Models of large deformations GDEF_LOG and GDEF_HYPO_ELAS

Summarized:

This document presents the hypoelastic formulations of large deformations used in Code_Aster for the constitutive laws of the type von Mises.

GDEF_LOG : this model, due to C.Miehe, N.Appel and M.Lambrecht [13] is a model of large deformations based to a logarithmic curve measure, with a particular stress tensor in duality. It is valid whatever the behavior in small strains and has the advantage of providing a symmetric tangent matrix. No kinematical modification of the local variables is necessary.

GDEF_HYPO_ELAS : this model is resulting from an approach due to Simo and Hughes [1]. It is based on the notion of a turned configuration objectifies in which the derivatives are carried out. This generic approach makes it possible to treat the constitutive laws with hardenings isotropic and kinematical, with or without viscosity, with formulation of a hypoelastic type.

These two models allow an integration incrémentalement objectifies constitutive laws like SIMO_MIEHE the model. However, like all the hypoelastic models, these constitutive laws in any rigor are limited to the weak elastic strain.

One and the illustrates in this document the capacities of these models advantages compared to the approximation PETIT_REAC.

Contents

1 Introduction

Most constitutive laws of Code_Aster are usable under the assumption of the small disturbances (HP), which makes it possible to confuse the geometrical configurations initial and current. However, when the strains become important (one in general fixes the limit at 5%), this assumption is not checked any more. The notions of derivatives particulate and partial are then different, and of this fact the constitutive laws formulated incrémentalement lose their objective character (independence of the mechanical state according to the observer); a tedious consequence is the possible evolution of the stresses for a rigid body motion, opposite with physics.

In order to find objectivity, essential thus to guarantee a good reliability of result, various strategies large deformations are possible. The object of this document is to present the formalisms set up in Code_Aster to treat the constitutive laws with hardenings isotropic and kinematical and criterion of Von Mises.

One presents firstly the formalism hypoelastic of Simo-Hughes [1] usable in Code_Aster *via* key word `DEFORMATION=GDEF_HYPO_ELAS` of operator `COMP_INCR`. One result presents then one allowing to check the objectivity of this integration and to check that the formalism `PETIT_REAC` is put at fault.

In a second part, one presents formalism `GDEF_LOG`, due to C.Miehe, N.Appel and M.Lambrecht [13] which is a model of large deformations based to a logarithmic curve measure, with a particular stress tensor in duality. It is valid some is the behavior in small strains and present the advantage of providing a symmetric tangent matrix. No kinematical modification of the local variables is necessary.

2 Algorithm GDEF_HYPO_ELAS

2.1 Elements of kinematics

the kinematical elements in the continuous case can be found for example in [3]. One will be interested here in the case directly discretized in time allowing to define the quantities used in the formalism presented in this document.

One considers a closed initial continuous field $\Omega_0 \subset \mathbb{R}^3$, whose each point is located by its coordinates $\boldsymbol{X} \in \Omega_0$, undergoing a strain field φ making it pass in the configuration Ω :

$$\varphi : \Omega_0 \rightarrow \Omega \subset \mathbb{R}^3 \quad \text{Eq1}$$

One will note $\boldsymbol{x} \in \Omega$ the coordinates of this point in the current configuration.
The deformed shape evolving in the course of time, one actually defines, by the means of the temporal discretization, a family of field φ_n corresponding each one to one time t_n of the history of evolution of the field.
In the case of the formalism large deformations treated here, it is necessary to introduce 4 configurations for the field and its evolution (cf Figure 1): initial Ω_0 configuration of reference (i.e. for which the strains are null), configuration Ω_n at the beginning of time step running $t_{n+1} = t_n + \Delta t$, configuration Ω_{n+1} at the end of this time step, and a configuration medium of time step $\Omega_{n+\frac{1}{2}}$, formalism being integrated with a rule of point medium.

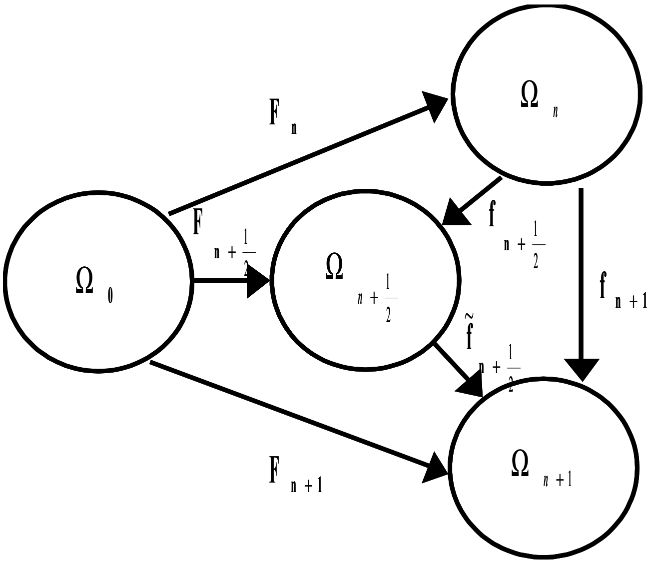


Figure 1: Configurations necessary and gradients of transformation

From these configurations, one defines the fields of displacements and the gradients of transformation to pass from the one to the other. The quantities making pass from the initial configuration to a given configuration are noted in capital letter $(\boldsymbol{U},\boldsymbol{F})$ and the quantities connecting two configurations deformed between them are noted into tiny $(\boldsymbol{u},\boldsymbol{f})$. Table 1 recapitulates the various quantities and their statements.

Configuration starting	Configuration of arrival	Gradient	Displacement of transformation
Ω_0	Ω_n	$\boldsymbol{U_n}$	$\boldsymbol{F_n}=\boldsymbol{I_d}+grad_0\boldsymbol{U_n}$
Ω_0	Ω_{n+1}	$\boldsymbol{U_{n+1}}$	$\boldsymbol{F_{n+1}}=\boldsymbol{I_d}+grad_0\boldsymbol{U_{n+1}}$
Ω_0	$\Omega_{n+\frac{1}{2}}$		$\boldsymbol{F}_{\boldsymbol{n+\frac{1}{2}}}=\frac{1}{2}(\boldsymbol{F_n}+\boldsymbol{F_{n+1}})$
Ω_n	$\Omega_{n+\frac{1}{2}}$		$\boldsymbol{f}_{\boldsymbol{n+\frac{1}{2}}} = \boldsymbol{I_d}+\frac{1}{2}grad_n\boldsymbol{u}$ $= \boldsymbol{F}_{\boldsymbol{n+\frac{1}{2}}}\cdot\boldsymbol{F_n^{-1}}$ $= \frac{1}{2}(\boldsymbol{f_{n+1}}+\boldsymbol{I_d})$
Ω_n	Ω_{n+1}	$\boldsymbol{u}$	$\boldsymbol{f_{n+1}} = \boldsymbol{I_d}+grad_n\boldsymbol{u} = \Delta\boldsymbol{F}$ $= \boldsymbol{F_{n+1}}\cdot\boldsymbol{F_n^{-1}}$
$\Omega_{n+\frac{1}{2}}$	Ω_{n+1}		$\tilde{\boldsymbol{f}}_{\boldsymbol{n+\frac{1}{2}}} = \boldsymbol{f_{n+1}}\cdot\boldsymbol{f}^{-1}_{\boldsymbol{n+\frac{1}{2}}}$ $= \boldsymbol{F_{n+1}}\cdot\boldsymbol{F}^{-1}_{\boldsymbol{n+\frac{1}{2}}}$

Table 1: summary of displacements and gradients of transformation

From these gradients of transformation, it is possible to define strain rates $\boldsymbol{L}$ as well as tensor rates of rotation $\boldsymbol{\omega}$ and strain rate $\boldsymbol{d}$:

$$\begin{cases} \boldsymbol{L}_. &= \dot{\boldsymbol{F}}_. \boldsymbol{F}_.^{-1} \\ \boldsymbol{d}_. &= \frac{1}{2}(\boldsymbol{L}_. + \boldsymbol{L}_.^T) \\ \boldsymbol{\omega}_. &= \frac{1}{2}(\boldsymbol{L}_. - \boldsymbol{L}_.^T) \end{cases} \quad \text{Eq 2}$$

with $.$ indicating the configuration n , $n+1$ or $n+\frac{1}{2}$.

The Eulerian strain tensor enters the configurations Ω_n and Ω_{n+1} results from these definitions:

$$\boldsymbol{e}_{n+1} = \frac{1}{2}[\boldsymbol{I}_d - (\boldsymbol{f}_{n+1} \boldsymbol{f}_{n+1}^T)^{-1}] \quad \text{Eq 3}$$

what makes it possible to write $\boldsymbol{d}_{n+\frac{1}{2}} = \frac{1}{\Delta t} \tilde{\boldsymbol{f}}^{-1}_{n+\frac{1}{2}} \boldsymbol{e}_{n+1} \tilde{\boldsymbol{f}}_{n+\frac{1}{2}}$ (strain rate is then quite related to the eulerian strain)

the last quantity to be introduced for our algorithm is the incremental displacement gradient, relating to the configuration $\Omega_{n+\frac{1}{2}}$ and defined by:

$$\boldsymbol{h}_{n+\frac{1}{2}} = grad_n \boldsymbol{u} \, \boldsymbol{f}^{-1}_{n+\frac{1}{2}} \quad \text{Eq 4}$$

This last makes it possible to determine the rate of rotation in the same configuration by the relation:

$$\boldsymbol{\omega}_{n+\frac{1}{2}} = \frac{1}{2\Delta t}[\boldsymbol{h}_{n+\frac{1}{2}} - \boldsymbol{h}^T_{n+\frac{1}{2}}] \quad \text{Eq5}$$

From these elements of kinematics, it is possible to define hypoelastic constitutive laws whose integration is objective in large deformations. The following paragraph presents this kind of formulation of the constitutive laws.

3 Hypo-elastoplastic constitutive laws

In this section, the phenomenologic class of model of plasticity (here independent of time) with hypoelasticity is considered. It constitutes an ad hoc *extension* of the writing of the models in small strains, which allows certain generics and represents an advantage in the context of a computer code: one will see in the chapter according to whether it is possible to carry out its numerical integration in a way equivalent to that of the small strains.

This class of models is to be opposed to the class hyper elastic, based on the thermodynamic approach of the continuums. In this context, a free energy, being able for example to be regarded as a function of the temperature and the strain of Green-Lagrange, is defined; the evolutions of the stresses and possibly of the local variables result from this. One can quote for example the case of the model hyper elastic of Signorini (cf [4]) in elasticity and the Simo-Miehe formalism in hyperelastoplasticity (cf [3]).

An hypo-elastoplastic constitutive law is generally built in five stages.
(I) Following the example of the additive decomposition of the small strains, strain rate $\boldsymbol{d}$ is first of all broken up into an elastic part and a plastic part:

$$\boldsymbol{d} = \boldsymbol{d}^e + \boldsymbol{d}^p \quad \text{Eq 6}$$

(II) a derivative of the stress of Kirchhoff $\boldsymbol{\tau}=det(\boldsymbol{F})\boldsymbol{\sigma}$ is then determined by an incremental relation function of the elastic rate of strain:

$$\mathring{\boldsymbol{\tau}}=\boldsymbol{C}:[\boldsymbol{d}-\boldsymbol{d}^p] \quad \text{Eq7}$$

with $\mathring{\boldsymbol{x}}$ a derivative objectifies to define and $\boldsymbol{C}$ the elasticity tensor.
(III) One builds a field of reversibility convex defining acceptable space of the stresses from a function f :

$$E_\tau=\{(\boldsymbol{\tau},\boldsymbol{q},\alpha)\in\boldsymbol{S}\,X\,\Re^{m+1}|f(\boldsymbol{\tau},\boldsymbol{q},\alpha)\leqslant 0\} \quad \text{Eq 8}$$

with $\boldsymbol{S}$ the space of the stresses and $\boldsymbol{q}$ all m the local variables representing the kinematic hardening of the material and the α variable scalars (including isotropic hardening).

(iv) the laws of evolution of these local variables follow a principle of normality (one considers only the associated constitutive laws here):

$$\begin{aligned}\boldsymbol{d}^p &= \gamma\frac{\partial f(\boldsymbol{\tau},\boldsymbol{q},\alpha)}{\partial\boldsymbol{\tau}}\\ \mathring{\boldsymbol{q}} &= -\gamma\,\boldsymbol{g}(\boldsymbol{\tau},\boldsymbol{q},\alpha)\end{aligned} \quad \text{Eq9}$$

with $\gamma\geqslant 0$ the plastic multiplier, $\frac{\partial f}{\partial\tau}(\boldsymbol{\tau},\boldsymbol{q},\alpha)$ defining the direction of yielding and $\boldsymbol{g}(\boldsymbol{\tau},\boldsymbol{q},\alpha)$ the evolution of the other local variables.

(v) the writing of the conditions of load/discharge, classically represented by Kuhn-Tucker and the condition of coherence:

$$\begin{cases}\gamma\geqslant 0\\ f(\boldsymbol{\tau},\boldsymbol{q},\alpha)\leq 0\\ \gamma f(\boldsymbol{\tau},\boldsymbol{q},\alpha)=0\end{cases} \quad \text{Eq 10}$$

This class of model is thus characterized by a strong analogy with the formalisms small strains, with an incremental writing of the stresses which is not without raising some difficulties of numerical integration: indeed, in order to prevent the evolutions of stresses by a rigid body motion, it is advisable to have an objective integration of the equation (Eq7).

The object of the following chapter is to describe the objective algorithm of integration used in Code_Aster, which had in Simo-Hughes.

3.1 Objective integration: algorithm GDEF_HYPO_ELAS

the preceding chapter defined the class of material hypo-elastic, of which one of the characteristics is to calculate the stresses incrémentalement. It is advisable to have an integration objectifies of it in order to during secure against an evolution of the stresses a rigid body motion.

For that, two large types of approaches exist: approaches by definition of an objective derivative (of Jaumann type or Binds), and the approaches by definition of an objective turned reference frame in which the derivatives are calculated. These approaches are in fact equivalent (see [1]) for the continuous problem, but the integration of objective derivatives often pose problem (cf [8]).

The algorithm describes here is based on a turned reference frame, which had in Simo-Hughes [1]. The idea is to use a description cancelling the effects of rotations in temporal derivatives. This kind of approach is formalized thus. That is to say $\tilde{\boldsymbol{\theta}}$ a skew-symmetric tensor of order 2 unspecified. One can build the equation of evolution of a clean orthogonal tensor $\boldsymbol{\Lambda}(\boldsymbol{X},t)\in SO_3$ from $\tilde{\theta}$ following form:

$$\dot{\boldsymbol{\Lambda}}=(\tilde{\boldsymbol{\theta}}\, o\, \varphi) \qquad \boldsymbol{\Lambda}|_{t=0}=\mathbf{1} \qquad \text{Eq11}$$

With $\tilde{\boldsymbol{\theta}}$ given, the solutions of (Eq11) generate a sub-group of the orthogonal special group (sub-group defined by a parameter), representing a turned local reference frame.

