-Kirchhoff stresses S , derives from the potential of Hencky-Von Put expressed using the strain of Green-Lagrange $\mathbf{E}$:

$$\mathbf{S}=\frac{\partial \Psi}{\partial E}(\mathbf{E})$$

Also let us point out the definitions of $\mathbf{E}$ and $\mathbf{S}$. One can also find additional information in [bib1].

$$\mathbf{F}=\mathbf{Id}+\mathbf{Grad}(\mathbf{u}) \quad \mathbf{E}=\frac{1}{2}\left({}^{T}FF-\mathbf{Id}\right)$$

$$S=\mathrm{Det}(\mathbf{F})\mathbf{F}^{-1}\boldsymbol{\sigma}^{T}\mathbf{F}^{-1}$$

Such a behavior model, known as hyper elastic, makes it possible in any rigor to take into account large deformations and large rotations. However, we limit ourselves to small strains, and this for two reasons. First of all, the adopted behavior model does not present the good properties (polyconvexity) to ensure the existence of solutions and does not control either important compressions. Then, the plastic behavior differs notably from a behavior hyper elastic as soon as the strains become appreciable. It is for these reasons that we chose to preserve the assumption of small strains, thus escaping the polemic from the large deformations.

2.2 Virtual wor of the external forces: assumption of the loads died

to deal with the problem of structural analysis hyper elastics, one seeks to write the equilibrium in variational form on the initial configuration. In particular, it is necessary to express the virtual wor of the external forces on this same initial configuration what requires the additional assumption of dead loads: it is supposed that the loading does not depend on the geometrical transformation. Typically, an imposed force is a dead load while the pressure is a following loading since it depends on the directional sense of the face of application, therefore of the transformation. Under this assumption, the virtual wor of the external forces is written like a linear form:

$$\delta W_{ext}.\boldsymbol{\delta}\,\mathbf{v}=\int_{\Omega_o}\rho_o F_i\,\delta v_i\,d\,\Omega_o+\int_{\partial_F \Omega_o} T_i^d\,\delta v_i\,dS_o$$

$\mathbf{F}$: voluminal loading

$\mathbf{T^d}$: surface loading being exerted on the edge $\partial_F \boldsymbol{\Omega}_\mathbf{o}$

2.3 Virtual wor of the internal forces

We will not give here a demonstration of the statements presented. For that, one will be able to refer to [bib1] and [R7.02.03]. There still, we choose the initial configuration like reference configuration, to express the work of the internal forces:

$$SW_{int}.\boldsymbol{\delta}\,\mathbf{v}=-\int_{\Omega_o} F_{ik} S_{kl}\,\delta v_{i,l}\,d\,\Omega_o$$

with: $\delta v_{i,l}=\dfrac{\partial dv_i}{\partial X_l}$

In the optics of a resolution by a method of Newton, it is important to also express the variation second of the virtual wor of the internal forces, namely:

$$d^2 W_{\text{int}}.\boldsymbol{\delta}\,\mathbf{u}.\boldsymbol{\delta}\,\mathbf{v}= -\int_{\Omega_o} \delta u_{i,k} S_{kl} \delta v_{i,l}\,d\,\Omega_o$$ Geometrical stiffness

$$\dots\ -\int_{\Omega_o} \delta u_{i,q} F_{ip} \frac{\partial^2 y}{\partial E_{pq} \partial E_{kl}} F_{jk} \delta v_{j,l}\,d\,\Omega_o$$ elastic Stiffness

2.4 variational Formulation

We now have at our disposal all the ingredients to write the variational formulation of problem:

$$\delta W_{\text{int}}.\boldsymbol{\delta}\,\mathbf{v}+SW_{ext}.\boldsymbol{\delta}\,\mathbf{v}=0\ ,\ \forall\boldsymbol{\delta}\,\mathbf{v}$$ kinematically admissible

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1.Bibliography E.: A nonlinear behavior model hyper elastic. Note intern EDF DER, HI-74/95/011/0, 1995.

4 Description of the versions of the document

Version Aster	Author (S) Organization (S)	Description of the modifications
3	3E.LorentzEDF-R&D/MMN	initial Text
10.1	J.M.Proix EDF-R&D/ AMA Change	of vocabulary: GREEN becomes GROT_GDEP