Code_Aster

Version 11

Titre : Formulation discrète du contact-frottement

Responsable : Mickael ABBAS

Date : 05/08/2013 Page : 1/60

Clé : R5.03.50   Révision : 11228

Discrete formulation of Summarized

contact-friction :

One describes in this document the numerical methods used to deal with the problems of contact/friction in large displacements in operator STAT_NON_LINE or DYNA_NON_LINE using a discrete formulation. The algorithms available make it possible to deal with the problem in an exact or approximate way, without restrictive choice on the subjacent mechanical problem (kinematical or constitutive laws nonlinear).

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1 Introduction

Two solids distinct are known as in contact when they share a common surface which is part of their border. To treat the unilateral contact consists in preventing that one of solids "does not cross" the other. Friction characterizes the tangential sliding of a solid compared to the other. In *Code_Aster*, only the solid friction of Coulomb is available.

The contact in discrete formulation in *Code_Aster* has the following characteristics:
•The models of contact and friction are drawn up from already discretized quantities: displacements and nodal forces on structures with a grid;
•The problem of contact/friction is solved by uncoupling it from the problem of the equilibrium of structure.

In this document one will present the various ingredients from problem:
•Description of the models of contact and friction;
•Geometrical pairing: it is the phase where one binds a node to another node or a node with a mesh to create a potential *couple* of contact, i.e. one locates the degrees of freedom for which one will write discrete relations of nonpenetration;
•Establishment of the problem of contact/friction;
•Description of the algorithms available to solve the problem of contact/friction.

This document is thus limited to methods known as GCP, FORCED, LAGRANGIAN and PENALIZATION.

The formulation known as "continuous" (FORMULATION='CONTINUE' in DEFI_CONTACT) been the object of the document [R5.03.52].
The formulation associated with elements XFEM (FORMULATION='XFEM' in DEFI_CONTACT) is discussed in [R5.03.53].

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2   Mechanics of the contact and of friction

2.1   Definition of the problem and notations

One considers two solids being able to come into rubbing contact, the contact zone is either specific, or linear, or surface. That is to say $\lfloor n \rfloor$ the outgoing norm on the surface of the one of solids in contact and $\lfloor u \rfloor$ the vector displacement between two solids. Then $g = \lfloor u \rfloor . \lfloor n \rfloor$ is project displacement on this norm, one will call it *clearance*. From the data of the stress of Cauchy, one defines the pression1Le[1] $p = \lfloor n \rfloor . \lfloor \sigma \rfloor . \lfloor n \rfloor$ and the shearing stress tangential $\lfloor r \rfloor = \lfloor \sigma \rfloor . \lfloor n \rfloor - p . \lfloor n \rfloor$ as being exerted by one of surfaces on the other.
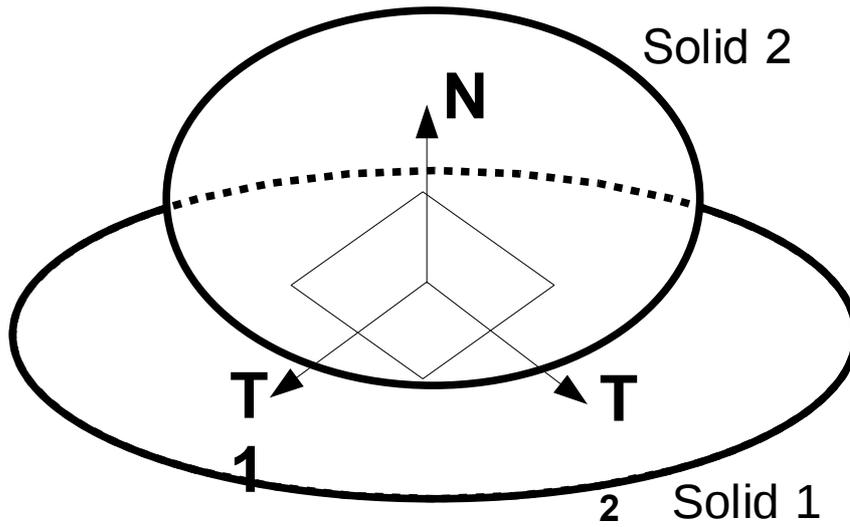
Appears 2.1-a : definition of the local coordinate system of contact.

The shear force has as a direction in the contact zone a vector $\lfloor t \rfloor$ located in the tangent plane $\left(\lfloor t_1 \rfloor , \lfloor t_2 \rfloor \right)$ indicated on the figure (2.1-a). The equation (1) 2) defines r the shearing stress exerted by solid (on the solid (1) per unit of contact surface.

$$\lfloor r \rfloor = \lfloor \sigma \rfloor . \lfloor n \rfloor - (\langle n \rangle . \lfloor \sigma \rfloor . \lfloor n \rfloor) . \lfloor n \rfloor = r . \lfloor t \rfloor \quad \text{with} \quad r = \lVert \lfloor r \rfloor \rVert \tag{1}$$

2.2   Conditions of contact of Signorini

One introduced the two variables defining the contact:
  ● The distance signed between two surfaces or "gap" g ;
  ● The contact pressure p ;
One then defines the three conditions of contact of geometrical
  ● Hertz-Signorini-Moreau Condition: impenetrability of the matter (Signorini-Hertz).

$$g \geq 0 \Leftrightarrow \begin{cases} g > 0 \ \text{pas de contact} \\ \quad g = 0 \ \text{contact} \end{cases} \tag{2}$$

  ● mechanical Condition: intensility (Signorini-Hertz)

$$p \geq 0 \Leftrightarrow \begin{cases} \quad p > 0 \ \text{contact} \\ p = 0 \ \text{pas de contact} \end{cases} \tag{3}$$

  ● energy Condition: complementarity/exclusion (Moreau)

---

1   term "pressure" is an abusive abbreviation of "density of force of contact"

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$$p \cdot g = 0 \Leftrightarrow \begin{cases} p = 0 & \text{décollement} \\ g = 0 & \text{contact} \end{cases} \tag{4}$$

Graphically, the three conditions are represented on the figure (2.2-a) where the shaded zones represent the zones excluded and the white areas or the black features represent the authorized zones.

Impenetrability   Intensilité   Complementarity
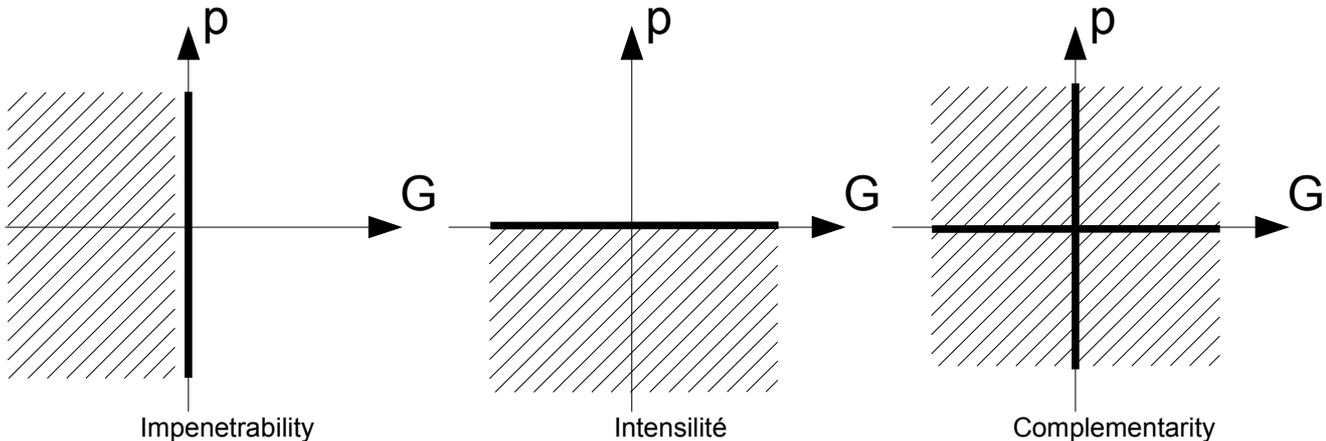
Appears 2.2-a: chart of the conditions of Hertz-Signorini-Moreau.

By combining the three conditions one obtains the graph of the figure (2.2-b).
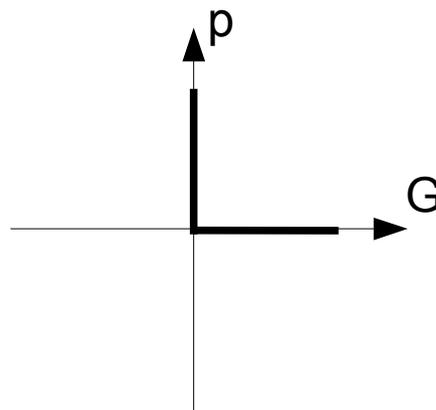
Appears 2.2-b : graph of the condition of unilateral contact.

The problem of contact thus posed introduced a relation *NON-univocal* (p is not *a function* of g), *semi-definite positive* and *NON-differentiable* in $p = g = 0$. It is about a problem mathematically difficult to treat. The contact is a phenomenon reversible and conservative for which one can introduce a natural energy and of which result does not depend on the way of loading, it is similar to the elastoplastic model of Hencky. The model of contact of Signorini is written:

$$\begin{cases} g \geq 0 & (a) \\ p \geq 0 & (b) \\ p \cdot g = 0 & (c) \end{cases} \tag{5}$$

Note:
• The contact is supposed without adhesion thanks to the second condition (intensity) .
• The third condition makes it possible the problem of unilateral contact to be well posed to be soluble by classical techniques of optimization under stresses (introduction of the conditions of Kuhn & Tucker) like the method of the active stresses.

2.3 Formulation of the problem of friction

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2.3.1   Definitions

the selected criteria of friction are form:

$$h(|r|) \leq 0 \tag{6}$$

where $h(|r|)$ is a convex function. The field of nonsliding is defined by the interior of the convex one. Two criteria of friction of the form $h(|r|) \leq 0$ are particularly used: the criterion of Tresca and the criterion of Coulomb.

2.3.2   The criterion of Tresca

the criterion of Tresca is defined by the following $h(|r|)$ function:

$$h(|r|) = \||r|\| - k \leq 0 \text{ with } k \text{ a constant data} \tag{7}$$

One notes C the convex disc of radius k centered in the beginning defined by:

$$C = \{|r| \text{ tel que } \||r|\| \leq k\} \tag{8}$$

the condition of nonsliding is then defined by the belonging of $|r|$ inside the disc C. In the event of sliding, for $|r|$ located on the border of C, the direction of sliding $|t|$ of $|\dot{u}|$ is given by the norm to the criterion in $|r|$, as indicated on Appears 2.3.2-a.
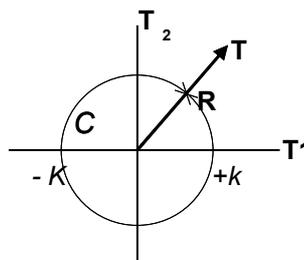
Appears 2.3.2-a : disc of friction for the criterion of Tresca.

What results in a form of the velocity of sliding similar to the plastic multiplier:

$$|\dot{u}_t| = \lambda . |r| \tag{9}$$

With $\lambda \geq 0$, i.e. the velocity of sliding is in the same direction as the shear stress.

2.3.3   The criterion of Coulomb

the criterion of Coulomb is defined by the following $h(|r|)$ function:

$$h(|r|, \mu, p) = \||r|\| - k(\mu, p) \leq 0 \text{ and } k = \mu . |p| \tag{10}$$

the value of k depends on the contact pressure p and of the coefficient of kinetic friction of Coulomb μ Thus defined, h is a cone. In the event of sliding, for $|r|$ located on the border of h, the direction of sliding $|t|$ of $|\dot{u}|$ is not given by the norm to the criterion in $|r|$, but by the norm with the convex disc C of radius $k = \mu . |p|$. The criterion of Tresca corresponds to a "slice" according to the orthogonal plane with the cone of Coulomb.

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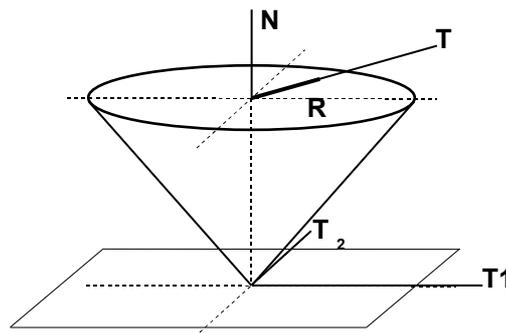
Appear 2.3.3-a : cone of friction for the criterion of Coulomb.

2.3.4 Application to the friction of Coulomb

Let us write the system of equations and of inequations having to be checked by these quantities in the case of the criterion of friction of Coulomb:

$$
\begin{cases}
g \geq 0 & (a) \\
p \geq 0 & (b) \\
p \cdot g = 0 & (c) \\
\|[r]\| - \mu . |p| \leq 0 & (d) \\
[\dot{u}_t] = \lambda . [r] & (e) \\
\lambda . (\|[r]\| - \mu . |p|) = 0 & (f) \\
\lambda \geq 0 & (g)
\end{cases}
\tag{11}
$$

the first set of equations and inequations (11) corresponds to management of the contact. The second batch (11) corresponds to the description of friction obeying the criterion of Coulomb. It utilizes several fields and the binds between them: normal pressure p, the shearing stress $[r]$ and tangent velocity $[\dot{u}_t]$. It can be understood as follows:

$$
\begin{cases}
\text{Si } \|[r]\| < \mu . |p| \text{ alors } \lambda = 0 \text{ et } [\dot{u}_t] = 0 & (a) \\
\text{Si } \|[r]\| = \mu . |p| \text{ alors } \lambda > 0 \text{ et } [\dot{u}_t] = \lambda . [r] & (b)
\end{cases}
\tag{12}
$$

On the figure (2.3.4-a) one represented the cone of Coulomb. Within the space of stresses, the force of rubbing contact can be only inside the cone of Coulomb: if it is strictly inside, the contact is adherent; if it is on the surface of the cone, the contact is slipping.
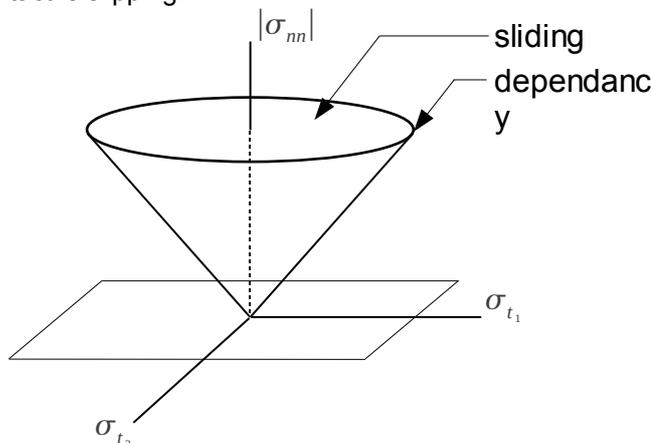
Appears 2.3.4-a : interpretation of the cone of friction.

One can give another representation of this criterion (see figure (2.3.4-b)).

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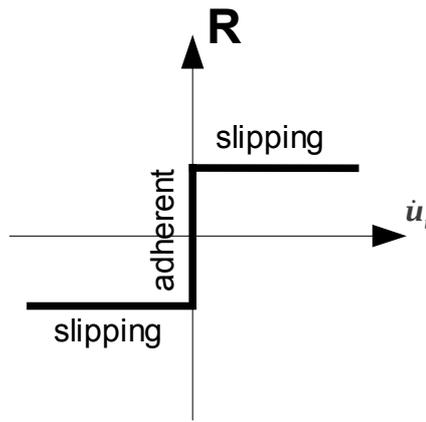
Appears 2.3.4-b : graph of the friction of Coulomb.

Friction induces the notion of threshold. The relation introduced by the friction of Coulomb is a NON-differentiable NON-univocal relation. Contrary to the contact, the relation is not associative2Par[2] and cannot derive directly from a natural energy. It will be also noticed that friction binds *the relative velocity* of two surfaces and not displacement. However one can establish that in static, if one solves the problem in incremental form (what is the case when one uses Newton as in *Code_Aster*), one can replace the relative velocity by the tangential displacement increment.

2.4   Formulation by differential inclusions

the NON-differentiable character of the relations of friction leads us to introduce the notion of differential under-inclusion.

One notes V all kinematically admissible displacements of the problem. The relation between the velocity of relative sliding $\lfloor\dot{u}\rfloor$ and the shearing stress r translates the two possible states of the system: not sliding or relative sliding following the normal direction in $\lfloor r\rfloor$ to the convex disc C. For the three criteria presented, the function $\lfloor\dot{u}_t\rfloor(r)$ and its reciprocal $\lfloor r(\dot{u}_t)\rfloor$ belong both to under - differential of two combined pseudopotentials $\Psi_c(r)$ and $\Psi_c^*(\dot{u}_t)$, so that one can write:

$$\lfloor\dot{u}\rfloor\in\partial\,\Psi_c(r) \text{ and } \lfloor r\rfloor\in\partial\,\Psi_c^*(\dot{u}) \tag{13}$$

the appearance of differential inclusions comes from the NON-differentiable character from the models from contact - friction. Indeed, Ψ_c indicates the indicating function of the disc convex C of radius k, centered in the beginning, previously definite. It is such as:

$$\Psi_c(r)=\begin{cases}0 \text{ si } \lfloor r\rfloor\in C\\+\infty \text{ sinon}\end{cases} \tag{14}$$

the under-differential $\partial\,\Psi_c(r)$ of the function Ψ_c in r merges with the norm external with C in $\lfloor r\rfloor$. $\Psi_c^*(\dot{u})=k\|\dot{u}\|$, where k is the threshold of friction resistance, Ψ_c^* is combined of Fenchel of the indicating function Ψ_c. Ψ_c^* of degree one is positively homogeneous. This function is interpreted like the density of power dissipated in the sliding. Using the notions of under-differential, one can establish the following relations for $\lfloor\dot{u}\rfloor$ formula $\lfloor r\rfloor$ formula:

---

2   against the friction of Tresca is well an associative model

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$$\begin{cases} \dot{\boldsymbol{u}} \in \partial\,\Psi_c(\boldsymbol{r}) \Leftrightarrow \begin{cases} \Psi_c(\boldsymbol{r})=0 \\ \dot{\boldsymbol{u}}.(\boldsymbol{r}'-\boldsymbol{r})\leq 0,\ \forall\,\boldsymbol{r}'\in C \end{cases} \\ \boldsymbol{r} \in \partial\,\Psi_c^*(\dot{\boldsymbol{u}}) \Leftrightarrow \begin{cases} \dot{\boldsymbol{u}}\in V \\ \boldsymbol{r}.(\dot{\boldsymbol{v}}-\dot{\boldsymbol{u}})\leq \Psi_c^*(\dot{\boldsymbol{v}})-\Psi_c^*(\dot{\boldsymbol{u}}),\ \forall\,\dot{\boldsymbol{v}}\in V \end{cases} \\ \Psi_c^*(\dot{\boldsymbol{u}})+\Psi_c(\boldsymbol{r})=\dot{\boldsymbol{u}}.\boldsymbol{r}=k\|\dot{\boldsymbol{u}}\| \end{cases}$$
(15)

Note:

• The two combined pseudopotentials presented are not - differentiable.

• Once known the normal reaction for the criterion of Coulomb, one is brought back locally to a criterion of friction of Tresca whose threshold is worth $k=\mu.|p|$.

• Adopted local criteria having a circular form one from of deduced that $\dot{\boldsymbol{u}}\in\partial\,\Psi_c(\boldsymbol{r})$ implies that there exists λ positive reality such as $\dot{\boldsymbol{u}}=\lambda\,\boldsymbol{r}$.

• The formulation of the problem of velocity suggests an incremental numerical resolution of the problem of friction. The resolution of the problem of equilibrium will thus be presented in incremental form.

2.5   Resolution of the problem of equilibrium

One considers two solids of total volume Ω of which contact surface is Γ_c . To simplify, one will suppose the existence of a differentiable strain energy to characterize the response of two separated solids with external requests (in fact, one can show that the results given hereafter are independent of this assumption). One notes V all the kinematically admissible fields of displacement, constrained by the respect of the conditions of contact and friction on the interface. The equilibrium of two solids in the absence of friction is written:

To find U field of displacement kinematically admissible such as

$$\left\{\boldsymbol{U}=\underset{\boldsymbol{v}\in V}{\text{argmin}}\left[\Phi(\varepsilon(\boldsymbol{v}))-W(\boldsymbol{v})\right]\right\}\Leftrightarrow\left\{\Phi(\boldsymbol{U})-W(\boldsymbol{U})\leq\Phi(\boldsymbol{v})-W(\boldsymbol{v}),\ \forall\,\boldsymbol{v}\in V\right\}$$
(16)

In elasticity, $\Phi(\boldsymbol{v})=\int_\Omega \varphi(\varepsilon(\boldsymbol{v})).d\Omega$ is strain energy. The function $W(\boldsymbol{v})$ represents the work of the external forces. A requirement (which becomes sufficient if Φ is strictly convex) so that this equilibrium is checked is that:

$$D\Phi(\boldsymbol{U})-DW(\boldsymbol{U})=D\Phi(\boldsymbol{U})-\boldsymbol{L}^{\text{ext}}=0$$
(17)

where D is the operator derived from Cakes and $\boldsymbol{L}^{\text{ext}}$ is the linear form associated with the external forces.

With the introduction of friction, the problem must be approached in incremental form. One is led (see 59 and 59) to the problem of minimization following on all $\bar{V}$ the kinematically admissible fields constrained by the respect of the conditions of contact and friction on the interface:

U known, to find $\Delta U\in\bar{V}$ such as

$$\boldsymbol{U}+\Delta\boldsymbol{U}=\underset{\Delta\boldsymbol{v}\in\bar{V}}{\text{argmin}}\left[\Phi(\varepsilon(\boldsymbol{U}+\Delta\boldsymbol{v}))+\Psi_c^*(\Delta\boldsymbol{v}_t)-W(\boldsymbol{U}+\Delta\boldsymbol{v})\right]$$
(18)

$\boldsymbol{U}+\Delta\boldsymbol{U}$ is thus solution of:

$$\underset{\Delta\boldsymbol{v}\in\bar{V}}{\min}\left[\int_\Omega \varphi(\varepsilon(\boldsymbol{U}+\Delta\boldsymbol{v}))d\Omega+\int_{\Gamma_c} k|\Delta\boldsymbol{v}_t|d\Gamma_c-W(\boldsymbol{U}+\Delta\boldsymbol{v})\right]$$
(19)

where $\Delta\boldsymbol{v}_t$ is the tangential component of the displacement increment relatif3[3] solid 2 compared to solid 1 along contact surface, with the conventions adopted with the §2.32.3.

By means of relations $\Psi_c^*(\Delta\boldsymbol{v}_t)=k|\Delta\boldsymbol{v}_t|$ and $\Psi_c^*(\Delta\boldsymbol{v}_t)\geq \boldsymbol{r}\,\Delta\boldsymbol{v}$ if $\boldsymbol{r}\in C$ one from of deduced that $\boldsymbol{U}+\Delta\boldsymbol{U}$ is solution of the problem of minmax following, on the space V of the kinematically admissible fields:

---

3    $\Delta\boldsymbol{v}_t$ is the tangential component of the relative displacement increment in quasi-static formulation, it becomes relative velocity of two surfaces in dynamic formulation.

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$$\underset{\Delta v \in V}{\text{Min}} \underset{r}{\text{Max}} \quad J(U + \Delta v, r) \tag{20}$$

where the functional calculus J is worth:

$$J(U + \Delta v, r) = \int_{\Omega} \phi(\varepsilon(U + \Delta v)).d\Omega + \int_{\Gamma_c} (r.\Delta v_t - \Psi_c(r)).d\Gamma_c - W(U + \Delta v) \tag{21}$$

the presence of the indicating function in this statement indicates that the shears r on contact surface Γ_c belong to the convex disc of friction C.

