Modelization élasto (visco) plastic with isotropic hardening in large deformations

Summarized

One describes here a thermoelastoplastic behavior model with isotropic hardening written in large deformations and proposed by Simo and Miehe. This model is available in command STAT_NON_LINE via the key word RELATION: "VMIS_ISOT_TRAC", "VMIS_ISOT_PUIS" or "VMIS_ISOT_LINE" under factor key word the COMP_INCR and with the key word DEFORMATION: "SIMO_MIEHE". A viscous version of this model is also proposed: "VISC_ISOT_TRAC" and "VISC_ISOT_LINE".

This model is established for the three-dimensional modelizations (3D), axisymmetric (AXIS) and in plane strains (D_PLAN).

One presents the writing and the digital processing of this model, as well as the associated variational formulation. It is about an eulerian variational formulation, with reactualization of the geometry and which takes account of the stiffness of behavior and the geometrical stiffness.

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1 Introduction

We present here a thermoelastoplastic constitutive law written in large deformations proposed by SIMO J.C and MIEHE C. [bib1] which tends in small strains towards model elastoplastic behavior with isotropic hardening and criterion of Von Mises, described in [R5.03.02]. The kinematical choices make it possible, as with the simple reactualization available via the key word `PETIT_REAC`, to treat large displacements and large deformations but also of large rotations in an exact way.
Specificities of this model are the following ones:

- just like in small strains, one supposes the existence of a slackened configuration, i.e. locally free of stress, which makes it possible to break up the total deflection into a thermo-elastic part and a plastic part,
- the decomposition of this thermo-elastic strain into cubes parts and plastic is not additive any more as in small strains (or for the model large deformations written in strain rate with for example a derivative of Jaumann) but multiplicative,
- the elastic strain are measured in the present configuration (deformed) while plastic strains are measured in the initial configuration,
- as in small strains, the stresses depend only on the thermoelastic strains,
- plastic strains are done with volume constant. The variation of volume is then only due to the thermoelastic strains,
- this model during leads its numerical integration to a model incrémentalement objective (cf [§3.3]) what makes it possible to obtain the exact solution in the presence of large rotations.

A viscous version of this model is also available (model in hyperbolic sine as in the case of the model of Rousselier ROUSS_VISC, cf [R5.03.07]).

Thereafter, one briefly points out some notions of mechanics in large deformations, then one presents the behavior models of the model and his numerical integration to treat the balance equations.

One proposes an eulerian variational formulation, with reactualization of the geometry. For this reason, one expresses the work of the internal forces and his variation (with an aim of a resolution by the method of Newton) for the continuous problem, which respectively provide after discretization by finite elements the vector of the internal forces and the tangent matrix.

Nota bene :

> *One will find in [bib2] or [bib3] a presentation deepened on the large deformations.*
> *This document is extracted from [bib4] where one makes a more detailed presentation of the elastoplastic model, of his numerical integration and where some examples of validation are given.*

2 Notations

One will note by:

$\mathbf{Id}$	stamp identity
$\mathrm{tr}\,A$	X traces $\mathbf{A}$
$\mathbf{A}^{\mathbf{T}}$	tensor transposed of $\mathbf{A}$
$\det A$	the determining tensor $\mathbf{A}$
$\langle X \rangle$	of positive part *of*
$\tilde{A}$	deviatoric part of the tensor $\mathbf{A}$ defined by $\tilde{A}=A-(\frac{1}{3}tr\,A)\mathbf{Id}$
:	doubly contracted product: $\mathbf{A}:\mathbf{B}=\sum_{i,j} A_{ij}\,B_{ij}=tr(\mathbf{A}\mathbf{B}^T)$
$\otimes$	tensor product: $(\mathbf{A}\otimes\mathbf{B})_{ijkl}=A_{ij}B_{kl}$
A_{eq}	equivalent value of von Mises defined by $A_{eq}=\sqrt{\frac{3}{2}\tilde{A}:\tilde{A}}$
$\nabla_{\mathbf{X}}\mathbf{A}$	gradient: $\nabla_X \mathbf{A}=\frac{\partial\mathbf{A}}{\partial\mathbf{X}}$
$\mathrm{div}_x\mathbf{A}$	divergence: $(\mathrm{div}_x\mathbf{A})_i=\sum_j \frac{\partial A_{ij}}{\partial x_j}$
λ,μ,E,ν,K	moduli of the isotropic elasticity
σ_y	elastic limit
α	coefficent of thermal thermal expansion
T	temperature
T_{ref}	reference temperature

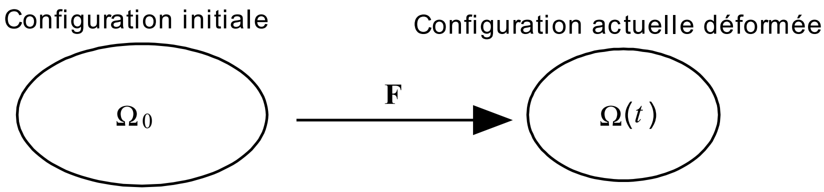
In addition, in the frame of a discretization in time, all the quantities evaluated at previous time are subscripted by $^-$, the quantities evaluated at time $t+\Delta t$ are not subscripted and the increments are indicated par. Δ One has as follows:

$$\Delta Q=Q-Q^-$$

3 Recalls on the large deformations

3.1 Kinematics

Let us consider a solid subjected to large deformations. That is to say Ω_0 the field occupied by solid before strain and $\Omega(t)$ the field occupied at time t by deformed solid.



Appear 3.1-a: Representation of the configuration initial and deformed

In the initial configuration Ω_0 , the position of any particle of solid is indicated by $\mathbf{X}$ (Lagrangian description). After strain, the position at the time t of the particle which occupied the position $\mathbf{X}$ before strain is given by the variable $\mathbf{x}$ (eulerian description).

The total motion of solid is defined, with $\mathbf{u}$ displacement, by:

$$\mathbf{x}=\hat{\mathbf{x}}(\mathbf{X},t)=\mathbf{X}+\mathbf{u}$$

To define the change of metric in the vicinity of a point, the tensor gradient of the transformation is introduced $\mathbf{F}$:

$$\mathbf{F}=\frac{\partial\hat{\mathbf{x}}}{\partial\mathbf{X}}=\mathbf{Id}+\nabla_{\mathbf{X}}\mathbf{u}$$

The transformations of the volume element and the density are worth:

$$d\,\Omega=Jd\,\Omega_o \quad \text{with} \quad J=\det F=\frac{\rho_o}{\rho}$$

where ρ_o and ρ are respectively the density in the configurations initial and current.