One can then define the stress of Kirchhoff and strain rate in this reference frame:

$$\boldsymbol{\Sigma} := \boldsymbol{\Lambda}^T \boldsymbol{\tau} \boldsymbol{\Lambda} \qquad \boldsymbol{D} = \boldsymbol{\Lambda}^T \boldsymbol{d} \boldsymbol{\Lambda} \qquad \text{Eq12}$$

the temporal derivative of the stress of Kirchhoff turned is then:

$$\dot{\boldsymbol{\Sigma}}=\boldsymbol{\Lambda}^T[\dot{\boldsymbol{\tau}}+\boldsymbol{\tau}\boldsymbol{\theta}-\boldsymbol{\theta}\boldsymbol{\tau}]\boldsymbol{\Lambda}:=\boldsymbol{\Lambda}^T\overset{\circ}{\boldsymbol{\tau}}\boldsymbol{\Lambda} \qquad \text{Eq13}$$

what reveals an objective derivative (from where equivalence enters the approaches). Choice of $\tilde{\boldsymbol{\theta}}$ depends objective derivative obtained.

Constitutive law of the hypo-elastoplastic type defined in paragraph 3 is rewritten in this reference frame:

$$\begin{array}{ll}(i) & \boldsymbol{D}:=\boldsymbol{\Lambda}^T\boldsymbol{D}\boldsymbol{\Lambda}\,;\boldsymbol{D}=\boldsymbol{D}^e+\boldsymbol{D}^p \\ (ii) & \dot{\boldsymbol{\Sigma}}=\boldsymbol{a}:[\boldsymbol{D}-\boldsymbol{D}^p] \\ (iii) & E_{\boldsymbol{\Sigma}}=\{(\boldsymbol{\Sigma},\boldsymbol{Q},\alpha)\in\boldsymbol{S}\,X\,\Re^m | F(\boldsymbol{\Sigma},\boldsymbol{Q},\alpha)\leqslant 0\} \\ (iv) & \boldsymbol{D}^p=\gamma\dfrac{\partial F(\boldsymbol{\Sigma},\boldsymbol{Q},\alpha)}{\partial\boldsymbol{\Sigma}}\,;\dot{\boldsymbol{Q}}=-\gamma\,\boldsymbol{g}(\boldsymbol{\Sigma},\boldsymbol{Q},\alpha) \\ (v) & \gamma\geqslant 0\,;F(\boldsymbol{\tau},\boldsymbol{q},\alpha)\geqslant 0\,;\gamma F(\boldsymbol{\tau},\boldsymbol{q},\alpha)=0\,;\gamma\dot{F}(\boldsymbol{\tau},\boldsymbol{q},\alpha)\end{array} \qquad \text{Eq14}$$

This system of equations is integrated numerically by an algorithm of point medium describes hereafter.

3.2 Algorithm of integration

(I) the first stage consists in discretizing temporally the equations (Eq14). A rule of classical point medium is first of all used for derivative of the stress of Kirchhoff in the turned reference (Eq14 (II)) :

$$\begin{aligned}\boldsymbol{\Sigma}_{n+1}-\boldsymbol{\Sigma}_n &= \Delta t\,\dot{\boldsymbol{\Sigma}}_{n+\frac{1}{2}} \\ &= \Delta t\,\boldsymbol{a}_{n+\frac{1}{2}}:\boldsymbol{D}_{n+\frac{1}{2}} \\ &= \boldsymbol{a}_{n+\frac{1}{2}}:\boldsymbol{\Lambda}^T_{n+\frac{1}{2}}[\Delta t\,\boldsymbol{d}_{n+\frac{1}{2}}]\boldsymbol{\Lambda}_{n+\frac{1}{2}} \\ &= \boldsymbol{a}_{n+\frac{1}{2}}:\boldsymbol{\Lambda}^T_{n+\frac{1}{2}}[\boldsymbol{f}^{\tilde{T}}_{n+\frac{1}{2}}\boldsymbol{e}_{n+1}\boldsymbol{f}^{\sim}_{n+\frac{1}{2}}]\boldsymbol{\Lambda}_{n+1}\end{aligned} \qquad \text{Eq15}$$

and, finally, while posing $\boldsymbol{e}^{\sim}_{n+\frac{1}{2}}=\boldsymbol{f}^{\tilde{T}}_{n+\frac{1}{2}}\boldsymbol{e}_{n+1}\boldsymbol{f}^{\sim}_{n+\frac{1}{2}}$:

$$\boldsymbol{\Sigma}_{n+1}=\boldsymbol{\Sigma}_n+\boldsymbol{a}_{n+\frac{1}{2}}:\boldsymbol{\Lambda}^T_{n+\frac{1}{2}}\boldsymbol{e}^{\sim}_{n+\frac{1}{2}}\boldsymbol{\Lambda}_{n+\frac{1}{2}} \qquad \text{Eq16}$$

the equation (Eq16) makes it possible to see that the algorithm is objective (invariance of the stress by rotation of rigid body):

$$\boldsymbol{f}_{n+1}\in SO_3\Leftrightarrow\boldsymbol{e}_{n+1}=\mathbf{0}\Leftrightarrow\boldsymbol{\Sigma}_{n+1}=\boldsymbol{\Sigma}_n \qquad \text{Eq17}$$

(II) This integration of the stress is then coupled with an algorithm of radial return completely similar to that used in small strains. The complete algorithm which rises for a criterion of the type von Mises with

isotropic and/or kinematical hardening is described in Box 1 hereafter. In practice, in the case of treated plasticity, the elastic matrix is constant, except introducing a command variable like the temperature; in this case however, it is the elastic matrix at the time t_{n+1} which is actually used.
One thus finds the same type of diagram as in [R5.03.02] or [R5.03.16].

Note:
The direct integration of the equation (Eq14) (II) by a derivative of the type Jaumann $\overset{\circ}{\boldsymbol{\sigma}}^J=\boldsymbol{a}:[\boldsymbol{D}-\boldsymbol{D}^p]=\dot{\boldsymbol{\sigma}}+\boldsymbol{\sigma}\theta-\theta\boldsymbol{\sigma}$ is necessarily explicit on the term of transport of derivative, generating secondary stresses which tend towards 0 with the step of load. It is this kind of difficulty which is circumvented by the use of an objective local coordinate system.

(I) Knowing the fields of displacement, compute gradients of transformation:

$$\boldsymbol{f}_{n+1} = \boldsymbol{I}_d+grad_n\boldsymbol{u} \quad \boldsymbol{f}_{n+\frac{1}{2}} = \boldsymbol{I}_d+\frac{1}{2}grad_n\boldsymbol{u}\ ,\ \tilde{\boldsymbol{f}}_{n+\frac{1}{2}} = \boldsymbol{f}_{n+1}\cdot\boldsymbol{f}^{-1}_{n+\frac{1}{2}}$$

(II) Increment of strain in the turned reference frame:

$$\tilde{\boldsymbol{e}}_{n+\frac{1}{2}}=\frac{1}{2}\boldsymbol{f}^{\tilde{T}}_{n+\frac{1}{2}}[\boldsymbol{I}_d-(\boldsymbol{f}_{n+1}\boldsymbol{f}^T_{n+1})^{-1}]_{n+1}\tilde{\boldsymbol{f}}_{n+1}$$

(III) relative Rotation tensors:

$$\boldsymbol{r}_{n+1}=\boldsymbol{\Lambda}_{n+1}\boldsymbol{\Lambda}^T_n\ ,\ \boldsymbol{r}_{n+\frac{1}{2}}=\boldsymbol{\Lambda}_{n+1}\boldsymbol{\Lambda}^T_{n+\frac{1}{2}}$$

(iv) Elastic Prediction:

$$\boldsymbol{\tau}^{pred}_{n+1}=\boldsymbol{r}_{n+1}\boldsymbol{\tau}_n\boldsymbol{r}^T_{n+1}+\boldsymbol{a}:[\boldsymbol{r}_{n+\frac{1}{2}}\tilde{\boldsymbol{e}}_{n+\frac{1}{2}}\boldsymbol{r}^T_{n+\frac{1}{2}}]$$

$$\boldsymbol{q}^{pred}_{n+1}=\boldsymbol{r}_{n+1}\boldsymbol{q}_n\boldsymbol{r}^T_{n+1}$$

$$\boldsymbol{\eta}^{pred}_{n+1}=dev[\boldsymbol{\tau}^{pred}_{n+1}-\boldsymbol{q}^{pred}_{n+1}]$$

$$f^{pred}_{n+1}=\|\boldsymbol{\eta}^{pred}_{n+1}\|-\sqrt{\frac{2}{3}}(\sigma_Y+K\alpha_n)$$

(v) plastic Correction if necessary:

SO $f^{pred}_{n+1}<0$ THEN $(\cdot)_{n+1}=(\cdot)^{pred}_{n+1}$

IF NOT

$$\Delta\gamma=\frac{\langle f^{pred}_{n+1}\rangle/2\mu}{1+\dfrac{H+K}{3\mu}}$$

$$\boldsymbol{\tau}_{n+1}=\boldsymbol{\tau}^{pred}_{n+1}-2\mu\Delta\gamma\frac{\boldsymbol{\eta}^{pred}_{n+1}}{\|\boldsymbol{\eta}^{pred}_{n+1}\|}$$

$$\boldsymbol{q}_{n+1}=\boldsymbol{q}^{pred}_{n+1}+\frac{2}{3}\mu\Delta\gamma H\frac{\boldsymbol{\eta}^{pred}_{n+1}}{\|\boldsymbol{\eta}^{pred}_{n+1}\|}$$

$$\alpha_{n+1}=\alpha_n+\sqrt{\frac{2}{3}}\Delta\gamma$$

FIN IF

Table 3.2-1 : Box 1 - algorithm of resolution

(III) the last point relates to the choice of the skew-symmetric tensor $\tilde{\boldsymbol{\theta}}$ and thus of the computation of the tensors rotations $\boldsymbol{\Lambda}$ which rise from the equation (Eq11).

3.3 Choice of the skew-symmetric tensor and integration

the choice which is made in Code_Aster is to use the tensor rate of rotation $\boldsymbol{\omega}$; this choice led to an algorithm equivalent to an objective derivative of Jaumann (but with an integration incrémentalement objectifies).

The stage (III) of Box 1 contains consequently the stages described hereafter in Box 2. The last stage of the Box 2, which constitutes the integration of the equation (Eq11), is related to the rule of the point medium in SO_3 . One will be able to refer to [1] for a demonstration of this relation.

(I)	incremental Displacement gradient $$\boldsymbol{h}_{n+\frac{1}{2}}=grad_n \boldsymbol{u}\, \boldsymbol{f}^{-1}_{n+\frac{1}{2}}$$
(II)	Tensor rate of rotation $$\boldsymbol{\omega}_{n+\frac{1}{2}}=\frac{1}{2\Delta t}[\boldsymbol{h}_{n+\frac{1}{2}}-\boldsymbol{h}^T_{n+\frac{1}{2}}]$$
(III)	Tensor rotation of the following increments $$\boldsymbol{\Lambda}_{n+\frac{1}{2}}=\exp[\frac{\Delta t}{2}\boldsymbol{\omega}_{n+\frac{1}{2}}]\boldsymbol{\Lambda}_n$$ $$\boldsymbol{\Lambda}_{n+1}=\exp[\Delta t\, \boldsymbol{\omega}_{n+\frac{1}{2}}]\boldsymbol{\Lambda}_n$$

Table 3.3-1 : Box 2 - Integration of the tensors rotation

Moreover, the computation of the exponential term in Boîte2 (III) is carried out in practice by the formula of Eulerian Rodrigues: $\exp(\theta)=1+\frac{\sin(\|\theta\|)}{\|\theta\|}+...$, cf [9]

3.4 Advantages and limitations of the method

the algorithm describes above has advantages and limitations which are pointed out here. A comparative study of the various approaches in large deformations can be found in [5].
Among the advantages let us quote the following:

- Generics of the algorithm: it makes it possible to treat a large number of constitutive laws without particular modifications; thus, the following models of Code_Aster are currently available, corresponding to hardenings isotropic and kinematical, like with viscoplasticity: `VMIS_ISOT_LINE`, `VMIS_ISOT_TRAC`, `VMIS_ISOT_PUIS`, `VMIS_CINE_LINE`, `VMIS_CIN2_MEMO`, `VMIS_ECMI_LINE`, `VMIS_ECMI_TRAC`, `VMIS_CIN1_CHAB`, `VMIS_CIN2_CHAB`, `LEMAITRE`, `VENDOCHAB`, `VISC_CIN1_CHAB`, `VISC_CIN2_CHAB`, `VISC_CIN2_MEMO`

- Similarity with the small strains: this generics is related to the similarity between the algorithm of radial return presented and that classically used in small strains; only the stresses and the variable kinematics of entry and output must be affected by a relative rotation of $\boldsymbol{r}$

- guaranteed Objectivity: as underlined by Eq17, the algorithm makes it possible to be secured against any evolution of stresses by a rigid body motion.

It presents some disadvantages all the same:

- It is first of all restricted with the isotropic and relatively weak elastic strain (hypoelasticity) ; at the more theoretical level, the hypoelastic models can lead to a non-zero dissipation during a closed cycle of elastic strain, which is not physical ; because of these disadvantages, one will prefer as much as possible formalism SIMO_MIEHE for the models with isotropic hardening.

- Only the constitutive laws of the type VMIS are treated: one does not treat cases of criterion of Tresca or Hill for example; moreover, for the models with kinematic hardening, it is necessary to transport the kinematical tensor at previous time in the objective reference frame at current time;

- The computing times are relatively important; this is due to an expensive algorithm and an asymmetric tangent matrix which is not optimal (see additional), which increases the nombre of iterations of Newton with convergence. It is noted that the number D" iterations with convergence is notably more important than that observed by means of the formalism very-elastic of Simo Miehe. The analytical study of the possibility of writing a tangent matrix for these models is described in appendix.

4 Algorithm GDEF_LOG

This algorithm, due to C.Miehe; N.Appel and M.Lambrecht [13] have the same selected advantages as those until now but they based on an energy formulation and the stiffness matrix is provided in [13].

4.1 Summary description

The model is based on a logarithmic strain defined by: $\boldsymbol{E}=\frac{1}{2}\log\left[\boldsymbol{F}^{\boldsymbol{T}}\boldsymbol{.}\boldsymbol{F}\right]$. The definition of this statement is provided in appendix 2.