2.6   Variational formulation

If φ is convex, the problem minmax to be solved puts in an equivalent way in the form:

To find $\Delta U \in V$ and $r \in C$, together independent variables such as

$$\delta J(U + \Delta U, r) \ni 0 \tag{22}$$

This amounts solving the system of equations with the following equilibrium:

$$\begin{cases} \int_{\Omega} \dfrac{\partial \varphi}{\partial \varepsilon}(\varepsilon(U + \Delta U)).\delta \varepsilon.d\Omega + \int_{\Gamma_c} r.\delta v_t.d\Gamma_c - L^{\text{ext}}.\delta v = 0 & (a) \\ \int_{\Gamma_c} (\delta r.\Delta U_t.d\Gamma_c - \partial \Psi_c(r)) \ni 0 & (b) \end{cases} \tag{23}$$

or in an equivalent way:

$$\begin{cases} \int_{\Omega} \dfrac{\partial \varphi}{\partial \varepsilon}(\varepsilon(U + \Delta U)).\delta \varepsilon.d\Omega + \int_{\Gamma_c} r.\delta v_t.d\Gamma_c - L^{\text{ext}}.\delta v = 0 & (a) \\ \Delta U_t \in \partial \Psi_c(r) & (b) \\ r = (\sigma_1.n).t \ \text{sur} \ \Gamma_c & (c) \end{cases} \tag{24}$$

As in the preceding section, L^{ext} is the linear form associated with the external forces. The linear form L^{frot} is associated with the shear forces exerted by solid 2 on the contact surface of solid 1. It will be also noted that the variational formulation makes it possible to find not only the balance equations of the system but also the belonging of ΔU_t to the under-differential of Ψ_c.

2.7   Stages of the resolution of the problem of contact-friction

the method of discrete resolution established in *Code_Aster* is founded on a writing of the relations of interpenetration on the nodes of the meshes in opposite, which implies:
1) A discrete description of contact surfaces (mesh);
2) The search of the minimal distance from projection and the position of this projection (operation known as of pairing);
3) The writing of the kinematic relations between the nodes;
4) Algorithms of resolution of the problem of contact/friction;
We will detail each one of these aspects in the continuation of the document.

For the formulations known as "discrete" (in opposition to the "continuous" formulation known as, to see [R5.03.52]), *Code_Aster* solves the problem of contact by methods which belong to the family of the methods called "method of the statutes" in the literature, with a decoupling of the total-room type.

The contact algorithms/friction act as two times:
1) Geometrical problem of pairing
•Location: definition of potential surfaces of contact (cf § 3.2) by the user [4] .

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4   L there not of mechanism automatic of location of surfaces in contact in *Code_Aster*

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- Pairing: determination of the potential couples of contact (cf § 3.4) by a research method of the minimum clearance enters a node and a facet (for pairing node/facet) or of the node nearest to another node (for nodal pairing).
- Kinematics: writing of the relation of nonpenetration by the determination of the direction of projection and the evaluating of the coefficients (cf § 19). The relation is written between the slave node and the main nodes.
2) Mechanical problem. Several types of algorithms are established in *Code_Aster* :
- An algorithm based on the method of the active stresses 59 usable in contact without friction only. It is that which is used by default and which corresponds to `ALGO_CONT = " FORCED "` .
- An alternative of the method of the active stresses using an iterative resolution by conjugate gradient project. `ALGO_CONT='GCP'`. Method reserved exclusively for the problems of contact without friction.
- An algorithm is available under `ALGO_CONT='LAGRANGIEN'`. It is similar to the method of the stresses more or less. It is usable also in the case of " `LAGRANGIAN " friction ALGO_FROT=`.
- An algorithm of resolution by regularization of the conditions of contact and/or friction, which one activates with `ALGO_CONT='PENALIZATION'` and `ALGO_FROT='PENALIZATION'` by choosing a coefficient of judicious penalization (what implies *a fortiori*, a parametric study on the value of this coefficient).
- A mixed algorithm (exact in contact and regularized in friction): `ALGO_CONT='LAGRANGIEN'` and `ALGO_FROT='PENALIZATION'`.

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3 Geometrical problem of pairing

3.1 Introduction

the first problem to treating the contact in the case of is to define in a way reliable and robust which parts of two surfaces are likely to make contact. It is the operation known as of pairing, commune with the mixed hybrid formulation known as "continuous" method presented in the document [R5.03.52].

There are two types of pairing available in *Code_Aster*:

• Nodal pairing;
• Master-slave pairing (or node-facet).

Nodal pairing (`APPARIEMENT=' NODAL'`) imposes that relative displacement between a slave node and the master node which is paired to him, project on the direction of the norm to the slave node, is lower than initial clearance in this direction. The use of this formulation is disadvised because it requires to have compatible meshes (nodes "opposite") which remains it during the strain (assumption of small slidings), and for which the norms Master and slave are about colinéaires. Without these assumptions, the made approximation becomes hazardous and it is preferable to use the node-facet formulation.

Master-slave pairing, selected by the key word `APPARIEMENT=' MAIT ESCL'`, does not grant a role equivalent to two surfaces: the surface (S_1) described under `GROUP_MA_MAIT` or `MAILLE_MAIT` is called surface Master and surface (S_2) is surface slave. The conditions of noninterpenetration express that nodes of surface slave (of stars on the figure (3.1-a)) do not penetrate in meshes of surface Master. One can see, on the other hand, that it is possible that the main nodes (rounds) penetrate in surface slave.
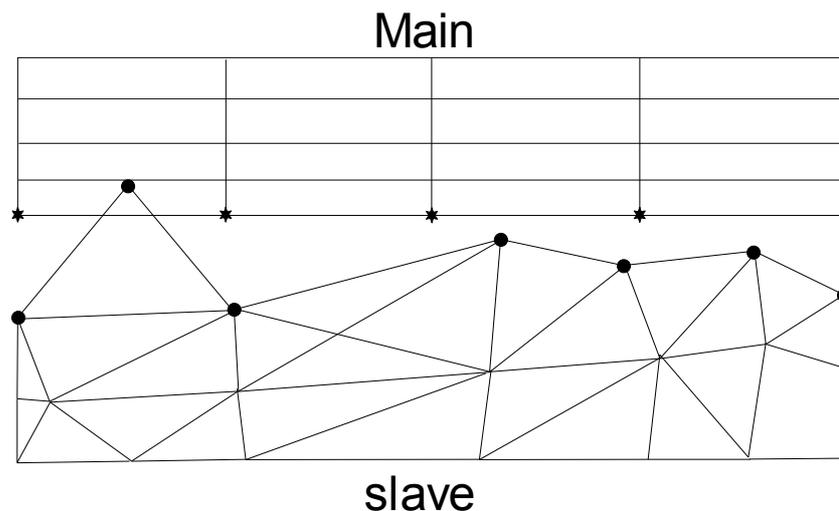
Appears 3.1-a : surface main and surfaces slave.

Note:

• The nodes slaves are by default all the nodes belonging to meshes of contact defining surface slave. Key words `SANS_NO` and `SANS_GROUP_NO` make it possible to give, zone by zone, a list of the nodes which must be removed list of the nodes slaves. That makes it possible to remove the nodes subjected to boundary conditions of Dirichlet incompatible with the contact (see § 56).

3.2 Definition of the potential zones of contact

Code_Aster not having mechanism of automatic control of the potential zones of contact, it is thus to the user to define *a priori* the zones of which he predicts that they will make contact. These zones must be sufficiently broad not to observe interpenetration. It should be noted that the phase of pairing is an inexpensive operation in general (much less expensive than the resolution of the mechanical problem of contact) and than one can thus define sufficiently wide zones without being likely to penalize the performances, subject respecting some care to ensure, in particular, the unicity of projections.

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One considers three solids of the figure (3.2-a), represented in 2D. One defined three possible zones of interpenetration between solids: a zone enters the solid A and the solid B, and two zones between solid B and solid C. The user, who defines these zones in the command file, supposes here that apart from these zones, there is no risk of interpenetration, taking into account the loading.
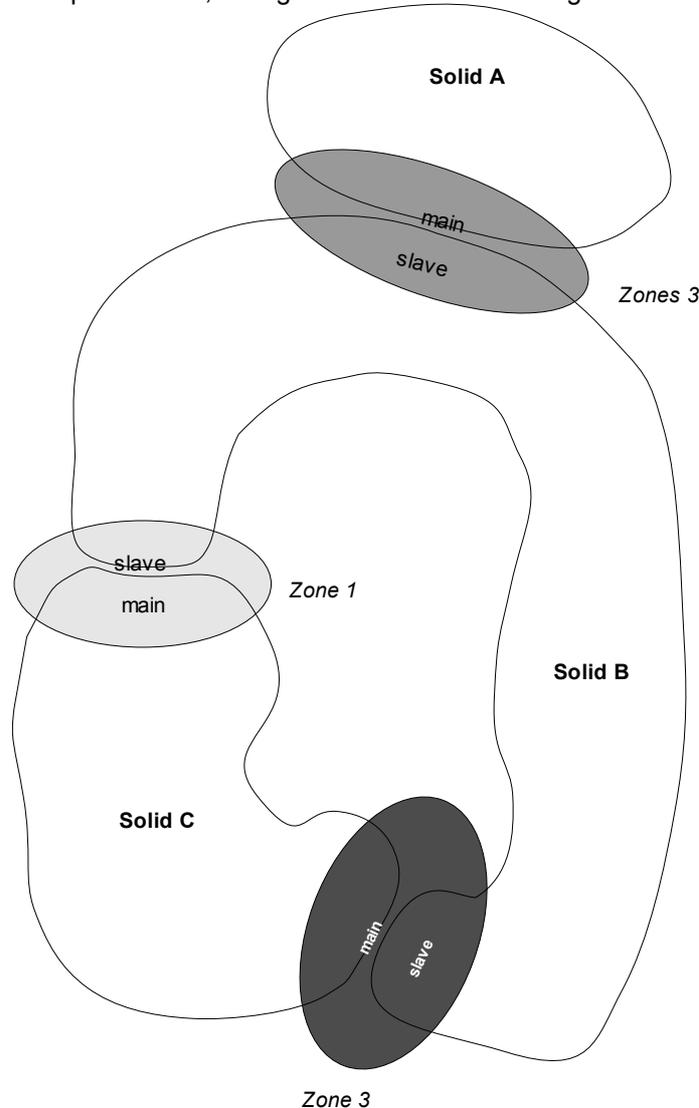
Appears 3.2-a : definition of three contact zones.

Each contact zone is defined in the operator `DEFI_CONTACT`, keyword factor `ZONE`. A zone is composed by definition of two surfaces which one seeks to prevent the interpenetration: first is defined under key word `GROUP_MA_MAIT` (or `MAILLE_MAIT`), the second under key word `GROUP_MA_ESCL` (or `MAILLE_ESCL`), i.e. by the data of meshes **of edge** which constitute them. These meshes are `SEG2` or `SEG3` for a 2D mesh, of the `TRIA3`, `TRIA6`, `QUAD4`, `QUAD8` or `QUAD9` for a mesh 3D.

Note:
•Meshes of the edge necessary ones to the contact will not be created by *Code_Aster* starting from the voluminal elements and must thus already exist in mesh file.
•The choice of surfaces which will be Masters or slaves is important. Information on this subject is available in documentation [U2.04.04].

3.2.1   Cas particulier of the contact for a cable or a beam in 3D

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It is possible in 3D to treat the contact between a mesh `SEG2` or `SEG3` (modelling a cable or a beam) and a surface. In this case, it is imperatively necessary to use the method of pairing "`MAIT_ESCL`" and to give the segments under key word `GROUP_MA_ESCL` (meshes slaves). The cross-section of the beam can be then taken into account by the use of key word `DIST_ESCL` (cf §3.63.6).

3.2.2 Case of pairing nodal

One must to choose to take as surface slave that which comprises less nodes (an error message will stop you if your surface Master contains less nodes than your surface slave), in order to maximize the chances to have an injective pairing (a master node is paired only with one slave node). The master node paired with each slave node is determined by a computation moreover nearer close explained in the §3.4.23.4.2. One uses the norm with the master node to write the relation of noninterpenetration.

Even in the case of nodal pairing, contact surfaces are defined in terms of meshes. The nodes slaves and Masters are then the nodes of meshes thus defined .

3.3 Directional sense of the norms

It is imperative that meshes of contact are defined so that the norm is outgoing: the connectivity of the segments must be defined in the order AB , that of the triangles in the order ABC , and that of the quadrangles in the order $ABCD$, as indicated on the figure 3.3-a. For a better reading of the drawing, one a little drew aside the edge mesh here being used in contact with the "face" of the voluminal element 2D or 3D on which it is pressed. One will be able to use for that the operator `MODI_MAILLAGE`, options `ORIE_PEAU_2D, ORIE_PEAU_3D` or `ORIE_COQUE`.

2D   3D
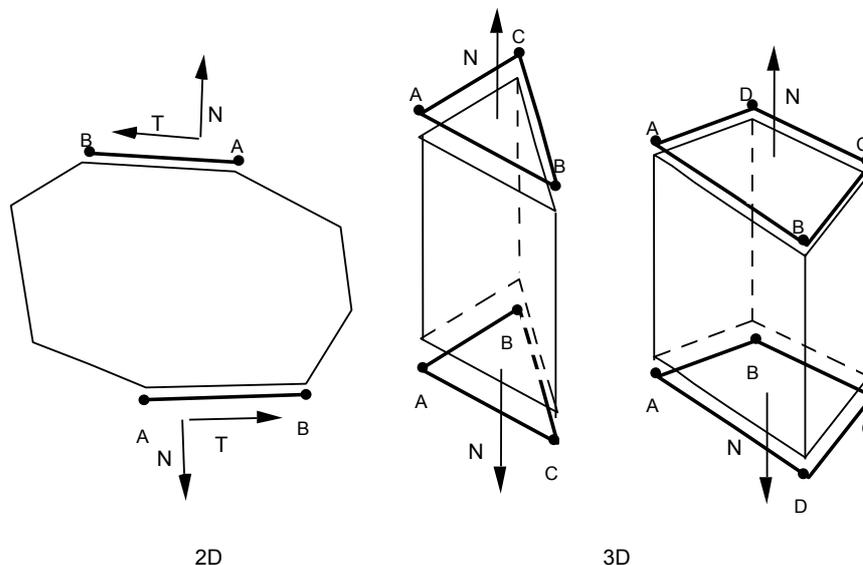
Appears 3.3-a : classification of meshes of contact to have an outgoing norm.

3.4 Algorithm of pairing

3.4.1 Principle

the algorithm of pairing proceeds in two times:
1. For each slave node one searches the master node nearest;
2. One seeks then the master mesh attached to the master node previously given which is closest to the slave node;

In the case of nodal pairing, one makes obviously only the first stage.

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3.4.2   Searching for nearer close to a node

This phase is common to both formulations: node/facet and node/node. The method used to search the master node nearest to a slave node is very simple: it is enough and to calculate the distance (in brought up to date geometry, cf3.7 §3.7) between the slave node main nodes candidates.

3.4.3   Of the mesh search main paired

the algorithm to search the master mesh which is paired with the slave node is the following:
• Knowing the master node nearest to the slave node P (see §3.4.23.4.2), one examines successively the meshes main ones containing this node.
• For each mesh thus located, one seeks the orthogonal project M of the slave node P on the master mesh.
• The mesh minimizing the distance $\overline{PM}$ is selected to be paired with the slave node.

Note:

- If the slave node is projected apart from a master mesh, one folds back his projection on this mesh under certain conditions (see § 3.4.5).
- So at least a projection takes place inside a master mesh, then this one will be preferred with a projection leading to a folding back (whatever the measured distance)
- If all projections of the slave node take place apart from their respective master mesh, the slave node is regarded as not paired and will thus be excluded from the phase of Computation

3.4.4   resolution of projection of the slave node on the master mesh

3.4.4.1   Notations

parts of surfaces $\partial \Omega^i$ likely to make contact at the time of the strain of two solids are noted γ_c^i. One supposes the existence of noted regular cards Θ^i describing surfaces γ_c^i. These cards are defined as follows:

$$\Theta^i: \quad \omega^i \rightarrow \overline{\Omega^i}$$
$$\left(\xi_1^i, \xi_2^i\right) \rightarrow p^i = \Theta^i\left(\xi_1^i, \xi_2^i\right) \tag{25}$$

where ω^i is a field limited (of reference) contents in $\mathbb{R}^2$ (it is the parametric space of reference of the finite element). In addition, one indicates by φ^i the transformation of solid B^i, definite by:

$$\varphi^i: \quad \overline{\Omega^i} \rightarrow \overline{\Omega_t^i}$$
$$\left[p^i\right] \rightarrow \left[x^i\right] \tag{26}$$

One always places itself at a time t built-in. Surface Master is noted γ_c^1 and surface slave is noted γ_c^2. One carries out pairing while searching, for any point $\left[x^i\right] = \varphi^i\left(p^i, t\right)$ border γ_c^1 the point $\left[\tilde{x}\right]$ of γ_c^2 nearest. That amounts solving the problem of optimization under stresses according to. For any point $\left[x^1\right] = \varphi^1\left(\Theta^1\left(\xi_1^1, \xi_2^1\right), t\right)$ with $\left(\xi_1^1, \xi_2^1\right) \in \omega^1$, and for any $t \geq 0$, to find $\tilde{\zeta}_t = \left(\xi_1^2, \xi_2^2\right) \in \omega^2$ such as:

$$\tilde{\zeta}_t = \underset{\left(\xi_1^2, \xi_2^2\right) \in \omega^2}{\operatorname{argmin}} \left\{ \frac{1}{2}. \left\| \varphi^1\left(\Theta^1\left(\xi_1^1, \xi_2^1\right), t\right) - \varphi^2\left(\Theta^2\left(\xi_1^2, \xi_2^2\right), t\right) \right\|^2 \right\} \tag{27}$$

the solution $\tilde{\zeta}_t$ is the position within the space of parametric reference of projection M slave node P on the master mesh.

3.4.4.2   Formulation of the problem of minimization

to solve the nonlinear problem (27), one proposes to use an algorithm of minimization of Newton. The problem is reformulated in the following way: for a given slave node one seeks the point (in parametric coordinates) which minimizes the distance to a master mesh given (that defined in the paragraph §3.4.316).
One notes f the norm L_2 distance between a slave node located in $\left[x_e\right]$ and a point $\left[x_m\right]$ pertaining to the master mesh in opposite:

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$$f\left(\lfloor x_m\rfloor\right)=\left\|\lfloor x_e\rfloor-\lfloor x_m\rfloor\right\|^2 \qquad (28)$$

One writes the functional calculus in parametric space, $\lfloor x_m\rfloor$ expressing itself starting from nb_{no} nodes $\lfloor x_{i,m}\rfloor$ of the mesh main paired:

$$f\left(\xi_1,\xi_2\right)=f\left(\lfloor\zeta\rfloor\right)=\left\|\lfloor x_e\rfloor-\sum_{i=1}^{nb_{no}}\phi_i\left(\lfloor\zeta\rfloor\right).\lfloor x_{i,m}\rfloor\right\|^2 \qquad (29)$$

It is this quantity which it is necessary to minimize. One applies the algorithm of Newton to the conditions of stationarity (Eulerian-Lagrange) of the problem of minimization thus stated, i.e. with the vectorial functional calculus $\lfloor\nabla f\left(\lfloor\zeta\rfloor\right)\rfloor$, statement of the development of Taylor to the first order, around the point $\lfloor\zeta_0\rfloor$:

$$\lfloor\nabla f\left(\lfloor\zeta\rfloor\right)\rfloor\underset{linéarisation}{=}\lfloor\nabla f\left(\lfloor\zeta_0\rfloor\right)\rfloor+\lfloor H\left(\zeta_0\right)\rfloor.\left(\lfloor\zeta\rfloor-\lfloor\zeta_0\rfloor\right) \qquad (30)$$

Where $\lfloor H\rfloor$ is the Hessienne matrix of second derivative of f and $\lfloor\nabla f\left(\lfloor\zeta\rfloor\right)\rfloor$ is its gradient. A minimum of f occurs when its gradient is null.

The iterative algorithm is written then:
1. To leave the initial point $\lfloor\zeta_{i=0}\rfloor$ on the master mesh. This starting point is simply selected in $\lfloor\zeta_{i=0}\rfloor=\lfloor0\rfloor$;
2. To evaluate the gradient $\lfloor\nabla f\left(\lfloor\zeta_i\rfloor\right)\rfloor$ and the Hessienne matrix $\lfloor H\left(\zeta_i\right)\rfloor$ in this point;
3. To calculate the direction of descent $\lfloor d\ \zeta\rfloor=-\lfloor H\left(\zeta\right)\rfloor^{-1}.\lfloor\nabla f\left(\lfloor\zeta\rfloor\right)\rfloor$
4. Compute the linear parameter of search α (see §3.4.4.4)
5. To calculate the following point such as $\lfloor\zeta_{i+1}\rfloor=\lfloor\zeta_i\rfloor+\alpha\lfloor d\ \zeta\rfloor$;
6. If the process converged, one stops, if not one buckles into 2.

Note:

- This problem is written without stresses;
- One cannot guarantee the unicity of the solution, the process of Newton will find the first point carrying out the conditions of stationarity.
- The existence of the gradient and the Hessienne matrix is assured if the mesh is sufficiently regular, which is always the case on a mesh too not distorted finite element.
- The iterative process stops by a criterion on the displacement increment in parametric space $\varepsilon=\dfrac{\sqrt{\langle\zeta_{i+1}\rangle\lfloor\zeta_{i+1}\rfloor-\langle\zeta_i\rangle\lfloor\zeta_i\rfloor}}{\sqrt{\langle\zeta_{i+1}\rangle\lfloor\zeta_{i+1}\rfloor}}$ with $\varepsilon\leqslant10^{-4}$ (nonmodifiable value by the user). This value being estimated compared to the parametric coordinates, it does not have there problems with the size of meshes and the units used.
- The maximum number of iterations of Newton is fixed at 200 (nonmodifiable value by the user).
- If the maximum number of iterations is reached, one selects as reiterated project which minimized the distance between the slave node and the master mesh (it is thus not perfectly the orthogonal project).
- There exists an alternative in which the direction of search is not estimated by the algorithm of Newton but a direction fixes given by the user, which can be useful in certain difficult cases (perfectly convex mesh which does not give a single projection for example). One activates it by keyword `TYPE_APPA='FIXE'` and the vector is given by `DIRE_APPA`.
- Projection on the elements segments in 3D (case of the beams or the cables) requires a definition of the norm by the user via keywords `VECT_MAIT/VECT_ESCL`.

3.4.4.3 Form of the tangent matrix and the residue

One gives below the statements of the intervening terms in the writing of the algorithm of Newton describes above. The functional calculus is pointed out:

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$$f\left(\xi_1, \xi_2\right)= f\left(\zeta\right)=\left\|\left(x_e\right)-\sum_{i=1}^{nb_{no}} \varphi_i\left(\zeta\right).\left(x_{i,m}\right)\right\|^2 \tag{31}$$

the gradient of this functional calculus:

$$\left(\nabla f\left(\zeta\right)\right)=\begin{Bmatrix}\dfrac{\partial f}{\partial \xi_1}\\[2mm]\dfrac{\partial f}{\partial \xi_2}\end{Bmatrix} \tag{32}$$

the terms are written explicitly:

$$\frac{\partial f}{\partial \xi_j}\left(\xi_1,\xi_2\right)=2\left(\left(x_e\right)-\sum_{i=1}^{nb_{no}}\phi_i\left(\left(\zeta\right)\right).\left(x_{i,m}\right)\right).\left(-\sum_{i=1}^{nb_{no}}\frac{\partial \phi_i}{\partial \xi_j}\left(\left(\zeta\right)\right).\left(x_{i,m}\right)\right) \tag{33}$$

the hessien of this functional calculus:

$$\left(H\left(\zeta\right)\right)=\begin{bmatrix}\dfrac{\partial^2 f}{\partial \xi_1^2} & \dfrac{\partial^2 f}{\partial \xi_1.\partial \xi_2}\\[3mm]\dfrac{\partial^2 f}{\partial \xi_1.\partial \xi_2} & \dfrac{\partial^2 f}{\partial \xi_2^2}\end{bmatrix} \tag{34}$$

the terms are written explicitly:

$$\frac{\partial^2 f}{\partial \xi_1^2}\left(\xi_1,\xi_2\right)=\left\|\sum_{i=1}^{nb_{no}}\frac{\partial \phi_i}{\partial \xi_j}\left(\left(\zeta\right)\right).\left(x_{i,m}\right)\right\|^2+2\left(\left(x_e\right)-\sum_{i=1}^{nb_{no}}\phi_i\left(\left(\zeta\right)\right).\left(x_{i,m}\right)\right).\left(-\sum_{i=1}^{nb_{no}}\frac{\partial^2 \phi_i}{\partial \xi_j^2}\left(\left(\zeta\right)\right).\left(x_{i,m}\right)\right) \tag{35}$$

And for the cross terms:

$$\begin{aligned}\frac{\partial^2 f}{\partial \xi_j \partial \xi_k}\left(\xi_1,\xi_2\right)\quad &=\left(\sum_{i=1}^{nb_{no}}\frac{\partial \phi_i}{\partial \xi_j}\left(\left(\zeta\right)\right).\left(x_{i,m}\right)\right).\left(\sum_{i=1}^{nb_{no}}\frac{\partial \phi_i}{\partial \xi_k}\left(\left(\zeta\right)\right).\left(x_{i,m}\right)\right)\\[2mm]&+2\left(\left(x_e\right)-\sum_{i=1}^{nb_{no}}\phi_i\left(\left(\zeta\right)\right).\left(x_{i,m}\right)\right).\left(-\sum_{i=1}^{nb_{no}}\frac{\partial^2 \phi_i}{\partial \xi_j.\partial \xi_k}\left(\left(\zeta\right)\right).\left(x_{i,m}\right)\right)\end{aligned} \tag{36}$$

3.4.4.4   linear Search in the algorithm of projection

to improve the robustness of the algorithm of projection, a linear phase of search is added in the algorithm of Newton (cf §3.4.4.2). It acts, being given a direction of descent $\left(d\,\zeta\right)$, to determine a parameter of advanced α which minimizes a functional calculus F associated with f.
The linear algorithm of search used (linear search with graining and rule of Armijo) is the same one as for the implicit integration of the constitutive laws [R5.03.14].