Various strain tensors can be obtained by eliminating rotation in the local transformation. For example, by directly calculating the variations length and angle (variation of the scalar product), one obtains:

$$\mathbf{E}=\frac{1}{2}(\mathbf{C}-\mathbf{Id}) \quad \text{with} \quad \mathbf{C}=\mathbf{F}^{\mathbf{T}}\mathbf{F}$$

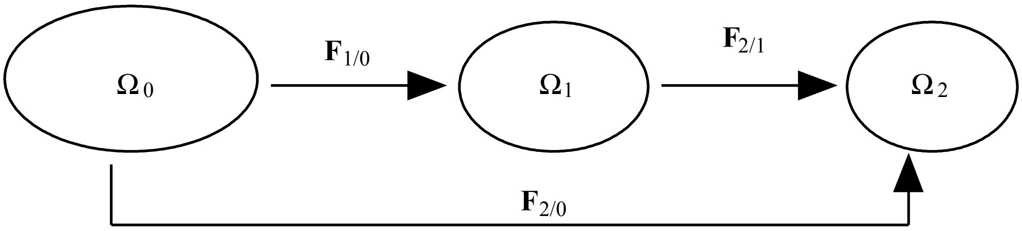
$$\mathbf{A}=\frac{1}{2}(\mathbf{Id}-\mathbf{b}^{-1}) \quad \mathbf{b}=\mathbf{F}\mathbf{F}^{T}$$

$\mathbf{E}$ and $\mathbf{A}$ are respectively the strain tensors of Green-Lagrange and Eulerian-Almansi and $\mathbf{C}$ $\mathbf{b}$, the tensors of right and left Cauchy-Green respectively.

In Lagrangian description, one will describe the strain by the tensors $\mathbf{C}$ or $\mathbf{E}$ because they are quantities defined on Ω_0 , and of eulerian description by the tensors $\mathbf{b}$ or $\mathbf{A}$ (definite on Ω).

Note:

That is to say a solid undergoing two successive transformations, for example the first transformation makes pass solid of the initial configuration Ω_0 to a configuration Ω_1 (tensor gradient $\mathbf{F_{1/0}}$ and vector displacement $\mathbf{u_{1/0}}$), then the second transformation of the configuration Ω_1 with Ω_2 (tensor gradient $\mathbf{F_{2/1}}$ and vector displacement $\mathbf{u_{2/1}}$).



The transition of the configuration Ω_0 with Ω_2 is given by the tensor gradient $\mathbf{F_{2/0}}$ (displacement $\mathbf{u_{2/0}}=\mathbf{u_{2/1}}+\mathbf{u_{1/0}}$) such as:

$$\mathbf{F_{2/0}}=\mathbf{F_{2/1}}\mathbf{F_{1/0}}$$

One obtains then, for example, for the strain tensor of Green-Lagrange $\mathbf{E}$

$$\mathbf{E_{2/0}}=\mathbf{F_{1/0}^T}\mathbf{E_{2/1}}\mathbf{F_{1/0}}+\mathbf{E_{1/0}}$$

where $\mathbf{E_{2/0}}$, $\mathbf{E_{1/0}}$ and $\mathbf{E_{2/1}}$ are the strains of Green-Lagrange of the configurations Ω_2 compared to Ω_0 associated with $F_{2/0}$, Ω_1 compared to Ω_0 associated with $F_{1/(}$ and Ω_2 compared to Ω_1 associated with $\mathbf{F_{2/1}}$, respectively.
This constitutes one of the difficulties encountered at the time of the writing of a constitutive law in large deformations because one cannot write any more one formula similar to that written in small strains, namely $\boldsymbol{\varepsilon}_{2/0}=\boldsymbol{\varepsilon}_{2/1}+\boldsymbol{\varepsilon}_{1/0}$ where $\boldsymbol{\varepsilon}$ is the linearized strain tensor total.
To find $\boldsymbol{\varepsilon}_{2/0}=\boldsymbol{\varepsilon}_{2/1}+\boldsymbol{\varepsilon}_{1/0}$ in small strains from the statement of $\mathbf{E_{2/0}}$, it is necessary to neglect all the terms of order 2 of $\nabla\mathbf{u_{2/0}}$, $\nabla\mathbf{u_{1/0}}$ and $\nabla\mathbf{u_{2/1}}$. In this case, one has $\mathbf{E_{2/0}}\simeq\boldsymbol{\varepsilon}_{2/0}$, $\mathrm{E}_{1/0}\simeq\boldsymbol{\varepsilon}_{1/0}$ and $\mathbf{F_{1/0}^T}\mathbf{E_{2/1}}\mathbf{F_{1/0}}\simeq\boldsymbol{\varepsilon}_{2/1}$.

3.2 Forced

For the model described here, the tensor of the stresses used is the eulerian tensor of Kirchhoff $\boldsymbol{\tau}$ defined by:

$$J\boldsymbol{\sigma}=\boldsymbol{\tau}$$

where $\boldsymbol{\sigma}$ is the eulerian tensor of Cauchy. The tensor $\boldsymbol{\tau}$ thus results from a “scaling” by the variation of volume of the tensor of Cauchy $\boldsymbol{\sigma}$; this is not the case of other stress tensors used (first and second tensor of Piola-Kirchhoff).

In eulerian description, the balance equations are given by:

$$\begin{aligned} &\mathrm{div}_x\boldsymbol{\sigma}+\rho\,\mathbf{f}=0 \text{ sur } \Omega \\ &\boldsymbol{\sigma}.\mathbf{n}=\mathbf{t^d} \quad \text{ sur } \partial\Omega^f \end{aligned}$$

where f is the volume force applied to the field Ω , $\mathbf{n}$ the norm external with the border $\partial\Omega^f$ and $\partial\Omega^f$ the part of the border of the field Ω where are applied the surface forces $\mathbf{t^d}$.