The stress $\boldsymbol{T}$ is defined in space logarithmic curve like dual of $\boldsymbol{E}$, so that the density of power mechanical is expressed $\boldsymbol{T}:\dot{\boldsymbol{E}}$. It is not a classical stress tensor, but one can deduce the usual tensors from them:

$\boldsymbol{\Pi}=\boldsymbol{T}:\boldsymbol{P}_{\Pi}$, with $\boldsymbol{P}_{\Pi}$ a tensor of projection (of order 4 in 3D) and $\boldsymbol{\Pi}$ the tensor of the Piola-Kirchhoff stresses of first species. The tensors of Cauchy $\boldsymbol{\sigma}$ and Kirchhoff $\boldsymbol{\tau}$ will be written in a usual way $J\boldsymbol{\sigma}=\boldsymbol{\tau}=\boldsymbol{\Pi}\,\boldsymbol{F}^{T}$

Indeed the mechanical power being written $\boldsymbol{T}:\dot{\boldsymbol{E}}=\boldsymbol{\Pi}:\dot{\boldsymbol{F}}$, one obtains:

$\boldsymbol{T}:\dot{\boldsymbol{E}}=\boldsymbol{T}:\frac{\partial\boldsymbol{E}}{\partial\boldsymbol{F}}:\dot{\boldsymbol{F}}=\boldsymbol{\Pi}:\dot{\boldsymbol{F}}=\boldsymbol{T}:\boldsymbol{P}_{\Pi}:\dot{\boldsymbol{F}}$ what defines the tensor of projection $\boldsymbol{P}_{\Pi}=\frac{\partial\boldsymbol{E}}{\partial\boldsymbol{F}}$.

One can also calculate the second tensor of Piola-Kirchhoff S according to $\boldsymbol{T}$:

$\boldsymbol{T}:\dot{\boldsymbol{E}}=\boldsymbol{S}:\dot{\boldsymbol{\Delta}}=\boldsymbol{S}:\frac{1}{2}\dot{\boldsymbol{C}}$, with $\boldsymbol{\Delta}=\frac{1}{2}\left(\boldsymbol{C}-\boldsymbol{I}_{d}\right)=\frac{1}{2}\left(\boldsymbol{F}^{T}\boldsymbol{F}-\boldsymbol{I}_{d}\right)$ the tensor of the strains of Green-Lagrange.

Then $\boldsymbol{T}:\dot{\boldsymbol{E}}=\boldsymbol{T}:\frac{\partial\boldsymbol{E}}{\partial\boldsymbol{C}}:\dot{\boldsymbol{C}}=\boldsymbol{S}:\frac{1}{2}\dot{\boldsymbol{C}}$ thus $\boldsymbol{S}=\boldsymbol{T}:\boldsymbol{P}$ with $\boldsymbol{P}=2\frac{\partial\boldsymbol{E}}{\partial\boldsymbol{C}}$

If selected physics is particular, the model allows however to keep the additive decomposition of the elastic strain and plastics classic in HP with $\boldsymbol{E}^{p}=\frac{1}{2}\log\left(\boldsymbol{F}^{pT}.\boldsymbol{F}^{p}\right)$.

Such a choice is always licit. That simply amounts adopting a definition for the elastic strain. However, this one proves to be coherent with a multiplicative decomposition, in the absence of rotation (coaxial situation). Moreover, the plastic incompressibility is assured because: $tr\,\boldsymbol{E}^{p}=\log J^{p}$.

The elastic strain energy ψ^{e} of the model also takes the same shape as that of the small strains, but by adopting the notions of stress and strain specific to this formalism:

$$\psi^{e}=\frac{1}{2}\|\boldsymbol{E}-\boldsymbol{E}^{p}\|_{E}^{2}=\frac{1}{2}\boldsymbol{T}:\boldsymbol{C}^{-1}:\boldsymbol{T}.$$

This formulation has certain advantages:

- as for GDEF_HYPO_ELAS , the kinematical dimension of the model is confined upstream and downstream from the integration of the behavior; this was one of the principal elements for the choice of the formalism; all the models of behavior available in small strains are *a priori* available, with condition of course that has a physical meaning (the hypoelastic large deformations are well adapted to the metal behaviors, and not to the behaviors of the concrete).
- so the model HP admits an energy statement, it will be the same for the model large deformations: the tangent matrix is thus symmetric, which was not the case of GDEF_HYPO_ELAS ;
- the only difficulty seems *a priori* concentrated in the definition of the logarithmic strain, but the article [13] provides a calculation algorithm distinguishing the difficult cases (multiple clean entities);
- the model, according to the examples presented by the authors, results very close to those obtained by a classical formalism give to multiplicative decomposition;

• the model can be wide with the cases of anisotropy (initial or induced), which can be an important source of motivation (in particular for the behavior of alloys containing Zirconium).

Moreover, the article [13] provides a form of the tangent matrix in the configuration using $\boldsymbol{\Pi}$ (called nominal); however, as it is based on a writing starting from the Piola-Kirchhoff stresses of first species, asymmetric, which classical and is never carried out in Code_Aster, one prefers here to compute: to use the second tensor of Piola-Kirchhoff the internal forces and the matrix tangent on the initial configuration, while referring to [15] for example.

4.2 Algorithm

Preprocessing :

$\boldsymbol{E}_{n+1}=\frac{1}{2}\log\left[\boldsymbol{F}_{n+1}^{T}.\boldsymbol{F}_{n+1}\right]=\frac{1}{2}\log\left[\boldsymbol{C}_{n+1}\right]$ calculated by spectral decomposition:

If $\lambda^{(i)}$ are the eigenvalues of $\boldsymbol{C}_{n+1}$ and the $\boldsymbol{N}^{(i)}$ associated eigenvectors, then selected strain measurement is written $\boldsymbol{E}_{n+1}=\frac{1}{2}\sum_{i=1,3}\log(\lambda^{(i)})\boldsymbol{N}^{(i)}\otimes\boldsymbol{N}^{(i)}$.

This measurement makes it possible to obtain an additive decomposition: $\boldsymbol{E}=\boldsymbol{E}^{e}+\boldsymbol{E}^{p}$, with $\boldsymbol{E}^{p}=\frac{1}{2}\log(\boldsymbol{F}^{pT}.\boldsymbol{F}^{p})$ and $\boldsymbol{E}^{e}=\frac{1}{2}\log(\boldsymbol{F}^{eT}.\boldsymbol{F}^{e})$

Moreover one can also write, between time n and time $n+1$:

$\boldsymbol{E}_{n+1}=\boldsymbol{E}_{n}+\Delta\boldsymbol{E}$

One calls then the constitutive law HP

It provides the tensor of the stresses $\boldsymbol{T}$, defined by $\boldsymbol{T}_{n+1}=\hat{\boldsymbol{T}}\left(\Delta\boldsymbol{E};\boldsymbol{E}_{n},\boldsymbol{T}_{n},\boldsymbol{\beta}_{n}\right)$ where $\boldsymbol{\beta}_{n}$ all the local variables at time and the n forced $\boldsymbol{T}_{n}$ at time It n
represents is thus necessary to recompute $\boldsymbol{T}_{n}$ according to the stresses of Cauchy $\boldsymbol{\sigma}_{n}$ stored at time n (forced which are written: $\boldsymbol{T}_{n}=\boldsymbol{S}_{n}:\boldsymbol{P}_{n}^{-1}$. However that requires the transformation of $\boldsymbol{\sigma}_{n}$ in $\boldsymbol{S}_{n}$ and the computation of $\boldsymbol{P}_{n}^{-1}$ which can be expensive. One thus chooses to store the tensors $\boldsymbol{T}$ as local variables.

Postprocessing:

The tensor of the Piola-Kirchhoff stresses of 2nd species is obtained by :

$\boldsymbol{S}_{n+1}=\boldsymbol{T}_{n+1}:\boldsymbol{P}_{n+1}$, with $\boldsymbol{P}_{n+1}=\partial_{C}\left(\boldsymbol{E}_{n+1}\right)$; this quantity is calculated *via* an algorithm presented in [13] and quoted in appendix 2.

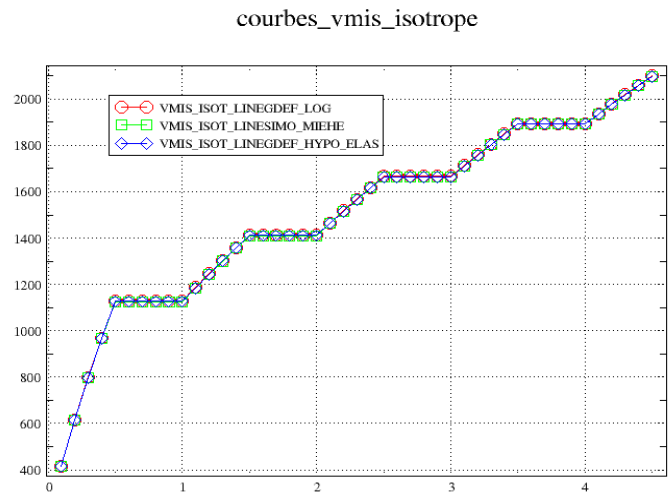
The tensor of the stresses of Cauchy is obtained par. $J\boldsymbol{\sigma}=\boldsymbol{\tau}=\boldsymbol{F}.\boldsymbol{S}.\boldsymbol{F}^{T}=\boldsymbol{F}.(\boldsymbol{T}:\boldsymbol{P}).\boldsymbol{F}^{T}$

One obtains the tangent modulus, in configuration known as “Lagrangian”, by derivative of $\boldsymbol{S}_{n+1}=\boldsymbol{T}_{n+1}:\boldsymbol{P}_{n+1}$: $\dot{\boldsymbol{S}}_{n+1}=\boldsymbol{C}_{n+1}^{ep}:\frac{1}{2}\dot{\boldsymbol{C}}_{n+1}$ with $\boldsymbol{C}_{n+1}^{ep}=\boldsymbol{P}_{n+1}^{T}:\boldsymbol{E}_{n+1}^{ep}:\boldsymbol{P}_{n+1}+\boldsymbol{T}_{n+1}:\boldsymbol{L}_{n+1}$ where $\boldsymbol{E}_{n+1}^{ep}=\frac{\partial\boldsymbol{T}_{n+1}}{\partial\boldsymbol{E}_{n+1}}$ the tangent operator resulting from the constitutive law represents and $\boldsymbol{L}_{n+1}$ is the tensor of a nature 6 (in 3D) defined by $\boldsymbol{L}_{n+1}=4\partial_{CC}^{2}\left(\boldsymbol{E}_{n+1}\right)$

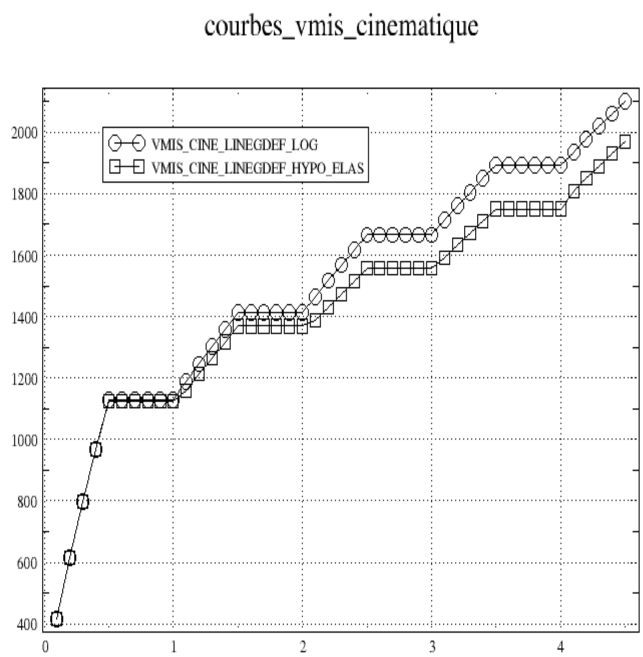
One can then calculate on this configuration the internal forces and the tangent matrix, on the initial configuration, as in [14].

The formalism is tested in particular on the case test of tension-rotation for elastoplastic models of Von Mises with isotropic or kinematical hardening (cf tests SSND106, SSND107).

As the figures show it below, objectivity is preserved (forced invariant by rotation in hardenings isotropic and kinematical) and the good accuracy (results identical to SIMO_MIEHE into isotropic).



InstantContrainte s



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4: Responses for the model `GDEF_LOG` in tension - rotationsContraintes

5 Validity of the models of large deformations

5.1 Identification of the parameters

It also should be specified that the formalism here presented and studied does not extend the validity of the constitutive laws in the field as of large deformations, it nothing but does propose an objective derivative of it. To clarify this matter, it is possible to consider the case of an elastoplastic model with linear isotropic hardening. This kind of model of plasticity is valid physically in small strains; its use is extended to the large deformations, but its physical validity can be precisely called in question.

Moreover, one identification made on tests in small strains must be potentially reconsidered; on figure 2, one presents a curve of tension modelled by a linear isotropic hardening: the tangent modulus must inevitably be defined compared to the beach of strain considered. In small strains, it seems more judicious to use E_{T1} ; if the strains are more important, it seems more judicious to use E_{T2} as value of the slope of hardening. But it is felt well that it is the physical validity of the model itself which should be reconsideration.

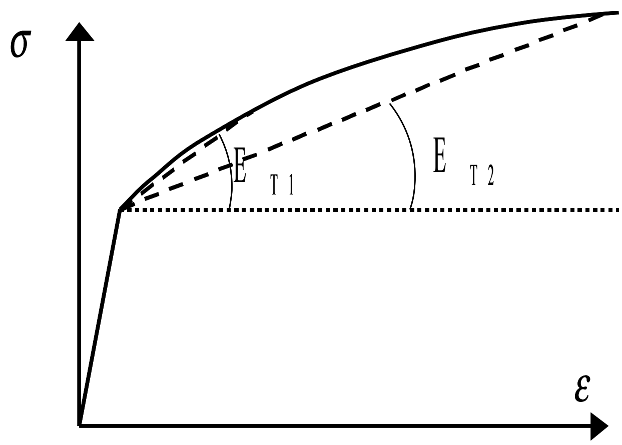

Figure 2: Identification in great or small strains

6 Comparison with PETIT_REAC

6.1 Approximation of the large deformations by PETIT_REAC

the principle of the formulation PETIT_REAC simply consists in reactualizing the geometry of the problem during iterations of Newton (and not at the end of each time step). This means that all the quantities intervening in the equations of the problem are evaluated on the present configuration. Anything else is not modified compared to the case small disturbances.

6.1.1 Kinematical description

kinematical description is the same one as that of the small disturbances. This means that increment of strain is calculated by:

$$\Delta\varepsilon=\frac{1}{2}(\nabla_{\Omega_i}(\Delta u)+\nabla^T_{\Omega_i}(\Delta u))$$

Table 1

Ω_i being reactualized configuration. The total deflection is then the sum of each one of these increments of linearized strain, calculated on different configurations. It is thus delicate to give him a physical meaning and it is better to use it like an indicator of the reached level of strain. The assumption of additive decomposition of the strains is applied.