The functional calculus F, associated with the functional calculus outdistances f defined in (31), is the following one:

$$F\left(\alpha\right)=\frac{1}{2}\left\|\nabla f\left(\left(\zeta\right)+\alpha\left(d\,\zeta\right)\right)\right\|^2 \tag{37}$$

to choose α, one does not go in fact not to seek to minimize the functional calculus exactly F but an approximation of this one (quadratic, cubic). To know which approximation to use, one leans on the rule of Armijo. For more details, one returns to [R5.03.14] and 59 which detail the implementation of such an algorithm.

For the linear search implemented in the algorithm of projection, the selected parameters (into tough) are the following:

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Parameter of the rule of Armijo	$\omega = 0,0001$
Limit min for the folding back	$\alpha_{min} = 0,1$
Limit max for the folding back	$\alpha_{max} = 0,5$
maximum Number of cubic interpolations	$it_{max} = 2$

These parameters imply in particular that if the step by default ($\alpha = 1$) does not satisfy the rule with Armijo, i.e. the direction of selected descent does not bring closer to the orthogonal project, then α will be with most equal to $0,5$. This choice makes it possible to support the robustness rather (to avoid following a direction of erroneous search) than the performance (the values α higher than 1).

For meshes of poor quality, with meshes distorted, the linear search showed its interest, finding the good project orthogonal on difficult cases where an algorithm without linear search failed.

3.4.5   Processing of projections except mesh

There exist projections of which result is sensitive to purely numerical parameters or whose existence and unicity mathematics are not guaranteed. Under certain conditions, one can detect of the contact between two surfaces whereas there is not. The problem comes initially from an incorrect and imperfect definition of surfaces likely to make contact. Of the contact let us take the case in 2D (contact surfaces are thus segments) where a slave node must be projected apart from surface Master (see figure 3.4.5-a):
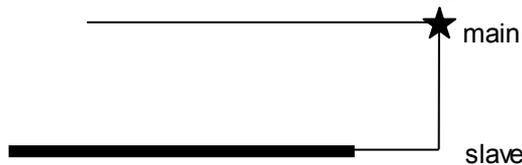
Appears 3.4.5-a : projection out of a master mesh.

The user can choose "Re-to project" this slave node on the prolonged master mesh or not to do it (see figure 3.4.5-b).

Re-projection
(one is in the tolerance)
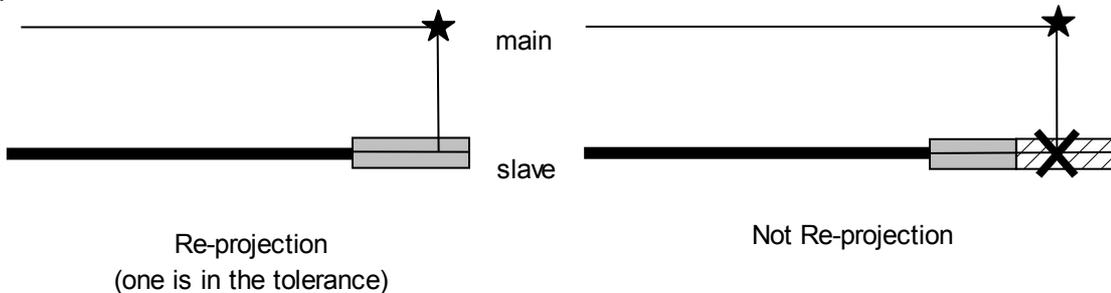
Not Re-projection

Appears 3.4.5-b : principle of the zone of tolerance for projection on a master mesh.

The limiting value of this Re-projection is fixed by the keyword `TOLE_PROJ_EXT` which of the mesh takes for argument the value (paid to the element of reference) of the extension main in which one authorizes Re-projection. By default, this value is fixed at 0.50 . What means that any slave node being projected at a distance higher than half the length of the mesh main Re-will not be projected. To prohibit Re-projection completely, it is enough to fix `TOLE_PROJ_EXT` at zero. This operator is valid in 2D and 3D. In 3D case, it is the extension of a surface mesh of contact, and, as one reasons in parametric space, the curvature of edges of the elements is well taken into account.

In the same way if contact surfaces are extended too much, any slave node located behind the master mesh is paired. The notion of "in front of-behind" is given by the directional sense of the norms given by the user (see §3.33.3). One can restrict this search for mesh paired with the parameter `TOLE_APPA` which meshes specifies the maximum distance from search of "the apparialbles" with the slave node.

3.5   Kinematic relations

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3.5.1 Definition of the contact matrix

One carries out a idealized modelization of the phenomenon of contact, in the sense that it line supposes the borders of the bodies perfectly defined by one or a surface: one then writes a condition of nondiscrete and linearized interpenetration 59. That is to say P a slave node, M its projection on the master mesh which was given during pairing. In 2D, this master mesh has two nodes (SEG2) or three nodes (SEG3). In 3D, it can have of them three, four, six, eight or nine (TRIA3, QUAD4, TRIA6, QUAD8, QUAD9). The displacement of the point M is a linear combination of displacements of the nodes of the finite element, with for coefficients the values of the shape functions Φ in M. We place if the master mesh is a SEG2 to simplify the talk. One has then :

$$\{u_M\} = [\Phi_A(M)].\{u_A\} + [\Phi_B(M)].\{u_B\} \tag{38}$$

Initially, one chooses to take as direction $[N]$ the outgoing norm of the mesh main (cf Appears 3.5.1-a).
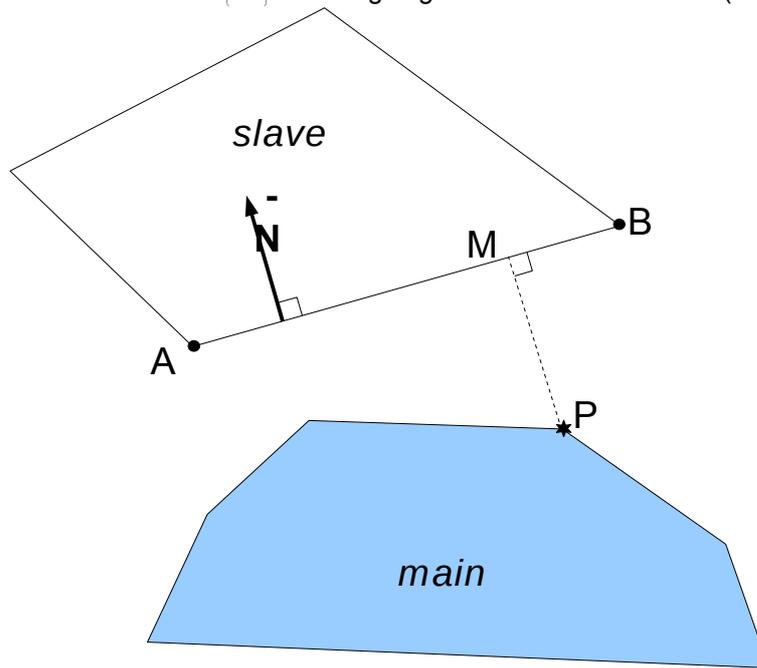
Appears 3.5.1-a : projection of a slave node on a mesh SEG2.

Normal clearance is written like the difference between the slave node P and its projection M on the master mesh:

$$\langle N \rangle . \left[\{u_P\} - \sum_{j=1}^{n_m} [\Phi_{B_j}(M)].\{u_{B_j}\} \right] = d \tag{39}$$

If one writes such a relation for all the couples of contact, one obtains the geometrical conditions of nonpenetration in matric form:

$$[A^c].\{u\} = d \tag{40}$$

the matrix $[A^c]$, called contact matrix, line contains one by couple of contact, and as many columns as of degrees of freedom physical of the problem. Let us suppose that one has two meshes contact of the type SEG2 , according to the diagram of Appear 3.5.1-b.

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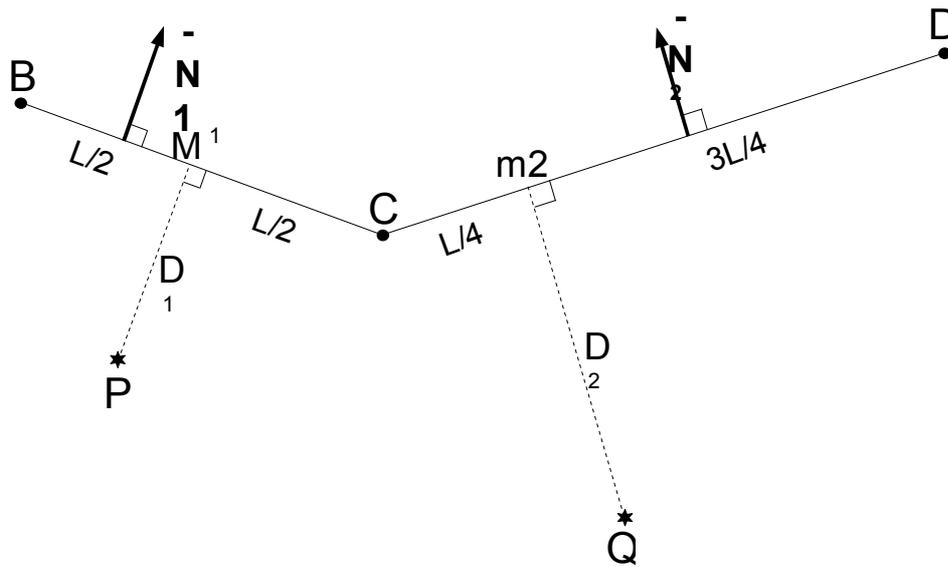
Appear 3.5.1-b : writing of the contact matrix A on an example.

If one notes for example u_B the displacement of the node B according to the direction x, v_B the displacement of the node formulates B according to the direction y, and d^1 the d^2 current clearances for the two couples:

$$[A^c].\begin{Bmatrix} u_P \\ v_P \\ u_Q \\ v_Q \\ u_B \\ v_B \\ u_C \\ v_C \\ u_D \\ v_D \end{Bmatrix} = \begin{Bmatrix} d^1 \\ d^2 \end{Bmatrix} \tag{41}$$

With the contact matrix $[A^c]$:

$$[A^c] = \begin{bmatrix} N_x^1 & N_y^1 & 0 & 0 & -\dfrac{1}{2}.N_x^1 & -\dfrac{1}{2}.N_y^1 & -\dfrac{1}{2}.N_x^1 & -\dfrac{1}{2}.N_y^1 & 0 & 0 \\ 0 & 0 & N_x^2 & N_y^2 & 0 & 0 & -\dfrac{3}{4}.N_x^2 & -\dfrac{3}{4}.N_y^2 & -\dfrac{1}{4}.N_x^2 & -\dfrac{3}{4}.N_y^2 \end{bmatrix} \tag{42}$$

One considered here only the degrees of freedom of the nodes implied in the contact; the matrix $[A^c]$ should be hollower. But in practice, one always reduces the contact matrix on the active degrees of freedom. Only the non-zero coefficients are thus stored.

3.5.2   Definition of the matrix of friction

the notion of contact matrix extends to the case from the tangential slidings, on the tangent level. It is the matrix of the kinematic relations of friction $[A^f]$.

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3.5.3 Choice of the norm

In the preceding paragraph, one chose to take as direction $\lfloor N \rfloor$ the outgoing norm of the mesh main. It is the behavior by default in *Code_Aster*. However, it is possible to choose other norms:
- Normal slave NORMALE='ESCL' ;
- An average enters the norm Master and normal slave NORMALE='MAIT_ESCL' ;

It is also possible to ask to use smoothed norms (LISSAGE='OUI'), i.e. that instead of using the norm Master at the point of projection, one can take a norm resulting from the linear interpolation between norms at nodes of the mesh main one.

In the same way, the computation of the norms is always done via the shape functions of the element, it is what one defines as "the true" norm or "automatic" norm. But it is possible to impose a norm on the master mesh, the mesh slave or on both in a different way:
- Directly (VECT_MAIT or VECT_ESCL = ' FIXE')
- Indirectly by use D `a trihedron (VECT_MAIT or VECT_ESCL='VECT_Y'). In this last case, the norm used will be the vector from the cross product between the tangent with the mesh and vector VECT_Y given.

Note:
• As regards a pairing of the node-facet type, the normal slave is calculated itself by lissage, the option of LISSAGE thus does not have an effect if NORMALE=' ESCL' is chosen ;
• The use of preset norms is compulsory in the case of beams;
• The choice of a norm other than the master mesh must be restricted with exceptional cases like when the mesh or the compatibility issues of the contact with the boundary conditions forced the user to coarsely take for Master a surface with a grid;
• The use of TYPE_APPA=' FIXE' for the search of the mesh main nearest (see § 16) does not prejudge a choice of the norm in the writing of the relation of nonpenetration, which remains with the choice of the user. But it is more coherent to choose a fixed norm (VECT_MAIT=' FIXE').

3.5.4 Coefficients of the contact matrix
3.5.4.1 standard Elements

the values of the shape functions $\Phi_{B_j}(M)$ of the main nodes at the point M for the different ones meshes from contact are standard (see [R1.01.01]).

3.5.4.2 Meshes QUAD8

meshes of quadrilateral type to eight nodes present a default. Indeed, the classical shape functions are not positive on all the field and lead to aberrant results when they are used in the contact. The principal symptom related to the use of the classical shape functions is the appearance of negative forces of contact NON-physics which causes oscillations (between the nodes tops and the nodes mediums).
To avoid this phenomenon, *Code_Aster* carries out the modification of element QUAD8 by imposing linear relations between nodes mediums and nodes tops and by means of finally the shape functions of the QUAD4.
In a general way, it is preferable to avoid using this kind of element and preferring to him complete quadratic elements like the QUAD9 , because one regards the QUAD8 as linear elements and one introduces besides the linear relations which can be awkward.

3.5.4.3 Elements of COQUE_3D

the elements of the type COQUE_3D are of the finite elements mixed nonisoparametric. They are based on the meshes quadratic ones of type QUAD9 (respectively TRIA7) but the medium node not carrying a degree of freedom of translation, one carries out a projection on a QUAD8 (respectively TRIA6) and one thus falls down on the defaults evoked in the preceding paragraph. These finite elements are thus disadvised in the problems of contact.

3.6 Introduction of a fictitious clearance

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One can want to model the contact between structures has certain characteristics ("hole" or "bump") which one does not wish to net (see figure 3.6-a).
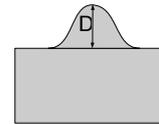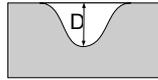
Appears 3.6-a : holes and bumps.

A solution consists in netting surface without these defaults and adding a distance to it given by the user (see figure 3.6-b).
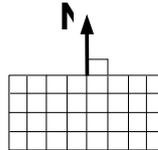
Appears 3.6-b : surface with a grid without default.

The value of clearance is corrected:

$$\langle N \rangle . \left(\left\lfloor u_P \right\rfloor - \left\lfloor u_M \right\rfloor \right) . = d - \left(d_e + d_m \right) \tag{43}$$

where d_e and d_m are given by the user respectively under key words DIST_MAIT and DIST_ESCL for each contact zone. These distances are signed : they represent the translation to be applied to the node of the mesh in the direction of the outgoing $\lfloor n \rfloor$ norm to obtain the point of real structure. These key words make it possible also of the contact to give an account between shells of which only mean surfaces are with a grid: d_e and d_m are worth then the half-thickness of the shells (positive values).

Note:
• If one uses DIST_MAIT and DIST_ESCL , it is necessary to take care of the visual interpretation of the results. If $\left(d_e + d_m \right) > 0$, the code will be able to announce of the contact whereas visualization shows a spacing of the two meshes. If $\left(d_e + d_m \right) < 0$, the code will be able to announce of the contact whereas visualization shows two interpenetrated meshes;
• To remember the signs, to think of:
   • $\left(d_e + d_m \right) > 0$: matter addition compared to the mesh
   • $\left(d_e + d_m \right) < 0$: ablation of matter compared to the mesh

options DIST_POUTRE and DIST_COQUE lean on the elementary characteristics defined in operator AFFE_CARA_ELEM to add fictitious clearance corresponding to the thickness (in the case of shells) or to the radius (in the case of beams with circular section).

3.7   Geometrical reactualization

In the frame of the modelization of contact in large displacements, the evolution of the geometry of surfaces plays a fundamental role. Indeed, it is it which potentially conditions the computation of the norms to the sides in contact and thus which conditions pairing carried out.

The geometrical reactualization is defined by the key word REAC_GEOM of the command DEFI_CONTACT. Its operation is the following:
• If REAC_GEOM= " SANS " : there is no geometrical reactualization. All computation is carried out on the initial configuration with initial pairing.
• If REAC_GEOM='CONTRÔLE' : NB_ITER_GEOM should be informed . Within **the same** step of load, one carries out NB_ITER_GEOM time the cycle *iteration until convergence, geometrical reactualization, pairing*.
• If REAC_GEOM='AUTOMATIQUE' : the decision to remake a geometrical pairing is made automatically by the software. The criterion is the following :

$$\frac{\| \Delta u^+ - \Delta u^- \|_\infty}{\| \Delta u^- \|_\infty} < \text{RESI\_GEOM} \tag{44}$$

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`RESI_GEOM` is worth 1% by default. If the infinite norm of displacement between two times, divided by the infinite norm of the displacement obtained since equilibrium is higher than `RESI_GEOM`, then one reactualizes. The infinite norm is defined by :

$$\|\Delta\boldsymbol{u}\|_{\infty} = \max\sqrt{u_x^2 + u_y^2 + u_z^2} \qquad (45)$$

Note:

• With the first iteration or in the event of rigid body motions (in dynamics), $\|\Delta\boldsymbol{u}^-\|_{\infty} = 0$. To avoid dividing by zero, the reactualization is forced ;
• The infinite norm is evaluated on all the nodes of the mesh (and not only on the nodes in contact) carrying degrees of freedom of displacement.

One can first of all notice that pairing is subjected to the geometrical phase of reactualization. Moreover, the fact of carrying out several times within the same step of load the cycle *iteration until convergence, geometrical reactualization, pairing* makes it possible to follow the evolution of the geometry of structure. It should indeed be stressed that this geometrical evolution is of the components nonlinear of a computation of contact in large displacements.

In the practice, one can advise the following choices for key word `REAC_GEOM` :
• For a computation in small displacements, `REAC_GEOM= " SANS "` . One works on the initial configuration.
• For computations in large displacements, either to use `REAC_GEOM='AUTOMATIQUE'` (default value), or to use `REAC_GEOM='CONTRÔLE'` and a value for `NB_ITER_GEOM` depending on the importance of the geometrical evolution of surfaces.
If the user does not leave with Code_Aster the possibility of managing the geometrical reactualization automatically, the code will inform it by an alarm if the automatic criterion (being worth 1%) is not ensured because of the choice of the user.

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4   Mechanical problem of contact/friction

the taking into account of contact-friction in a mechanical problem has two consequences:
–Modification of the balance equation to take into account the reactions of contact-friction to the interfaces;
–The application of additional models governing the contact and friction (see § 2 to compute: these reactions but also to impose conditions on the kinematics.
For the contact without friction, the conditions of Signorini are brought back to a problem of optimization under classical stress (Kuhn & Tucker). On the other hand, for the friction of Coulomb, one cannot write equivalent problem of optimization without making assumption (S) additional (S).

In *Code_Aster*, the discrete *methods* of resolution of the problem of contact/friction are founded on an approach *decoupled* between the equilibrium and the model of contact/friction. After the resolution of the problem of mechanics *without* contact-friction, one corrects the solution (kinematical and reactions) by applying the law of Signorini-Coulomb. This strategy makes it possible not to make any other assumption on the nature of the mechanical problem, whether it is the kinematics or the behavior models. However, it is essential that the stiffness matrix of the problem without contact is *symmetric*.

Note:
•The statement "to do a calculation with contact" wants to say that one writes the relations of nonpenetration, but does not imply that there is effective contact for the loading considered. However it is the resolution of the problem of contact-friction which is most expensive.
•The contact acting like correction on the results resulting from a classical mechanical computation, it is essential that the problem without contact is mechanically well posed and numerically soluble. In particular, possible motions of rigid bodies must be presumedly eliminated except resolution of the problem of contact.

4.1   Mechanical problem without contact/friction

the resolution of a nonlinear problem in operator `STAT_NON_LINE` (or `DYNA_NON_LINE`) is described in detail in the document [R5.03.01]. With each time step i, one seeks to check the total equilibrium of structure:

$$\begin{cases} \left[L_i^{\text{int}}(\boldsymbol{u}_i)\right] + \left[\boldsymbol{B}\right]^T . \left[\boldsymbol{\lambda}_i\right] = \left[L_i^{\text{méca}}(\boldsymbol{u}_i)\right] \\ \left[\boldsymbol{B}\right].\left[\boldsymbol{u}_i\right] = \left[\boldsymbol{u}_i^d\right] \end{cases} \tag{46}$$

In order to avoid overloading the equations, the assumption is made that the limiting conditions are eliminated and where one thus does not have a matrix $\left[\boldsymbol{B}\right]$, one thus seeks to solve:

$$\left[L_i^{\text{int}}(\boldsymbol{u}_i)\right] = \left[L_i^{\text{méca}}(\boldsymbol{u}_i)\right] \tag{47}$$

This nonlinear problem is solved by an iterative method of Newton-Raphson type which has as characteristics:
•A loading *division* a priori in "time step" (noted by the index i);
•A linearization of the problem of equilibrium by the method of Newton (the iterations being noted by the index n).