3.3 Objectivity

When a constitutive law in large deformations is written, one must check that this model is objective, i.e. invariant by any change of spatial reference frame of the form:

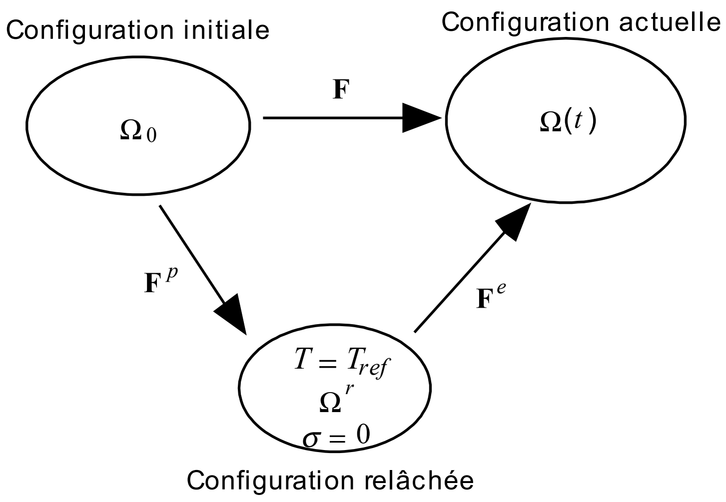
$$\mathbf{x}^{*}=\mathbf{c}(t)+\mathbf{Q}(t)\,\mathbf{x}$$

where $\mathbf{Q}$ is an orthogonal tensor which represent the rotation of the reference frame and $\mathbf{c}$ a vector which represents the translation.
More concretely, if one carries out a traction test in the direction $\mathbf{e}_1$, for example, followed by a rotation of 90° around $\mathbf{e}_3$, which amounts carrying out a traction test according to $\mathbf{e}_2$, then the danger with a nonobjective constitutive law is not to find a uniaxial tensor of the stresses in the direction $\mathbf{e}_2$ (what is in particular the case with the kinematics `PETIT_REAC`).

4 Presentation of the model of kinematical

4.1 behavior Aspect

This model supposes, just like in small strains, the existence of a slackened configuration Ω^r , i.e. locally free of stress, which makes it possible then to break up the total deflection into cubes elastic parts and plastic, this decomposition being multiplicative.
Thereafter, one will note by $\mathbf{F}$ the tensor gradient which makes pass from the initial configuration Ω_0 to the present configuration $\Omega(t)$, by $\mathbf{F}^\mathbf{p}$ the tensor gradient which makes pass from the configuration Ω_0 to the slackened configuration Ω^r , and $\mathbf{F}^\mathbf{e}$ of the configuration Ω^r with $\Omega(t)$. The index p refers to the plastic part, the index e with the elastic part.



Appear 4.1-a: Decomposition of the tensor gradient $\mathbf{F}$ in an elastic and $\mathbf{F}^\mathbf{e}$ plastic part $\mathbf{F}^\mathbf{p}$

By composition of motions, one obtains following multiplicative decomposition:

$$\mathbf{F}=\mathbf{F}^\mathbf{e}\,\mathbf{F}^\mathbf{p}$$

The elastic strain are measured in the present configuration with the eulerian tensor of left Cauchy-Green $\mathbf{b}^\mathbf{e}$ and the plastic strains in the initial configuration by the tensor $\mathbf{G}^\mathbf{p}$ (Lagrangian description). These two tensors are defined by:

$$\mathbf{b}^e=\mathbf{F}^e\mathbf{F}^{eT} \ , \ \mathbf{G}^p=(\mathbf{F}^{pT}\mathbf{F}^p)^{-1} \quad \text{from where} \quad \mathbf{b}^e=\mathbf{F}\mathbf{G}^p\mathbf{F}^T$$

The model presented is written in order to distinguish the isochoric terms from the terms of change of volume. One introduces for that the two following tensors:

$$\bar{\mathbf{F}}=J^{-1/3}\mathbf{F} \quad \text{and} \quad \bar{\mathbf{b}}^e=J^{-2/3}\mathbf{b}^e \quad \text{with} \quad J=\det\mathbf{F}$$

By definition, one a: $\det\bar{\mathbf{F}}=1$ and $\det\bar{\mathbf{b}}^e=1$.

4.2 Behavior models

the model presented is a model thermoélasto (visco) plastic with isotropic hardening which tends under the assumption of the small strains towards the model [R5.03.02] with criterion of Von Mises (it is the plastic model). Plastic strains are done with constant volume so that:

$$J^p=\det F^p=1 \quad \text{from where} \quad J=J^e=\det\mathbf{F}^e$$

the behavior models are given by:

- the definition of the free energy hyper elastic:

  $\psi=\psi_{ther}(T, J)+\psi_{elas}(J, \bar{\mathbf{b}}^e)+K(\alpha)$,

  with in particular $\psi_{elas}(J, \bar{\mathbf{b}}^e)=\frac{1}{2}\frac{E}{3(1-2\nu)}\left[\frac{1}{2}(J^2-1)-\ln J\right]+\frac{\mu}{2}(tr\,\bar{\mathbf{b}}^e-3)$

  and $\psi_{ther}(T, J)=-3\alpha\Delta T\left(J-\frac{1}{J}\right)$

- thermo-elastic relation stress-strain:

$$\tilde{\boldsymbol{\tau}}=\mu\tilde{\bar{\mathbf{b}}}^e$$
$$\operatorname{tr}\boldsymbol{\tau}=\frac{3K}{2}(J^2-1)-\frac{9K}{2}\alpha(T-T_{ref})(J+\frac{1}{J})$$

- threshold of plasticity (it is admitted that it is expressed with the stresses of Kirchhoff):

$$f=\tau_{eq}-R(p)-\sigma_y$$

  where R is the isotropic variable of hardening, function of the cumulated plastic strain *p.*

- models of flow:

$$\bar{\mathbf{F}}\dot{\mathbf{G}}^p\bar{\mathbf{F}}^T=-\dot{\lambda}\frac{3}{\tau_{eq}}\tilde{\boldsymbol{\tau}}\,\bar{\mathbf{b}}^e=-3\dot{\lambda}\left(\frac{1}{3}\operatorname{tr}\,\bar{\mathbf{b}}^e+\frac{\tilde{\boldsymbol{\tau}}}{\mu}\right)\frac{\tilde{\boldsymbol{\tau}}}{\tau_{eq}}$$
$$\dot{p}=\dot{\lambda}$$

For the model from plasticity, the plastic multiplier is obtained by writing the condition of coherence $\dot{f}=0$ and one a:

$$\dot{p}\geq 0, f\leq 0 \text{ et } \dot{p}\,f=0$$

In the viscous case, one takes $\dot{p}$ equalizes with:

$$\dot{p}=\dot{\varepsilon}_0\left[\mathrm{sh}\left(\frac{\langle f\rangle}{\sigma_0}\right)\right]^m$$

where $\dot{\varepsilon}_0$, σ_0 and m are the viscosity coefficients. Let us announce that this model is reduced to a model of the type Norton when the 2 materials parameters $\dot{\varepsilon}_0$ and σ_0 are very large.