6.1.2 Elastoplastic behavior model

In the statement of the relation elastic stress-strains, one saw (eq.7) the need for using an objective derivative: $\overset{\circ}{\boldsymbol{\sigma}}=\boldsymbol{C}:[\dot{\boldsymbol{\varepsilon}}-\dot{\boldsymbol{\varepsilon}}^p]$. With `PETIT_REAC` one replaces objective derivative by simple derivative in time: it is thus not objective. Consequently, the use of `PETIT_REAC` is thus not appropriate to large rotations but it is it with the large deformations, under certain conditions [10]:

- very small increments,
- very small rotations (what implies a quasi-radial loading)
- elastic strain small in front of plastic strains,
- isotropic behavior.

6.1.3 Balance and stamps tangent

In term of finite elements, the resolution by `PETIT_REAC` implies with each step of load the resolution of the same nonlinear system as in small strains [11]:

$$L^{\text{int}}(u_i, t_i) + B^T.\lambda_i = L^{\text{ext}}(t_i)$$
$$B.u_i = u^d(t_i)$$

Table 6.1.3-1

With the difference close the internal forces are formally calculated by:

$$L^{\text{int}}(u_i, t_i)=Q^T(u_i):\sigma$$

Table 6.1.3-2

where the operator Q depends on displacements. In this frame, the computation of the tangent matrix carries out to:

$$K_i^n=\frac{\partial L^{\text{int}}}{\partial u}\,|_{(u_i^n, t_i)}=Q(u):\frac{\partial \sigma}{\partial u}\,|_{(u_i^n, t_i)}+\frac{\partial Q(u)}{\partial u}\,|_{(u_i^n, t_i)}:\sigma$$

Table 6.1.3-3

the first term is the contribution of the behavior, similar to what was presented in small transformations, to the difference which this contribution is evaluated here in present configuration. The second term is the contribution of the geometry which is not present in small transformations.
In the frame of the resolution `PETIT_REAC`, this term is not present in the computation of the tangent matrix. One thus has:

$$K_i^n=Q(u):\frac{\partial \sigma}{\partial u}\,|_{(u_i^n, t_i)}$$

Table 6.1.3-4

the absence of the geometrical contribution in the tangent matrix can sometimes make convergence difficult.

6.2 Comparison on an example

the formalism `PETIT_REAC` (cf [6]) is based on an actualization of the geometry for the computation of the increment of strain before integrating the behavior of way identical to the small strains. This allows a simple processing of the large deformations, but in a very approximate way, not objectifies and being able to generate great errors.

To be convinced some, let us consider the alternate tension-rotation of a cube; for more details on the test, one will refer to [7] for example.

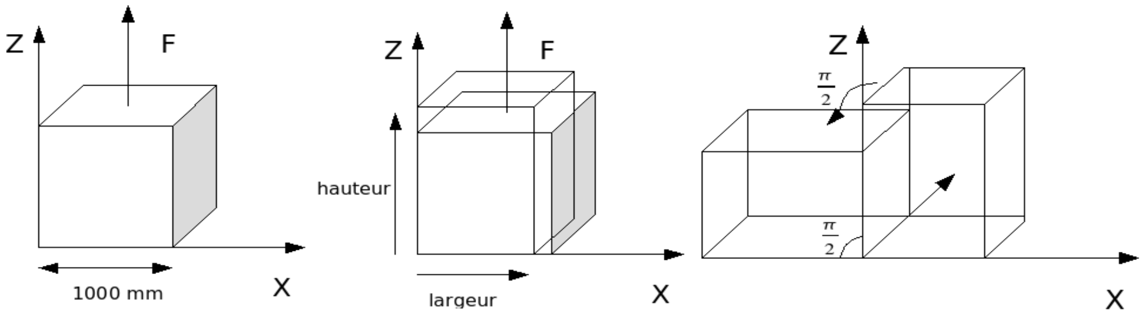

Figure 3: Example of tension rotation of a cube

During the phases of rotation, the stress must remain constant: a rigid body motion does not generate stresses (in static all at least and without viscosity).

The figure below PETIT_REAC presents the response obtained with `a behavior` VMIS_ISOT_LINE for `the strains` and `GDEF_HYPO_ELAS`. The type of strain `PETIT_REAC` is put at fault whereas `GDEF_HYPO_ELAS` is valid (and provides a response identical to `SIMO_MIEHE`).

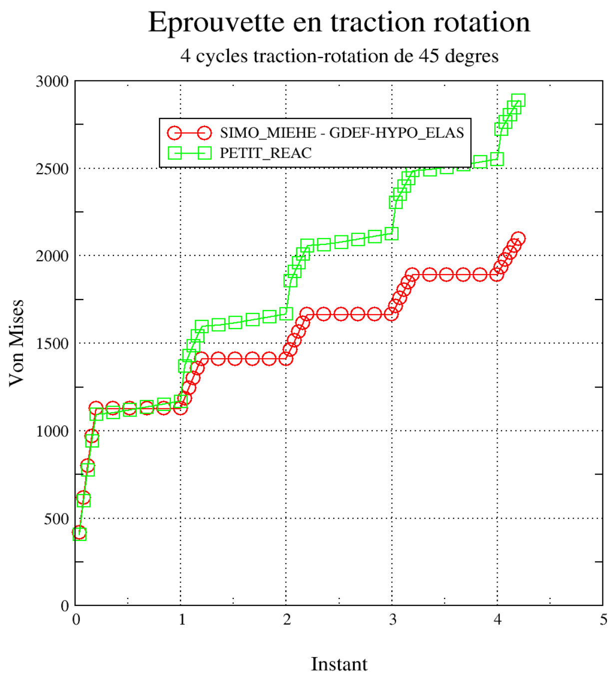

Figure 4: Von Mises stress in tension-rotation

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8 of the versions of the document Version

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R.BARGELLINI	, J.M.PROIX R & D /AMA J.COLOIGNER Univ . Rennes initial Text	10.4 R.BARGELLINI
,	J.M.PROIX R & D /AMA Addition of the description	of the algorithm using derivative of Dregs, of the computation of the tangent operator, and algorithm GDEF_LOG Table 8-1 Appendix 1

: tangent 8-1

9 operator for GDEF_HYPO_ELAS In the reference [1

], two algorithms are proposed to solve the equations of elastoplasticity with a hypo formulation - elastic: that presented in this document (GDEF_HYPO_ELAS), and a simpler algorithm, using an approximation of derivative of Dregs, which we describe here. GDEF_HYPO_ELAS (derivative

of Jaumann) SIMO_HUGHES_1 (derivative	of Dregs) Knowing the fields
of displacements, one calculates the following quantities: , Definition of $f_{n+1} = I_d + grad_n u$ rotation tensors $f_{n+\alpha} = I_d + \alpha\, grad_n u = (1-\alpha) I_d + \alpha f_{n+1}$ $\tilde{f}_{n+\alpha} = f_{n+1} \cdot f^{-1}_{n+\alpha}$	
1 incremental gradient of displacement 2 Rates of rotation and $h_{n+1/2} = grad_n u\, f^{-1}_{n+1/2} = I_d - f^{-1}_{n+1/2}$ clean ω rotation tensor orthogonal 3/Tensor Θ $\Theta = \Delta t\, \omega_{n+1/2} = \frac{1}{2}[h_{n+1/2} - h^T_{n+1/2}] = f^{-T}_{n+1/2} - f^{-1}_{n+1/2}$ and of rotation, Λ Nothing Definition r $\Lambda_{n+1} = \exp[\Theta]\Lambda_n$ $\Lambda_{n+1/2} = \exp[\Theta/2]\Lambda_n$ $r_{n+1} = \Lambda_{n+1}\Lambda^T_n$ strains $r_{n+1/2} = \Lambda_{n+1}\Lambda^T_{n+1/2}$	
: , Definition of the strains $e_{n+1} = \frac{1}{2}[I_d - (f_{n+1} f^T_{n+1})^{-1}]$ $\tilde{e}_{n+\alpha} = \tilde{f}^T_{n+\alpha} e_{n+1} \tilde{f}_{n+\alpha}$	**: , Given for the integration** $e_{n+1} = \frac{1}{2}[I_d - (f_{n+1} f^T_{n+1})^{-1}]$ $\tilde{e}_{n+\alpha} = \tilde{f}^T_{n+\alpha} e_{n+1} \tilde{f}_{n+\alpha}$
of behavior GDEF_HYPO_ELAS (derivative $\begin{cases} \sigma_n = r_{n+1} \tau_n r^T_{n+1} & \text{contrainte à l'instant n} \\ \varepsilon_n = [r_{n+\frac{1}{2}} \tilde{e}_{n+\frac{1}{2}} r^T_{n+\frac{1}{2}}] & \text{déformations} \\ x_n = r_{n+1} q_n r^T_{n+1} & \text{variables cinématiques} \end{cases}$	
of Jaumann) **Table 9-1: Box 2**	

– Two 9-1 of resolution Some is the value

of, These two algorithms α are incrémentalement objective: let us consider a pure rotation: then Principle of the computation $\boldsymbol{e}_{n+1}=\frac{1}{2}[\boldsymbol{I}_d-(\boldsymbol{f}_{n+1}\boldsymbol{f}^T_{n+1})^{-1}]=0$ of $\boldsymbol{\tau}_{n+1}=\boldsymbol{r}_{n+1}\boldsymbol{\tau}_n\boldsymbol{r}^T_{n+1}$

9.1 the tangent matrix: variational writing and general statement This paragraph is generic

for all the formalisms large deformations and constitutive laws. It is inspired in particular by [5] and makes it possible to define the tangent matrix, whose computation in the case as of formalisms described before will be then detailed. The frame of the analysis
is obviously that of the finished transformations. The purpose is to solve the definite mechanical problem by the equilibrium. This comes down to determine field of displacement acceptable which cancels the functional calculus $\boldsymbol{U}$ of the virtual wors, which can be written $G(\boldsymbol{U},\boldsymbol{V})$, in the initial configuration: with the space of acceptable

$$\forall \boldsymbol{V}\in\boldsymbol{V}^{\boldsymbol{0}}_{ad},\ G(\boldsymbol{U},\boldsymbol{V})=\int_{\Omega_0}\left(\boldsymbol{P}:\boldsymbol{grad}_{\boldsymbol{0}}\boldsymbol{V}-\boldsymbol{b}\,\boldsymbol{V}\right)d\,\Omega_0-\int_{\partial\Omega_0}\boldsymbol{t}.\boldsymbol{V}\,d\,\Omega_0=0$$

$\boldsymbol{V}^{\boldsymbol{0}}_{ad}$ displacements, and the voluminal forces $\boldsymbol{b}$ $\boldsymbol{t}$ and surfaces and the first tensor of $\boldsymbol{P}$ the Piola-Kirchhoff stresses. By comparison, its equivalent
in small strains is: In particular, the notion

$$\forall \boldsymbol{V}\in\boldsymbol{V}^{\boldsymbol{0}}_{ad},\ G(\boldsymbol{U},\boldsymbol{V})=\int_{\Omega}\left(\boldsymbol{\sigma}:\boldsymbol{grad}\,\boldsymbol{V}-\boldsymbol{b}\,\boldsymbol{V}\right)d\,\Omega-\int_{\partial\Omega}\boldsymbol{t}.\boldsymbol{V}\,d\,\Omega$$

of configuration of field is in small strains without object, since all are equivalent. The dependence of comes from it from the dependence $\boldsymbol{G}$ $\boldsymbol{U}$ from in through the gradient $\boldsymbol{P}$ $\boldsymbol{U}$ of transformation. In order to solve $\boldsymbol{F}=\boldsymbol{I}_d+\boldsymbol{grad}_{\boldsymbol{0}}\boldsymbol{U}$ the problem
, one linearizes it compared to $G(\boldsymbol{U},\boldsymbol{V})=0\,\forall\,\boldsymbol{V}\in\boldsymbol{V}^{\boldsymbol{0}}_{ad}$ the unknown around an arbitrary state $\boldsymbol{U}$ definite par. the linearized problem $\boldsymbol{U}^*$ then consists in determining the field which cancels the linearized $\delta\boldsymbol{U}$ functional calculus of the virtual wors, i.e.: with directional derivative

$$\text{trouver }\delta\boldsymbol{U}\text{ tel que }\forall\boldsymbol{V}\in\boldsymbol{V}^{\boldsymbol{0}}_{ad}\ ,\ L(\delta\boldsymbol{U},\boldsymbol{V})=G(\boldsymbol{U}^*,\boldsymbol{V})+DG(\boldsymbol{U}^*,\boldsymbol{V})[\delta\boldsymbol{U}]=0$$

$DG(\boldsymbol{U}^*,\boldsymbol{V})[\delta\boldsymbol{U}]$ of at the point in the direction $G(\boldsymbol{U},\boldsymbol{V})$. One $\boldsymbol{U}^*$ introduces to solve $\delta\boldsymbol{U}$
the problem an infinitesimal displacement around; the gradient $\epsilon\,\delta\boldsymbol{U}$ of transformation $\boldsymbol{U}^*$ is written then like a function of such as, and the directional ε derivative $\boldsymbol{F}(\varepsilon)=\boldsymbol{F}(\boldsymbol{U}^*)+\varepsilon\,\delta\boldsymbol{U}$ of is defined by: By $G(\boldsymbol{U},\boldsymbol{V})$ observing

$$\begin{aligned}DG(\boldsymbol{U}^*,\boldsymbol{V})[\delta\boldsymbol{U}]\ &=\frac{\mathrm{d}}{\mathrm{d}\,\varepsilon}\left[\int_{\Omega_0}\left(\boldsymbol{P}(\boldsymbol{F}(\varepsilon)):\boldsymbol{grad}_{\boldsymbol{0}}\boldsymbol{V}-\boldsymbol{b}\boldsymbol{V}\right)d\,\Omega_0-\int_{\partial\Omega_0}\boldsymbol{t}.\boldsymbol{V}\,d\,\Omega_0\right](\varepsilon=0)\\&=\frac{\mathrm{d}}{\mathrm{d}\,\varepsilon}\left[\int_{\Omega_0}\left(\boldsymbol{P}(\boldsymbol{F}(\varepsilon)):\boldsymbol{grad}_{\boldsymbol{0}}\boldsymbol{V}\right)d\,\Omega_0\right](\varepsilon=0)\end{aligned}$$

the composed derivative rules simply, one can express this directional derivative as follows: with the material tangent

$$DG(\boldsymbol{U}^*,\boldsymbol{V})[\delta\boldsymbol{U}]=\int_{\Omega_0}\ \boldsymbol{A}:\boldsymbol{grad}_{\boldsymbol{0}}\delta\boldsymbol{U}:\boldsymbol{grad}_{\boldsymbol{0}}\boldsymbol{V}\,d\,\Omega_0$$

matrix $\boldsymbol{A}=\frac{\partial\boldsymbol{P}}{\partial\boldsymbol{F}}\left(\boldsymbol{F}(\boldsymbol{U}^*)\right)$. This then makes it possible to define
the linearized equation of the virtual wors in the hardware configuration: to find such as, For reasons $\delta\boldsymbol{U}$ of $\forall\boldsymbol{V}\in\boldsymbol{V}^{\boldsymbol{0}}_{ad}$ simplicity

$$\int_{\Omega_0}\boldsymbol{A}:\boldsymbol{grad}_{\boldsymbol{0}}\delta\boldsymbol{U}:\boldsymbol{grad}_{\boldsymbol{0}}\boldsymbol{V}\,d\,\Omega_0=-\int_{\Omega_0}\left(\boldsymbol{P}:\boldsymbol{grad}_{\boldsymbol{0}}\boldsymbol{V}-\boldsymbol{b}\,\boldsymbol{V}\right)d\,\Omega_0+\int_{\partial\Omega_0}\boldsymbol{t}.\boldsymbol{V}\,d\,\Omega_0$$