The unknowns are calculated in an incremental way. From $\left[\boldsymbol{u}_{i-1}\right]$, solution satisfying the equilibrium in t_{i-1}, one determines $\left[\Delta\boldsymbol{u}_i\right]$ which makes it possible to obtain the solution in t_i:

$$\left[\boldsymbol{u}_i\right] = \left[\boldsymbol{u}_{i-1}\right] + \left[\Delta\boldsymbol{u}_i\right] \tag{48}$$

the increment $\left[\Delta\boldsymbol{u}_i\right]$ is initially estimated by linearizing the problem compared to time around $\left(\left[\boldsymbol{u}_{i-1}\right], t_{i-1}\right)$ (phase of prediction or Eulerian). Then one uses a method of Newton or one of his alternatives to solve the equation (47) in an iterative way: one calculates a continuation $\left[\delta\boldsymbol{u}_i^n\right]$ where the exhibitor n is the number of the iteration. To simplify, one will not make distinction between the phase of prediction and the phase of correction of Newton . One writes finally:

$$\left[\boldsymbol{u}_i^n\right] = \left[\boldsymbol{u}_{i-1}\right] + \left[\Delta\boldsymbol{u}_i^{n-1}\right] + \left[\delta\boldsymbol{u}^n\right] \tag{49}$$

One places itself at the time t_i and the iteration of Newton n. The following notations were used:

• $\left[\boldsymbol{u}_i^n\right]$: displacements at time t_i and with the iteration of Newton n ;

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- $\lfloor u_{i-1} \rfloor$: displacement at time t_{i-1}. This solution observes the equilibrium condition of structure;

- $\lfloor \delta u^n \rfloor$: increment of displacements for the iteration of Newton n ;

- $\lfloor \Delta u_i^{n-1} \rfloor$: increment of the displacements cumulated since the beginning of time step, before the iteration of Newton n ;

After linearization of (47) , for the iteration of Newton n , one introduces [5] the mechanical tangent matrix $\lfloor K^{m,n-1} \rfloor$ and one must then find $\lfloor \delta u^n \rfloor$ such as:

$$\lfloor K^{m,n-1} \rfloor . \lfloor \delta u^n \rfloor = \lfloor L_i^{\text{méca},n-1} \rfloor - \lfloor L_i^{\text{int},n-1} \rfloor \tag{50}$$

4.2   Modification of the balance equation with contact/friction

to ensure the equilibrium of structure, it is necessary to introduce forces of contact/friction. In the most general possible way, one writes that the mechanical external loading is modified by the forces of contact/friction :

$$\lfloor L_i^{\text{méca}}(u_i) \rfloor = \lfloor L_i^{\text{ext}}(u_i) \rfloor - \lfloor L_i^{\text{c}}(u_i) \rfloor - \lfloor L_i^{\text{f}}(u_i) \rfloor \tag{51}$$

Where:

- $\lfloor L_i^{\text{ext}}(u_i) \rfloor$ is the vector of the external forces (conditions of Neumann);

- $\lfloor L_i^{\text{c}}(u_i) \rfloor$ is the vector of the forces of contact;

- $\lfloor L_i^{\text{f}}(u_i) \rfloor$ is the vector of the frictional forces ;

A priori, all these quantities are nonlinear because they depend on the vector displacement $\lfloor u_i \rfloor$ (one speaks about "following" loadings). While injecting (51) in (47), one thus seeks to solve:

$$\lfloor L_i^{\text{int}}(u_i) \rfloor = \lfloor L_i^{\text{ext}}(u_i) \rfloor - \lfloor L_i^{\text{c}}(u_i) \rfloor - \lfloor L_i^{\text{f}}(u_i) \rfloor \tag{formulate5 \\ 252}$$

) each iteration of Newton n , one linearizes the equation (52) compared to $\lfloor u_i^n \rfloor$. This process introduces the tangent matrix $\lfloor K^{m,n-1} \rfloor$ which will contain the contributions resulting from the linearization from the internal forces and outsides and the matrix $\lfloor K^{\text{cf},n-1} \rfloor$ which will contain the contributions resulting from the linearization from the forces from contact/friction , one must then find $\lfloor \delta u^n \rfloor$ such as:

$$(\lfloor K^{m,n-1} \rfloor + \lfloor K^{\text{cf},n-1} \rfloor) . \lfloor \delta u^n \rfloor = \lfloor L_i^{\text{ext},n-1} \rfloor - \lfloor L_i^{\text{int},n-1} \rfloor - \lfloor L_i^{\text{c},n-1} \rfloor - \lfloor L_i^{\text{f},n-1} \rfloor \tag{53}$$

4.3   Models of contact/friction

the models of Signorini and of Coulomb inequalities and equalities imply. The discrete formulation available in *Code_Aster* consists in modifying the relations of inequality in relations of equality. With this intention, there are two possible methods:
1.Dualiser models of contact friction;
2.To regularize the models of contact friction.

4.3.1   Kinematics
4.3.1.1   Contact matrixes/friction

In the paragraph §3.5.13.5.1, we saw how to write the kinematical conditions of contact by introducing the contact matrix $\lfloor A^c \rfloor$. Same way, but by considering the kinematical conditions of friction (on the tangent level), one will evaluate the matrixes of sliding $\lfloor A^g \rfloor$ and of dependancy $\lfloor A^a \rfloor$. These quasi-full and rectangular matrixes are calculated by the evaluating of the kinematic relations between the slave node and the paired master mesh. The construction of the kinematic relations is done at the time of the geometrical procedure of pairing which intervenes at the beginning of each time step and with each geometrical reactualization. For reasons of performances, one does not use and one builds only the under-part as of these matrixes

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5   exists several ways choose and calculate this matrix. For more details to see [R5.03.01]

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corresponding to the activation of the various thresholds (contact or sliding), within the algorithms of resolution of the contact/friction.

These matrixes depend on the geometrical reactualization **and** the under-iteration of contact (actualization of the statutes in the dualized methods).

At the total level, matrixes not depending on the solution without contact of the mechanical problem.

4.3.1.2 Unilateral conditions of contact

the relation of nonpenetration consists in saying that relative displacement according to a given direction cannot exceed initial clearance $\left[d_{\text{ini}}^{\text{c}}\right]$, measured on the mesh, in this direction. The unilateral condition of contact is written (see § 3.5.1) :

$$\left[A^{\text{c}}\right].\left[u_i\right]\leq\left[d_{\text{ini}}^{\text{c}}\right] \tag{54}$$

This equation translates the fact that any motion of structure must be done with the respect of the condition of nonpenetration, or that the displacement of the nodes of contact surface is lower than initial clearance $\left[d_{\text{ini}}^{\text{c}}\right]$.

After resolution of the mechanical problem by the method of Newton, the condition of nonpenetration becomes:

$$\left[A^{\text{c}}\right].\left(\left[u_{i-1}\right]+\left[\Delta u_i^{n-1}\right]+\left[\delta u^n\right]\right)\leq\left[d_{\text{ini}}^{\text{c}}\right] \tag{55}$$

This condition can be written in an iterative way:

$$\left[A^{\text{c}}\right].\left[\delta u^n\right]\leq\left[d^{\text{c},n-1}\right] \tag{56}$$

With $\left[d^{\text{c},n-1}\right]$ the clearance evaluated before the current iteration of Newton n :

$$\left[d^{\text{c},n-1}\right]=\left[d_{\text{ini}}^{\text{c}}\right]-\left[A^{\text{c}}\right].\left(\left[u_{i-1}\right]+\left[\Delta u_i^{n-1}\right]\right) \tag{57}$$

4.3.1.3 Conditions of dependancy

In dependancy, the nodes do not move on time step, i.e.:

$$\left[A^{\text{a}}\right].\left(\left[u_i\right]-\left[u_{i-1}\right]\right)=0 \tag{58}$$

$\left[A^{\text{a}}\right]$ is the matrix of the nodes in adherent contact, i.e. the under-part of the matrix of friction $\left[A^{\text{f}}\right]$ (see § 3.5.1) applied to the nodes in adherent contact.

4.3.1.4 Conditions of sliding

the shearing stress $\left[r_i\right]$ is colinéaire with the tangent direction of sliding, that is to say:

$$\left[A^{\text{g}}\right].\left(\left[u_i\right]-\left[u_{i-1}\right]\right)=\lambda.\left[r_i\right] \tag{59}$$

$\left[A^{\text{g}}\right]$ is the matrix of the nodes in slipping contact, i.e. the under-part of the matrix of friction $\left[A^{\text{f}}\right]$ (see § 3.5.1) applied to the nodes in slipping contact. One a:

$$\left[A^{\text{g}}\right].\left(\left[\Delta u_i^{n-1}\right]+\left[\delta u^n\right]\right)=\lambda.\left[r_i\left(\left[u_i\right]\right)\right] \tag{60}$$

the equation (60) depends on final displacement $\left[u_i\right]$. The conditions of sliding are introduced with a multiplier of Lagrange $\left[\mu^{\text{g}}\right]$ such as:

$$\left[\mu^{\text{g}}\left(u_i\right)\right]=\mu.\left[\left|\mu^{\text{c}}\left(u_i\right)\right|\right].\left[t^n\right] \tag{61}$$

With t^n the unit vector of the direction of sliding which is worth in 3D:

$$\left[t\right]=\frac{\left[A^{\text{g}}\right].\left(\left[\Delta u_i^{n-1}\right]+\left[\delta u^n\right]\right)}{\left\|\left[A^{\text{g}}\right].\left(\left[\Delta u_i^{n-1}\right]+\left[\delta u^n\right]\right)\right\|} \tag{62}$$

That is to say still:

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$$\left[\mu^{\mathrm{g}}(u_i)\right]=\left[k^{\mathrm{g}}(u_i)\right].\left[t^n\right] \text{ with } \left[k^{\mathrm{g}}(u_i)\right]=\mu.\left[\left\|\mu^{\mathrm{c}}(u_i)\right\|\right] \tag{63}$$

the multiplier of Lagrange is nonlinear and depends on the solution $\left[u_i\right]$. In 2D, it is also nonlinear but the multiplier does not depend on the direction of sliding, one thus has:

$$\left[\mu^{\mathrm{g}}(u_i)\right]_{2D}=\mu.\left[\left\|\mu^{\mathrm{c}}(u_i)\right\|\right]=\left[k^{\mathrm{g}}(u_i)\right] \tag{64}$$

4.3.2 Dualisation

In the case of the dualisation, one uses an test-ERREUR algorithm which applies a priori the state of a connection and which checks its state after application of the model of Signorini-Coulomb. To take into account the stresses (one speaks about the stresses in the meaning of conditions of "restriction" and not in the mechanical meaning of the term) bearing on the field of displacements or the forces, one utilizes them in the equations through Lagrange multipliers (as that can be made in *Code_Aster* for the kinematical boundary conditions). Three sets of Lagrange multipliers are introduced:

- $\left[\mu^{\mathrm{c}}\right]$ relating to the conditions of contact;
- $\left[\mu^{\mathrm{a}}\right]$ relating to the conditions of dependancy;
- $\left[\mu^{\mathrm{g}}\right]$ relating to the conditions of sliding.

By writing the equilibrium of structure, one gives the following interpretation of the Lagrange multipliers:

- $\left[L_i^{\mathrm{c}}(u_i)\right]=\left[A^{\mathrm{c}}\right]^T.\left[\mu_i^{\mathrm{c}}\right]$ represents the nodal forces of contact;
- $\left[L_i^{\mathrm{a}}(u_i)\right]=\left[A^{\mathrm{a}}\right]^T.\left[\mu_i^{\mathrm{a}}\right]$ represents the nodal forces of dependancy;
- $\left[L_i^{\mathrm{g}}(u_i)\right]=\left[A^{\mathrm{g}}\right]^T.\left[\mu_i^{\mathrm{g}}\right]$ represents the nodal forces of sliding.

Except the typical case of the sliding, we will see that the dualisation of the conditions of contact/friction does not require a process of linearization.

4.3.3 Regularization

the principle of the regularization is to amend the laws of contact-friction to obtain some easier to handle, for example, than the relations become univocal and differentiable (to be able to apply the method of Newton for example). One thus introduces assumptions which make NON-exact the model compared to the models of Signorini-Coulomb.

4.3.3.1 Regularization of the conditions of contact

the principle of the regularization is to modify the graph of the model of contact in order to remove the NON-univocal character of the relation of contact.
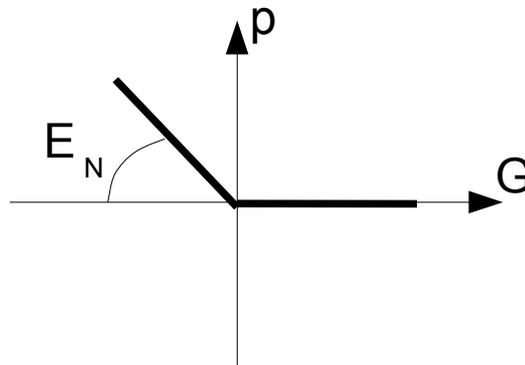
4.3.3.1 4.3.3.1-a : regularization of the condition of contact.

It is pointed out that the condition of NON-interpenetration is written in iterative form:

$$\left[A^{\mathrm{c}}\right].\left[\delta u^n\right]\leq\left[d^{\mathrm{c},n-1}\right] \tag{65}$$

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the idea of the regularization is to penalize the situation for which there is interpenetration. The interpenetration $\lfloor h^n \rfloor$ is worth:

$$\lfloor h^n \rfloor = \lfloor A^c \rfloor . \lfloor \delta u^n \rfloor - \lfloor d^{c,n-1} \rfloor > 0 \tag{66}$$

In other words, one writes that the force of contact is all the more large as the interpenetration $\lfloor h^n \rfloor$ is important, from where the regularized form of the force of contact:

$$\left[L_i^{c,n} \right]_{\text{regu}} = E_N . [A^c]^T . \lfloor h^n \rfloor^+ \text{ with } E_N > 0 \tag{67}$$

E_N is the coefficient of regularization (or penalization) of the contact and one notes $\square^+$ the positive part of a quantity. The coefficient of penalization of contact is interpreted as the come out from stiffness E_N which is opposed to the penetration of the slave node in surface Master. The larger it is, the less there is interpenetration and the higher the recoil force is.

4.3.3.2   Regularization of the condition of dependancy

the principle of the regularization of the dependancy is to modify the graph of the model of Coulomb . Initially, one removes the NON-univocal character of the model in the adherent part.
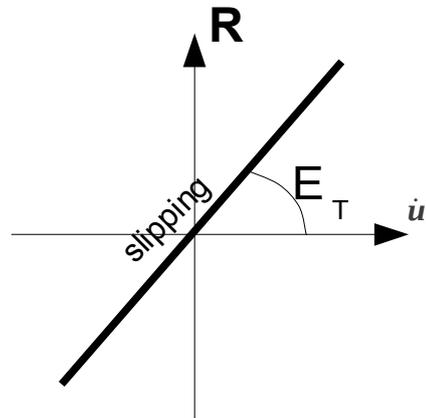
4.3.3.2 4.3.3.2-a : regularization of the condition of dependancy – Models of Tresca.

One establishes the regularized form of the force of dependancy:

$$\left[L_i^{a,n} \right]_{\text{regu}} = E_T . [A^a]^T . [A^a] . \left(\lfloor \Delta u_i^{n-1} \rfloor + \lfloor \delta u^n \rfloor \right) \text{ with } E_T > 0 \tag{68}$$

E_T is the coefficient of regularization (or penalization) of friction . The notion of dependancy strictly speaking will thus disappear, all the nodes slip . One sees on the figure (4.3.3.2-a) that this regularization is not satisfactory because one did nothing but transform the problem of Coulomb into problem of Tresca: the force is proportional to the relative sliding . To approach the model of Coulomb, should be added an additional inequation:

$$\left\| E_T . [A^a] . \left(\lfloor \Delta u_i^{n-1} \rfloor + \lfloor \delta u^n \rfloor \right) \right\| \leq \mu . |p(u)| \tag{69}$$

This additional inequation modifies the graph (see figure (4.3.3.2-b)). It results from it a form approximate from the model of Coulomb, in which one will define the adherent nodes compared to the threshold of contact, product of the coefficient of Coulomb μ and of the contact pressure $p(u)$. The threshold can be approximate (penalization of the contact) or exact (dualisation of the contact). The approximate character of the model of Coulomb results in the fact that the nodes known as "adherent" will slip more especially as the coefficient of regularization will be low. Physical interpretation is thus less direct than in the case of the contact, which mainly explains the difficulty of finding in practice a value satisfactory of this coefficient. However the made mistake ("distorts" detection of the threshold of sliding) is often not very important compared to the assumptions of modelization. As this method of regularization converges much better than the exact case [6], one advises with

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6   to model the friction of Coulomb exactly, it is rather recommended to use the continuous method, to see [R5.03.52]

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the users as often as possible to use it. There remains a non-linearity which relates to the "slipping" part of the model. The processing of this non-linearity is deferred to the § 4.3.4 .
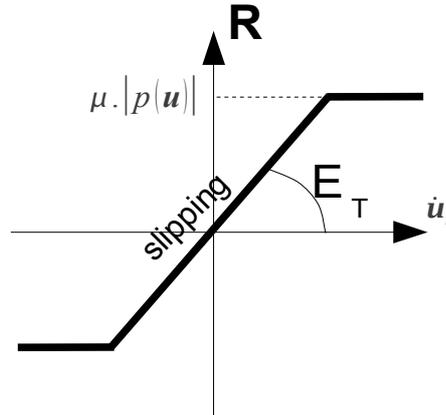
4.3.3.2 4.3.3.2-b : regularization of the condition of dependancy – Models of Coulomb.

4.3.4   Modelization of the sliding

the conditions of sliding are always introduced with a multiplier of Lagrange $\left[\boldsymbol{\mu}^{\mathrm{g}}\right]$ which is expressed (in 3D) from (63):

$$\left[\boldsymbol{\mu}^{\mathrm{g}}(\boldsymbol{u}_i)\right]=\left[\boldsymbol{k}^{\mathrm{g}}(\boldsymbol{u}_i)\right].\left[\boldsymbol{t}^n\right] \tag{70}$$

What reveals the two unknowns who depend both on the solution: the contact pressure in the matrix of the thresholds of sliding $\left[\boldsymbol{k}^{\mathrm{g}}\right]$ and the direction of sliding $\left[\boldsymbol{t}^n\right]$. In practice, however, it is considered that the knowledge of the threshold of sliding $\left[\boldsymbol{k}^{\mathrm{g},n-1}\right]$ is acquired with the preceding iteration $(n-1)$, which amounts being brought back to a criterion of Tresca for each iteration. A convergence the threshold is obviously fixed: there are not thus more differences between the thresholds during iterations. $\left[\boldsymbol{\mu}^{\mathrm{g},n}\right]$ is thus approximated compared to the state of the preceding iteration $(n-1)$:

$$\left[\boldsymbol{\mu}^{\mathrm{g},n}\right]\approx\left[\boldsymbol{k}^{\mathrm{g},n-1}\right].\frac{\left[A^{\mathrm{g}}\right].\left(\left[\Delta\boldsymbol{u}_i^{n-1}\right]+\left[\delta\boldsymbol{u}^n\right]\right)}{\left\|\left[A^{\mathrm{g}}\right].\left(\left[\Delta\boldsymbol{u}_i^{n-1}\right]+\left[\delta\boldsymbol{u}^n\right]\right)\right\|}=\left[\boldsymbol{k}^{\mathrm{g},n-1}\right].\left[\boldsymbol{t}^n\right] \tag{71}$$

One also has the threshold of sliding to the preceding iteration:

$$\left[\boldsymbol{k}^{\mathrm{g},n-1}\right]=\mu.\left[\left\|\boldsymbol{\mu}^{\mathrm{c},n-1}\right\|\right] \tag{72}$$

4.3.5   Linearizations

to solve the nonlinear problem, it is necessary to linearize the quantities which depend on $\left[\boldsymbol{u}_i^n\right]$ in the balance equation (52). The writing of the forces of contact/friction introduced into certain cases of the nonlinear quantities which it is thus advisable to linearize to be able to apply the algorithm of Newton. In the following table, one counts the cases where the linearization is necessary according to the type of algorithm.

	Dualisation	Regularization
$\left[\boldsymbol{L}_i^{\mathrm{c},n}\right]$	No linearization	Linearization
$\left[\boldsymbol{L}_i^{\mathrm{a},n}\right]$	No linearization	Linearization
$\left[\boldsymbol{L}_i^{\mathrm{g},n}\right]$	Linearization	Linearization

4.3.5.1   Linearization of the forces of contact

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One regularizes the condition of contact from the statement (67) of the force of contact, when one is in the phase of interpenetration, i.e. while supposing $\left| h^n \right| > 0$:

$$\left[L_i^{c,n} \right]_{\text{regu}} = E_N . \left[A^c \right]^T . \left(\left[A^c \right] . \left(\delta u^n \right) - \left(d^{c,n-1} \right) \right) \tag{73}$$

While linearizing, one obtains:

$$\left[L_i^{c,n} \right]_{\text{regu}} \underset{\text{linéarisation}}{\rightarrow} \left[\hat{L}_i^{c,n} \right]_{\text{regu}} = \left[K_i^{c,n-1} \right] . \left(\delta u^n \right) + \left(L_i^{c,n-1} \right) \tag{74}$$

$\left[K_i^{c,n-1} \right]$ makes a new contribution to the tangent matrix of the problem, it is the tangent matrix "of contact". It is worth obviously:

$$\left[K_i^{c,n-1} \right] = E_N . \left[A^{c,n-1} \right]^T . \left[A^{c,n-1} \right] \tag{75}$$

And $\left(L_i^{c,n-1} \right)$ contributes to the second member, it is worth:

$$\left(L_i^{c,n-1} \right) = - E_N . \left[A^{c,n-1} \right]^T . \left(d^{c,n-1} \right) \tag{76}$$

One recalls that the coefficient of penalization of contact is interpreted as the come out from stiffness E_N which is opposed to the penetration of the slave node in surface Master. The larger it is, the less there is interpenetration and the more the recoil force is raised, but this coefficient also intervenes in the tangent matrix, which modifies its conditioning and returns the resolution of the more difficult linear system. When the contact is not activated (i.e. if $\left| h^n \right| \leq 0$), the matrix and the second member vector are null.

4.3.5.2 Linearization of the forces of dependancy

One regularizes the condition of dependancy from the statement (68) of the force d'adhérence, when one is in the phase of dependancy (in the meaning defined in the §4.3.3.2), i.e. by supposing that the inequality (69) is strictly respected:

$$\left[L_i^{a,n} \right]_{\text{regu}} = E_T . \left[A^a \right]^T . \left[A^a \right] . \left(\left(\Delta u_i^{n-1} \right) + \left(\delta u^n \right) \right) \tag{77}$$

While linearizing, one obtains:

$$\left[L_i^{a,n} \right]_{\text{regu}} \underset{\text{linéarisation}}{\rightarrow} \left[\hat{L}_i^{a,n} \right]_{\text{regu}} = \left[K_i^{a,n-1} \right] . \left(\delta u^n \right) + \left(L_i^{a,n-1} \right) \tag{78}$$

$\left[K_i^{a,n-1} \right]$ makes a new contribution to the tangent matrix of the problem, it is the tangent matrix "of dependancy". It is worth obviously:

$$\left[K_i^{a,n-1} \right] = E_T . \left[A^{a,n-1} \right]^T . \left[A^{a,n-1} \right] \tag{79}$$

And $\left(L_i^{a,n-1} \right)$ contributes to the second member, it is worth:

$$\left(L_i^{a,n-1} \right) = E_T . \left[A_a^{n-1} \right]^T . \left(\Delta u_i^{n-1} \right) \tag{80}$$

One recalls that the coefficient of penalization of contact is interpreted as the come out from stiffness E_T which is opposed to the relative sliding of two surfaces . The larger it is, the more the recoil force is raised, but this coefficient also intervenes in the tangent matrix, which modifies its conditioning and returns the resolution of the more difficult linear system.

4.3.5.3 Linearization of the forces of sliding

We saw in the § 4.3.4 that the forces of sliding depended on final displacement $\left(u_i \right)$ by the threshold of Tresca, but that we made the assumption that information with the iteration $(n-1)$ was enough to solve the model of Coulomb [7] while writing:

$$\left(\mu^{g,n} \right) = \left[k^{g,n-1} \right] . \left(t^n \right) \tag{81}$$

---

[7] 7L' idea to transform the problem of Coulomb in a succession of problems of Tresca is very largely justified in the literature (see 59 for example) and mainly used in the commercial codes. It is also the strategy retained in the version "not fixes" algorithm in continuous formulation (see [R5.03.52]).