It is pointed out that:

$$\bar{\mathbf{b}}^e=J^{-2/3}\,\mathbf{b}^{\mathbf{e}}$$

$$\bar{\mathbf{F}}=J^{-1/3}\,\mathbf{F}$$

and that the partition of the strains is written:

$$\bar{\mathbf{b}}^e=\bar{\mathbf{F}}\,\mathbf{G}^{\mathbf{p}}\,\bar{\mathbf{F}}^T$$

For metallic materials where the ratio τ_{eq}/μ is small in front of 1, the form of the flow model can be approximate by:

$$\bar{\mathbf{F}}\,\dot{\mathbf{G}}^p\,\bar{\mathbf{F}}^T=-\dot{\lambda}\,\mathrm{tr}\;\bar{\mathbf{b}}^{\mathbf{e}}\frac{\tilde{\boldsymbol{\tau}}}{\tau_{eq}}+O\left(\frac{\tau_{eq}}{\mu}\right) \qquad \text{éq.4.2-1}$$

where $O\left(\frac{\tau_{eq}}{\mu}\right)$ is negligible in front of the first term.

It is this last statement which is established in *the Code_Aster*.

Note:

If the strains are small, one a:

$$J\simeq 1+\mathrm{tr}\,\boldsymbol{\varepsilon}$$
$$\mathbf{b}^{\mathbf{e}}\simeq\mathbf{Id}+2\boldsymbol{\varepsilon}^{\mathbf{e}}$$
$$\mathbf{G}^{\mathbf{p}}\simeq\mathbf{Id}-2\boldsymbol{\varepsilon}^{\mathbf{p}}$$

where $\boldsymbol{\varepsilon}$ is the total deflection, $\boldsymbol{\varepsilon}^{\mathbf{e}}$ elastic strain and $\boldsymbol{\varepsilon}^{\mathbf{p}}$ plastic strain in small strains.

By replacing these three statements in the equations of the constitutive law presented here, one finds well the model classical thermoelastoplastic with isotropic hardening and criterion of Von Mises.

4.3 Correction of elastic strain energy in the presence of thermal

the statement of energy hyper elastic ψ_{elas} raises some difficulties. Indeed, it depends on J ; in the presence of thermal strain, J included a thermal component which disturbs the statement and which it is advisable to correct.

The approach of the correction of the energy of Simo-Miehe in the presence of thermal is the following one:

1. cancellation of energy in pure thermal when the stress is null;
2. resolution of the equation in J obtained, which one calls J_0 the solution.
3. final computation of the elastic strain energy of Simo-Miehe in the presence of thermal:

$$\psi_{elas}^{corrigee}=\psi_{elas}\left(J,\bar{\mathbf{b}}^{\mathbf{e}}\right)+\psi_{ther}\left(T,J\right)+\psi_{elas}\left(J,\bar{\mathbf{b}}^{\mathbf{e}}\right)-\psi_{elas}\left(J_{0,}\bar{\mathbf{b}}^{\mathbf{e}}=\boldsymbol{I_d}\right)-\psi_{ther}\left(T,J_0\right)$$

The cancellation of the stress in pure thermal leads to the equation:

$$\operatorname{tr}\boldsymbol{\tau}=0\Leftrightarrow\left(J^2-1\right)-3\alpha(T-T_{ref})(J+\frac{1}{J})=0\Leftarrow\{\begin{matrix}J^3-J-3\alpha\Delta T\left(J^2-1\right)\\ J\neq0\end{matrix}$$

This equation, under the assumption $3\alpha\Delta T\ll1$, has solutions close to -1,0 and 1. Only largest is physically acceptable. One thus poses:

$$J_0=MAX\{J\,tq\,J^3-J-3\alpha\Delta T\left(J^2-1\right)=0\}$$

The energy corrected in the presence of thermal is written finally:

$$\psi_{elas}^{corrigee}=\psi_{elas}\left(J,\bar{\mathbf{b}}^{\mathbf{e}}\right)+\psi_{ther}\left(T,J\right)+\psi_{elas}\left(J,\bar{\mathbf{b}}^{\mathbf{e}}\right)-\psi_{elas}\left(J_{0,}\bar{\mathbf{b}}^{\mathbf{e}}=\boldsymbol{I_d}\right)-\psi_{ther}\left(T,J_0\right)$$

4.4 Choice of the function of hardening

This behavior model is available in operator `STAT_NON_LINE`, under factor key word `the COMP_INCR` and argument "`SIMO_MIEHE`" of factor key word `the DEFORMATION`. One can choose for the function of hardening, a linear hardening or provide a curve of tension. Five relations can be used.

```
RELATION =        "VMIS_ISOT_TRAC"
               /  "VMIS_ISOT_PUIS"
               /  "VMIS_ISOT_LINE"
               /  "VISC_ISOT_TRAC"
               /  "VISC_ISOT_LINE"
```

For a purely thermo-elastic behavior, the user chooses argument "`VMIS_ISOT_LINE`" for example, with `SY` very large (the behavior is then hyper elastic); for an isotropic hardening given by a curve of tension, the user chooses argument "`VMIS_ISOT_TRAC`" in the plastic case or "`VISC_ISOT_TRAC`" in the viscous case and for a linear isotropic hardening, argument "`VMIS_ISOT_LINE`" in the plastic case or "`VISC_ISOT_LINE`" in the viscous case. For an elastoplastic behavior whose curve of hardening (rational curve of tension) is given by a model in power, form

$R(p)=\sigma_y+\sigma_y\left(\frac{E}{a\sigma_y}p\right)^{\frac{1}{n}}$, the user chooses argument "VMIS_ISOT_PUIS".

The various characteristics of the material are indicated in operator DEFI_MATERIAU ([U4.23.01]) under the key words:

- ELAS some is the model (one gives the Young modulus, the Poisson's ratio and possibly the thermal coefficient of thermal expansion),
- TENSION for "VMIS_ISOT_TRAC" and "VISC_ISOT_TRAC" (rational curve of tension is given),
- ECRO_PUIS for "VMIS_ISOT_PUIS" (one gives the parameters of the model power),
- ECRO_LINE for "VMIS_ISOT_LINE" and "VISC_ISOT_LINE" (one gives the elastic limit and the hardening slope),
- VISC_SINH for "VISC_ISOT_TRAC" and "VISC_ISOT_LINE" (one gives the three viscosity coefficients).