, it is preferable to remain in initial configuration for the definition of the integrals (in particular for the definition of the volume forces and surface) and to calculate the gradients in the present configuration; in fact besides the choice had been retained for the integration of the formalism of Simo-Miehe (see [6]). One thus makes use of the relation (with the gradient compared to the $\boldsymbol{grad_0\, a} = \boldsymbol{grad_n\, a}\,.\boldsymbol{F}$ $\boldsymbol{grad_n}$ present configuration) to write directional derivative as follows: Moreover, the use

$$DG\left(\boldsymbol{U}^*,\boldsymbol{V}\right)\left[\delta\boldsymbol{U}\right] = \int_{\Omega_0} \boldsymbol{A} : \left[\left(\boldsymbol{grad_n}\,\delta\boldsymbol{U}\right).\boldsymbol{F}\right] : \left[\left(\boldsymbol{grad_n}\boldsymbol{V}\right).\boldsymbol{F}\right] d\,\Omega_0$$

of the first Piola-Kirchhoff stress, asymmetric, is problematic for the definition of the internal forces and the tangent matrixes; one thus makes use of his definition to express, in indicielle $\boldsymbol{P} = J\,\boldsymbol{\sigma}\,\boldsymbol{F}^{-T} = \boldsymbol{\tau}\,\boldsymbol{F}^{-T}$ notation $\boldsymbol{A}$: because for a tensor of

$$A_{ijkl} = \frac{\partial P_{ij}}{\partial F_{kl}} = \frac{\partial \tau_{ip} F^{-1}_{jp}}{\partial F_{kl}} = \frac{\partial \tau_{ip}}{F_{kl}} F^{-1}_{jp} - \tau_{ip} F^{-1}_{jk} F^{-1}_{lp}$$

order 2. From where: with $\boldsymbol{X}$ the tangent $\frac{\partial X^{-1}_{jp}}{\partial X_{kl}} = -X^{-1}_{jk} X^{-1}_{lp}$

matrix

$$\begin{aligned}
DG\left(\boldsymbol{U}^*,\boldsymbol{V}\right)\left[\delta\boldsymbol{U}\right] &= \int_{\Omega_0} A_{ijkl}\left[\left(\boldsymbol{grad_n}\boldsymbol{\delta U}\right)_{kr} F_{rl}\right]\left[\left(\boldsymbol{grad_n}\boldsymbol{V}\right)_{is} F_{sj}\right] d\,\Omega_0 \\
&= \int_{\Omega_0} \left[\frac{\partial \tau_{ip}}{\partial F_{kl}} F^{-1}_{jp} - \tau_{ip} F^{-1}_{jk} F^{-1}_{lp}\right]\left[\left(\boldsymbol{grad_n}\boldsymbol{\delta U}\right)_{kr} F_{rl}\right]\left[\left(\boldsymbol{grad_n}\boldsymbol{V}\right)_{is} F_{sj}\right] d\,\Omega_0 \\
&= \int_{\Omega_0} \left[\frac{\partial \tau_{ip}}{\partial F_{kl}} F^{-1}_{jp} F_{rl} F_{sj} - \tau_{ip} F^{-1}_{jk} F^{-1}_{lp} F_{rl} F_{sj}\right]\left[\left(\boldsymbol{grad_n}\boldsymbol{\delta U}\right)_{kr}\right]\left[\left(\boldsymbol{grad_n}\boldsymbol{V}\right)_{is}\right] d\,\Omega_0 \\
&= \int_{\Omega_0} \left[\frac{\partial \tau_{is}}{\partial F_{kl}} F_{rl} - \tau_{ir}\delta_{ks}\right]\left[\left(\boldsymbol{grad_n}\boldsymbol{\delta U}\right)_{kr}\right]\left[\left(\boldsymbol{grad_n}\boldsymbol{V}\right)_{is}\right] d\,\Omega_0 \\
&= \int_{\Omega_0} \boldsymbol{C} : \left(\boldsymbol{grad_n}\boldsymbol{\delta U}\right) : \left(\boldsymbol{grad_n}\boldsymbol{V}\right) d\,\Omega_0
\end{aligned}$$

of

$$C_{iskr} = \frac{\partial \tau_{is}}{\partial F_{kl}} F_{rl} - \tau_{ir}\delta_{ks}$$

the problem. This tangent matrix is made up of two distinct terms. Second is called “geometrical stiffness”, and first is related on the formalism of large deformations used, and possibly to the constitutive law: it is him whom it is necessary here to calculate. It will be noted that this statement is in agreement with [6]; from an integration point of view of code, that makes it possible not to have to modify this one too basically insofar as it is possible to use the routines dedicated to Simo-Miehe. It is pointed out that in small

strains, the tangent matrix is written simply. Computation of the tangent $C^{HPP}_{iskr} = \frac{\partial \sigma_{is}}{\partial \varepsilon_{kr}}$ matrix

9.2 for the two formalisms As announced in introduction

, this paragraph is voluntarily very detailed so that an attentive reader can correct the possible errors of reasoning and/or computation. The preceding paragraph

arrives at the conclusion which the term with calculating is. If one discretizes temporally $\frac{\partial\boldsymbol{\tau}}{\partial\boldsymbol{F}}.\boldsymbol{F}^{\boldsymbol{T}}$ the configurations as presented of Figure 1, he is written indifferently: It is thus necessary mainly

$$\frac{\partial\boldsymbol{\tau}}{\partial\boldsymbol{F}_{n+1}}.\boldsymbol{F}^{\boldsymbol{T}}_{n+1} = \frac{\partial\boldsymbol{\tau}}{\partial\boldsymbol{f}_{n+1}}.\boldsymbol{f}^{\boldsymbol{T}}_{n+1}$$

to evaluate the computation of derivative of the stress of Kirchhoff. The formalisms presented and considered here use the elastoplastic constitutive laws written in small strains. In this frame, one always has, where represent all $\boldsymbol{\tau}_{n+1}=g\left(\boldsymbol{\sigma}_n,\boldsymbol{\varepsilon}_n,\boldsymbol{x}_n;\beta_n\right)$ β_n the scalar local variables of isotropic hardening and the variable tensorial $\boldsymbol{x}_n$ of kinematic hardening; the statement of the tangent behavior is then sought in the form: For the two algorithms

$$\frac{\partial\boldsymbol{\tau}_{n+1}}{\partial\boldsymbol{f}_{n+1}}=\frac{\partial\boldsymbol{\tau}_{n+1}}{\partial\boldsymbol{\varepsilon}_n}\cdot\frac{\partial\boldsymbol{\varepsilon}_n}{\partial\boldsymbol{f}_{n+1}}+\frac{\partial\boldsymbol{\tau}_{n+1}}{\partial\boldsymbol{\sigma}_n}\cdot\frac{\partial\boldsymbol{\sigma}_n}{\partial\boldsymbol{f}_{n+1}}+\frac{\partial\boldsymbol{\tau}_{n+1}}{\partial\boldsymbol{x}_n}\cdot\frac{\partial\boldsymbol{x}_n}{\partial\boldsymbol{f}_{n+1}}$$

considered here, it is thus necessary to evaluate each term: : provided by the constitutive law

1. $\frac{\partial\boldsymbol{\tau}_{n+1}}{\partial\boldsymbol{\varepsilon}_n}$ called in small strains. : specific to each
2. $\frac{\partial\boldsymbol{\varepsilon}_n}{\partial\boldsymbol{f}_{n+1}}$ algorithm, it contains however a general term: : not currently calculated $\frac{\partial\tilde{\boldsymbol{e}}_{n+\alpha}}{\partial\boldsymbol{f}_{n+1}}$
3. $\frac{\partial\boldsymbol{\tau}_{n+1}}{\partial\boldsymbol{\sigma}_n}$ by the constitutive laws. In elasticity, it is worth the identity. In plasticity, its exact statement is specific to each constitutive law. : specific to each
4. $\frac{\partial\boldsymbol{\sigma}_n}{\partial\boldsymbol{f}_{n+1}}$ algorithm. And in the case of a kinematic hardening

: : there still specific

5. $\frac{\partial\boldsymbol{\tau}_{n+1}}{\partial\boldsymbol{x}_n}$ to each constitutive law: specific to each
6. $\frac{\partial\boldsymbol{x}_n}{\partial\boldsymbol{f}_{n+1}}$ algorithm. Two aspects are to be put

forward now. First of all the feeling which the computation of the tangent matrix will be long, with large numbers of terms; from a numerical point of view, this is likely to have a consequent cost. Moreover, the presence of two narrower terms with the constitutive law called by the formalisms: there one loses in advance the generic character of the formalisms, which was one of their strong point for computation him even; moreover, it is understood why the authors themselves advance that “the notion of consistent tangent modulus is not available for the class of formalisms” considered here (see [1], Remark 8.3.2, pp 292) the continuation presents successively
the computation of each term. Term 1 This term is

directly $\frac{\partial\boldsymbol{\tau}_{n+1}}{\partial\boldsymbol{\varepsilon}_n}$

resulting from the constitutive law in small strains; no computation is thus necessary. Term 2 This term, as

specified $\frac{\partial\boldsymbol{\varepsilon}_n}{\partial\boldsymbol{f}_{n+1}}$

front, depends on the formalism considered. It will be seen however that it contains general terms. For algorithm SIMO

9.2.1 _HUGHES_1 For this `algorithm, one`

A. From where Two derivatives $\boldsymbol{\varepsilon}_n=\tilde{\boldsymbol{e}}_{n+\alpha}=\tilde{\boldsymbol{f}}^T_{n+\alpha}\boldsymbol{e}_{n+1}\tilde{\boldsymbol{f}}_{n+\alpha}$
are thus

$$\frac{\partial \boldsymbol{\varepsilon}_n}{\partial \boldsymbol{f}_{n+1}}=\frac{\partial \tilde{\boldsymbol{e}}_{n+\alpha}}{\partial \boldsymbol{f}_{n+1}}=\frac{\partial \tilde{\boldsymbol{f}}^T_{n+\alpha}}{\partial \boldsymbol{f}_{n+1}}\boldsymbol{e}_{n+1}\tilde{\boldsymbol{f}}_{n+\alpha}+\tilde{\boldsymbol{f}}^T_{n+\alpha}\frac{\partial \boldsymbol{e}_{n+1}}{\partial \boldsymbol{f}_{n+1}}\tilde{\boldsymbol{f}}_{n+\alpha}+\tilde{\boldsymbol{f}}^T_{n+\alpha}\boldsymbol{e}_{n+1}\frac{\partial \tilde{\boldsymbol{f}}_{n+\alpha}}{\partial \boldsymbol{f}_{n+1}}$$

with calculating here, of which one is necessary twice; we will clarify them. , because By recapitulating

- $$\begin{aligned}\frac{\partial(\boldsymbol{e}_{n+1})_{ij}}{\partial(\boldsymbol{f}_{n+1})_{kl}} &= -\frac{1}{2}\tilde{\boldsymbol{f}}^T_{n+\alpha}\frac{\partial\left(\boldsymbol{f}^{-T}_{n+1}\boldsymbol{f}^{-1}_{n+1}\right)}{\partial \boldsymbol{f}_{n+1}}\tilde{\boldsymbol{f}}_{n+\alpha}\\ &= \frac{1}{2}\left[(\boldsymbol{f}^{-1}_{n+1})_{li}(\boldsymbol{f}^{-1}_{n+1})_{mj}(\boldsymbol{f}^{-1}_{n+1})_{mk}+(\boldsymbol{f}^{-1}_{n+1})_{lj}(\boldsymbol{f}^{-1}_{n+1})_{mi}(\boldsymbol{f}^{-1}_{n+1})_{mk}\right]\\ &= \frac{1}{2}\left[(\boldsymbol{f}^{-1}_{n+1})_{li}(\boldsymbol{f}^{-1}_{n+1})_{mj}+(\boldsymbol{f}^{-1}_{n+1})_{lj}(\boldsymbol{f}^{-1}_{n+1})_{mi}\right](\boldsymbol{f}^{-1}_{n+1})_{mk}\end{aligned}$$
- $$\begin{aligned}\frac{\partial(\tilde{\boldsymbol{f}}_{n+\alpha})_{ij}}{\partial(\boldsymbol{f}_{n+1})_{kl}} &= \frac{\partial[(\boldsymbol{f}_{n+1})_{ip}(\boldsymbol{f}^{-1}_{n+\alpha})_{pj}]}{\partial(\boldsymbol{f}_{n+1})_{kl}}=\delta_{ik}\delta_{pl}(\boldsymbol{f}^{-1}_{n+\alpha})_{pj}+(\boldsymbol{f}_{n+1})_{ip}\frac{\partial(\boldsymbol{f}^{-1}_{n+\alpha})_{pj}}{\partial(\boldsymbol{f}_{n+1})_{kl}}\\ &= \delta_{ik}\delta_{pl}(\boldsymbol{f}^{-1}_{n+\alpha})_{pj}+(\boldsymbol{f}_{n+1})_{ip}\frac{\partial(\boldsymbol{f}^{-1}_{n+\alpha})_{pj}}{\partial(\boldsymbol{f}_{n+\alpha})_{rs}}\frac{\partial(\boldsymbol{f}_{n+\alpha})_{rs}}{\partial(\boldsymbol{f}_{n+1})_{kl}}\\ &= \delta_{ik}(\boldsymbol{f}^{-1}_{n+\alpha})_{lj}-(\boldsymbol{f}_{n+1})_{ip}(\boldsymbol{f}^{-1}_{n+\alpha})_{pr}(\boldsymbol{f}^{-1}_{n+\alpha})_{sj}\alpha\,\delta_{rk}\delta_{sl}\\ &= \left[\delta_{ik}-\alpha(\boldsymbol{f}_{n+1})_{ip}(\boldsymbol{f}^{-1}_{n+\alpha})_{pk}\right](\boldsymbol{f}^{-1}_{n+\alpha})_{lj}\end{aligned}$$