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Nevertheless, it is necessary to prepare what will occur to the following iteration $(n+1)$ if the threshold is not correct and will require a new iteration. With the next iteration of Newton, solution displacement will be:

$$\left[u_i^{n+1}\right]=\left[u_{i-1}\right]+\left[\Delta u_i^n\right]+\left[\delta u^{n+1}\right]=\left[u_{i-1}\right]+\left[\Delta u_i^{n-1}\right]+\left[\delta u^n\right]+\left[\delta u^{n+1}\right] \tag{82}$$

It is thus necessary to linearize the quantity (81) in $(n+1)$:

$$\left[\mu^{\mathrm{g},n+1}\right]=\left[k^{\mathrm{g},n}\right].\left[t^{n+1}\right] \tag{83}$$

the equation (83) is form $\dfrac{[h(x^{n+1})]}{\|[h(x^{n+1})]\|}$ with $\left[x^{n+1}\right]=\left[x^n\right]+\left[\delta x^{n+1}\right]$. The linearization is written:

$$\left[k\right].\frac{\left[h\left(x^{n+1}\right)\right]}{\left\|\left[h\left(x^{n+1}\right)\right]\right\|}\underset{linéarisation}{\rightarrow}\left[k\right].\left(\frac{\left[h\left(x^n\right)\right]}{\left\|\left[h\left(x^n\right)\right]\right\|}+\frac{1}{\left\|\left[h\left(x^n\right)\right]\right\|}.\left(\left[I\right]-\frac{\left[h\left(x^n\right)\right]\left\langle\left[h\left(x^n\right)\right]\right\rangle}{\left\|\left[h\left(x^n\right)\right]\right\|^2}\right).\left[\delta x^{n+1}\right]\right) \tag{84}$$

One applies (84) to the statement (83) with:

$$\left[x^n\right]\leftarrow\left[\Delta u_i^{n-1}\right]+\left[\delta u^n\right]\text{ and }\left[x^{n+1}\right]\leftarrow\left[\Delta u_i^{n-1}\right]+\left[\delta u^n\right]+\left[\delta u^{n+1}\right]$$

$$\left[\delta x^{n+1}\right]\leftarrow\left[\delta u^{n+1}\right]$$

$$\left[h\left(x^n\right)\right]\leftarrow\left[A^{\mathrm{g}}\right].\left(\left[\Delta u_i^{n-1}\right]+\left[\delta u^n\right]\right)\text{ and }\left[h\left(x^{n+1}\right)\right]\leftarrow\left[A^{\mathrm{g}}\right].\left(\left[\Delta u_i^{n-1}\right]+\left[\delta u^n\right]+\left[\delta u^{n+1}\right]\right) \tag{85}$$

$$\left[k\right]\leftarrow\left[k^{\mathrm{g},n}\right]$$

One linearizes (83):

$$\left[\mu^{\mathrm{g},n+1}\right]\underset{linéarisation}{\rightarrow}\left[\hat{\mu}^{\mathrm{g},n+1}\right] \tag{86}$$

One starts by noting $\left[g^{\mathrm{t},n}\right]$ the tangential sliding relating to the iteration n :

$$\left[g^{\mathrm{t},n}\right]=\left[A^{\mathrm{g}}\right].\left(\left[\Delta u_i^{n-1}\right]+\left[\delta u^n\right]\right) \tag{87}$$

One obtains:

$$\left[\hat{\mu}^{\mathrm{g},n+1}\right]=\left[k^{\mathrm{g},n}\right].\left(\frac{\left[g^{t,n}\right]}{\left\|\left[g^{t,n}\right]\right\|}+\frac{1}{\left\|\left[g^{t,n}\right]\right\|}.\left(\left[I\right]-\frac{\left[g^{t,n}\right]\left\langle\left[g^{t,n}\right]\right\rangle}{\left\|\left[g^{t,n}\right]\right\|^2}\right).\left[\delta u^{n+1}\right]\right) \tag{88}$$

If one notes:

$$\left[\rho^{\mathrm{g},n}\right]=\left[k^{\mathrm{g},n}\right].\frac{\left[g^{t,n}\right]}{\left\|\left[g^{t,n}\right]\right\|} \tag{89}$$

And:

$$\left[B^{\mathrm{g},n}\right]=\frac{\left[k^{\mathrm{g},n}\right]}{\left\|\left[g^{t,n}\right]\right\|}.\left(\left[I\right]-\frac{\left[g^{t,n}\right]\left\langle\left[g^{t,n}\right]\right\rangle}{\left\|\left[g^{t,n}\right]\right\|^2}\right) \tag{90}$$

One can write (88) in the following condensed form:

$$\left[\hat{\mu}^{\mathrm{g},n+1}\right]=\left[B^{\mathrm{g},n}\right].\left[\delta u^{n+1}\right]+\left[\rho^{\mathrm{g},n}\right] \tag{91}$$

the force of sliding expresses itself according to the multiplier of Lagrange:

$$\left[L_i^{\mathrm{g},n+1}\right]=\left[A^{\mathrm{g}}\right]^T.\left[\mu^{\mathrm{g},n+1}\right] \tag{92}$$

the matrix $\left[A^{\mathrm{g}}\right]$ not depending on displacement, the force of sliding linearized can be written:

$$\left[L_i^{\mathrm{g},n+1}\right]\underset{linéarisation}{\rightarrow}\left[\hat{L}_i^{\mathrm{g},n+1}\right]=\left[A^{\mathrm{g}}\right]^T.\left[\hat{\mu}^{\mathrm{g},n+1}\right]=\left[K^{\mathrm{g},n}\right].\left[\delta u^{n+1}\right]+\left[L^{\mathrm{g},n}\right] \tag{93}$$

With the matrix of sliding $\left[K^{\mathrm{g},n}\right]$ such as:

$$\left[K^{\mathrm{g},n}\right]=\left[A^{\mathrm{g}}\right]^T.\left[B^{\mathrm{g},n}\right] \tag{94}$$

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And the vector:

$$\left[L^{g,n}\right]=\left[A^g\right]^T.\left[\rho^{g,n}\right] \tag{95}$$

$\left[K^{g,n}\right]$ makes a new contribution to the tangent matrix of the problem, it is the tangent matrix "of sliding". It is worth obviously:

$$\left[K^{g,n}\right]=\frac{\left[A^g\right]^T.\left[k^{g,n}\right]}{\|\left[g^{t,n}\right]\|}.\left([I]-\frac{\left[g^{t,n}\right]\langle g^{t,n}\rangle}{\|\left[g^{t,n}\right]\|^2}\right) \tag{96}$$

the second part of the statement is preceded by the sign $-$, the effect of this contribution is particularly destabilizing for the total behavior of the tangent matrix to the system, more particularly when one is far from the equilibrium and thus at the beginning of the resolution with each new time step. One thus decides to take it into account only partially by affecting it of a coefficient $\theta\in[0,1]$ that one can modify via keyword COEF_MATR_FROT :

$$\left[\tilde{K}_\theta^{g,n}\right]=\frac{\left[A^g\right]^T.\left[k^{g,n}\right]}{\|\left[g^{t,n}\right]\|}.\left([I]-\theta.\frac{\left[g^{t,n}\right]\langle g^{t,n}\rangle}{\|\left[g^{t,n}\right]\|^2}\right) \tag{97}$$

One advises to use an initial value of 0.5 for this coefficient and to decrease it if convergence is not obtained. If $\theta=0$ convergence always seems to be obtained but is particularly slow. When one is close to the solution, it is on the other hand very useful to have a value of this coefficient equal to 1 in order to accelerate convergence. That is done automatically in the code when residue RESI_GLOB_RELA is lower than 10^{-3}. One replaces $\left[K^{g,n}\right]$ by $\left[\tilde{K}_\theta^{g,n}\right]$.

The second member is worth finally:

$$\left[L^{g,n}\right]=\left[A^g\right]^T.\left[k^{g,n}\right].\frac{\left[g^{t,n}\right]}{\|\left[g^{t,n}\right]\|} \tag{98}$$

In 2D, the multiplier of Lagrange for the sliding is approximated by:

$$\left[\mu^g(u_i)\right]_{2D}=\mu.\left[\|\mu^c(u_i)\|\right]=\left[k^g(u_i)\right] \tag{99}$$

One can show $\left[K^{g,n}\right]_{2D}=0$. And thus:

$$\left[\hat{L}_i^{g,n+1}\right]_{2D}=\left[A^g\right]^T.\left[\rho^{g,n}\right] \tag{100}$$

4.3.6   general Algorithms

the discrete methods of resolution of the problem of contact/friction are founded on a decoupled *approach* between the equilibrium and the contact/friction. The contact/friction (noted C in the second column) is treated after each iteration of Newton of the total problem (noted G in the second column) .

Without taking into account of the contact, one will note the vectors solutions with $\sim$, for example:

$$\left[\tilde{u}_i^n\right]=\left[u_{i-1}\right]+\left[\Delta u_i^{n-1}\right]+\left[\delta\tilde{u}^n\right] \tag{101}$$

4.3.6.1   Case of the dualisation

the general procedure for the dualized case is the following one:

With the iteration of Newton n		
1	R e s o l u	of the problem of equilibrium without contact formulates, equation (53)	$\rightarrow\left[\delta\tilde{u}^n\right]$

	ti o n		
2	G	Updated of displacements without contact	$[\tilde{u}_i^n]=[u_{i-1}]+[\Delta u_i]+[\delta \tilde{u}^n]$
3	C	Modification of displacements to observe the conditions of contact-friction	$[\delta \tilde{u}^n]\rightarrow[\delta u^n]$
4	C	Updated of displacements with taking into account of contact-friction	$[u_i^n]=[u_{i-1}]+[\Delta u_i^{n-1}]+[\delta u^n]$
5	C	Computation of the forces of contact/friction	$[L_i^{c,n}]$ formulates $[L_i^{f,n}]$
6	6	Computation possible of matrixes (case of the sliding)	$[K^{g,n+1}]$
7	7	Computation of the internal forces and outsides with modified displacements	$[L_i^{ext,n}]$ and $[L_i^{int,n}]$
8	G	Checking of the equilibrium	

the matrix $[K^{cf,n+1}]$ are calculated only in the case of the sliding. It will not necessarily be useful, all will depend on the convergence of Newton and the contact algorithm/friction (threshold of Tresca converged on the problem of Coulomb).

4.3.6.2   Case of the regularization

the general procedure for the regularized case (penalization) is the following one:

With the iteration of Newton n			
1	G	Resolution of the problem of equilibrium without contact, equation (53)	$\rightarrow[\delta \tilde{u}^n]$
2	G	Updated of displacements without contact	$[\tilde{u}_i^n]=[u_{i-1}]+[\Delta u_i^{n-1}]+[\delta \tilde{u}^n]$
3	C	Computation of the forces of contact/friction	$[L_i^{c,n}]$ formulates $[L_i^{f,n}]$
4	4	Computation of the modified matrixes	$[K^{cf,n+1}]$
5	G	Computation of the internal forces and outsides	$[L_i^{ext,n}]$ formulates $[L_i^{int,n}]$
6	6	Checking of the equilibrium	

the contact algorithm/friction does not modify displacements, it is the total system which will modify them. What implies that for the penalized methods, one will always need at least two iterations of Newton, even in the elastic case: an iteration to solve the problem of equilibrium without contact/friction and a second iteration to integrate the conditions of contact-friction which will modify the total system.

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5   Algorithmic resolution – Contact without friction

5.1   Connections of contact

Each slave node *potentially* in contact has a statute whose algorithms will have to determine nature. A slave node is called "potential connection of contact" or more simply "connection". The connection term which alludes to the fact that the condition of contact is result of the imposition of a kinematic relation on *the degrees of freedom* of displacement of the slave node. These connections are joined together in various sets:

- Ξ^1 is all possible connections (active and nonactive);
- Ξ^{nc} is all the nodes slaves which are not in contact (nonactive connections);
- Ξ^c is all the nodes indeed in contact (active connections).

One thus has the following relations between the sets:

- $\Xi^c \cap \Xi^{nc} = \emptyset$ because the nodes are in contact or not;
- $\Xi^1 = \Xi^c \oplus \Xi^{nc}$ because the nodes potentially in contact are it or not;

5.2   The methods dualized in contact
5.2.1   Balances structure in the presence of contact

One recalls that the balance equation in the presence of contact is written :

$$\left[L_i^{\text{int}}(u_i) \right] = \left[L_i^{\text{ext}}(u_i) \right] - \left[L_i^{\text{c}} \right]$$

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) the dualized contact, the force of contact is written according to the multiplier of Lagrange of contact:

$$\left[L_i^{\text{c}} \right] = \left[L_i^{\text{c}} \right]_{\text{dual}} = [A^{\text{c}}]^T . \left[\mu_i^{\text{c}} \right] \tag{103}$$

After linearization of the balance equation (102) , one introduces the tangent matrix $\left[K^{\text{m},n-1} \right]$ which will contain the contributions resulting from the linearization from the internal forces and outsides and the matrix $\left[K^{\text{c},n-1} \right]$ for the forces of contact , one finds $\left[\delta \tilde{u}^n \right]$, increment of solution of the problem of equilibrium to the iteration n but without application of the model of contact :

$$\left(\left[K^{\text{m},n-1} \right] + \left[K^{\text{c},n-1} \right] \right) . \left[\delta \tilde{u}^n \right] = \left[L_i^{\text{ext},n-1} \right] - \left[L_i^{\text{int},n-1} \right] - \left[L_i^{\text{c},n-1} \right] \tag{104}$$

We saw that the forces of contact in the dualized case do not depend on displacement (see § 4.3.5) . There is thus $\left[K^{\text{c},n-1} \right] = 0$. What enables us to express the equilibrium of structure with taking into account of the forces of contact (by noting $[K] = \left[K^{\text{m},n-1} \right]$ and $[F] = \left[L_i^{\text{ext},n-1} \right] - \left[L_i^{\text{int},n-1} \right] - \left[L_i^{\text{c},n-1} \right]$ to reduce) :

$$[K] . \left[\delta \tilde{u}^n \right] = [F] \tag{105}$$

the solution $\left[\tilde{u}_i^n \right]$ is obtained after equilibrium and before application of the conditions of contact. She is written:

$$\left[\tilde{u}_i^n \right] = \left[u_{i-1} \right] + \left[\Delta u_i^{n-1} \right] + \left[\delta \tilde{u}^n \right] \tag{106}$$

to obtain the solution $\left[\tilde{u}_i^n \right]$, us did not apply the conditions of contact (model of Signorini) for the current iteration of Newton n. On the other hand, the reactions of contact calculated with the iteration of Newton preceding $n-1$ are well taken into account in $[F]$. To completely solve the problem of equilibrium with contact/friction, with the iteration n, it is necessary to apply the law of Signorini, which is expressed by the following system :

$$\begin{cases} \left[A^{\text{c}} \right] . \left[u_i^n \right] \le \left[d_{\text{ini}}^{\text{c}} \right] & (a) \\ \left[\mu_i^{\text{c}} \right] \ge 0 & (b) \\ \left\langle \mu_i^{\text{c}} \right\rangle . \left(\left[A^{\text{c}} \right] . \left[u_i^n \right] \right) = 0 & (c) \end{cases} \tag{107}$$

One points out the interpretation of the conditions of Signorini:

- The equation (107) represents the geometrical conditions of noninterpenetration, component inequality being understood by component (each line is relative to a potential couple of contact).

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- The equation (107) expresses the absence of opposition to separation (contact surfaces can know only compressions), it is the condition known as of intensility.
- The equation (107) is the compatibility condition. When for a given connection the multiplier of Lagrange is non-zero, there is contact and thus clearance is null. When clearance is non-zero (two surfaces are not in contact), the associated multiplier must be null (not compression).

The application of the model of Signorini will modify the displacement increment $\left[\delta \tilde{u}^n\right]$ (which becomes $\left[\delta u^n\right]$), from where the solution obtained after equilibrium and application of the model of contact which S" written:

$$\left[u_i^n\right]=\left[u_{i-1}\right]+\left[\Delta u_i^{n-1}\right]+\left[\delta u^n\right] \tag{108}$$

5.2.2 System reduced to active connections

We will write the system allowing to completely solve the problem of equilibrium with taking into account of the model of Signorini. The idea is to transform the inequalities of the system (107) into equalities. One starts by of the contact evaluating the clearance given by the displacement calculated before the correction $\left[\tilde{d}^{c,n}\right]$, for all connections [8] :

$$\left|\left[\tilde{d}^{c,n}\right]\right|_{\Xi^1}=\left|\left[d_{\text{ini}}^c\right]-\left[A^c\right].\left(\left[u_{i-1}\right]+\left[\Delta u_i^{n-1}\right]+\left[\delta \tilde{u}^n\right]\right)\right|_{\Xi^1} \tag{109}$$

One says that a connection J is active if its clearance $\left|\left[\tilde{d}^{c,n}\right]\right|_{J \in \Xi^1}$ is negative, which indicates an interpenetration (the condition of contact is not observed) , this connection thus becomes a connection of contact and thus makes it possible to define the whole *initially* applied of connections of contact Ξ^c :

$$\Xi^c=\left\{J \in \Xi^1 \middle| \left|\left[\tilde{d}^{c,n}\right]\right|_J<0\right\} \tag{110}$$

One applies that, for these active connections, effective clearance will be null, and that thus the inequality $\left|\left[A^c\right].\left[u_i^n\right]\right|_{\Xi^1} \le \left|\left[d_{\text{ini}}^c\right]\right|_{\Xi^1}$ becomes an equality for the group active connections:

$$\left|\left[A^c\right].\left[u_i^n\right]=\left[d_{\text{ini}}^c\right]\right|_{\Xi^c} \tag{111}$$

If one uses $\left[d^{c,n-1}\right]$, the clearance evaluated before the current iteration of Newton:

$$\left[d^{c,n-1}\right]=\left[d_{\text{ini}}^c\right]-\left[A^c\right].\left(\left[u_{i-1}\right]+\left[\Delta u_i^{n-1}\right]\right) \tag{112}$$

With:

$$\left[\tilde{d}^{c,n}\right]=\left[d^{c,n-1}\right]-\left[A^c\right].\left[\delta \tilde{u}^n\right] \tag{113}$$

the equality (111) is written finally:

$$\left|\left[A^c\right].\left[\delta u^n\right]=\left[d^{c,n-1}\right]\right|_{\Xi^c} \tag{114}$$

the "mixed" system inequality/equality (containing the balance equation and the conditions of Signorini) transforms itself finally into simple system which treats only equalities, on the basis of applied connection of contact Ξ^c :

$$\begin{cases}[K].\left[\delta u^n\right]+\left[A^c\right]^T.\left[\mu_i^c\right]=\left[F\right] \\ \left|\left[A^c\right].\left[\delta u^n\right]=\left[d^{c,n-1}\right]\right|_{\Xi^c}\end{cases} \tag{115}$$

One seeks the unknowns $\left[\delta u^n\right]$ and $\left[\mu_i^c\right]$ who are solutions of the following system:

$$\begin{bmatrix}[K] & [A^c]^T \\ [A^c] & [0]\end{bmatrix}.\begin{Bmatrix}\left[\delta u^n\right] \\ \left[\mu_i^c\right]\end{Bmatrix}=\begin{Bmatrix}\left[F\right] \\ \left[d^{c,n-1}\right]\end{Bmatrix} \Leftrightarrow [K^c].\begin{Bmatrix}\left[\delta u^n\right] \\ \left[\mu_i^c\right]\end{Bmatrix}=\begin{Bmatrix}\left[F\right] \\ \left[d^{c,n-1}\right]\end{Bmatrix} \tag{116}$$

One will notice that the dualized formulation of the condition of NON-interpenetration translates the stationarity of Lagrangian L thus definite:

---

8   note by $\left|\ \right|_\square$ the operations which one makes on a subset of connections such as defined in the § 5.1 .

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$$L\left(\left[\delta\,\boldsymbol{u}^n\right],\left[\boldsymbol{\mu}_i^{\mathrm{c}}\right]\right)=\frac{1}{2}.\langle\delta\,\boldsymbol{u}^n\rangle.[\boldsymbol{K}].\left[\delta\,\boldsymbol{u}^n\right]-\langle\delta\,\boldsymbol{u}^n\rangle.[\boldsymbol{F}]+\langle\boldsymbol{\mu}_i^{\mathrm{c}}\rangle.\left([A^{\mathrm{c}}].\left[\delta\,\boldsymbol{u}^n\right]-\left[\boldsymbol{d}^{\mathrm{c},n-1}\right]\right)$$ (117)

This stationarity expresses itself in the form of the problem of POINT-saddles:

$$\min_{\left[\delta\,\boldsymbol{u}^n\right]}\max_{\left[\boldsymbol{\mu}_i^{\mathrm{c}}\right]\geq 0}L\left(\left[\delta\,\boldsymbol{u}^n\right],\left[\boldsymbol{\mu}_i^{\mathrm{c}}\right]\right)$$ (118)

5.2.3   the method of the active stresses – `ALGO_CONT=' CONTRAINTE'`

It acts to generalize the approach of the preceding paragraph by adopting an iterative algorithm which works in three phases:
1.To make assumptions on the state of connections (transformations inequalities → equality);
2.To solve the new system thus created;
3.To check the initial assumptions and to buckle (possibly) in 1.
One will be able to find a description complete of the method with the theoretical justifications necessary in 59 and 59. The principle is the following: a set of stresses known as active are applied, which correspond to a null clearance (the relation inequality becomes an equality); one solves the system of equations obtained in this subspace, and one looks at if the starting postulate were justified. If the selected whole were too small (active connections had been forgotten), one adds with the whole connection violating more the condition of NON-interpenetration; if the selected whole were too large (presumedly active connections are not it in fact not), one removes from the group the most improbable connection *i.e that* whose multiplier of Lagrange violates more the condition of intensility. The fact of removing or of adding only one connection with each iteration of the method twice guarantees convergence in a number finished of iterations inferior or equal to the maximum number of connections. In
elasticity, at the end of the iterations of active stresses, there is one result converged within the meaning of Newton. In plasticity or if the geometry is reactualized, it is not the case because several iterations of Newton are necessary to obtain the equilibrium. After each iteration of Newton, one launches the algorithm of active stresses to satisfy the conditions with contact. Thus, in elasticity, one will necessarily converge for each step in an iteration if REAC _GEOM= "SANS ". Writing

5.2.3.1   of the iterative problem One

leaves the increment obtained without treating the contact formulates $\left[\delta\,\tilde{\boldsymbol{u}}^n\right]$ one carries out the iterations of active stresses formulates k "with clean convergence of this algorithm. Convergence within the meaning of the active stresses is obtained lorsqu" no connection does not violate the kinematical condition (inequality (107 107) and when the associated Lagrange multipliers all are positive (inequality (107 107) . One notes formula k the iterations of active stresses. The starting solution without correction of the contact is formula $\left[\delta\,\tilde{\boldsymbol{u}}^n\right]$ the increment added by the new iteration of contact is formula $\left[\delta_k\right]$ notes: formulate

$$\begin{aligned}\left[\delta\,u_0^n\right]&=\left[\delta\,\tilde{u}^n\right]\\\left[\delta_0\right]&=[0]\\\left[\delta\,u_k^n\right]&=\left[\delta\,u_{k-1}^n\right]+\left[\delta_k\right]\end{aligned}$$ 119

places itself at the iteration of contact. k One seeks to solve the system (115 115 : formulate

$$\begin{cases}[\boldsymbol{K}].\left[\delta\,u_k^n\right]+[A_k^{\mathrm{c}}]^T.\left[\boldsymbol{\mu}_i^{\mathrm{c}}\right]=[\boldsymbol{F}]\\\left[[A_k^{\mathrm{c}}].\left[\delta\,u_k^n\right]=\left[\boldsymbol{d}^{\mathrm{c},n-1}\right]\right]_{\Xi_k^{\mathrm{c}}}\end{cases}$$ 120

notes that the multiplier of Lagrange of the contact formula $\left[\boldsymbol{\mu}_i^{\mathrm{c}}\right]$ is not solved in an incremental way but in a total way (on time step). While injecting (119 119 into this system, one obtains: formulate

$$\begin{cases}[\boldsymbol{K}].\left[\delta\,u_{k-1}^n\right]+[\boldsymbol{K}].\left[\delta_k\right]+[A_k^{\mathrm{c}}]^T.\left[\boldsymbol{\mu}_i^{\mathrm{c}}\right]=[\boldsymbol{F}]\\\left[[A_k^{\mathrm{c}}].\left(\left[\delta\,u_{k-1}^n\right]+\left[\delta_k\right]\right)=\left[\boldsymbol{d}^{\mathrm{c},n-1}\right]\right]_{\Xi_k^{\mathrm{c}}}\end{cases}$$ 121

the first equation can rewrite itself: formulate

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$$\left\{\delta_k\right\}=\left[K\right]^{-1}.\left\{F\right\}-\left\{\delta u_{k-1}^n\right\}-\left[K\right]^{-1}.\left[A_k^c\right]^T.\left\{\mu_i^c\right\} \tag{122}$$

taking account that formula $\left\{\delta \tilde{u}^n\right\}=\left[K\right]^{-1}.\left\{F\right\}$ one simplifies: (123

$$\left\{\delta_k\right\}=\left\{\delta \tilde{u}^n\right\}-\left\{\delta u_{k-1}^n\right\}-\left[K\right]^{-1}.\left[A_k^c\right]^T.\left\{\mu_i^c\right\} \tag{123}$$

the applied state of active connections: (124

$$\left.\left|\left[A_k^c\right].\left(\left\{\delta u_{k-1}^n\right\}+\left\{\delta_k\right\}\right)=\left\{d^{c,n-1}\right\}\right|_{\Xi_k^c} \tag{124}$$

by taking again the statement of formula $\left\{d^{c,n-1}\right\}$ by (57)57: formulate

$$\left.\left|\left[A_k^c\right].\left\{\delta u_{k-1}^n\right\}+\left[A_k^c\right].\left\{\delta_k\right\}\right|_{\Xi_C^k}=\left.\left|\left\{d_{ini}^c\right\}-\left[A_k^c\right].\left(\left\{u_{i-1}\right\}+\left\{\Delta u_i^{n-1}\right\}\right)\right|_{\Xi_k^c} \tag{125}$$

uses the statement of formula $\left\{\delta_k\right\}$ by (122122: formulate

$$\left.\left|\left[A_k^c\right].\left\{\delta \tilde{u}^n\right\}-\left[A_k^c\right].\left[K\right]^{-1}.\left[A_k^c\right]^T.\left\{\mu_i^c\right\}\right|_{\Xi_k^c}=\left.\left|\left\{d_{ini}^c\right\}-\left[A_k^c\right].\left(\left\{u_{i-1}\right\}+\left\{\Delta u_i^{n-1}\right\}\right)\right|_{\Xi_k^c} \tag{126}$$

final, the Lagrange multipliers for $\left\{\mu_i^c\right\}$ connections of contact are solutions of the following system: formulate

$$\left.\left|-\left[A_k^c\right].\left[K\right]^{-1}.\left[A_k^c\right]^T.\left\{\mu_i^c\right\}=\left\{\tilde{d}^{c,n}\right\}\right|_{\Xi_k^c} \tag{127}$$

notices that (127127 the complement of Schur of $\left[K\right]_{\text{schur}}=\left[A_k^c\right].\left[K\right]^{-1}.\left[A_k^c\right]^T$ the matrix. $\left[K^c\right]$ One can finally calculate the displacement increments with $\left\{\delta_k\right\}$: formulate

$$\left\{\delta_k\right\}=\sum_{l=0}^{k-1}\left\{\delta_l\right\}-\left[K\right]^{-1}.\left[A_k^c\right]^T.\left\{\mu_i\right\}^c \tag{128}$$

the resolution of (127127 the most expensive part in time computation of the algorithm. All the effectiveness of the strategy consists in using the fact that one solves this system only on all active connections , Ξ_k^c one thus uses two properties. The computation

• complement of Schur formulates $\left[K\right]_{\text{schur}}$ calls on a factorization of the type formulates LDL^T which makes it possible to save time, because such a factorization has the property remarkable to be incremental, i.e. the addition of a connection does not oblige to rebuild factorized since the beginning but only the part which one modifies.

