Constitutive law in large rotations and small strains

Summarized:

One describes here the formulation adopted for treating large rotations and small strains. This formulation is valid for all the constitutive laws defined under COMP_INCR of the command STAT_NON_LINE and provided with the modelizations three-dimensional (3D), axisymmetric (AXIS), in plane strains (D_PLAN) and plane stresses (C_PLAN).
This functionality is selected via the key word DEFORMATION = "GROT_GDEP" under COMP_INCR.

Contents

1 Some definitions

One point out here some definitions of tensors related to the large deformations.
One calls tensor gradient of the transformation $\mathbf{F}$, the tensor which makes pass from the initial configuration Ω_0 to the deformed present configuration $\Omega(t)$.

$$\mathbf{F}=\frac{\partial\hat{\mathbf{x}}}{\partial\mathbf{X}}=\mathbf{Id}+\nabla_{\mathbf{X}}\mathbf{u} \text{ with } \mathbf{x}=\hat{\mathbf{x}}(\mathbf{X},t)=\mathbf{X}+\mathbf{u} \qquad \textbf{éq 1-1}$$

where $\mathbf{X}$ is the position of a point in Ω_0 , $\mathbf{x}$ the position of this same point after strain in $\Omega(t)$ and $\mathbf{u}$ displacement.
Various strain tensors can be obtained by eliminating rotation in the local transformation. This can be done of two ways, either by means of the theorem of polar decomposition, or by directly calculating the variations length and angle (variation of the scalar product).
Of Lagrangian description is obtained (i.e. on the initial configuration):

- By polar decomposition:

$$\mathbf{F}=\mathbf{R}\,\mathbf{U} \qquad \textbf{éq 1-2}$$

where $\mathbf{R}$ is the rotation tensor (orthogonal) and $\mathbf{U}$ the strain tensor pure right (symmetric and definite positive).

- By a direct computation of the strains:

$$\mathbf{E}=\frac{1}{2}(\mathbf{C}-\mathbf{Id}) \text{ with } \mathbf{C}=\mathbf{F}^T\mathbf{F} \qquad \textbf{éq 1-3}$$

where $\mathbf{E}$ is the strain tensor of Green-Lagrange and $\mathbf{C}$ the tensor of right Cauchy-Green.
The tensors $\mathbf{U}$ and $\mathbf{C}$ are connected by the following relation:

$$\mathbf{C}=\mathbf{U}^2 \qquad \textbf{éq 1-4}$$

2 Assumption of the small strains and large rotations

When the strains are small, it does not have there fundamental difficulties in writing the constitutive laws: the various models “large deformations” lead to the same model “small strains”, and this as well for isotropic behaviors as anisotropic. Only the difficulty of a geometrical nature related to finished rotation remains.

To write the model in large rotations and small strains, one leaves polar decomposition $\mathbf{F}$ is $\mathbf{F}=\mathbf{R}\,\mathbf{U}$. As the tensor $\mathbf{U}$ is a strain tensor pure and in addition small, one can calculate, by a constitutive law small strains, the tensor of the stresses $\boldsymbol{\sigma}^*$ associated with this history in strain $\mathbf{U}$. It is then enough to subject to this tensor $\boldsymbol{\sigma}^*$, rotation $\mathbf{R}$ to obtain the tensor of the stresses $\boldsymbol{\sigma}$ associated with the history in strain $\mathbf{F}$, as follows:

$$\boldsymbol{\sigma}=\mathbf{R}\,\boldsymbol{\sigma}^*\mathbf{R}^T \qquad \textbf{éq 2-1}$$

One can summarize this diagram as follows:

$$\mathbf{F}\rightarrow\mathbf{U}=\mathbf{Id}+\boldsymbol{\varepsilon}\overset{ldc\,HPP}{\rightarrow}\boldsymbol{\sigma}^{*}\rightarrow\boldsymbol{\sigma}=\mathbf{R}\,\boldsymbol{\sigma}^{*}\,\mathbf{R}^{T}$$ **éq the 2-2**

disadvantage of this computation channel is that it requires the polar decomposition of $\mathbf{F}$. Two assumptions are made then to avoid it.
On the one hand, to avoid the computation of $\mathbf{U}$, one can approach the strain HP $\boldsymbol{\varepsilon}$, by the strain of Green $\mathbf{E}$, by benefiting owing to the fact that the strains are small:

$$\mathbf{E}=\frac{1}{2}(\mathbf{F}^{T}\mathbf{F}-\mathbf{Id})=\frac{1}{2}(\mathbf{U}-\mathbf{Id})(\mathbf{U}+\mathbf{Id})=\boldsymbol{\varepsilon}+\frac{1}{2}\boldsymbol{\varepsilon}^{2}\approx\boldsymbol{\varepsilon}$$ **éq 2-3**

One from of then deduced $\boldsymbol{\sigma}^{*}$ by the constitutive law “small strains”.
In addition, the same way to avoid the computation of $\mathbf{R}$, one can approach the tensor of the stresses HP $\boldsymbol{\sigma}^{*}$ by the second tensor of Piola-Kirchhoff $\mathbf{S}$:

$$\mathbf{S}=J\,\mathbf{F}^{-1}\boldsymbol{\sigma}\,\mathbf{F}^{-T}=\mathrm{Det}(\mathbf{U})\,\mathbf{U}^{-1}\boldsymbol{\sigma}^{*}\mathbf{U}^{-1}=\boldsymbol{\sigma}^{*}+\boldsymbol{\sigma}^{*}\,\mathrm{O}(\boldsymbol{\varepsilon})\approx\boldsymbol{\sigma}^{*}$$ **éq 2-4**

One from of deduced then $\boldsymbol{\sigma}$ by:

$$\boldsymbol{\sigma}=\frac{1}{J}\mathbf{F}\mathbf{S}\mathbf{F}^{T}$$ **éq 2-5**

Finally, in the presence of large rotations and to small strains, it is enough to write the constitutive law “small strains” with, as starter, the history of the strains of Green $\mathbf{E}$, and in output, the history of the Piola-Kirchhoff stresses $\mathbf{S}$. This approach is valid as well for isotropic constitutive laws as anisotropic.
As for the adapted variational formulation, it is about that adopted in very-elasticity (behavior `ELAS`, `ELAS_VMIS_XXX` under `COMP_ELAS` with the strains of the type `GROT_GDEP)`. For more details, one will refer to the associated reference document [R5.03.20]. It is necessary however to be sure that the problem studied induced many small strains because if not one cannot make any more simplifications [éq 2-3] and [éq 2-4]. Without this assumption, the variation with a plastic behavior increases quickly with the intensity of the strains.

3 Bibliography

1.CANO V., LORENTZ E., “Introduction into *the Code_Aster* of a model of behavior in elastoplastic large deformations with isotropic hardening”, intern EDF DER, HI-74/98/006/0, 1998 Notes

4 Description of the versions of the document

Version Aster	Author (S) Organization (S)	Description of the modifications
6	V.Cano EDF-R&D/AMA	initial Text
10.1	J.M.Proix EDF-R&D/ AMA Change	of GREEN in GROT_GDEP