Note:

The user must make sure well that "experimental" curve of tension used, either directly, or to deduce some the hardening slope is well given in the plane forced rational $\sigma=F/S$ - logarithmic strain $\ln(1+\Delta l/l_0)$ where l_0 is the initial length of the useful part of the test-tube, Δl the variation length after strain, F the applied force and S current surface. It will be noticed that $\sigma=F/S=\frac{F}{S_0}\frac{l_0}{l}\frac{1}{J}$ from where $\tau=J\sigma=\frac{F}{S_0}\frac{l_0}{l}$. In general, it is well the quantity $\frac{F}{S_0}\frac{l_0}{l}$ which is measured by the experimenters and this gives the stress of Kirchhoff directly used in the model of Simo and Miehe.

4.5 Stresses and local variables

the stresses are the stresses of Cauchy $\boldsymbol{\sigma}$, thus calculated on the present configuration (six components in 3D, four in 2D).
The local variables produced in *the Code_Aster* are:

- $V1$, cumulated plastic strain p ,
- $V2$ with $V7$: opposite of the elastic strain $\bar{\mathbf{b}}^e$
- $V8$ the indicator of plasticity (0 if the last calculated increment is elastic, 1 if not).

Note:

If the user wants to possibly recover strains in postprocessing of his computation, it is necessary to trace the strains of Green-Lagrange $\mathbf{E}$, which represents a measurement of the strains in large deformations (options EPSG_ELGA or EPSG_ELNO). The classical linearized $\boldsymbol{\varepsilon}$ strains measure strains under the assumption of the small strains and do not have a meaning in large deformations.

4.6 Field of application

the choice of a kinematics DEFORMATION: "PETIT_REAC" also allows to treat a thermoelastoplastic constitutive law with isotropic hardening and criterion of Von Mises in large deformations. The model is written in small strains and the taking into account of the large deformations is done by reactualizing the geometry.
Between the model presented here (SIMO_MIEHE) and PETIT_REAC,

- there is no difference if the strains are small
- it does not have there a difference if the strains are large but small rotations
- there are differences if rotations are important.

In particular, the solution obtained with the kinematics PETIT_REAC can deviate notably from the exact solution in the presence of large rotations and this whatever the size from time step chosen by the user, contrary to kinematics SIMO_MIEHE.

4.7 Integration of the constitutive law

In the case of an incremental behavior, factor key word COMP_INCR, knowing the stress $\boldsymbol{\sigma}^-$, the cumulated plastic strain p^-, the trace divided by three of the strain tensor elastics $\frac{1}{3}\mathrm{tr}\,\bar{\mathbf{b}}^{e-}$, displacements $\mathbf{u}^-$ and $\Delta\mathbf{u}$ the temperatures T^- and T, one seeks to determine $(\boldsymbol{\sigma}, p, \frac{1}{3}\mathrm{tr}\,\bar{\mathbf{b}}^e)$.

Displacements being known, the gradients of the transformation of Ω_0 with Ω^-, noted $\mathbf{F}^-$, and of Ω^- with Ω, noted $\Delta\mathbf{F}$, are known.

The implicit *discretization* of the model gives:

$$\mathbf{F}=\Delta\mathbf{F}\,\mathbf{F}^-$$
$$J=\det\mathbf{F}$$
$$\bar{\mathbf{F}}=J^{-1/3}\,\mathbf{F}$$
$$\bar{\mathbf{b}}^e=\bar{\mathbf{F}}\,\mathbf{G}^{\mathbf{p}}\,\bar{\mathbf{F}}^T$$
$$J\boldsymbol{\sigma}=\boldsymbol{\tau}$$
$$\tilde{\boldsymbol{\tau}}=\mu\,\tilde{\bar{\mathbf{b}}}^e$$
$$\frac{1}{3}\mathrm{tr}\,\boldsymbol{\tau}=\frac{1}{2}K(J^2-1)-\frac{3}{2}K\alpha(T-T_{ref})(J+\frac{1}{J})$$
$$f=\boldsymbol{\tau}_{eq}-R(p^-+\Delta p)-\sigma_y$$
$$\bar{\mathbf{F}}(\mathbf{G}^{\mathbf{p}}-\mathbf{G}^{\mathbf{p}-})\bar{\mathbf{F}}^T=-\mathrm{tr}\,\bar{\mathbf{b}}^e\frac{\tilde{\boldsymbol{\tau}}}{\tau_{eq}}\Delta p \text{ from where } \bar{\mathbf{b}}^e=\bar{\mathbf{F}}\,\mathbf{G}^{\mathbf{p}-}\,\bar{\mathbf{F}}^T-\mathrm{tr}\,\bar{\mathbf{b}}^e\frac{\tilde{\boldsymbol{\tau}}}{\tau_{eq}}\Delta p$$

In the plastic case: $\Delta p\geq 0, f\leq 0$ et $f\,\Delta p=0$

In the viscous case: $\langle\tau_{eq}-R(p^-+\Delta p)-\sigma_y\rangle-\sigma_0\,\mathrm{sh}^{-1}\left[\left(\frac{\Delta p}{\dot{\varepsilon}_0\Delta t}\right)^{\frac{1}{m}}\right]=0$

Note:

> *This formulation is incrémentalement objective because the only incremental tensorial quantity which intervenes in the discretization is* $\dot{\mathbf{G}}^p$ *. As* $\mathbf{G}^{\mathbf{p}}$ *and* $\mathbf{G}^{\mathbf{p}-}$ *are measured on the same configuration, i.e. the initial configuration, the discretization of* $\dot{\mathbf{G}}^p$ *, is* $\Delta\mathbf{G}^{\mathbf{p}}=\mathbf{G}^{\mathbf{p}}-\mathbf{G}^{\mathbf{p}-}$ *, is incrémentalement objective.*

One introduces $\boldsymbol{\tau}^{Tr}$, the tensor of Kirchhoff which results from an elastic prediction (Tr: trial, in English test):

$$\tilde{\boldsymbol{\tau}}^{Tr}=\mu\,\tilde{\bar{\mathbf{b}}}^{eTr}$$

where

$$\bar{\mathbf{b}}^{eTr}=\bar{\mathbf{F}}\,\mathbf{G}^{\mathbf{p}-}\,\bar{\mathbf{F}}^T=\Delta\bar{\mathbf{F}}\,\bar{\mathbf{b}}^{e-}\,\Delta\bar{\mathbf{F}}^T,\ \Delta\bar{\mathbf{F}}=(\Delta J)^{-1/3}\Delta\mathbf{F} \text{ and } \Delta J=\det(\Delta\mathbf{F})$$