• The resolution (gone up descent/) is done also only on all active connections formulates Ξ_k^c Validity

5.2.3.2 of the active connection set selected At the end of

each iteration of active stress, it is appropriate to check if the formula group Ξ_k^c quite correct. That is to say connection, $J \in \Xi^1$ three situations are possible:
1) Relative   displacement   compensates   for   initial   clearance   formulates
$$\left.\left|\left[A_k^c\right].\left(\left\{u_{i-1}\right\}+\left\{\Delta u_i^{n-1}\right\}+\left\{\delta u_k^n\right\}\right)\right|_{J \in \Xi^1}=\left.\left|\left\{d_{\text{ini}}^c\right\}\right|_{J \in \Xi^1}$$
2) Relative   displacement   is   lower   than   initial   clearance   formulates
$$\left.\left|\left[A_k^c\right].\left(\left\{u_{i-1}\right\}+\left\{\Delta u_i^{n-1}\right\}+\left\{\delta u_k^n\right\}\right)\right|_{J \in \Xi^1}<\left.\left|\left\{d_{\text{ini}}^c\right\}\right|_{J \in \Xi^1}$$
3) Relative   displacement   is   higher   than   initial   clearance   formulates
$$\left.\left|\left[A_k^c\right].\left(\left\{u_{i-1}\right\}+\left\{\Delta u_i^{n-1}\right\}+\left\{\delta u_k^n\right\}\right)\right|_{J \in \Xi^1}>\left.\left|\left\{d_{\text{ini}}^c\right\}\right|_{J \in \Xi^1}$$

The situation is (3) prohibited because it corresponds to a violation of the condition of NON-interpenetration. The situation corresponds (1) to a connection known as active, the situation with (2) a connection not - active . At

the beginning of the iteration of k the algorithm, one had applied a set of active connections formulates Ξ_k^c One found an increment possible formulates $\left\{\delta_k\right\}$ unknowns under these assumptions and one now will check that this increment is compatible with the assumptions. In practice, that consists in making two checks:

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1) Is the formula group Ξ_k^c too small? It is checked that nonactive supposed connections do not violate the condition of NON-interpenetration, if not one activates one of them, that which violates more the condition.

2) Is the formula group Ξ_k^c too large? It is checked that presumedly active connections are associated with multipliers of contact formulates $\left[\mu_i^c\right]$ or null, if not one disables of it that which violates more the condition.

Is the formula group Ξ_k^c too small?

To check that the formula group Ξ_k^c is not too small, one will calculate for all presumedly inactive connections, the following quantity: formulate

$$\rho_J = \left[\frac{\left[d_{\text{ini}}^c\right] - \left[A_k^c\right].\left(\left[u_{i-1}\right] + \left[\Delta u_i^{n-1}\right] + \left[\delta u_k^n\right]\right)}{\left[A_k^c\right].\left[\delta_k\right]}\right]_{J \in \Xi_k^{\text{nc}}} = \left[\frac{\left[d^{c,n-1}\right] - \left[A_k^c\right].\left[\delta u_k^n\right]}{\left[A_k^c\right].\left[\delta_k\right]}\right]_{J \in \Xi_k^{\text{nc}}} \qquad 129$$

are two cases: If

1), $\left[\left[A_k^c\right].\left[\delta_k\right]\right]_{J \in \Xi_k^{\text{nc}}} < 0$ clearance for connection J will increase, and thus presumedly inactive connection remains in this state, formula ρ_J strictly higher than formula 1 S

2)I formulates $\left[\left[A_k^c\right].\left[\delta_k\right]\right]_{J \in \Xi_k^{\text{nc}}} > 0$ clearance for connection formula J will decrease , and thus connection supposed not - active will be activated, formula ρ_J lower or equal to formula 1 One

thus examines formula $\bar{\rho} = \underset{J}{Min}\,\rho_j$ all connections formulates J nonactive. If formula $\bar{\rho} < 1$ that indicates that a connection at least is violated (situation formulates (3) : one then adds with the list of active connections the number of the connection whose interpenetration is largest, i.e. that which carries out the minimum of formula ρ_J one writes formula $\left[\delta u_{k+1}^n\right] = \left[\delta u_k^n\right] + \bar{\rho}.\left[\delta_k\right]$ corresponds to a null clearance for added connection).

The algorithm used is presented: If

If not $\Xi_k^c = \Xi^1$			
	$\bar{\rho} = 1$		
Buckles			
	$\bar{\rho} = 1$		
	on connections Computation $J \in \Xi_k^{\text{nc}}$		
		of So $\alpha_J = \left[\left[A_k^c\right].\left[\delta_k\right]\right]_J$	
		Fine $\alpha_J < 0$	
			$\rho_J = \dfrac{\left[\left[d^{c,n-1}\right] - \left[A_k^c\right].\left[\delta u_k^n\right]\right]_J}{\alpha_J}$
			$\bar{\rho} = \min\left(\rho_J, \bar{\rho}\right)$
			$J_{min} = J$
	Buckles In		

output of this algorithm, there will be the number of the connection which violates more the condition of nonpenetration formulates J_{min} the value formulates $\bar{\rho}$ does he $\left[J_{min}, \bar{\rho}\right] = IsPetit\left(\Xi_k^c\right)$

the formula Ξ_k^c is too large?

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The second checking consists in wondering whether all active connections are too large. One places oneself now in the case where, $\bar{\rho} \geq 1$ i.e. one knows that formula Ξ_k^c is not too small. Then: If

- no connection is active, the method converged towards a state without contact; If
- there are presumedly active connections: If
  - all the Lagrange multipliers are $\left[\mu_i^c\right]$ positive or null, one also converged towards a state with effective contact; If
  - there exist negative Lagrange multipliers $\left[\mu_i^c\right]$, corresponding connections should not be active: one withdraws from all active connections the connection whose negative multiplier is largest in absolute value. In

output of this algorithm, there will be the number of the connection which violates more the condition of intensity: formulate $\left[J_{max}\right] = IsGrand\left(\Xi_k^c\right)$ Algorithm

5.2.3.3 With

final, the algorithm for the active stresses is the following: Computation

Ini	$\delta u_0^n = \delta \tilde{u}^n$			
	of Evaluating $\left\|\left[\tilde{d}^{c,n}\right]\right\|_{\Xi^1}$			
	of and $\Xi_0^c = \left\{J \in \Xi^1 \left\| \left[\tilde{d}^{c,n}\right]\right\|_J < 0\right\}$ formulates $\left[A_0^c\right]$			
B_k	on the active stresses formulates $k = 1, Iter_{max}$			
		$\left[\delta_{k-1}\right] = \left[\delta \tilde{u}^n\right] - \left[\delta u_{k-1}^n\right]$ $\Xi_k^c \leftarrow \Xi_{k-1}^c$ formulates $\left[A_k^c\right] \leftarrow \left[A_{k-1}^c\right]$		
		Computation $\Xi_k^c \neq \varnothing$		
			of Factorization $\left[K_k\right]_{schur} = \left[A_k^c\right].\left[K\right]^{-1}.\left[A_k^c\right]^T$	
			of Resolution $\left[\left[K_k\right]_{schur}\right]_{\Xi_k^c}$	
			of If $\left[\left[K_k\right]_{schur}.\mu_i^c = \left[\tilde{d}^{c,n}\right]\right]_{\Xi_k^c} \rightarrow \left[\left[\mu_i^c\right]\right]_{\Xi_k^c}$	
			$\left[\delta_k\right] = \left[\delta_{k-1}\right] - \left[K\right]^{-1}.\left[A_k^c\right]^T.\left[\mu_i^c\right]$	
		$\left[J_{min}, \bar{\rho}\right] = IsPetit\left(\Xi_k^c\right)$		
		$\left[\delta u_k^n\right] = \left[\delta u_{k-1}^n\right] + \bar{\rho}.\left[\delta_k\right]$		
		If not $\bar{\rho} < 1$		
			$\Xi_k^c \leftarrow \Xi_k^c + \left[J_{min}\right]$	
		If		
			Fine $\Xi_k^c = \varnothing$	
				formula formulates F
			$\left[J_{max}\right] = IsGrand\left(\Xi_k^c\right)$	
			Fine $\left[J_{max}\right] = 0$	
				Goto formulates F

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			formulates
			$\Xi_k^c \leftarrow \Xi_k^c - \lfloor J_{max} \rfloor$
B_k	FIN $k = k+1$		
F	Computation		
	of formula $\lfloor L_i^c \rfloor = [A^c]^T . \lfloor \mu_i^c \rfloor$		

5.2.4 the method of the Conjugate gradient Project – ALGORITHME _CONT=' GCP'

the method of resolution presented in this part is an application of the algorithm of the Conjugate gradient Project. It is very precisely the iterative version of the method of the Active Stresses presented in the preceding part. Reformulation

5.2.4.1 of the problem of contact One

recalls that the system to be solved with each iteration of Newton is n the following: formulate

$$\begin{cases} [K].\lfloor \delta u^n \rfloor + [A^c]^T . \lfloor \mu_i^c \rfloor = \lfloor F \rfloor \\ [A^c].\lfloor \delta u^n \rfloor \leq \lfloor d^{c,n-1} \rfloor \end{cases}$$

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comes from the dualisation of the conditions of contact and it translates the stationarity of Lagrangian thus L defined: formulate

$$L\left(\lfloor \delta u^n \rfloor , \lfloor \mu_i^c \rfloor \right) = \frac{1}{2} . \langle \delta u^n \rangle . [K].\lfloor \delta u^n \rfloor - \langle \delta u^n \rangle . \lfloor F \rfloor + \langle \mu_i^c \rangle . \left([A^c].\lfloor \delta u^n \rfloor - \lfloor d^{c,n-1} \rfloor \right)$$

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stationarity can express itself in the form of the problem of POINT-saddles: formulate

$$\min_{\lfloor \delta u^n \rfloor} \max_{\lfloor \mu_i^c \rfloor \geq 0} L\left(\lfloor \delta u^n \rfloor , \lfloor \mu_i^c \rfloor \right)$$

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the minimum of compared to L is δu_i known; it has as a statement: and

$$\lfloor \delta u^n \rfloor = [K]^{-1} . \left(\lfloor F \rfloor - [A^c]^T . \lfloor \mu_i^c \rfloor \right) \quad \text{does not formulate} \quad \langle \delta u^n \rangle = \left(\langle F \rangle - \langle \mu_i^c \rangle . [A^c]\right) . [K]^{-1}$$

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does not have thus any more but to make maximization from L ratio with. $\lfloor \mu_i^c \rfloor \geq 0$ Knowing that to maximize a functional calculus is equivalent J minimizing, $-J$ one is reduced to the problem of minimization according to: (134

$$\min_{\lfloor \mu_i^c \rfloor \geq 0} H\left(\lfloor \mu_i^c \rfloor \right)$$

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the functional calculus H is written: formulate

$$H\left(\lfloor \mu_i^c \rfloor \right) = \frac{1}{2} . \langle \mu_i^c \rangle . [A^c].[K]^{-1}.[A^c]^T . \lfloor \mu_i^c \rfloor + \langle \mu_i^c \rangle . \left(\lfloor d^{c,n-1} \rfloor - [A^c].\lfloor \delta \tilde{u}^n \rfloor \right) + \frac{1}{2} . \langle F \rangle . [K]^{-1} . \lfloor F \rfloor$$

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statement is the dual form of the problem of contact: it utilizes the field of Lagrange multipliers and $\lfloor \mu_i^c \rfloor$ any more the field of displacement. $\lfloor \delta u^n \rfloor$ The problem to be solved is now a minimization under stress of positivity of the unknown. The method of the Conjugate gradient Project is a simple and effective method for this kind of problem. One will note: formulate
  • $\lfloor Z \rfloor$ the direction of search formulates
  • $\lfloor r \rfloor$ the under-gradient and formulates $\lfloor r^p \rfloor$ packaged version Searches

5.2.4.2 linear In

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an algorithm of conjugate gradient, it is necessary to estimate a step of advance. Two alternatives of linear search are available, acceptable or not. They are represented graphically on the 5.2.4.2 figure 5.2.4.2-a The acceptable alternative, which forces to remain in the convex field acceptable armature a single resolution by iteration; it leads to a rather regular convergence. The non-acceptable alternative, which authorizes to leave the acceptable field for there reprojeter then, induced two resolutions per iteration; it leads to a rather erratic convergence but generally faster than the acceptable method. Appear

5.2.4.2 5.2.4.2-a of the linear search. Here

the algorithm of the linear search: State

RechLine		
Ini	of contact formulates $[A^c] \leftarrow [A^c_k]$ formulates $\Xi^c \leftarrow \Xi^c_k$	
	of the second member Resolution $[F] = [A^c]^T . [Z]$	
	of Computation $[K].[\delta a] = [F] \rightarrow [\delta a]$	
	of If $\alpha = \dfrac{\langle r \rangle . [r^p]}{\langle \delta a \rangle . [F]}$	
	ADMISSIBLE `formulates`	
		$\forall J \in \Xi^1 : \text{Si } [Z]_J < 0 \ \text{alors } \alpha = \min\limits_{J \in \Xi^1} \left(\alpha, -\dfrac{[\mu^c]_J}{[Z]_J} \right)$
		$[\mu^c_{k+1}] = [\mu^c_k] + \alpha . [Z]$
		$[\delta u_{k+1}] = [\delta u_k] - \alpha . [\delta a]$
	`formulates`	
		$[\mu^c_{k+1}] = [\mu^c_k] + \alpha . [Z]$
		$[\mu^c_{k+1}] \leftarrow \max\left([\mu^c_{k+1}], 0 \right)$
		of the second member formulates $[F] = [A^c]^T . [\mu^c]$
		of formula $[K].[\delta a] = [F] \rightarrow [\delta a]$
		$[\delta u_{k+1}] = [\delta \tilde{u}^n] - [\delta a]$
F	In	

output of this algorithm, one will and the have displacement Lagrange multipliers after computation of the step of advance: formulate $\left([\delta u_{k+1}], [\mu^c_{k+1}] \right) = RechLine\left([Z], [r], [r^p] \right)$ This algorithm understands two resolutions but by means of the matrix formulates $[K]$ factorized (it is the total matrix of the problem of equilibrium), one thus makes only one descent-increase, which is little expensive.

The non-acceptable alternative is obliged to recompute a displacement increment compatible with the stress of positivity on the multipliers of contact. Pre

5.2.4.3   - conditioning In

the algorithm of the GCP, it is mentioned an optional call to a preconditioner. The goal of this preconditioner is to accelerate the convergence of the method. Its definition comes from the following considerations of functional analysis: in the phase of update formulates $[\mu^c_{k+1}] = [\mu^c_k] + \alpha . [Z_k]$ if one did not call on the preconditioner, the

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field formula $\left[\boldsymbol{\mu}_k^{\mathrm{c}}\right]$ to formula $H^{-1/2}(\Gamma_{\mathrm{c}})$ formula $\left[\boldsymbol{Z}_k\right]$ to formula $H^{1/2}(\Gamma_{\mathrm{c}})$ This sum thus does not have a mathematical meaning. For him to give one of them, it is necessary to send formula $\left[\boldsymbol{Z}_k\right]$ formula $H^{-1/2}(\Gamma_{\mathrm{c}})$ is the operation of preconditionning. Knowing that formula $\left[\boldsymbol{Z}_k\right]$ is obtained by the statement formulates $\left[\boldsymbol{Z}_k\right]=\left[\boldsymbol{r}_k^p\right]+\gamma_k \cdot \left[\boldsymbol{Z}_{k-1}\right]$ is on formula $\left[\boldsymbol{r}_k^p\right]$ one will operate. It is what is made by the preconditioner by solving the following auxiliary problem: (136

$$\left|\begin{array}{l} \left\|[\boldsymbol{K}] \cdot [\boldsymbol{a}] + \left[\boldsymbol{A}_k^{\mathrm{c}}\right]^T \cdot \left[\boldsymbol{r}_k^p\right]\right\|_{\Xi_k^{\mathrm{c}}} = 0 \\ \left\|\left[\boldsymbol{A}_k^{\mathrm{c}}\right] \cdot [\boldsymbol{a}] = \left[\boldsymbol{r}_k^p\right]\right\|_{\Xi_k^{\mathrm{c}}} \end{array}\right.$$

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thus solves a problem of displacement imposed on the part where the contact is effective and one recovers the reactions to the fixed support formulates $\left[\boldsymbol{r}_k^p\right]$ belong well to formula $H^{-1/2}(\Gamma_{\mathrm{c}})$ In the terminology of the decomposition of field, it is a preconditioner of Dirichlet.

To solve this auxiliary problem, one uses also an algorithm of conjugate gradient but without projection because no stress of positivity appears there. This iterative approach authorizes an approximate resolution so as to save computing time.

- $[\boldsymbol{X}]$ direction of search;
- $[\boldsymbol{s}]$ the under-gradient. Here

the algorithm of the pre-conditioner: formulate

PreCond			
Ini	$[\boldsymbol{a}_0]=0$		
	of contact and $[\boldsymbol{A}^{\mathrm{c}}] \leftarrow [\boldsymbol{A}_k^{\mathrm{c}}]$ If $\Xi^{\mathrm{c}} \leftarrow \Xi_k^{\mathrm{c}}$		
	Fine $\Xi^{\mathrm{c}}=\emptyset$		
		Goto formulates F	
B_p	on Computation $p=1, Iter_{max}$		
		of the gradient formulates $[\boldsymbol{s}_p]=[\boldsymbol{A}^{\mathrm{c}}] \cdot [\boldsymbol{a}_{p-1}]-[\boldsymbol{r}]$	
		of the residue If $\varepsilon=\max\limits_{J \in \Xi^c}[\boldsymbol{s}_p]$	
		Fine $\varepsilon < \varepsilon_{precond}$	
			Goto formulates F
		or $p=1$ reactualization formulates	
			$[\boldsymbol{X}_p]=[\boldsymbol{s}_p]$
		, conjugation formulates	
			$\beta=\dfrac{\langle \boldsymbol{s}_p \rangle \cdot [\boldsymbol{s}_p]}{\langle \boldsymbol{s}_{p-1} \rangle \cdot [\boldsymbol{s}_{p-1}]}$
			$[\boldsymbol{X}_p]=[\boldsymbol{s}_p]+\beta \cdot [\boldsymbol{X}_{p-1}]$
		of the second member formulates $[\boldsymbol{F}]=\left\|[\boldsymbol{A}^{\mathrm{c}}]^T \cdot [\boldsymbol{X}_p]\right\|_{\Xi^c}$	
		of formula $[\boldsymbol{K}] \cdot [\delta \boldsymbol{a}]=[\boldsymbol{F}] \rightarrow [\delta \boldsymbol{a}]$	
		of the complement of Schur formulates $[\delta \boldsymbol{a}]_{schur}=\left\|[\boldsymbol{A}^c] \cdot [\delta \boldsymbol{a}]\right\|_{\Xi^c}$	

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	of the step of advance formulates $\alpha = \dfrac{\langle s_p \rangle . \lfloor s_p \rfloor}{\langle F \rangle . \lfloor \delta a \rfloor_{schur}}$	
	under-gradient formulates $\lfloor s_p \rfloor = \lfloor s_{p-1} \rfloor + \alpha . \lfloor X_p \rfloor$	
	displacement formulates $\lfloor a_p \rfloor = \lfloor a_{p-1} \rfloor - \alpha . \lfloor \delta a \rfloor$	
B_k	Buckles formula $k = k+1$	
F	formulates	
	$\lfloor r_k^p \rfloor = \lfloor s_p \rfloor$: formulate $(max_J(r_k^p, 0) \, si \, \mu_J < 0)$	

output of this algorithm, one will have packaged under-gradient: formulate $\lfloor r_k^p \rfloor = PreCond(\lfloor r_k \rfloor)$ Algorithm

5.2.4.4 Here

the total algorithm for method GCP: formulate

Ini	formula $\langle \mu_{i-1}^c \rangle . \lfloor \mu_{i-1}^c \rfloor \neq 0$ formula $[K].\lfloor v \rfloor = [A_{i-1}^c]^T . \lfloor \mu_{i-1}^c \rfloor \rightarrow \lfloor v \rfloor$ formula $\langle \mu_{i-1}^c \rangle . \lfloor \mu_{i-1}^c \rfloor = 0$ formula $\lfloor v \rfloor = \lfloor 0 \rfloor$ of Evaluating $\delta u_0^n = \tilde{\delta} u^n - v$ of formula $\Xi_0^c = \{ J \in \Xi^1 | | \lfloor \tilde{d}^{c,n} \rfloor_J < 0 \}$
B_k	on formula $k = 1, Iter_{max}$
	$\Xi_k^c \leftarrow \Xi_{k-1}^c$
	of the under-gradient Projection $\lfloor r_k \rfloor = ([A_k^c].\lfloor \delta u_{k-1}^n \rfloor - \lfloor d^{c,n-1} \rfloor)$
	of the under-gradient formulates $\lfloor r_k \rfloor = \underset{J \in \Xi^1}{max}(\lfloor r_k \rfloor, 0)$
	of the residue formulates $\varepsilon = \underset{J \in \Xi^1}{max} \lfloor r_k \rfloor$
	formulates $\lfloor r_k^p \rfloor = PreCond(\lfloor r_k \rfloor)$
	Computation
	of the coefficient formulates $\gamma^k = \dfrac{\langle r_k \rangle . \lfloor r_k^p \rfloor - \langle r_k \rangle . \lfloor r_{k-1}^p \rfloor}{\langle r_{k-1} \rangle . \lfloor r_{k-1}^p \rfloor}$
	$\lfloor Z_k \rfloor = \lfloor r_k \rfloor + \gamma^k . \lfloor Z_{k-1} \rfloor$
	Search formulates $\{ \lfloor \delta u_{k+1} \rfloor, \lfloor \mu^c \rfloor \} = RechLine(\lfloor Z \rfloor, \lfloor r \rfloor, \lfloor r^p \rfloor)$
B_k	formula $k = k+1$
F	One

can make the following comments on the algorithm: At the time

• of the linear search, the resolution of the system formulates $[K].\lfloor \delta a \rfloor = [A^c]^T.\lfloor Z \rfloor$ the computation of the

term, $\langle \delta a \rangle.[A^c]^T.\lfloor Z \rfloor$ one perceives that the algorithm presented is the iterative version of the method of

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the active stresses. Indeed, if one clarifies the term formulates $\langle \delta a \rangle . [A^c]^T . [Z]$ obtains. $\langle Z \rangle . ([A_c] . [K]^{-1} . [A_c]^T) . [Z]$ One finds the complement of Schur formulates $[K]_{schur} = [A^c] . [K]^{-1} . [A^c]^T$ is explicitly built in the method of the active stresses. As in all the iterative methods, the Conjugate gradient Project does not build this operator but uses his effect on a vector (here on) $[Z]$;

• To be really effective, this algorithm must be used with a direct solver or a preconditioned iterative solver by method LDLT_SP . With

○ a direct solver, the matrix of the system being already factorized, each resolution is summarized with an descent-increase: With

○ a preconditioned iterative solver by method LDLT_SP , each resolution summarizes itself with some iterations of the iterative solver. One

• combines with each iteration.