  these two $\boldsymbol{f}_{n+\alpha}=(1-\alpha)\boldsymbol{I}_d+\alpha\boldsymbol{f}_{n+1}$

results, one can thus write the first term (the derivative of the strain being calculated in a dedicated routine, one leaves it in the statements without clarifying it): For algorithm GDEF_HYPO_ELAS

$$\begin{aligned}\frac{\partial(\boldsymbol{\varepsilon}_n)_{ij}}{\partial(\boldsymbol{f}_{n+1})_{kl}} &= \frac{\partial\left((\tilde{\boldsymbol{f}}^T_{n+\alpha})_{ip}(\boldsymbol{e}_{n+1})_{pq}(\tilde{\boldsymbol{f}}_{n+\alpha})_{qj}\right)}{\partial(\boldsymbol{f}_{n+1})_{kl}}\\ &= \frac{\partial(\tilde{\boldsymbol{f}}_{n+\alpha})_{pi}}{\partial(\boldsymbol{f}_{n+1})_{kl}}(\boldsymbol{e}_{n+1})_{pq}(\tilde{\boldsymbol{f}}_{n+\alpha})_{qj}+(\tilde{\boldsymbol{f}}_{n+\alpha})_{pi}\frac{\partial(\boldsymbol{e}_{n+1})_{pq}}{\partial(\boldsymbol{f}_{n+1})_{kl}}(\tilde{\boldsymbol{f}}_{n+\alpha})_{qj}+(\tilde{\boldsymbol{f}}_{n+\alpha})_{pi}(\boldsymbol{e}_{n+1})_{pq}\frac{\partial(\tilde{\boldsymbol{f}}_{n+\alpha})_{qj}}{\partial(\boldsymbol{f}_{n+1})_{kl}}\\ &= (\tilde{\boldsymbol{f}}_{n+\alpha})_{pi}\frac{\partial(\boldsymbol{e}_{n+1})_{pq}}{\partial(\boldsymbol{f}_{n+1})_{kl}}(\tilde{\boldsymbol{f}}_{n+\alpha})_{qj}+\left[\delta_{tk}-\alpha(\boldsymbol{f}_{n+1})_{tr}(\boldsymbol{f}^{-1}_{n+\alpha})_{rk}\right](\boldsymbol{e}_{n+1})_{st}\left[(\boldsymbol{f}^{-1}_{n+\alpha})_{li}(\tilde{\boldsymbol{f}}_{n+\alpha})_{sj}+(\tilde{\boldsymbol{f}}_{n+\alpha})_{si}(\boldsymbol{f}^{-1}_{n+\alpha})_{lj}\right]\end{aligned}$$

9.2.2 For this `algorithm, one`

always takes and the strain is written $\alpha=\frac{1}{2}$: From where Compared to $\boldsymbol{\varepsilon}_n=\left[\boldsymbol{r}_{n+\frac{1}{2}}\tilde{\boldsymbol{e}}_{n+\frac{1}{2}}\boldsymbol{r}^T_{n+\frac{1}{2}}\right]$
SIMO_HUGHES

$$\frac{\partial(\boldsymbol{\varepsilon}_n)_{qp}}{\partial(\boldsymbol{f}_{n+1})_{kl}}=\frac{\partial(\boldsymbol{r}_{n+1/2})_{qa}}{\partial(\boldsymbol{f}_{n+1})_{kl}}(\tilde{\boldsymbol{e}}_{n+1/2})_{ab}(\boldsymbol{r}_{n+1/2})_{pb}+\frac{\partial(\tilde{\boldsymbol{e}}_{n+1/2})_{ab}}{\partial(\boldsymbol{f}_{n+1})_{kl}}(\boldsymbol{r}_{n+1/2})_{qa}(\boldsymbol{r}_{n+1/2})_{pb}+(\boldsymbol{r}_{n+1/2})_{qa}(\tilde{\boldsymbol{e}}_{n+1/2})_{ab}\frac{\partial(\boldsymbol{r}_{n+1/2})_{pb}}{\partial(\boldsymbol{f}_{n+1})_{kl}}$$

_1, this term `is longer` because it utilizes the derivatives of the rotation tensors, with and. One thus $\boldsymbol{r}_{n+1/2}=\boldsymbol{\Lambda}_{n+1}\boldsymbol{\Lambda}^T_{n+1/2}$ calculates $\boldsymbol{\Lambda}_{n+1}=\exp[\boldsymbol{\Theta}]\boldsymbol{\Lambda}_n$: $\boldsymbol{\Lambda}_{n+1/2}=\exp[\boldsymbol{\Theta}/2]\boldsymbol{\Lambda}_n$

To finish the computation of

$$
\begin{aligned}
\frac{\partial(\boldsymbol{r}_{n+1/2})_{qa}}{\partial(\boldsymbol{f}_{n+1})_{kl}} &= \frac{\partial(\Lambda_{n+1})_{qc}}{\partial(\boldsymbol{f}_{n+1})_{kl}}(\boldsymbol{\Lambda}^{-1}_{n+1/2})_{ca}+(\Lambda_{n+1})_{qc}\frac{\partial(\boldsymbol{\Lambda}^{-1}_{n+1/2})_{ca}}{\partial(\boldsymbol{f}_{n+1})_{kl}}\\
&= \frac{\partial\exp(\boldsymbol{\Theta})_{qd}}{\partial(\boldsymbol{f}_{n+1})_{kl}}(\boldsymbol{\Lambda}_n)_{dc}(\boldsymbol{\Lambda}^{-1}_{n+1/2})_{ca}+(\boldsymbol{\Lambda}_{n+1})_{qc}\frac{\partial(\boldsymbol{\Lambda}^{-1}_{n+1/2})_{ca}}{\partial(\boldsymbol{\Lambda}_{n+1/2})_{mn}}\frac{\partial(\boldsymbol{\Lambda}_{n+1/2})_{mn}}{\partial(\boldsymbol{f}_{n+1})_{kl}}\\
&= \frac{\partial\exp(\boldsymbol{\Theta})_{qd}}{\partial(\boldsymbol{\Theta})_{mn}}\frac{\partial(\boldsymbol{\Theta})_{mn}}{\partial(\boldsymbol{f}_{n+1/2})_{rs}}\frac{\partial(\boldsymbol{f}_{n+1/2})_{rs}}{\partial(\boldsymbol{f}_{n+1})_{kl}}(\boldsymbol{\Lambda}_n)_{dc}(\boldsymbol{\Lambda}^{-1}_{n+1/2})_{ca}\\
&\quad -(\boldsymbol{\Lambda}_{n+1})_{qc}(\boldsymbol{\Lambda}^{-1}_{n+1/2})_{cm}(\boldsymbol{\Lambda}^{-1}_{n+1/2})_{na}\frac{\partial\exp(\boldsymbol{\Theta}/2)_{mw}}{\partial(\boldsymbol{\Theta}/2)_{rs}}\frac{\partial(\boldsymbol{\Theta}/2)_{rs}}{\partial(\boldsymbol{f}_{n+1/2})_{de}}\frac{\partial(\boldsymbol{f}_{n+1/2})_{de}}{\partial(\boldsymbol{f}_{n+1})_{kl}}(\boldsymbol{\Lambda}_n)_{wn}\\
&= \frac{1}{2}\Bigg[\frac{\partial\exp(\boldsymbol{\Theta})_{qd}}{\partial(\boldsymbol{\Theta})_{mn}}\frac{\partial(\boldsymbol{\Theta})_{mn}}{\partial(\boldsymbol{f}_{n+1/2})_{rs}}\delta_{rk}\,\delta_{sl}(\boldsymbol{\Lambda}_n)_{dc}(\boldsymbol{\Lambda}^{-1}_{n+1/2})_{ca}\\
&\quad -(\boldsymbol{\Lambda}_{n+1})_{qc}(\boldsymbol{\Lambda}^{-1}_{n+1/2})_{cm}(\boldsymbol{\Lambda}^{-1}_{n+1/2})_{na}\frac{\partial\exp(\boldsymbol{\Theta}/2)_{mw}}{\partial(\boldsymbol{\Theta}/2)_{rs}}\frac{\partial(\boldsymbol{\Theta}/2)_{rs}}{\partial(\boldsymbol{f}_{n+1/2})_{de}}\delta_{dk}\,\delta_{el}(\boldsymbol{\Lambda}_n)_{wn}\Bigg]\\
&= \frac{1}{2}\Bigg[\frac{\partial\exp(\boldsymbol{\Theta})_{qd}}{\partial(\boldsymbol{\Theta})_{mn}}\frac{\partial(\boldsymbol{\Theta})_{mn}}{\partial(\boldsymbol{f}_{n+1/2})_{kl}}(\boldsymbol{\Lambda}_n)_{dc}(\boldsymbol{\Lambda}^{-1}_{n+1/2})_{ca}\\
&\quad -(\boldsymbol{\Lambda}_{n+1})_{qc}(\boldsymbol{\Lambda}^{-1}_{n+1/2})_{cm}(\boldsymbol{\Lambda}^{-1}_{n+1/2})_{na}\frac{\partial\exp(\boldsymbol{\Theta}/2)_{mw}}{\partial(\boldsymbol{\Theta}/2)_{rs}}\frac{\partial(\boldsymbol{\Theta}/2)_{rs}}{\partial(\boldsymbol{f}_{n+1/2})_{kl}}(\boldsymbol{\Lambda}_n)_{wn}\Bigg]
\end{aligned}
$$

this term, it misses two more derivatives. therefore, the derivative of

- $(\boldsymbol{\Theta})=\boldsymbol{f}^{-T}_{n+1/2}-\boldsymbol{f}^{-1}_{n+1/2}$ exponential

$$
\begin{aligned}
\frac{\partial(\boldsymbol{\Theta})_{mn}}{\partial(\boldsymbol{f}_{n+1/2})_{rs}}=\frac{\partial(\boldsymbol{\Theta})_{mn}}{\partial(\boldsymbol{f}^{-1}_{n+1/2})_{tu}}\frac{\partial(\boldsymbol{f}^{-1}_{n+1/2})_{tu}}{\partial(\boldsymbol{f}_{n+1/2})_{rs}} &= -(\delta_{nt}\,\delta_{mu}-\delta_{mt}\,\delta_{nu})(\boldsymbol{f}^{-1}_{n+1/2})_{tr}(\boldsymbol{f}^{-1}_{n+1/2})_{su}\\
&= (\boldsymbol{f}^{-1}_{n+1/2})_{mr}(\boldsymbol{f}^{-1}_{n+1/2})_{sn}-(\boldsymbol{f}^{-1}_{n+1/2})_{nr}(\boldsymbol{f}^{-1}_{n+1/2})_{sm}
\end{aligned}
$$

- tensorial. This one is presented in Appendix; it will be retained especially here that it is iterative and asks one enough a large number of computations. This term is thus of

a complex writing and inevitably of a cost high enough for this algorithm. Term 3 This term, commun run

with the two $\dfrac{\partial\boldsymbol{\tau}_{n+1}}{\partial\boldsymbol{\sigma}_n}$

formalisms, prevents the tangent matrix from being of single exact statement for any behavior. Indeed, it would be necessary that each routine of integration of the behaviors provides it, which is not currently the case. The algorithms presented and studied thus lose one their major attractions which was their generics. Nevertheless, one can calculate his statement for example in the case of the model of Von Mises with isotropic and kinematical hardening linear. The criterion is

form: with the solution $f(\boldsymbol{\tau}_{n+1},\boldsymbol{x}_{n+1},p_{n+1})=\|dev(\boldsymbol{\tau}_{n+1}-\boldsymbol{x}_{n+1})\|-\sqrt{\frac{2}{3}}(\sigma_y+K\,p_{n+1})\le 0$

is obtained $x_{n+1}=C\,\boldsymbol{\varepsilon}_{n+1}$

by: with and This kind of $\boldsymbol{\tau}_{n+1}=\boldsymbol{\tau}^{pred}_{n+1}-2\mu\,\Delta\gamma\,\boldsymbol{n}=\sigma_n+\boldsymbol{a}:\Delta\boldsymbol{\varepsilon}_{n+1}-2\mu\,\Delta\gamma\,\boldsymbol{n}$ algorithm

$\boldsymbol{n}=\dfrac{dev(\boldsymbol{\tau}^{pred}_{n+1}-\boldsymbol{x}_n)}{\|dev(\boldsymbol{\tau}^{pred}_{n+1}-\boldsymbol{x}_n)\|}$ $\Delta\gamma=\dfrac{\|dev(\boldsymbol{\tau}^{pred}_{n+1}-\boldsymbol{x}_n)\|-\sqrt{\frac{2}{3}}(\sigma_y+K\,p_n)}{2\mu+\frac{2}{3}K+C}$ is $\Delta\gamma=\sqrt{\frac{3}{2}}\Delta p$

completely similar to the method used in small strains (radial return), which makes it possible not to modify structure of the code and was selection criteria of these algorithms. In this case, One sees intervening

three

$$\begin{aligned}\frac{\partial \boldsymbol{\tau}_{n+1}}{\partial \boldsymbol{\sigma}_n} &= \frac{\partial \boldsymbol{\tau}_{n+1}}{\partial \boldsymbol{\tau}_{n+1}^{pred}}\frac{\partial \boldsymbol{\tau}_{n+1}^{pred}}{\partial \boldsymbol{\sigma}_n}=\frac{\partial \boldsymbol{\tau}_{n+1}}{\partial \boldsymbol{\tau}_{n+1}^{pred}}\boldsymbol{I}_d=\boldsymbol{I}_d-2\mu\frac{\partial(\Delta\gamma\boldsymbol{n})}{\partial(\boldsymbol{\tau}_{n+1}^{pred})^{dev}}\frac{\partial(\boldsymbol{\tau}_{n+1}^{pred})^{dev}}{\partial \boldsymbol{\tau}_{n+1}^{pred}}\\ &= \boldsymbol{I}_d-2\mu\left(\frac{\partial(\Delta\gamma)}{\partial(\boldsymbol{\tau}_{n+1}^{pred})^{dev}}\boldsymbol{n}+\Delta\gamma\frac{\partial(\boldsymbol{n})}{\partial(\boldsymbol{\tau}_{n+1}^{pred})^{dev}}\right)\frac{\partial(\boldsymbol{\tau}_{n+1}^{pred})^{dev}}{\partial \boldsymbol{\tau}_{n+1}^{pred}}\end{aligned}$$

distinct terms: the derivative of the deviator

- : the derivative of the plastic $\dfrac{\partial(\sigma^{dev})_{ij}}{\partial\sigma_{kl}}=\dfrac{\partial\left(\sigma_{ij}-\dfrac{tr\,\sigma}{3}\delta_{ij}\right)}{\partial\sigma_{kl}}=\dfrac{1}{2}\left(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\right)-\dfrac{1}{3}\delta_{ij}\delta_{kl}$