One obtains $\bar{\mathbf{b}}^{e-}$ starting from the stresses $\boldsymbol{\tau}^-$ by the thermo-elastic relation stress-strain and of the trace of the tensor of the elastic strain.

$$\bar{\mathbf{b}}^{e-}=\frac{\tilde{\boldsymbol{\tau}}^{-}}{\mu^{-}}+\frac{1}{3}\mathrm{tr}\ \bar{\mathbf{b}}^{e-}$$

Note:

The interest of this formulation is that it is not necessary to calculate the plastic strain $\mathbf{G}^{\mathbf{p}-}$ which would oblige us to reverse the gradient of the transformation $\bar{\mathbf{F}}$. One needs to only know $\bar{\mathbf{F}}\mathbf{G}^{\mathbf{p}-}\bar{\mathbf{F}}^{T}$.

If $\boldsymbol{\tau}_{eq}^{Tr}<R(p^{-})+\sigma_{y}$, one remains elastic. In this case, one a:

$$p=p^{-}\ ,\ \boldsymbol{\tau}=\tilde{\boldsymbol{\tau}}^{Tr}+\frac{1}{3}\mathrm{tr}\,\boldsymbol{\tau}^{Tr}\mathbf{Id}\ \text{ and }\ \frac{1}{3}\mathrm{tr}\ \bar{\mathbf{b}}^{e}=\frac{1}{3}\mathrm{tr}\ \bar{\mathbf{b}}^{eTr}$$

if not, one obtains:

$\mathrm{tr}\ \bar{\mathbf{b}}^{e}=\mathrm{tr}\,\bar{\mathbf{b}}^{eTr}$, thanks to simplification on the flow model: $\bar{\mathbf{b}}^{e}=\bar{\mathbf{b}}^{e^{Tr}}-\mathrm{tr}\,\bar{\mathbf{b}}^{e}\dfrac{\tilde{\boldsymbol{\tau}}}{\tau_{eq}}\Delta p$

By taking the deviatoric parts of this equation, and by multiplying them by μ one leads to:

$$\tilde{\boldsymbol{\tau}}^{Tr}=\tilde{\boldsymbol{\tau}}\left(1+\frac{\mu\,\mathrm{tr}\ \bar{\mathbf{b}}^{eTr}\Delta p}{\tau_{eq}}\right)$$

While calculating the equivalent stress, one brings back oneself to a nonlinear scalar equation in Δp :

$$\tau_{eq}^{Tr}-\tau_{eq}-\mu\,\mathrm{tr}\,\bar{\mathbf{b}}^{eTr}\Delta p=0$$

In the plastic case : $\tau_{eq}=\sigma_{y}+R(p^{-}+\Delta p)$, which leads to Δp solution of the equation:

$$\tau_{eq}^{Tr}-\sigma_{y}-R(p^{-}+\Delta p)-\mu\,\mathrm{tr}\,\bar{\mathbf{b}}^{eTr}\Delta p=0$$

In the viscoplastic case: $\tau_{eq}=\sigma_{y}+R(p^{-}+\Delta p)+\sigma_{0}\mathrm{sh}^{-1}\left[\left(\dfrac{\Delta p}{\dot{\varepsilon}_{0}\Delta t}\right)^{\frac{1}{m}}\right]$, which leads to Δp solution of the equation:

$$\tau_{eq}^{Tr}-\sigma_{y}-R(p^{-}+\Delta p)-\sigma_{0}\mathrm{sh}^{-1}\left[\left(\frac{\Delta p}{\dot{\varepsilon}_{0}\Delta t}\right)^{\frac{1}{m}}\right]-\mu\,\mathrm{tr}\,\bar{\mathbf{b}}^{eTr}\Delta p=0$$

If hardening linear, or is given by a point by point defined curve of tension, therefore closely connected per pieces, the equation to be solved is linear. The solution directly is obtained. Δp In the other cases, the resolution is carried out in *Code_Aster* by a method of the secants with interval of search (cf [R5.03.05]). Integration can be controlled by parameters `RESI_INTE_RELA` and `ITER_INTE_MAXI`.

Once Δp known, one can then deduce the tensor from it from Kirchhoff, that is to say:

$$\boldsymbol{\tau}=\frac{\tau_{eq}}{\tau_{eq}^{Tr}}\tilde{\boldsymbol{\tau}}^{Tr}+\left[\frac{K}{2}(J^{2}-1)-\frac{3\mathrm{K}}{2}(T-T_{ref})(J+\frac{1}{J})\right]\mathbf{Id}$$

Once calculated the cumulated plastic strain, the tensor of the stresses and the tangent matrix, one carries out a correction on the trail of tensor of the elastic strain $\bar{\mathbf{b}}^e$ to take account of the plastic incompressibility, which is not preserved with the simplification made on the flow model [éq 4.2.1]. This correction is by means of carried out a relation between the invariants of $\bar{\mathbf{b}}^e$ and $\tilde{\bar{\mathbf{b}}}^e$ by exploiting the plastic condition of incompressibility $J^p=1$ (or in an equivalent way $\det \bar{\mathbf{b}}^e=1$). This relation is written:

$$x^3-\bar{J}_2^e x-(1-\bar{J}_3^e)=0$$

with $\bar{J}_2^e=\frac{1}{2}\bar{b}_{eq}^{e^2}=\frac{\tau_{\text{eq}}^2}{2\mu^2}$, $\bar{J}_3^e=\det\tilde{\bar{\mathbf{b}}}^e=\det\frac{\tilde{\boldsymbol{\tau}}}{\mu}$ and $x=\frac{1}{3}\operatorname{tr}\bar{\mathbf{b}}^e$

the solution of this cubic equation makes it possible to obtain $\operatorname{tr}\bar{\mathbf{b}}^e$ and consequently the thermo-elastic strain $\bar{\mathbf{b}}^{e-}$ with time step according to. If this equation admits several solutions, one takes the solution nearest to the solution of time step preceding. It is besides why one stores in a local variable $\frac{1}{3}\operatorname{tr}\bar{\mathbf{b}}^e$.