5.2.5 Lagrangian method – ALGORITHME _CONT=' LAGRANGIEN'

the method known as "LAGRANGIAN " of the contact is an alternative of the method of the active stresses, with three nuances near:

○ the phase of checking of the group too deep east made in once at the end of the process, and not connection by connection; one

○ keeps information on active connections of an iteration of Newton to the other, but one forgets it of a reactualization of the geometry to the other;

○ initially active connections are evaluated on the basis of clearance BEFORE the current iteration of Newton: one uses formula $[d^{c,n-1}]$ than formula $[\tilde{d}^{c,n}]$ but the system solved with each iteration (of statute rather than of stress activates) always uses formula $[\tilde{d}^{c,n}]$ Because of

these modifications, a fixed convergence criterion of point is added: the contact algorithm is converged only when the number of active connections of an iteration of Newton to the other is the same one. It is the third point which makes it possible "to keep" the memory of active connections of an iteration of Newton to the other. Here

the algorithm used: computation

Ini	$\delta u_0^n = \delta \tilde{u}^n$ of If $\left\| [d^{c,n-1}] \right\|_{\Xi^1}$			
	first iteration of Newton after a pairing (geometrical reactualization) Evaluating			
		of formula $\Xi_0^c = \left\{ J \in \Xi^1 \,\middle|\, \left\| [d^{c,n-1}] \right\|_J < 0 \right\}$ formula $[A_0^c]$		
	formula			
		$\Xi_0^c \leftarrow \Xi^{c,n-1}$ formula $[A_0^c] \leftarrow [A^{c,n-1}]$		
B_k	on the statutes formulates $k=1, Iter_{max}$			
	$[\delta_{k-1}] = [\delta \tilde{u}^n] - [\delta u_{k-1}^n]$ $\Xi_k^c \leftarrow \Xi_{k-1}^c$ formulates $[A_k^c] \leftarrow [A_{k-1}^c]$			
	formula $\Xi_k^c \neq \varnothing$			
		of formula $[K_k]_{schur} = [A_k^c] . [K]^{-1} . [A_k^c]^T$		
		of formula $\left\| [K_k]_{schur} \right\|_{\Xi_k^c}$		
		of formula $\left\| [K_k]_{schur} . [\mu_i^c] = [\tilde{d}^{c,n}] \right\|_{\Xi_k^c} \rightarrow \left\| [\mu_i^c] \right\|_{\Xi_k^c}$		
		$[\delta_k] = [\delta_{k-1}] - [K]^{-1} . [A_k^c]^T . [\mu_i^c]$		

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		$\left[J_{min},\bar{\rho}\right]=IsPetit\left(\Xi_k^c\right)$	
		$\left[\delta u_k^n\right]=\left[\delta u_{k-1}^n\right]+\bar{\rho}.\left[\delta_k\right]$	
		formula $\bar{\rho}<1$	
			$\Xi_k^c \leftarrow \Xi_k^c + \left[J_{min}\right]$
		Fine	
			Goto formulates F
B_k	formula $k=k+1$		
F	formulates		
	$\left[J_{max}\right]=IsGrand\left(\Xi_k^c\right)$		
	formula $\left[J_{max}\right]=0$		
		the algorithm converged If not	
		the algorithm did not converge (one will need an iteration of Newton moreover) Computation	
	of formula $\left[L_i^c\right]=\left[A^c\right]^T.\left[\mu_i^c\right]$		

the algorithm of the active stresses, it does not have there the result theoretical one of convergence in a finished number of iterations of contact.

5.3 Method penalized in contact – ALGORITHME `_CONT='`
`PENALIZATION'` Balances

5.3.1 structure in the presence of contact One

recalls that the balance equation in the presence of contact is written: in the case of

$$\left[L_i^{int}(u_i)\right]=\left[L_i^{ext}(u_i)\right]-\left[L_i^c\right]$$
137

the penalized contact, the force of contact is written: formulate

$$\left[L_i^c\right]=\left[L_i^{c,n}\right]_{regu}=E_N.\left[A^c\right]^T.\left(\left[A^c\right].\left[\delta u^n\right]-\left[d^{c,n-1}\right]\right)$$
138

linearization of the balance equation (137 137 , one introduces the tangent matrix formulates $\left[K^{m,n-1}\right]$ will contain the contributions resulting from the linearization from the internal forces and outsides and the matrix formulates $\left[K^{c,n-1}\right]$ the forces of contact: formulate

$$\left[K_i^{c,n-1}\right]=E_N.\left[A^{c,n-1}\right]^T.\left[A^{c,n-1}\right]$$
139

the second being worth member: formulate

$$\left[L_i^{c,n-1}\right]=-E_N.\left[A^{c,n-1}\right]^T.\left[d^{c,n-1}\right]$$
140

the algorithm, we saw with the § 4.3.6.2 4.3.6.2 it acts as correction of the problem of equilibrium without contact. When

we are in the contact algorithm, with the iteration of Newton formulates n we evaluate formula $\left[K_i^{c,n+1}\right]$

formula $\left[L_i^{c,n}\right]$ which will be used only with the following iteration of Newton. There is a shift of an iteration: one needs necessarily at least two iterations of Newton to solve a problem with penalized contact . Algorithm

5.3.2 Computation

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Ini	of Evaluating $\left\|\left[\tilde{\boldsymbol{d}}^{c,n}\right]\right\|_{\Xi^1}$
	of formula $\Xi^c = \left\{ J \in \Xi^1 \middle\| \left[\tilde{\boldsymbol{d}}^{c,n}\right]\middle|_J < 0 \right\}$ formula $\left[A^{c,n}\right]$
	formulates $\left\|\left[\boldsymbol{\mu}^{c,n}\right]\right\|_{\Xi^c} = -E_N . \left\|\left[\tilde{\boldsymbol{d}}^{c,n}\right]\right\|_{\Xi^c}$
	member formulates $\left[\boldsymbol{L}_i^{c,n}\right] = \left[A^{c,n}\right]^T . \left\|\left[\boldsymbol{\mu}^{c,n}\right]\right\|_{\Xi^c}$
	makes $\left[\boldsymbol{K}^{c,n+1}\right] = E_N . \left[A^{c,n}\right]^T . \left[A^{c,n}\right]$

•The quantities formulates $\left[K_i^c\right]$ formula $\left[L_i^c\right]$ are evaluated only for active connections. For the rest of the unknowns, one initializes to zero;
•The modification of system N" does not introduce new variables compared to the problem without contact; There
•is no index in k the various quantities (in particular all active connections or the contact matrix) because it is not an iterative algorithm; For
•the computation of the matrix, one proceeds in two times: evaluatings of the elementary matrixes, then assembly. But these two operations are specific with the discrete contact, one does not use the mechanism elementary computations/generic assembly of Code_Aster.
•The choice of the coefficient of penalization is crucial: too much weak, of the interpenetrations will be observed, too extremely, the conditioning of the tangent matrix is degraded. Algorithmic

6   resolution – Contact with friction There

exist four distinct algorithms of the contact to treat the case with friction of Coulomb:
•Completely regularized method: contact and friction are penalized ALGORITHME _CONT=' PENALIZATION'
and ALGORITHME _FROT=' PENALIZATION';
•Partially regularized method: dualized contact and friction penalized ALGORITHME _CONT=' LAGRANGIEN'
and ALGORITHME _FROT=' PENALIZATION';
•Method completely dualized in 2D: contact and friction are dualisés ALGORITHME _CONT=' LAGRANGIEN'
and ALGORITHME _FROT=' LAGRANGIEN';
•Method completely dualized in 3D: contact and friction are dualisés ALGORITHME _CONT=' LAGRANGIEN'
and ALGORITHME _FROT=' LAGRANGIEN'. Connections

6.1   of friction One

adds two sets compared to those defined in the §5.15.1 to take into account the statute of friction of a slave
node: formula

* Ξ^{a} all the adherent nodes of contact; formula
* Ξ^{g} all the slipping nodes of contact. One

thus has the following relations between the sets: formulate

* $\Xi^{\mathrm{c}}=\Xi^{\mathrm{a}}\oplus\Xi^{\mathrm{g}}$ the nodes in contact are either slipping, or adherent; formulate
* $\Xi^{\mathrm{a}}\subseteq\Xi^{\mathrm{c}}$ formulates $\Xi^{\mathrm{g}}\subseteq\Xi^{\mathrm{c}}$ only the nodes in contact can be adherent or slipping. Kinematics

6.2   and tangent clearance We

saw in the § 4.3 4.3.1 friction uses kinematical quantities similar to the case of the contact alone, by introducing
the notion of "tangent" clearance (sliding $\left[\tilde{g}^{\mathrm{t},n}\right]$ relative of surfaces in the case of non-adhesion), this tangent
clearance is defined by: formulate

$$\left[\tilde{g}^{\mathrm{t},n}\right]=\left[A^{\mathrm{f}}\right].\left(\left[\Delta u_i^{n-1}\right]+\left[\delta\tilde{u}^n\right]\right) \qquad\qquad 141$$

uses the matrix of friction (see $\left[A^{\mathrm{f}}\right]$ § 3.5 3.5.2 The use of the quantity formulates $\left[\delta\tilde{u}^n\right]$ formula $\left[\delta u^n\right]$ is
justified owing to the fact that one solves a problem of Tresca, therefore without modifying clearance during the
process of resolution of the problem of sliding (see § 4.3 4.3.4 Note:

In
•the algorithms presented in the continuation of the document, one generally does not make distinction
between the slipping part and the adherent part of the nodes on the level them writings, to reduce the notations.

6.3   The methods dualized in contact and friction Balances
6.3.1   structure in the presence of contact/friction One

recalls that the balance equation in the presence of contact and of friction is written: formulate

$$\left[L_i^{\mathrm{int}}(u_i)\right]=\left[L_i^{\mathrm{ext}}(u_i)\right]-\left[L_i^{\mathrm{c}}(u_i)\right]-\left[L_i^{\mathrm{f}}(u_i)\right] \qquad\qquad 142$$

linearization of the balance equation (142 142 , one introduces the tangent matrix formulates $\left[K^{\mathrm{m},n-1}\right]$ will
contain the contributions resulting from the linearization from the internal forces and outsides, the matrix
formulates $\left[K^{\mathrm{c},n-1}\right]$ the forces of contact and the matrix formulates $\left[K^{\mathrm{f},n-1}\right]$ the frictional forces. One finds
formula $\left[\delta\tilde{u}^n\right]$ the increment of solution of the problem of equilibrium to the iteration formulates n without
application of the model of contact/friction, by solving the following system: formulate

$$\left(\left[K^{\mathrm{m},n-1}\right]+\left[K^{\mathrm{c},n-1}\right]+\left[K^{\mathrm{f},n-1}\right]\right).\left[\delta\tilde{u}^n\right]=\left[L_i^{\mathrm{ext},n-1}\right]-\left[L_i^{\mathrm{int},n-1}\right]-\left[L_i^{\mathrm{c},n-1}\right]-\left[L_i^{\mathrm{f},n-1}\right] \qquad 143$$

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saw that the forces of contact in the dualized case do not depend on displacement (see § 4.3 4.3.5 . One thus has. $\left[K^{\mathrm{c},n-1}\right]=0$ However, for friction, even in the dualized case, the forces of sliding make a non-zero contribution, but only in 3D (see § 4.3.5.3 4.3.5.3 : formulate

$$\left[K^{\mathrm{f},n-1}\right]=\left[K^{\mathrm{g},n-1}\right]_{3D}=\left[K^{\mathrm{g},n}\right]_{2D} \quad \text{formulates} \quad \left[K^{\mathrm{f},n-1}\right]=\left[K^{\mathrm{g},n-1}\right]_{2D}=0 \tag{144}$$

enables us to express the equilibrium of structure with taking into account of the forces of contact friction : formulate

$$\left[K\right].\left\{\delta \tilde{u}^n\right\}=\left\{F\right\} \tag{145}$$

: formulate

$$\left[K\right]=\left(\left[K^{\mathrm{m},n-1}\right]+\left[K^{\mathrm{f},n-1}\right]\right) \tag{146}$$

: formulate

$$\left\{F\right\}=\left\{L_i^{\mathrm{ext},n-1}\right\}-\left\{L_i^{\mathrm{int},n-1}\right\}-\left\{L_i^{\mathrm{c},n-1}\right\}-\left\{L_i^{\mathrm{f},n-1}\right\} \tag{147}$$

the solution formula $\left\{\tilde{u}_i^n\right\}$ is obtained after equilibrium and before application of the conditions of contact/friction. She is written: formulate

$$\left\{\tilde{u}_i^n\right\}=\left\{u_{i-1}\right\}+\left\{\Delta u_i^{n-1}\right\}+\left\{\delta \tilde{u}^n\right\} \tag{148}$$

to obtain the solution formulates $\left\{\tilde{u}_i^n\right\}$ did not apply the laws of contact/friction (model of Signorini/Coulomb) for the current iteration of Newton formula n , the reactions of contact/friction calculated with the iteration of Newton preceding formula $n-1$ are well taken into account in formula $\left\{F\right\}$ To completely solve the problem of equilibrium with contact/friction, with the iteration formulates n is necessary to apply the law of Signorini, which is expressed by the following system: formulate

$$\begin{cases} \left[A^{\mathrm{c}}\right].\left\{u_i^n\right\}\le\left\{d_{\mathrm{ini}}^{\mathrm{c}}\right\} & (a) \\ \left\{\mu_i^{\mathrm{c}}\right\}\ge 0 & (b) \\ \left\langle\mu_i^{\mathrm{c}}\right\rangle.\left(\left[A^{\mathrm{c}}\right].\left\{u_i^n\right\}\right)=0 & (c) \end{cases} \tag{149}$$

the friction law: formulate

$$\begin{cases} \left\|\left\{r\right\}\right\|-\mu.\left\|\left\{\mu_i^{\mathrm{c}}\right\}\right\|\le 0 & (a) \\ \left[A^{\mathrm{g}}\right].\left(\left\{\Delta u_i^{n-1}\right\}+\left\{\delta u^n\right\}\right)=\lambda.\left\{\mu_i^{\mathrm{g}}\right\} & (b) \\ \lambda.\left(\left\|\left\{r\right\}\right\|-\mu.\left\|\left\{\mu_i^{\mathrm{c}}\right\}\right\|\right)=0 & (c) \\ \lambda\ge 0 & (d) \end{cases} \tag{150}$$

the application of the model of Signorini/Coulomb will modify the displacement increment formulates $\left\{\delta \tilde{u}^n\right\}$ becomes formula $\left\{\delta u^n\right\}$ from where the solution obtained after equilibrium and *application* of the model of contact friction which S" written: formulate

$$\left\{u_i^n\right\}=\left\{u_{i-1}\right\}+\left\{\Delta u_i^{n-1}\right\}+\left\{\delta u^n\right\} \tag{151}$$

6.3.2   reduced to active connections We

will write the system allowing to completely solve the problem of equilibrium with taking into account of the model of Signorini/Coulomb. The way transform the inequalities of the system (6.3 6.3.1 into equalities was exposed in the § 5.2 5.2.2 one does not return above. It is pointed out simply that the resulting system to solve is: formulate

$$\begin{cases} \left[K\right].\left\{\delta u^n\right\}+\left[A^{\mathrm{c}}\right]^T.\left\{\mu_i^{\mathrm{c}}\right\}=\left\{F\right\} \\ \left[\left[A^{\mathrm{c}}\right].\left\{\delta u^n\right\}=\left\{d^{\mathrm{c},n-1}\right\}\right]_{\Xi^{\mathrm{c}}} \end{cases} \tag{152}$$

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an equivalent way, one supposes to know adherent connections. The inequality (150 150 is transformed then into equality and the system to be solved is the following: formulate

$$\begin{cases} [K].\{\delta u^n\}+[A^a]^T.\{\mu_i^a\}=\{F\} \\ [[A^a].\{\delta u^n\}=\{\tilde{g}^{t,n}\}]_{\Xi^a} \end{cases}$$

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solves the two systems simultaneously: formulate

$$\begin{cases} [K].\{\delta u^n\}+[A^c]^T.\{\mu_i^c\}+[A^a]^T.\{\mu_i^a\}=\{F\} \\ [[A^c].\{\delta u^n\}=\{d^{c,n-1}\}]_{\Xi^c} \\ [[A^a].\{\delta u^n\}=\{\tilde{g}^{t,n}\}]_{\Xi^a} \end{cases}$$

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what one writes in a compact way: formulate

$$\begin{cases} [K].\{\delta u^n\}+[A^{c+a}]^T.\{\mu_i^{c+a}\}=\{F\} \\ [[A^{c+a}].\{\delta u^n\}=\{d^{c+a}\}]_{\Xi^{c+a}} \end{cases}$$

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the vector formula $\{d^{c+a}\}$ and $\{d^{c,n-1}\}$ $\{\tilde{g}^{t,n}\}$ the vector formula $\{\mu_i^{c+a}\}$ formula $\{\mu_i^c\}$ formula $\{\mu_i^a\}$ Writing

6.3.3 of the iterative problem One

leaves the increment obtained without treating the contact/friction formulates $\{\delta\tilde{u}^n\}$ one carries out the iterations of statutes formulates k "with clean convergence of this algorithm. Convergence within the meaning of the statutes is obtained lorsqu" no connection does not violate the kinematical conditions and when the associated Lagrange multipliers are good sign (systems (149 149 and (147 147 The starting solution without correction of the contact/friction is and $\{\delta\tilde{u}^n\}$ the increment added by the new iteration is. $\{\delta_k\}$ One notes: formulate

$$\begin{cases} \{\delta u_k^n\}=\{\delta\tilde{u}^n\}+\{\delta_k\} \\ \{\delta u_0^n\}=\{\delta\tilde{u}^n\} \\ \{\delta_0\}=\{0\} \\ \{\delta u_k^n\}=\{\delta u_{k-1}^n\}+\{\delta_k\} \end{cases}$$

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places itself at the iteration of contact formulates k One seeks to solve the system (155 155 : just like

$$\begin{cases} [K].\{\delta u_k^n\}+[A^{c+a}]^T.\{\mu_i^{c+a}\}=\{F\} \\ [[A^{c+a}].\{\delta u_k^n\}=\{d^{c+a}\}]_{\Xi^{c+a}} \end{cases}$$

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the multiplier of Lagrange of the contact formula $\{\mu_i^c\}$ the Lagrange multipliers of friction formula $\{\mu_i^a\}$ are not solved in an incremental way but in a total way (on time step). While injecting (156 156 into this system, we obtain a system similar to that obtained in the § 5.2.3.1 5.2.3.1 do not remake the demonstrations, it is equivalent). Finally, the Lagrange multipliers formulates $\{\mu_i^c\}$ formulates $\{\mu_i^a\}$ connections of contact and of friction are solutions of the following system: formulate

$$\left[-[A_k^{c+a}].[K]^{-1}.[A_k^{c+a}]^T.\{\mu_i^{c+a}\}=\{\tilde{d}^{c+a,n}\}\right]_{\Xi_k^{c+a}}$$

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the vector formula $\{\mu_i^{c+a}\}$ formula $\{\mu_i^c\}$ formula $\{\mu_i^a\}$ the vector formula $\{\tilde{d}^{c+a,n}\}$ and $\{\tilde{d}^{c,n}\}$ 9Le $-\{\tilde{g}^{t,n}\}$

[9] One can finally calculate the displacement increments formulates $\{\delta_k\}$: formulate

$$\{\delta_k\}=-\sum_{l=0}^{k-1}\{\delta_l\}-[K]^{-1}.[A_k^{c+a}]^T.\{\mu_i\}^{c+a}$$

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---

9 sign comes $-$ owing to the fact that the initial tangential sliding from two surfaces null east, by definition.

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the strategies of reduction of the times computing developed in the case without friction (see § 5.2 5.2.2 apply to the case with friction.

The method dualized in contact/friction has two alternatives:
–The alternative 2D which does not understand any modification of the tangent matrixes;
–The alternative 3D which modifies the tangent matrix of the system; Indeed
, we saw in the § 4.3.5.3 4.3.5.3 3D case, because of NON-knowledge of the direction of friction in the tangent plane, a matrix during the linearization of the forces of sliding introduced.
The algorithms presented here are an extension to the case of the friction of the methods of statute: one applies the state of contact of connections and the state of friction (sliding dependancy/), one calculates a state, one checks the assumptions and one buckles as much as the state is not stabilized. As in the LAGRANGIAN case `without` friction, one does not have theoretical convergence of results. Moreover, these algorithms are not very robust and often expensive. One recommends CONTINUE to use rather the formulation `of the` `contact` /friction (see [R5.03.52]), or, in last spring, the versions regularized (PENALIZATION) . Checking

6.3.4   of the sliding joints During

the application of the algorithm of the statutes to friction, it is necessary to evaluate slipping supposed connections so are it really. Two strategies are used: In
–2D, the checking is done during iterations of contact; In
–3D, the checking is made when connections of contact were established and that the algorithm converged. We saw in the § 4.3.5.3 4.3.5.3 3D case, because of NON-knowledge of the direction of friction in the tangent plane, a point fixes was introduced in the form of a computation on the threshold of Tresca. The checking of the sliding joints corresponds to question to know if Ξ^a is not too small.
The algorithm used is presented: If

formula $\Xi_k^g \neq \emptyset$			
	on connections formulates $J \in \Xi_k^c$		
		of formula $\alpha_J = \left\lbrack \left\lbrack A_k^a \right\rbrack . \left(\left\lbrack \Delta u_i^{n-1} \right\rbrack + \left\lbrack \delta u_k^n \right\rbrack \right) \right\rbrack_J$	
		formula $\alpha_J < 0$	
			$\rho_J = \dfrac{\left\lbrack \left\lbrack d^{c,n-1} \right\rbrack - \left\lbrack A_k^c \right\rbrack . \left\lbrack \delta u_k^n \right\rbrack \right\rbrack_J}{\alpha_J}$
			$\bar{\rho} = \min \left(\rho_J, \bar{\rho} \right)$
			$J_{min} = J$
	Buckles In		

output of this algorithm, there will be the number of the connection which violates more the condition of nonpenetration formulates J_{min} the value formulates $\bar{\rho}$ 2D $\left\lbrack J_{min}, \bar{\rho} \right\rbrack = IsPetit\left(\Xi_k^c \right)$

6.3.5   Case Just like

in the Lagrangian case without friction, when it is possible (not reactualization of pairing), one preserves the memory of active connections of contact (group formulates Ξ^c but also connections of friction 10 [10] formulates Ξ^f . formula

Ini	$\delta u_0^n = \delta \tilde{u}^n$
	of and $\left\lbrack d^{c,n-1} \right\rbrack_{\Xi^l}$ formulates $\left\lbrack \tilde{g}^{t,n} \right\rbrack_{\Xi^l}$ formula $\left\lbrack \tilde{d}^{c+a,n} \right\rbrack$

---

10 , one rather preserves adherent connections but Ξ^a like one A. $\Xi^f = \Xi^c = \Xi^g \cup \Xi^a$ (group

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			first iteration of Newton after a pairing (geometrical reactualization) Evaluating
			of formula $\Xi^c_0=\left[J\in\Xi^1\middle|\left[\left[d^{c,n-1}\right]\right]_J<0\right]$ formula $\left[A^c_0\right]$ formula $\left[A^f_0\right]$
			connections are supposed to be adherent: formulate $\Xi^a_0=\Xi^c_0$ formulates $\Xi^g_0=\emptyset$
	formulates		
			$\Xi^c_0\leftarrow\Xi^{c,n-1}$ formulates $\left[A^c_0\right]\leftarrow\left[A^{c,n-1}\right]$ $\Xi^f_0\leftarrow\Xi^{f,n-1}$ formula $\left[A^f_0\right]\leftarrow\left[A^{f,n-1}\right]$
B_k	on the statutes formulates $k=1,Iter_{max}$		
			$\left[\delta_{k-1}\right]=\left[\delta\tilde{u}^n\right]-\left[\delta u^n_{k-1}\right]$ $\Xi^c_k\leftarrow\Xi^c_{k-1}$ formulates $\left[A^c_k\right]\leftarrow\left[A^c_{k-1}\right]$ $\Xi^f_k\leftarrow\Xi^f_{k-1}$ formula $\left[A^f_k\right]\leftarrow\left[A^f_{k-1}\right]$
		formula $\Xi^c_k\neq\emptyset$	
			of formula $\left[K_k\right]_{\text{schur}}=\left[A^{c+a}_k\right].\left[K\right]^{-1}.\left[A^{c+a}_k\right]^T$
			of formula $\left[\left[K_k\right]_{schur}\right]_{\Xi^{c+a}_k}$
			of formula $\left[\left[K_k\right]_{schur}.\left[\mu^{c+a}_i\right]=\left[d^{c+a}\right]\right]_{\Xi^{c+a}_k}\rightarrow\left[\left[\mu^{c+a}_i\right]\right]_{\Xi^{c+a}_k}$
			$\left[\delta_k\right]=\left[\delta_{k-1}\right]-\left[K\right]^{-1}.\left[A^{c+a}_k\right]^T.\left[\mu^{c+a}_i\right]$
		$\left[J_{min},\bar{\rho}\right]=IsPetit\left(\Xi^c_k\right)$	
		$\left[\delta u^n_k\right]=\left[\delta u^n_{k-1}\right]+\bar{\rho}.\left[\delta_k\right]$	
		formula $\bar{\rho}<1$	
			$\Xi^c_k\leftarrow\Xi^c_k+\left[J_{min}\right]$
		Fine	
			Goto formulates F
B_k	formula $k=k+1$		
F	formulates		
	$\left[J_{max}\right]=IsGrand\left(\Xi^c_k\right)$		
	formula $\left[J_{max}\right]=0$		
		the algorithm converged If not	
		the algorithm did not converge (one will need an iteration of Newton moreover) Computation	
	of formula $\left[L^c_i\right]=\left[A^c\right]^T.\left[\mu^c_i\right]$		