- multiplier: the derivative of the norm $\dfrac{\partial(\Delta\gamma)}{\partial(\boldsymbol{\tau}_{n+1}^{pred})_{kl}^{dev}}=\dfrac{1}{2\mu+\dfrac{2}{3}K+C}\dfrac{(\boldsymbol{\tau}_{n+1}^{pred})_{kl}^{dev}}{\left\|(\boldsymbol{\tau}_{n+1}^{pred})^{dev}\right\|}=\dfrac{1}{2\mu+\dfrac{2}{3}K+C}\boldsymbol{n}_{kl}$

- on the surface of load: With final, by noting $\dfrac{\partial(\boldsymbol{n}_{ij})}{\partial(\boldsymbol{\tau}_{n+1}^{pred})_{kl}^{dev}}=\dfrac{1}{\left\|(\boldsymbol{\tau}_{n+1}^{pred})^{dev}\right\|}\left(\dfrac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{jk}\delta_{il})-\boldsymbol{n}_{ij}\boldsymbol{n}_{kl}\right)$

the first tensor identity $\boldsymbol{I}_d^{4s}$ of a nature 4 defined by: , one obtains for this term $\boldsymbol{I}_{d\,ijkl}^{4s}=\dfrac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{jk}\delta_{il})$:

This statement is unfortunately not

$$\frac{\partial(\boldsymbol{\tau}_{n+1})_{ij}}{\partial(\boldsymbol{\sigma}_n)_{kl}}=(\boldsymbol{I}_d^{4s})_{ijkl}-2\mu\left(\frac{1}{2\mu+\dfrac{2}{3}K+C}\boldsymbol{n}_{ij}\boldsymbol{n}_{mn}+\frac{\Delta\gamma}{\left\|\boldsymbol{\tau}_{n+1}^{pred}\right\|}\left((\boldsymbol{I}_d^{4s})_{ijmn}-\boldsymbol{n}_{ij}\boldsymbol{n}_{mn}\right)\right)\left((\boldsymbol{I}_d^{4s})_{mnkl}-\frac{1}{3}\delta_{mn}\delta_{kl}\right)$$

valid for nonlinear hardenings… Term 4 This term is

again $\dfrac{\partial\boldsymbol{\sigma}_n}{\partial\boldsymbol{f}_{n+1}}$

specific to each algorithm considered here. For the algorithm Simo

9.2.1 _Hughes_1 In this algorithm,

the stress of Cauchy is written From where: For algorithm $\boldsymbol{\sigma}_n=\boldsymbol{f}_{n+\alpha}\boldsymbol{\tau}_n\boldsymbol{f}_{n+\alpha}^T$
GDEF_HYPO_ELAS

$$\begin{aligned}\left(\frac{\partial\boldsymbol{\sigma}_n}{\partial\boldsymbol{f}_{n+1}}\right)_{ijkl} &= \frac{\partial(\boldsymbol{f}_{n+\alpha})_{ip}(\boldsymbol{\tau}_n)_{pq}(\boldsymbol{f}_{n+\alpha})_{jq}}{\partial(\boldsymbol{f}_{n+1})_{kl}}=\alpha\delta_{ik}\delta_{pl}(\boldsymbol{\tau}_n)_{pq}(\boldsymbol{f}_{n+\alpha})_{jq}+(\boldsymbol{f}_{n+\alpha})_{ip}(\boldsymbol{\tau}_n)_{pq}\alpha\delta_{jk}\delta_{ql}\\ &= \alpha\delta_{ik}(\boldsymbol{\tau}_n)_{lq}(\boldsymbol{f}_{n+\alpha})_{jq}+(\boldsymbol{f}_{n+\alpha})_{ip}(\boldsymbol{\tau}_n)_{pl}\alpha\delta_{jk}\\ &= \alpha(\boldsymbol{\tau}_n)_{lq}\left(\delta_{ik}(\boldsymbol{f}_{n+\alpha})_{jq}+(\boldsymbol{f}_{n+\alpha})_{iq}\delta_{jk}\right)\end{aligned}$$

9.2.2 In this `algorithm`,

the stress of Cauchy is thus written: And like, and by means of $\boldsymbol{\sigma}_n = \boldsymbol{r}_{n+1}\boldsymbol{\tau}_n \boldsymbol{r}_{n+1}^T$

$$\frac{\partial(\boldsymbol{\sigma}_n)_{qp}}{\partial(\boldsymbol{f}_{n+1})_{kl}} = \frac{\partial\left((\boldsymbol{r}_{n+1})_{qa}(\boldsymbol{\tau}_n)_{ab}(\boldsymbol{r}_{n+1})_{pb}\right)}{\partial(\boldsymbol{f}_{n+1})_{kl}} = \frac{\partial(\boldsymbol{r}_{n+1})_{qa}}{\partial(\boldsymbol{f}_{n+1})_{kl}}(\boldsymbol{\tau}_n)_{ab}(\boldsymbol{r}_{n+1})_{pb} + (\boldsymbol{r}_{n+1})_{qa}(\boldsymbol{\tau}_n)_{ab}\frac{\partial(\boldsymbol{r}_{n+1})_{pb}}{\partial(\boldsymbol{f}_{n+1})_{kl}}$$

$(\boldsymbol{r}_{n+1})_{qa} = (\boldsymbol{\Lambda}_{n+1})_{qx}(\boldsymbol{\Lambda}_n)_{ax} = \exp[\boldsymbol{\Theta}]_{qb}(\boldsymbol{\Lambda}_n)_{bx}(\boldsymbol{\Lambda}_n)_{ax}$ preceding results, With the final one: The derivative

$$\begin{aligned}\frac{\partial(\boldsymbol{r}_{n+1})_{qa}}{\partial(\boldsymbol{f}_{n+1})_{kl}} &= \frac{\partial\exp[\boldsymbol{\Theta}]_{qb}}{\partial(\boldsymbol{\Theta})_{mn}}\frac{\partial(\boldsymbol{\Theta})_{mn}}{\partial(\boldsymbol{f}_{n+1/2})_{rs}}\frac{\partial(\boldsymbol{f}_{n+1/2})_{rs}}{\partial(\boldsymbol{f}_{n+1})_{kl}}(\boldsymbol{\Lambda}_n)_{bx}(\boldsymbol{\Lambda}_n)_{ax}\\ &= \frac{1}{2}\frac{\partial\exp[\boldsymbol{\Theta}]_{qc}}{\partial(\boldsymbol{\Theta})_{mn}}\left(-(\boldsymbol{f}^{-1}_{n+1/2})_{nk}(\boldsymbol{f}^{-1}_{n+1/2})_{lm} + (\boldsymbol{f}^{-1}_{n+1/2})_{mk}(\boldsymbol{f}^{-1}_{n+1/2})_{\ln}\right)(\boldsymbol{\Lambda}_n)_{cx}(\boldsymbol{\Lambda}_n)_{ax}\end{aligned}$$

of exponential

$$\begin{aligned}\frac{\partial(\boldsymbol{\sigma}_n)_{qp}}{\partial(\boldsymbol{f}_{n+1})_{kl}} &= \frac{1}{2}(\boldsymbol{\tau}_n)_{ab}\left(-(\boldsymbol{f}^{-1}_{n+1/2})_{nk}(\boldsymbol{f}^{-1}_{n+1/2})_{lm} + (\boldsymbol{f}^{-1}_{n+1/2})_{mk}(\boldsymbol{f}^{-1}_{n+1/2})_{\ln}\right)(\boldsymbol{\Lambda}_n)_{cx}\\ &\quad -\left((\boldsymbol{r}_{n+1})_{pb}\frac{\partial\exp[\boldsymbol{\Theta}]_{qc}}{\partial(\boldsymbol{\Theta})_{mn}}(\boldsymbol{\Lambda}_n)_{ax} + (\boldsymbol{r}_{n+1})_{qa}\frac{\partial\exp[\boldsymbol{\Theta}]_{pc}}{\partial(\boldsymbol{\Theta})_{mn}}(\boldsymbol{\Lambda}_n)_{bx}\right)\end{aligned}$$

tensorial is given in Appendix. Term 5 This term, present

only $\frac{\partial\boldsymbol{\tau}_{n+1}}{\partial\boldsymbol{x}_n}$

in the case of a kinematic hardening, is of the same type as the term, in the sense that it is $\frac{\partial\boldsymbol{\tau}_{n+1}}{\partial\boldsymbol{\sigma}_n}$ specific to each constitutive law with kinematic hardening and thus prevents the writing of a generic tangent matrix for the formalisms. Let us give its statement
for example in the case of the model of Von Mises to linear kinematic hardening. One has then: with and

From where which requires $\boldsymbol{\tau}_{n+1} = \boldsymbol{\sigma}_n + \boldsymbol{a}:\Delta\boldsymbol{\varepsilon}_{n+1} - 2\mu\Delta\gamma\,\boldsymbol{n}$ $\boldsymbol{n} = \frac{\boldsymbol{\xi}^{pred}}{\|\xi_{pred}\|}$ $\boldsymbol{\xi}^{pred} = dev(\boldsymbol{\sigma}_n + \boldsymbol{a}:\Delta\boldsymbol{\varepsilon}_{n+1} - \boldsymbol{x}_n)$

the computation $\frac{\partial\boldsymbol{\tau}_{n+1}}{\partial\boldsymbol{x}_n} = -2\mu\frac{\partial(\Delta\gamma\,\boldsymbol{n})}{\partial\boldsymbol{\xi}^{pred}}\frac{\partial\boldsymbol{\xi}^{pred}}{\partial\boldsymbol{x}_n^{dev}}\frac{\partial(\boldsymbol{x}_n)^{dev}}{\partial\boldsymbol{x}_n}$ of four terms: the derivative of the deviator

- , the same form as for isotropic hardening: the derivative of the predictor $\frac{\partial(\boldsymbol{x}_n^{dev})_{ij}}{\partial(\boldsymbol{x}_n)_{kl}} = \frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) - \frac{1}{3}\delta_{ij}\delta_{kl}$
- : the derivative of the multiplier $\frac{\partial\boldsymbol{\xi}^{pred}}{\partial\boldsymbol{x}_n^{dev}} = -\boldsymbol{I}_d$
- , from where the derivative $\Delta\gamma = \frac{\|\xi^{pred}\| - \sqrt{\frac{2}{3}}(\sigma_y + K\,p_n)}{2\mu + \frac{2}{3}K + C}$ of the norm $\frac{\partial(\Delta\gamma)}{\partial(\boldsymbol{\xi}^{pred})_{kl}} = \frac{1}{2\mu + \frac{2}{3}K + C}\boldsymbol{n}_{kl}$

- on the surface of load: By combining the various $\frac{\partial(\boldsymbol{n}_{ij})}{\partial(\boldsymbol{\xi}^{pred})^{dev}_{kl}}=\frac{1}{\|\boldsymbol{\xi}^{pred}\|}\left(\frac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{jk}\delta_{il})-\boldsymbol{n}_{ij}\boldsymbol{n}_{kl}\right)$

relations, one obtains finally: Following the example of the term,

$$\frac{\partial(\boldsymbol{\tau}_{n+1})_{ij}}{\partial(\boldsymbol{x}_n)_{kl}}=2\mu\left(\frac{1}{2\mu+\frac{2}{3}K+C}\boldsymbol{n}_{ij}\boldsymbol{n}_{mn}+\frac{\Delta\gamma}{\|\boldsymbol{\xi}^{pred}\|}\left((\boldsymbol{I}^{4s}_d)_{ijmn}-\boldsymbol{n}_{ij}\boldsymbol{n}_{mn}\right)\right)\left((\boldsymbol{I}^{4s}_d)_{mnkl}-\frac{1}{3}\delta_{mn}\delta_{kl}\right)$$

the statement given $\frac{\partial\boldsymbol{\tau}_{n+1}}{\partial\boldsymbol{\sigma}_n}$ here is valid only for linear kinematic hardening. Term 6 This last term

exists $\frac{\partial\boldsymbol{x}_n}{\partial\boldsymbol{f}_{n+1}}$

obviously only for kinematic hardening. It depends moreover of the selected formalism. For SIMO_HUGHES_1 In this case,

9.2.1 one has, from where

: . It is noticed $\boldsymbol{x}_n=\boldsymbol{f}_{n+\alpha}\boldsymbol{q}_n\boldsymbol{f}^T_{n+\alpha}$ that this $\left(\frac{\partial\boldsymbol{x}_n}{\partial\boldsymbol{f}_{n+1}}\right)_{ijkl}=\frac{\partial(\boldsymbol{f}_{n+\alpha})_{ip}(\boldsymbol{q}_n)_{pq}(\boldsymbol{f}_{n+\alpha})_{jq}}{\partial(\boldsymbol{f}_{n+1})_{kl}}$

statement is same form as for. By analogy, one thus $\frac{\partial(\boldsymbol{\sigma}_n)_{ij}}{\partial(\boldsymbol{f}_{n+1})_{kl}}$

obtains: For GDEF_HYPO_ELAS In

$$\frac{\partial(\boldsymbol{x}_n)_{ij}}{\partial(\boldsymbol{f}_{n+1})_{kl}}=\alpha(\boldsymbol{x}_n)_{lq}\left(\delta_{ik}(\boldsymbol{f}_{n+\alpha})_{jq}+(\boldsymbol{f}_{n+\alpha})_{iq}\delta_{jk}\right)$$

9.2.2 this case one has, from where

: , which has $\boldsymbol{x}_n=\boldsymbol{r}_{n+1}\boldsymbol{q}_n\boldsymbol{r}^T_{n+1}$ the same $\frac{\partial\boldsymbol{x}_n}{\partial\boldsymbol{f}_{n+1}}=\frac{\partial(\boldsymbol{r}_{n+1}\boldsymbol{q}_n\boldsymbol{r}^T_{n+1})}{\partial\boldsymbol{f}_{n+1}}$ form again as: One thus obtains by

analogy $\frac{\partial\boldsymbol{\sigma}_n}{\partial\boldsymbol{f}_{n+1}}$

: Assessment Two important points

$$\frac{\partial(\boldsymbol{x}_n)_{qp}}{\partial(\boldsymbol{f}_{n+1})_{kl}}=\frac{1}{2}(\boldsymbol{q}_n)_{ab}\left(-(\boldsymbol{f}^{-1}_{n+1/2})_{nk}(\boldsymbol{f}^{-1}_{n+1/2})_{lm}+(\boldsymbol{f}^{-1}_{n+1/2})_{mk}(\boldsymbol{f}^{-1}_{n+1/2})_{\ln}\right)(\boldsymbol{\Lambda}_n)_{cx}$$
$$\left((\boldsymbol{r}_{n+1})_{pb}\frac{\partial\exp[\boldsymbol{\Theta}]_{qc}}{\partial(\boldsymbol{\Theta})_{mn}}(\boldsymbol{\Lambda}_n)_{ax}+(\boldsymbol{r}_{n+1})_{qa}\frac{\partial\exp[\boldsymbol{\Theta}]_{pc}}{\partial(\boldsymbol{\Theta})_{mn}}(\boldsymbol{\Lambda}_n)_{bx}\right)$$

are to be retained mainly: the statements are heavy

- , with many terms with calculating, which forecasts important computing times, because of certain terms
- dependant on the constitutive law used, it is not possible to give a generic statement for the tangent matrixes. Consequently, unless programming for each constitutive law the computation which corresponds to him, the tangent matrix cannot be exact. One will find in the ref.