5 Variational formulation

Insofar as the stresses provided by the constitutive law are eulerian, one chooses a variational formulation written on the present configuration (eulerian) and not on the initial configuration, that is to say:

$$\underbrace{\int_\Omega \sigma \nabla_x \delta\mathbf{v}\, d\Omega}_{\mathbf{F}_{\text{int}}.\delta\mathbf{v}}=\underbrace{\int_\Omega \rho\,\mathbf{f}\,\delta\mathbf{v}\,d\Omega+\int_{\partial\Omega^f}\mathbf{t}^{\mathbf{d}}\,\delta\mathbf{v}\,\mathrm{dS}}_{\mathbf{F}_{\text{ext}}.\delta\mathbf{v}} \qquad \delta\mathbf{v}\ \text{Kinematically admissible}$$

We are interested only here in work of the internal forces and its variation in optics of a resolution by the method of Newton. One will find in [feeding-bottle 4] the demonstration of the statements presented.

5.1 Case of the continuum

One rewrites here the work of the internal forces in indicielle form, that is to say:

$$\mathbf{F}_{\text{int}}.\delta\mathbf{v}=\int_\Omega \sigma_{\text{ij}}\frac{\partial\delta v_i}{\partial x_j}d\Omega$$

We need also to express the variation of the internal forces in the present configuration Ω is:

$$\delta\mathbf{F}_{\text{int}}.\delta\mathbf{u}.\delta\mathbf{v}=\int_\Omega\left[\sigma_{ij}\frac{\partial\delta u_p}{\partial x_p}-\sigma_{ik}\frac{\partial\delta u_j}{\partial x_k}\right]\frac{\partial\delta v_i}{\partial x_j}d\Omega \quad \text{rigidité géométrique}$$
$$+\int_\Omega\left[\frac{\partial\sigma_{ij}}{\partial\Delta F_{pq}}\frac{\partial\delta u_p}{\partial x_q^-}\right]\frac{\partial\delta v_i}{\partial x_j}d\Omega \qquad \text{rigidité de comportement}$$

¿

where $\mathbf{x}^-$ are the punctual coordinates on the configuration Ω^- .

5.2 Discretization by finite elements

One discretizes displacements $\mathbf{u}$ and the virtual displacements $\mathbf{v}$ by finite elements. The notations are the following ones, by adopting the summation convention of the repeated indices:

$$u_i(x)=N^n(x)\,U_i^n \qquad \frac{\partial u_i}{\partial x_j}=D_j^n(x)\,U_i^n \qquad \frac{\partial u_i}{\partial x_j^-}=D_j^{-n}(x)\,U_i^n$$

where:

$N^n(x)$ is the shape function associated with the node n
U_i^n , component *I* of the nodal displacement of the node n
$D_j^n(x)$, the components of the gradient of the shape functions on the configuration Ω
$D_j^{-n}(x)$, the components of the gradient of the shape functions on the configuration Ω^-

One obtains for the vector of the internal forces:

$$(F_{\text{int}})_i^n=\int_\Omega \sigma_{ij}\,D_j^n\,d\,\Omega$$

and for the tangent matrix, which is not a priori symmetric:

$$K_{i\ p}^{n\,m}=\int_\Omega \left[D_p^m \sigma_{ij} D_j^n - D_k^m \sigma_{ik} D_p^n\right] d\,\Omega + \int_\Omega \left[D_q^{-n} \frac{\partial \sigma_{ij}}{\partial \Delta F_{pq}} D_j^n\right] d\,\Omega$$

In the case of a two-dimensional modelization (plane strain), the statements of the vector of the internal forces and tangent matrix are identical to this ready that the indices corresponding to the components only vary from 1 to 2.

In the case of an axisymmetric modelization, by numbering the axes in the order (r,z,θ), the vector of the internal forces is written:

$$(F_{\text{int}}^{axi})_\alpha^n=\int_\Omega \left[\sigma_{\alpha\beta} D_\beta^n + \sigma_{33} \frac{N^n}{r} \delta_{\alpha 1}\right] d\,\Omega \quad \alpha\in\{1,2\}\ ,\ \beta\in\{1,2\}$$

and the tangent matrix:

$$\left[\mathbf{K}^{axi}\right]=\left[\mathbf{K}\right]+\left[\mathbf{K}^{corr}\right]$$

with:

$$\left[\mathbf{K}_{(1)}^{\mathbf{corr}}\right]_{1\beta}^{nm}=\int_\Omega \frac{N^n}{r}\sigma_{\beta\gamma} D_\gamma^m\,d\,\Omega+\int_\Omega \frac{N^n}{r^-}\frac{\partial \sigma_{\beta\gamma}}{\partial \Delta F_{33}} D_\gamma^m\,d\,\Omega$$

$$\left[\mathbf{K}_{(2)}^{\mathbf{corr}}\right]_{\alpha 1}^{nm}=\int_\Omega D_\alpha^n \sigma_{33}\frac{N^m}{r}\,d\,\Omega+\int_\Omega D_\gamma^{-n}\frac{\partial \sigma_{33}}{\partial \Delta F_{\alpha\gamma}}\frac{N^m}{r}\,d\,\Omega$$

$$\left[\mathbf{K}_{(3)}^{\mathbf{corr}}\right]_{11}^{nm}=\int_\Omega \frac{N^n}{r^-}\frac{\partial \sigma_{33}}{\partial \Delta F_{33}}\frac{N^m}{r}$$

From an algorithmic point of view, the tangent elementary matrix $\mathbf{K}$ is not symmetric a priori. The total resolution will thus be made by default with an asymmetric solver. It is however possible to symmetrize the total matrix tangent before resolution (key word `solver`), which makes it possible to save time computation but can degrade convergence.

5.3 Form of the tangent matrix of the behavior

One gives the form of the tangent matrix here (option FULL_MECA during iterations of Newton, option RIGI_MECA_TANG for the first iteration). This one is obtained by linearizing the system of equations which governs the constitutive law. We give here the final result of this linearization. One will find in [bib4] the detail of this computation.