6.4 the method penalized in contact and in friction

6.4.1 structure in the presence of contact One Balances

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points out only the balance equation in the presence of contact is written: in the case of

$$\left[\boldsymbol{L}_i^{\text{int}}(\boldsymbol{u}_i)\right]=\left[\boldsymbol{L}_i^{\text{ext}}(\boldsymbol{u}_i)\right]-\left[\boldsymbol{L}_i^{\text{c}}\right] \qquad 160$$

the penalized contact, the force of contact is written: formulate

$$\left[\boldsymbol{L}_i^{\text{c}}\right]=\left[\boldsymbol{L}_i^{\text{c},n}\right]_{\text{regu}}=E_N.\left[\boldsymbol{A}^{\text{c}}\right]^T.\left(\left[\boldsymbol{A}^{\text{c}}\right].\left(\delta\boldsymbol{u}^n\right)-\left(\boldsymbol{d}^{\text{c},n-1}\right)\right) \qquad 161$$

linearization of the balance equation (160 160 , one introduces the tangent matrix formulates $\left[\boldsymbol{K}^{\text{m},n-1}\right]$ will contain the contributions resulting from the linearization from the internal forces and outsides and the matrix formulates $\left[\boldsymbol{K}^{\text{c},n-1}\right]$ the forces of contact: formulate

$$\left[\boldsymbol{K}_i^{\text{c},n-1}\right]=E_N.\left[\boldsymbol{A}^{\text{c},n-1}\right]^T.\left[\boldsymbol{A}^{\text{c},n-1}\right] \qquad 162$$

the second being worth member: formulate

$$\left[\boldsymbol{L}_i^{\text{c},n-1}\right]=-E_N.\left[\boldsymbol{A}^{\text{c},n-1}\right]^T.\left[\boldsymbol{d}^{\text{c},n-1}\right] \qquad 163$$

the algorithm, we saw with the § 4.3.6.2 4.3.6.2 it acts as correction of the problem of equilibrium without contact. When

we are in the contact algorithm, with the iteration of Newton formulates n we evaluate formula $\left[\boldsymbol{K}_i^{\text{c},n+1}\right]$ formula $\left[\boldsymbol{L}_i^{\text{c},n}\right]$ which will be used only with the following iteration of Newton. There is a shift of an iteration: one needs necessarily at least two iterations of Newton to solve a problem with penalized contact . Algorithm

6.4.2   This

method is simplest to implement. We saw in § 4.3.5.14.3.5.1 the regularization of the conditions of contact adds two new contributions: formulate

- $\left[\boldsymbol{K}_i^{\text{c}}\right]$ the total tangent matrix; formulate

- $\left[\boldsymbol{L}_i^{\text{c}}\right]$ the second member. For

friction, it is the same for the conditions of dependancy (§ 4.3.5.2 4.3.5.2 : formulate

- $\left[\boldsymbol{K}_i^{\text{a}}\right]$ the total tangent matrix; formulate

- $\left[\boldsymbol{L}_i^{\text{a}}\right]$ the second member. But

also for the conditions of sliding (§ 4.3.5.3 4.3.5.3 formulate

- $\left[\tilde{\boldsymbol{K}}_\theta^{\text{g}}\right]$ the total tangent matrix; formulate

- $\left[\boldsymbol{L}_i^{\text{g}}\right]$ the second member. Concerning

the algorithm, we saw with the § 4.3.6.2 4.3.6.2 it acts as correction of the problem of equilibrium without contact friction. When we are in the algorithm of contact friction, with the iteration of Newton formulates n we evaluate the quantities imposing the conditions of contact/friction which will be used that with the following iteration of Newton. What gives us the following algorithm, with formula n the iteration of Newton: formulate

Ini	of formula $\left[\left(\tilde{\boldsymbol{d}}^{\text{c},n}\right)\right]_{\Xi^{\text{l}}}$ formula $\left[\left(\tilde{\boldsymbol{g}}^{\text{t},n}\right)\right]_{\Xi^{\text{l}}}$		
	of formula $\Xi^{\text{c}}=\left\{J\in\Xi^{\text{l}}\left|\left[\left(\tilde{\boldsymbol{d}}^{\text{c},n}\right)\right]_J<0\right.\right\}$		
	formulates $\left[\left(\boldsymbol{\mu}^{\text{c},n}\right)\right]_{\Xi^{\text{c}}}=-E_N.\left[\left(\tilde{\boldsymbol{d}}^{\text{c},n}\right)\right]_{\Xi^{\text{c}}}$		
	member formulates $\left[\boldsymbol{L}_i^{\text{c},n}\right]=\left[\boldsymbol{A}^{\text{c},n}\right]^T.\left[\left(\boldsymbol{\mu}^{\text{c},n}\right)\right]_{\Xi^{\text{c}}}$		
	formula $\Xi^{\text{c}}\neq\emptyset$		
		formulates $\left[\boldsymbol{K}^{\text{c},n+1}\right]=E_N.\left[\boldsymbol{A}^{\text{c},n}\right]^T.\left[\boldsymbol{A}^{\text{c},n}\right]$	
		of the norm D U sliding formulates $\left\|\left\|\left(\tilde{\boldsymbol{g}}^{\text{t},n}\right)\right\|\right\|_{\Xi^{\text{c}}}$	

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		of formula $\big[\big[\boldsymbol{\lambda}^{\mathrm{f}}\big]\big]_{\varXi^{\mathrm{c}}}=max\big(0,\mu\,.\,\big[\big[\boldsymbol{\mu}^{\mathrm{c},n}\big]\big]_{\varXi^{\mathrm{c}}}\big)$
		of formula $\big[\big[\boldsymbol{a}\big]\big]_{\varXi^{\mathrm{c}}}=\bigg[\bigg(\big\|\big[\tilde{\boldsymbol{g}}^{\mathrm{t},n}\big]\big\|-\frac{1}{E_T}\,.\,\big[\boldsymbol{\lambda}^{\mathrm{f}}\big]\bigg)\bigg]_{\varXi^{\mathrm{c}}}$
		of formula $\varXi^{\mathrm{a}}=\big\{J\in\varXi^{\mathrm{c}}\big|\big[\big[\boldsymbol{a}\big]\big]_J\leq0\big\}$
		of formula $\varXi^{\mathrm{g}}=\big\{J\in\varXi^{\mathrm{c}}\big|\big[\big[\boldsymbol{a}\big]\big]_J>0\big\}$
		N œud S adherent S formulates $\big[\big[\boldsymbol{\mu}^{\mathrm{f},n}\big]\big]_{J\in\varXi^{\mathrm{a}}}=\sqrt{E_T}$
		N œud S slipping S formulates $\big[\big[\boldsymbol{\mu}^{\mathrm{f},n}\big]\big]_{J\in\varXi^{\mathrm{g}}}=\sqrt{\dfrac{\big[\big[\boldsymbol{\lambda}^{\mathrm{f}}\big]\big]_J}{\big[\big\|\big[\tilde{\boldsymbol{g}}^{\mathrm{t},n}\big]\big\|\big]_J}}$
		vector formulates $\big[\big[\boldsymbol{v}\big]\big]_{\varXi^{\mathrm{c}}}=\big[\boldsymbol{A}^{\mathrm{f},n}\big]^T\,.\,\big[\boldsymbol{\mu}^{\mathrm{f},n}\big]_{\varXi^{\mathrm{c}}}$ formulates $\big[\big[\boldsymbol{v}\big]\big]_{\varXi^{\mathrm{nc}}}=0$
		first left matrix friction formulates $\big[\boldsymbol{K}^{\mathrm{f}_1}\big]=\big[\boldsymbol{v}\big]\,.\,\langle\boldsymbol{v}\rangle$
		of the second member of friction formulates $\big[\boldsymbol{L}_i^{\mathrm{f},n}\big]=\big[\boldsymbol{K}^{\mathrm{f}_1}\big]\,.\,\big(\big[\Delta\,\boldsymbol{u}_i^{n-1}\big]+\big[\delta\,\tilde{\boldsymbol{u}}^n\big]\big)$
		of the norm D U sliding formulates $\big\|\big[\tilde{\boldsymbol{g}}^{\mathrm{t},n}\big]\big\|_{\varXi^{\mathrm{c}}}$
		of formula $\big[\big[\boldsymbol{\lambda}^{\mathrm{f}}\big]\big]_{\varXi^{\mathrm{c}}}=max\big(0,\mu\,.\,\big[\big[\boldsymbol{\mu}^{\mathrm{c},n}\big]\big]_{\varXi^{\mathrm{c}}}\big)$
		of formula $\big[\big[\boldsymbol{a}\big]\big]_{\varXi^{\mathrm{c}}}=\bigg[\bigg(\big\|\big[\tilde{\boldsymbol{g}}^{\mathrm{t},n}\big]\big\|-\frac{1}{E_T}\,.\,\big[\boldsymbol{\lambda}^{\mathrm{f}}\big]\bigg)\bigg]_{\varXi^{\mathrm{c}}}$
		of formula $\varXi^{\mathrm{a}}=\big\{J\in\varXi^{\mathrm{c}}\big|\big[\big[\boldsymbol{a}\big]\big]_J\leq0\big\}$
		of formula $\varXi^{\mathrm{g}}=\big\{J\in\varXi^{\mathrm{c}}\big|\big[\big[\boldsymbol{a}\big]\big]_J>0\big\}$
		N œud S adherent S formulates $\beta_{J\in\varXi^{\mathrm{a}}}=0$
		N œud S slipping S formulates $\beta_{J\in\varXi^{\mathrm{c}}}=\sqrt{\dfrac{1}{\big[\big[\boldsymbol{\lambda}^{\mathrm{f}}\big]\,.\,\big\|\big[\tilde{\boldsymbol{g}}^{\mathrm{t},n}\big]\big\|\big]_{J\in\varXi^{\mathrm{c}}}}}$
		of formula θ
		vector formulates $\big[\big[\boldsymbol{w}\big]\big]_{\varXi^{\mathrm{c}}}=\sqrt{\theta}\,.\,\big[\beta_{J\in\varXi^{\mathrm{c}}}\,.\,\big[\boldsymbol{A}^{\mathrm{f},n}\big]\,.\,\big[\boldsymbol{L}_i^{\mathrm{f},n}\big]\big]_{\varXi^{\mathrm{c}}}$ formulates $\big[\big[\boldsymbol{w}\big]\big]_{\varXi^{\mathrm{nc}}}=0$
		second part stamps friction formulates $\big[\boldsymbol{K}^{\mathrm{f}_2}\big]=\big[\boldsymbol{w}\big]\,.\,\langle\boldsymbol{w}\rangle$
		Matrix formulates $\big[\boldsymbol{K}^{\mathrm{f}}\big]=\big[\boldsymbol{K}^{\mathrm{f}_1}\big]-\big[\boldsymbol{K}^{\mathrm{f}_2}\big]$

uses not bad tricks of computation in this algorithm. One gives a value different of formula $\big[\boldsymbol{\mu}^{\mathrm{f}}\big]$ the dependancy and the sliding but one uses the same vector formulates $\big[\boldsymbol{v}\big]$ makes it possible to find the two matrixes (sliding and dependancy). Indeed, for the dependancy, one a: formulates

$$\big[\big[\boldsymbol{K}^{\mathrm{f}_1}\big]\big]_{\varXi^{\mathrm{a}}}=\big[\boldsymbol{v}\big]\,.\,\langle\boldsymbol{v}\rangle=\big[\boldsymbol{A}^{\mathrm{f}}\big]^T\,.\,\sqrt{E_T}\,.\,\sqrt{E_T}\,.\,\big[\boldsymbol{A}^{\mathrm{f}}\big] \qquad 164$$

thus finds the form of the matrix of dependancy (79 79 for the adherent nodes: formulate

$$\big[\big[\boldsymbol{K}^{\mathrm{f}_1}\big]\big]_{\varXi^{\mathrm{a}}}=\big[\boldsymbol{K}_i^{\mathrm{a}}\big]=E_T\,.\,\big[\boldsymbol{A}^{\mathrm{a}}\big]^T\,.\,\big[\boldsymbol{A}^{\mathrm{a}}\big] \qquad 165$$

the sliding: formulate

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$$\left\|\left[\boldsymbol{K}^{f_1}\right]\right\|_{\Xi^g} = \left(\boldsymbol{v}\right).\left\langle\boldsymbol{v}\right\rangle = \left[\boldsymbol{A}^f\right]^T.\sqrt{\frac{\left(\boldsymbol{\lambda}^f\right)}{\left\|\left(\tilde{\boldsymbol{g}}^{t,n}\right)\right\|}}.\sqrt{\frac{\left\langle\boldsymbol{\lambda}^f\right\rangle}{\left\|\left(\tilde{\boldsymbol{g}}^{t,n}\right)\right\|}}.\left[\boldsymbol{A}^f\right] \qquad 166$$

finds the statement of the first part of the matrix of sliding (97 97 for the slipping nodes forces: formulate

$$\left\|\left[\boldsymbol{K}^{f_1}\right]\right\|_{\Xi^g} = \frac{\left[\boldsymbol{A}^g\right]^T.\left[\boldsymbol{k}^{g,n}\right]}{\left\|\left(\boldsymbol{g}^{t,n}\right)\right\|} \qquad 167$$

the matrix of the thresholds of Tresca: formulate

$$\left[\boldsymbol{k}^g\right] = \left(\boldsymbol{\lambda}^f\right).\left\langle\boldsymbol{\lambda}^f\right\rangle \qquad 168$$

to find to it second part of the matrix of sliding, one uses the statement D U vector formulates $\left(\boldsymbol{w}\right)$ formulate

$$\left(\boldsymbol{w}\right) = \sqrt{\theta}.\beta.\left[\boldsymbol{A}^f\right].\left[\boldsymbol{K}^{f_1}\right].\left(\left(\Delta\boldsymbol{u}_i^{n-1}\right) + \left(\delta\tilde{\boldsymbol{u}}^n\right)\right) \text{ formula } \beta = \sqrt{\frac{1}{\left(\boldsymbol{\lambda}^f\right).\left\|\left(\tilde{\boldsymbol{g}}^{t,n}\right)\right\|}} \qquad 169$$

the form of the matrix formulates $\left\|\left[\boldsymbol{K}^{f_1}\right]\right\|_{\Xi^g}$ (166 166 and the definition of the tangential sliding in (87 87 the product (tensorial) of formula $\left(\boldsymbol{w}\right)$ him even makes it possible to find the second part of the matrix of sliding (97 97 for the slipping nodes forces: formulate

$$\left\|\left[\boldsymbol{K}^{f_2}\right]\right\|_{\Xi^g} = \left(\boldsymbol{w}\right).\left\langle\boldsymbol{w}\right\rangle = \theta.\frac{\left[\boldsymbol{A}^g\right]^T.\left[\boldsymbol{k}^{g,n}\right]}{\left\|\left(\boldsymbol{g}^{t,n}\right)\right\|}.\frac{\left(\boldsymbol{g}^{t,n}\right)\left\langle\boldsymbol{g}^{t,n}\right\rangle}{\left\|\left(\boldsymbol{g}^{t,n}\right)\right\|^2} \qquad 170$$

used the product (168 168 which explains the multiplier formulates β in the vector formulates $\left(\boldsymbol{w}\right)$ Note:

• The modification of system N" does not introduce new variables compared to the problem without contact/friction; There
• is no index formulates k the various quantities (in particular all active connections or matrixes of contact friction) because it is not an iterative algorithm; For
• the choice of formula θ it is the user who chooses it (parameter COEF _MATR_FROT), but this coefficient is put at zero as long as the residue of equilibrium (RESI_GLOB_RELA) is lower than formula 10^{-3} Theoretical

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7 results of convergence For

the problems without friction, one will find in [1 59 a demonstration of convergence for the method of the active stresses. For

the problems with friction, of the results of convergence with unicity of the solution to the discretized problem are established in [1 59 low values of the coefficient of kinetic friction of Coulomb. The results are got by means of an algorithm of point fixes associated with a method of Lagrange multipliers. For each problem of solved contact, one studies the problem of associated friction. Once this one solved, one solves a new problem of contact and so on. These methods are however different from those presented here and one cannot thus of results present theoretical convergence for these last.

The condition of fastening of the points which come in contact is particularly important to ensure the convergence of the method with Lagrange multipliers. Indeed when a point returns to the contact during reiterated its tangential displacement remains free. A condition of nonsliding would be far too constraining. The algorithm would oscillate then between two states with or without contact in examples of the type of that presented in [V6.04.105]. The point which is attached is thus regarded as free from the point of view of the sliding. One can then calculate the normal reaction as well as the tangential reaction by means of the assumption of sliding and one estimated of the displacement increment of initial sliding.

The use of the penalization makes it possible to avoid these oscillations while making it possible to slacken the stresses on the preceding system. Coupled with a method of Lagrange multipliers for the friction part, pathology announced is found above (oscillations of adherent threshold slipping/). Recutting

7.1 of time step On

the theoretical level, the convergence of the method of the active stresses is ensured in a finished number of iterations. In practice, certain numerical artefacts can make this convergence delicate. Also a strategy it was developed to ensure the robustness of the algorithm. During

computations of contact, in particular if the steps of load carried out are too large, of the undesirable phenomena can appear:
- The contact matrix is singular , Oscillation
- of the method of the active stresses: a node is detected alternatively "stuck" then "taken off".

To mitigate these difficulties, the following strategy was adopted. If:
- The contact matrix is singular,
- the nombre of iterations of active stresses is higher than a limit which depends amongst potential connections. This number is fixed at twice the nombre total of nodes slaves for the method of the active stresses, and at ITER_CONT_MULT `time` the nombre total of nodes slaves for the other methods. Then

one redécoupe time step the i.e *one returns* to the preceding step of load and instead of trying to reach the level of loading following in a step as one has just done it, one makes some several (For more precise details on this functionality of operator STAT_NON_LINE , `to see` documentation [U4.51.03]). Compatibility

7.2 with the boundary conditions of Dirichlet In the case of

the methods with Lagrange multipliers, one can observe incompatibilities with the fact of imposing boundary conditions of the Dirichlet type. Indeed, it is necessary that physically the problem has a meaning. One cannot deal with a problem of contact in the direction of the axis if all z the points have a following null displacement. As z we will see it, to deal such with a problem led to a singularity of the matrixes of the type formulates $\left[A^c\right].\left[K\right]^{-1}.\left[A^c\right]^T$ the processing of the boundary conditions of Dirichlet by double lagrange of Code_Aster . *Writing*

7.2.1 of the boundary conditions While

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taking as a starting point the documentation of reference [R5.03.01] of STAT_NON_LINE , `the dualisation` of the boundary conditions of Dirichlet results to the following system of equations in solving: formulate

$$\begin{cases} [K].\{\delta u\}+[B]^T.\{\delta\lambda\}=\{L_i^{\text{int}}\}-\{L_i^{\text{ext}}\} \\ [B].\{u_i\}=\{u_i^d\}-[B].\{u_{i-1}\} \end{cases}$$

One171

notes the stiffness matrix $\lceil C \rfloor$ of the system such as: (172)

$$[C]=\begin{bmatrix} [K] & [B]^T \\ [B] & [0] \end{bmatrix}$$

This172

matrix has a reverse of the form: (173)

$$[C]^{-1}=\begin{bmatrix} [E] & [F] \\ [F]^T & [G] \end{bmatrix}$$

such as173

: . One $[E].[B]^T=[0]$ checks thus that for each boundary condition one has l the property formulates $[E].[B_l]^T=[0]$

7.2.2 to the problem of contact the matrix

formula $[A^c].[K]^{-1}.[A^c]^T$ be also written formula $[A^c].[E].[A^c]^T$ the vectors of connection utilize $[A]$ only the degrees of freedom of displacement. It from of

• results whereas if a vector of connection of J the matrix formula $[A]$ a linear combination of the boundary conditions of the Dirichlet type it checks the following property: formulate $[E].[A_J]^T=[0]$ The matrix formula $[A^c].[E].[A^c]^T$ singular because it has a column of zeros. In practice, without particular processing, one finishes in the code on a message of stop of the standard stop ON MATRIX OF CONTACT-FRICTION SINGULAR. THE DETECTION of these singular columns was put in work in the code in order to eliminate from the relations of contact-friction this kind of relations and to avoid the stop previously described. It from of

• results whereas if a vector of connection of J the matrix formula $[A]$ a linear combination of the boundary conditions of the Dirichlet type and is written, it $[A_J]=\sum\alpha_i.[B_i]+[\overline{A}_J]$ checks the following property: formulate $[E].[A_J]^T=[E].[\overline{A}_J]^T$ then have a matrix formulates $[A^c].[E].[A^c]^T$ because it has two identical lines. This detection is not for the moment not available in the code and one finishes in the code on a message of stop of the standard stop on contact matrix - friction singular. Note:

This compatibility issue

between contact-friction and the boundary conditions does not appear with the regularized methods insofar as one adds stiffness with the total stiffness and that one does not make elimination as in the computation of the lagranges. Conclusion

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8   Of the discrete

modelizations of contact-friction with slip surfaces 1D and 2D were established in Code_Aster . *These* modelizations usable with STAT_NON_LINE and DYNA_NON_LINE are accessible under DEFI_ CONTACT. The modelizations

suggested lean on the meshes of surfaces coming in contact and make it possible to retranscribe node with node the conditions of contact friction between surfaces after discretization of the corresponding variational formulation. The method extends then without difficulty of small displacements to the case from large displacements. Indeed, the absence of use of finite elements, between surfaces being able to come in contact, avoids the great distortion of the latter, in the case of large displacements. One can then use either of the conditions of direct connections nodes to nodes for initially compatible meshes, or conditions of connections nodes to nodes balanced according to an approach by projection of the master-slave type for incompatible meshes. In

the case of slip surfaces 1D one could develop an algorithm using only Lagrange multipliers. The finished convergence of this kind of algorithm is proven for the unilateral contact without friction and in the case with friction for low values of the coefficient of kinetic friction of Coulomb In the case of slip surfaces 2D, the rubbing contact is treated either by dualisation or by regularization with various mixings. Bibliography

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9   "

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is resulting from the fusion of the documents [R5.03.50] and [R5.03.51]. [R5.03

.50] (Old version) unilateral Contact by kinematical conditions Version

Aster Author	(S) or contributor (S), organization Description	of modifications 5.0 N
. Tardieu	, I.Vautier 8.5 M	
. Abbas	, N.Tardieu EDF-R &D/AMA Extension	to the dynamics, addition of the penalization. 9.4 M
. Abbas	N.Tardieu Addition	of the description of the algorithm of the GCP Addition of the description of the method of projection [R5.03

.51] discrete Contact-Friction in 2D and 3D Version

Aster Author	(S) or contributor (S), organization Description	of modifications 5.0 P.
Massin	Friction	of Coulomb or Tresca, method derived from the active stresses in 2D, penalization for friction in 3D, large displacements 7.1
N.Tardieu	, P. Massin Extension	of the methods 2D to the situations 3D 8.5
Mr. Abbas	Addition	of the reactualization geometrical (§7.2) and updated of the document [R5.03

.50] (New version) discrete Formulation of contact-friction Version

Aster Author	(S) or contributor (S), organization Description	of the modifications 10.1
M.Abbas	, T. De Soza Fusion	of [R5.03.50] and [R5.03.51] 10.5
M.Abbas	various	Corrections, modifications of the references for new command DEFI_ CONTACT 11.0
M.Abbas	Simplifications	of the equations, details on the initialization of the algorithms, rewriting of the algorithms, removal of the advice (see document U2) 11.3
M.Abbas	, S.Mercier Corrections	of shells in the algorithms 11.4
M.Abbas	, T. De Soza Addition	of the linear search in the algorithm of projection