[15], an assessment of the implementation of these statements in terms of performances and velocity of convergence through examples. Appendix 2: Computation of

10 the logarithmic strains Notations: indicate

10.1 a tensor of

A order 2, and a tensor of order 4 we $\bar{A}$ adopt in

the implementation the notation of Voigt, (see for example [16]) defined by: where the components relating

$$A=\begin{Bmatrix} A_{11} \\ A_{22} \\ A_{33} \\ \sqrt{2}A_{12} \\ \sqrt{2}A_{13} \\ \sqrt{2}A_{23} \end{Bmatrix} \quad \bar{A}=\begin{Bmatrix} A_{1111} & A_{1122} & A_{1133} & \sqrt{2}A_{1112} & \sqrt{2}A_{1123} & \sqrt{2}A_{1113} \\ A_{2211} & A_{2222} & A_{2233} & \sqrt{2}A_{2212} & \sqrt{2}A_{2223} & \sqrt{2}A_{2213} \\ A_{3311} & A_{3322} & A_{3333} & \sqrt{2}A_{3312} & \sqrt{2}A_{3323} & \sqrt{2}A_{3313} \\ \sqrt{2}A_{1211} & \sqrt{2}A_{1222} & \sqrt{2}A_{1233} & 2A_{1212} & 2A_{1223} & 2A_{1213} \\ \sqrt{2}A_{1311} & \sqrt{2}A_{1322} & \sqrt{2}A_{1333} & 2A_{1312} & 2A_{1323} & 2A_{1313} \\ \sqrt{2}A_{2311} & \sqrt{2}A_{2322} & \sqrt{2}A_{2333} & 2A_{2312} & 2A_{2323} & 2A_{2313} \end{Bmatrix}$$

to the notation of Voigt will be indicated by a Greek letter. There are then the following

properties: The reverse of a tensor
$\|A_{ij}\|=\|A_{\alpha}\|$

$$A:B=A_{ij}B_{ij}=A_{\alpha}B_{\alpha} \quad \bar{A}:B=A_{ijkl}B_{kl}=A_{\alpha\beta}B_{\beta} \quad \bar{A}:\bar{B}=A_{ijkl}B_{klmn}=A_{\alpha\beta}B_{\beta\gamma}$$

of a nature 4 comprising of minor symmetries () is written: and with $A_{ijrs}=A_{jirs}=A_{ijsr}$ the notation of

$\bar{A}:\bar{A}^{-1}=\bar{I}_d \quad A_{ijrs}A^{-1}_{rskl}=I_{ijkl}=\frac{1}{2}\left(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\right)$ Voigt: Statement of the stresses $A_{\alpha\gamma}A^{-1}_{\gamma\beta}=I_{\alpha\beta}=\delta_{\alpha\beta}$

10.2 in Lagrangian configuration the power of the internal forces

is written: with what makes it possible

$p_{\text{int}}=T:\dot{E}=S:\bar{P}^{-1}\dot{E}$ to calculate $\bar{P}=2\dfrac{\partial E}{\partial C}$: or To compute: the tensor $S=T:P \quad S_{ij}=T_{kl}:P_{klij}$

of the stresses of Cauchy, it is enough to write: Typical case of the plane stresses $\sigma=\dfrac{1}{det\,F}F.S.F^T$

: In this case, one entirely

does not know: indeed, the component $det\,F$ of the strain tensor zz logarithmic curves is unknown, because dependant E on the constitutive law. While limiting oneself to the behaviors such as (plastic incompressibility $det\,F^P=0$), one has then: . According to [16] one can $det\,F=det\,F^e$ calculate
this statement: where represents the elastic $det\,F^e=\exp\left(E^e_{xx}+E^e_{yy}+E^e_{zz}\right)$ E^e part of the logarithmic strains, known for any elastoplastic or élasto-viscoplastic constitutive law by the Hooke's law: . Indeed the definition $E^e=\Lambda^{-1}T$

of the logarithmic strains: conduit with being $E_{ij}=\frac{1}{2}\sum_{k=1,3}\log(\lambda^{(k)})N_i^{(k)}\otimes N_j^{(k)}$ eigenvalues $det\,F^e=\sqrt{det\,F^T F}=\sqrt{\lambda_1\lambda_2\lambda_3}$ λ_i of. Thus from where result $F^T F$. Statement $\log(det\,F^e)=\frac{1}{2}\log(\lambda_1)+\log(\lambda_2)+\log(\lambda_3)$ of the tangent

10.3 operator in Lagrangian configuration By deriving the statement

compared to time: $S=T:P$ maybe, with and what

$$\dot{S}=\dot{T}:\bar{P}+T:\dot{\bar{P}}=\left(\frac{\partial T}{\partial E}:\dot{E}\right):\bar{P}+T:\left(\frac{\partial \bar{P}}{\partial C}:\dot{C}\right)=\left[\frac{\partial T}{\partial E}:\left(\frac{\partial E}{\partial C}:\dot{C}\right)\right]:\bar{P}+T:\left(\frac{\partial \bar{P}}{\partial C}:\dot{C}\right)$$ defines

$$\dot{S}=\left(\bar{P}^T:\bar{E}^p:\bar{P}+T:\bar{\bar{L}}\right):\tfrac{1}{2}\dot{C}\quad \bar{\bar{L}}=4\frac{\partial^2 E}{\partial C\,\partial C}$$ the tangent $\bar{E}^P=\frac{\partial T}{\partial E}$

operator who checks or $\bar{\bar{C}}^{ep}=\left(\bar{P}^T\bar{E}^p\bar{P}+T:\bar{\bar{L}}\right)$, according to $\dot{S}=\bar{\bar{C}}^{ep}:\frac{1}{2}\dot{C}$

the strains of Green-Lagrange: The statement of this $\Delta=\frac{1}{2}\left(C-I_d\right)$ tangent

$$\frac{\partial S}{\partial \Delta}=\frac{\partial S}{\partial C}:\frac{\partial C}{\partial \Delta}=2\frac{\partial S}{\partial C}=\bar{\bar{C}}^{ep}$$

operator as well as the tensor of the stresses, both in Lagrangian configuration, allow, for the computation of the internal forces, to use a variational formulation in initial configuration, as in [R5.03.20] for example. One writes the equilibrium

in variational form on the initial configuration. , kinematically admissible

$$\delta W_{\text{int}}.\boldsymbol{\delta}\,\mathbf{v}+SW_{ext}.\boldsymbol{\delta}\,\mathbf{v}=0\quad\forall\,\boldsymbol{\delta}\,\mathbf{v}$$ Under the assumption that

the loading does not depend on the geometrical transformation, the virtual wor of the external forces is written like a linear form: : voluminal loading

$$\delta W_{ext}.\boldsymbol{\delta}\,\mathbf{v}=\int_{\Omega_o}\rho_o F_i\,\delta v_i\,d\Omega_o+\int_{\partial_F\Omega_o}T_i^d\,\delta v_i\,dS_o$$

$\mathbf{F}$: surface loading
$\mathbf{T^d}$ being exerted on edge There still, we choose $\partial_F\boldsymbol{\Omega}_\mathbf{o}$

the initial configuration like reference configuration, to express the work of the internal forces [R5,03,20]. [R7.02.03]. with: In the optics of

$$SW_{int}.\boldsymbol{\delta}\,\mathbf{v}=-\int_{\Omega_o}F_{ik}S_{kl}\,\delta v_{i,l}\,d\Omega_o$$ a resolution $\delta v_{i,l}=\frac{\partial dv_i}{\partial X_l}$

by a method of Newton, it is important to also express the variation second of the virtual wor of the internal forces, namely: Geometrical stiffness

$$d^2W_{\text{int}}.\boldsymbol{\delta}\,\mathbf{u}.\boldsymbol{\delta}\,\mathbf{v}=-\int_{\Omega_o}\delta u_{i,k}S_{kl}\,\delta v_{i,l}\,d\Omega_o$$ elastic Stiffness effective

$$\ldots \; -\int_{\Omega_o} \delta u_{i,q} F_{ip} \left(\frac{\partial \boldsymbol{S}}{\partial \boldsymbol{\Delta}} \right)_{pqkl} F_{jk} \delta v_{j,l} d\Omega_o$$

Computation of the logarithmic strains

10.4 They are defined by

: (in any rigor, it $\boldsymbol{E}=\frac{1}{2}\log(\boldsymbol{C})=\frac{1}{2}\log(\boldsymbol{F}^{\boldsymbol{T}}.\boldsymbol{F})$ would be necessary to add the metric tensor in the case of an initial configuration defined in a space different from Euclidean space (case of the shells for example)). To simplify the writings, we will place ourselves in the case of an Euclidean initial configuration, the components of the vectors and tensors being written in a reference orthonormé 3D. The restriction on 2D is immediate. The computation of the logarithmic strain

can be done only in the clean reference. It is thus necessary to determine
the 3 eigenvalues and eigenvectors solutions $\lambda^{(i)}$ of the problem $\boldsymbol{N}^{(i)}$ to the eigenvalues One can then calculate $\boldsymbol{C}\,\boldsymbol{N}^{(i)}=\lambda^{(i)}\boldsymbol{N}^{(i)}$

the 3 values: The logarithmic strains $e^{(i)}=\frac{1}{2}\log(\lambda^{(i)})$
are then transported within the space of origin by: because the function is

$\boldsymbol{E}_{ij}=\frac{1}{2}\sum_{k=1,3}\log(\lambda^{(k)})\boldsymbol{N}_i^{(k)}\otimes\boldsymbol{N}_j^{(k)}$ an isotropic function $\log$ of the tensor [16]. For postprocessing $\boldsymbol{C}$

, i.e. the computation of the tensor of the stresses and the tangent operator, it is necessary to calculate the quantities and with and, and defined $\bar{\boldsymbol{P}}=2\frac{\partial \boldsymbol{E}}{\partial \boldsymbol{C}}$ $\boldsymbol{T}:\bar{\bar{\boldsymbol{L}}}$

$$\bar{\boldsymbol{P}}=\sum_{i=1,3}\tfrac{1}{2}d^{(i)}\boldsymbol{N}^{(i)}\otimes\boldsymbol{N}^{(i)}\otimes\boldsymbol{M}^{(ii)}+\sum_{i=1,3}\sum_{j\neq i}^{3}\theta_{ij}\boldsymbol{N}^{(i)}\otimes\boldsymbol{N}^{(j)}\otimes\boldsymbol{M}^{(ij)}$$

$$\boldsymbol{T}:\bar{\bar{\boldsymbol{L}}}=\sum_{i}^{3}\tfrac{1}{4}f^{(i)}\zeta^{(ii)}\boldsymbol{M}^{(ii)}\otimes\boldsymbol{M}^{(ii)}+\sum_{i}^{3}\sum_{j\neq i}^{3}\sum_{k\neq i,k\neq j}^{3}2\eta\,\zeta^{(ij)}\boldsymbol{M}^{(ik)}\otimes\boldsymbol{M}^{(ij)}$$
$$+\sum_{i}^{3}\sum_{j\neq i}2\,\xi^{(ij)}[\zeta^{(ij)}(\boldsymbol{M}^{(ij)}\otimes\boldsymbol{M}^{(jj)}+\boldsymbol{M}^{(jj)}\otimes\boldsymbol{M}^{(ij)})+\zeta^{(jj)}\boldsymbol{M}^{(ij)}\otimes\boldsymbol{M}^{(ij)}]$$

by $d^{(i)}=\frac{1}{\lambda^{(i)}}$ $f^{(i)}=\frac{-2}{(\lambda^{(i)})^2}$ $\zeta^{(ij)}=\boldsymbol{T}:\boldsymbol{N}^{(i)}\otimes\boldsymbol{N}^{(j)}$ $\boldsymbol{M}_{\boldsymbol{ab}}^{(\boldsymbol{ij})}=\boldsymbol{N}_{\boldsymbol{a}}^{(i)}\boldsymbol{N}_{\boldsymbol{b}}^{(j)}+\boldsymbol{N}_{\boldsymbol{a}}^{(j)}\boldsymbol{N}_{\boldsymbol{b}}^{(i)}$

: $\theta^{(ij)}$ if $\xi^{(ij)}$ all $\eta^{(ij)}$ the eigenvalues

•are different: , if two eigenvalues $\theta^{(ij)}=\frac{\left(e^{(i)}-e^{(j)}\right)}{\left(\lambda^{(i)}-\lambda^{(j)}\right)}$ $\xi^{(ij)}=\frac{\left(\theta^{(ij)}-\frac{1}{2}d^{(j)}\right)}{\left(\lambda^{(i)}-\lambda^{(j)}\right)}$

$$\eta=\sum_{i}^{3}\sum_{j\neq i}^{3}\sum_{k\neq i,k\neq j}^{3}\frac{e^{(i)}}{2(\lambda^{(i)}-\lambda^{(j)})(\lambda^{(i)}-\lambda^{(k)})}$$

•are equal: : if the three eigenvalues

$$\lambda^{(i)}=\lambda^{(j)}\neq\lambda^{(k)} \quad \theta^{(ij)}=\theta^{(ji)}=\frac{1}{2}d^{(j)} \quad \xi^{(ij)}=\xi^{(ji)}=\frac{1}{8}f^{(j)} \quad \eta=\xi^{(ki)}$$

$$\theta^{(mn)}=\frac{\left(e^{(m)}-e^{(n)}\right)}{\left(\lambda^{(m)}-\lambda^{(n)}\right)} \quad \xi^{(mn)}=\frac{\left(\theta^{(mn)}-\frac{1}{2}d^{(n)}\right)}{\left(\lambda^{(m)}-\lambda^{(n)}\right)} \quad n=k\,,\ m\in\{i\,,j\} \quad \text{ou} \quad m=k\,,\ n\in\{i\,,j\}$$

•are equal: : $\lambda^{(i)}=\lambda^{(j)}=\lambda^{(k)} \quad \theta^{(ij)}=\frac{1}{2}d^{(j)} \quad \xi^{(ij)}=\frac{1}{8}f^{(j)} \quad \eta=\frac{1}{8}f^{(j)}$