One poses:

$$J=\det \mathbf{F}\ ,\ J^{-}=\det \mathbf{F}^{-}\ \text{ and }\ \Delta J=\det \Delta \mathbf{F}$$

• For the option FULL_MECA , one a:

$$\mathbf{A}=\frac{\partial \boldsymbol{\sigma}}{\partial \Delta \mathbf{F}}=\frac{(\Delta J)^{-1/3}}{J}\mathbf{H}-\frac{1}{3J\Delta J}(\mathbf{H}\Delta \bar{\mathbf{F}})\otimes \mathbf{B}-\frac{J^{-}}{J^{2}}\boldsymbol{\tau}\otimes \mathbf{B}$$
$$+\frac{J^{-}}{J}\left[KJ-\frac{3}{2}K\alpha(T-T_{ref})(1-J^{-2})\right]\mathbf{Id}\otimes \mathbf{B}$$

where **B** is worth:

$$\begin{aligned}
B_{11}&=\Delta F_{22}\Delta F_{33}-\Delta F_{23}\Delta F_{32}\\
B_{22}&=\Delta F_{11}\Delta F_{33}-\Delta F_{13}\Delta F_{31}\\
B_{33}&=\Delta F_{11}\Delta F_{22}-\Delta F_{12}\Delta F_{21}\\
B_{12}&=\Delta F_{31}\Delta F_{23}-\Delta F_{33}\Delta F_{21}\\
B_{21}&=\Delta F_{13}\Delta F_{32}-\Delta F_{33}\Delta F_{12}\\
B_{13}&=\Delta F_{21}\Delta F_{32}-\Delta F_{22}\Delta F_{31}\\
B_{31}&=\Delta F_{12}\Delta F_{23}-\Delta F_{22}\Delta F_{13}\\
B_{23}&=\Delta F_{31}\Delta F_{12}-\Delta F_{11}\Delta F_{32}\\
B_{32}&=\Delta F_{13}\Delta F_{21}-\Delta F_{11}\Delta F_{23}
\end{aligned}$$

H and $\mathbf{H}\Delta \bar{\mathbf{F}}$ are given by:

In the elastic case ($f<0$):

$$H_{\mathrm{ijkl}}=\mu\left(\delta_{\mathrm{ik}}\bar{b}_{\mathrm{lp}}^{e-}\,\Delta \bar{F}_{\mathrm{jp}}+\Delta \bar{F}_{\mathrm{ip}}\bar{b}_{\mathrm{pl}}^{e-}\,\delta_{\mathrm{jk}}\right)-\frac{2\mu}{3}\delta_{\mathrm{ij}}\Delta \bar{F}_{\mathrm{kp}}\bar{b}_{\mathrm{lp}}^{e-}$$

and

$$\mathbf{H}\Delta \bar{\mathbf{F}}=2\mu\,\tilde{\bar{\mathbf{b}}}^{\mathrm{eTr}}$$

In the plastic case ($f=0$) or viscoplastic:

$$H_{\mathrm{ijkl}}=\frac{\mu}{a}(\delta_{\mathrm{ik}}\bar{b}_{\mathrm{lp}}^{e-}\,\Delta \bar{F}_{\mathrm{jp}}+\Delta \bar{F}_{\mathrm{ip}}\bar{b}_{\mathrm{pl}}^{e-}\,\delta_{\mathrm{jk}})$$
$$-2\mu\left[\frac{\delta_{\mathrm{ij}}}{3\mathrm{a}}+\frac{\bar{R}\Delta p\,\tilde{\tau}_{\mathrm{ij}}}{\tau_{\mathrm{eq}}(\bar{R}+\mu\,\mathrm{tr}\,\bar{b}^{\mathrm{eTr}})}\right]\Delta \bar{F}_{\mathrm{kp}}\bar{b}_{\mathrm{lp}}^{e-}$$
$$+\frac{3\mu^{2}\,\mathrm{tr}\,\bar{b}^{\mathrm{eTr}}(\bar{R}\Delta p-\tau_{\mathrm{eq}})}{a\tau_{\mathrm{eq}}^{3}(\bar{R}+mu\,\mathrm{tr}\,\bar{b}^{\mathrm{eTr}})}\tilde{\tau}_{\mathrm{ij}}\tilde{\tau}_{\mathrm{kq}}\Delta \bar{F}_{\mathrm{qp}}\bar{b}_{\mathrm{lp}}^{e-}$$

and

$$\mathbf{H}\Delta \bar{F}=\frac{2\mu}{a}\bar{\mathbf{b}}^{eTr}-2\mu\,\mathrm{tr}\,\bar{\mathbf{b}}^{eTr}\left[\frac{\mathbf{Id}}{3\mathrm{a}}+\frac{\bar{R}\Delta p\tilde{\boldsymbol{\tau}}}{\tau_{\mathrm{eq}}(\bar{R}+\mu\,\mathrm{tr}\,\bar{\mathbf{b}}^{eTr})}\right]+\frac{3\mu^2\,\mathrm{tr}\,\bar{\mathbf{b}}^{eTr}(\bar{R}\Delta p-\tau_{eq})}{a\tau_{eq}^3(\bar{R}+\mu\,\mathrm{tr}\,\bar{\mathbf{b}}^{eTr})}(\tilde{\boldsymbol{\tau}}:\bar{\mathbf{b}}^{eTr})\tilde{\boldsymbol{\tau}}$$

where $a=\frac{\tau_{eq}^{Tr}}{\tau_{eq}}$

and $$\bar{R}=\mathrm{R}'(p)+\underbrace{\sigma_0\times\left(1+\left(\frac{\Delta p}{\dot{\varepsilon}_0\Delta t}\right)^2\right)^{\frac{-1}{2}}\times\frac{1}{m\left(\dot{\varepsilon}_0\Delta t\right)^{\frac{1}{m}}}\times(\Delta p)^{1/m-1}}_{\text{uniquement cas visqueux}},$$

$R'(p)$ being the derivative of isotropic hardening compared to the cumulated plastic strain p .

- For the option `RIGI_MECA_TANG`, it acts of the same statements as those given for `FULL_MECA` but with $\Delta p=0$ and all the variables and coefficients of the material taken to time t^- (in theory, it would be necessary in the viscous case, to take the statements of `FULL_MECA` in the elastic case, all the variables being taken at time t^-). In particular, one will have $\Delta\bar{\mathbf{F}}=\mathbf{Id}$.

6 Bibliography

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7 Description of the versions of the document

Version Aster	Author (S) or contributor (S), organization	Description of the modifications
4.4	V.Cano, E.Lorentz EDF/R & D /AMA	initial Text
6.3	V.Cano, E.Lorentz EDF/R & D /AMA	Card-indexes 6396
7.4	S.Michel-Ponnelle EDF/R & D /AMA	Card-indexes 8000: addition of viscosity
9.4	J.M.Proix EDF/R & D/ AMA Addition of	VMIS_ISOT_PUIS .