Elements of contact derived from a continuous hybrid formulation

Abstract:

This document describes the way in which elements of rubbing contact are derived from a continuous hybrid formulation of problems of contact between solids (2D or 3D) in great transformations and specifies the strategy of resolution used.

One uses an original formulation called stabilized Lagrangian formulation which makes it possible to find the classical cases of the literature (Lagrangian, Lagrangian increased, penalization) by a wise choice of its parameters.

This formulation is available in command DEFI_CONTACT under the name "CONTINUE".

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Array of the illustrations

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1 Introduction

There exists, for the processing of the problem of contact-friction, an important difference between the "discrete" formulations known as and the formulation known as "continuous". The implementations in the computer codes are founded on models such those developed by Hallquist [1], Alart and Curnier [2], Laursen and Simo ([3]).

The two essential objects of this document are, on the one hand, the description of derivative of elements of contact from a variational mixed formulation of a problem of contact between three-dimensional, deformable solids and undergoing great transformations, and, on the other hand, the detail of the strategy of resolution.

This document gradually builds the problem of contact-friction established in *Code_Aster* under the name of "continuous method for contact-friction" activable via keyword FORMULATION=' CONTINUE' in DEFI_CONTACT, initially proposed by Ben-Dhia and Zarroug [4].

One will find an abundant bibliography at the end of the document. In particular, four theses supported on the subject by M.Zarroug [5], C.Zammali [6], M.Torkhani [7] and D.Kudawoo [9].

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2 Kinematical

2.1 kinematics of solids

One considers two deformable $B^i(i=1,2)$ solids in rubbing contact. These two solids occupy in their initial configuration the dependancy of two fields Ω_0^1 and Ω_0^2 of $\mathbb{R}^3$ and, in their current configuration at time t, the dependancy of Ω_t^1 and Ω_t^2 (also understood in $\mathbb{R}^3$), respectively. One supposes that, in their initial configuration, these two solids are in a natural state, i.e. without residual stresses or predeformations. Let us consider a material particle M^i of solid B^i, located by the vector $\boldsymbol{p}^i$ in the initial configuration. One indicates by $\boldsymbol{\varphi}_t^i$ the transformation at the time of t solid B^i, this application transports the particle M^i towards the position $\boldsymbol{x}_t^i$ in the current configuration. One a:

$$\begin{aligned}\boldsymbol{\varphi}_t^i(\boldsymbol{p}^i,t):\quad & \Omega_0^i\times[0,t]\to\Omega_t^i\subset\mathbb{R}^3\\ & \boldsymbol{p}^i\to\boldsymbol{x}_t^i=\boldsymbol{\varphi}_t^i(\boldsymbol{p}^i,t)\end{aligned}\qquad(1)$$

During their motion, the solids can come into contact, as indicated on Figure 1. The border $\Gamma^i=\partial\Omega^i$ of each solid B^i is broken up into parts Γ_u^i, Γ_g^i and Γ_c^i in the initial configuration, whose intersections are empty two to two:

$$\begin{aligned}\Gamma_u^i\cap\Gamma_g^i&=\emptyset\\ \Gamma_u^i\cap\Gamma_c^i&=\emptyset\\ \Gamma_c^i\cap\Gamma_g^i&=\emptyset\\ \Gamma_u^i\cup\Gamma_g^i\cup\Gamma_c^i&=\Gamma^i\end{aligned}\qquad(2$$

borders 2) become deformed in the current configuration in γ_u^i, γ_g^i and γ_c^i. The solid B^i is embedded on Γ_u^i and subjected to one density nominal of surface forces noted $\boldsymbol{g}^i$ on the part Γ_g^i. In addition, one notes $\boldsymbol{f}^i$ the voluminal field of density of forces applied to solids B^i. The parts of surfaces Γ^i likely to make contact at the time of the strain of two solids are noted Γ_c^i. These surfaces are not known *a priori* (it is one of non-linearities of the problem to be solved as it further will be seen).

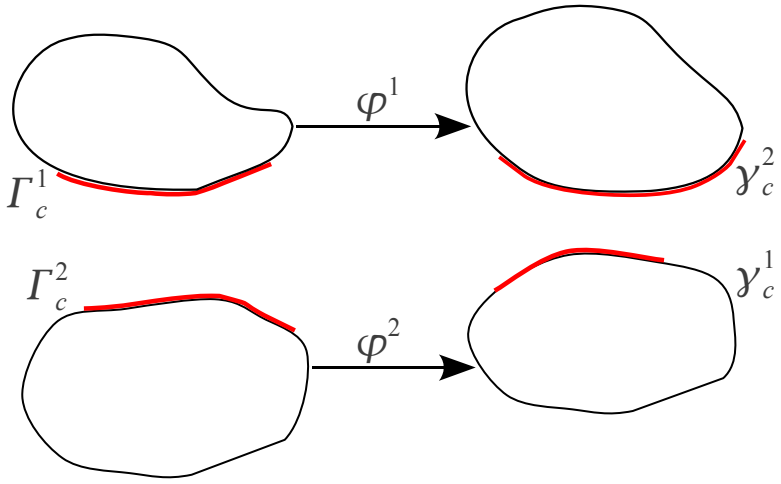

Figure 1: Description of the various configurations of the mechanical problem

One supposes the existence of noted regular cards $\boldsymbol{\Phi}^i$ describing surfaces Γ^i. These cards are defined as follows:

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$$\begin{aligned}\boldsymbol{\Phi}^i(\boldsymbol{\zeta}):\quad & \omega\subset\mathbb{R}^2\rightarrow\Omega^i\subset\mathbb{R}^3\\ & \boldsymbol{\zeta}(\zeta_1,\zeta_2)\rightarrow\boldsymbol{p}^i=\boldsymbol{\Phi}^i(\boldsymbol{\zeta})\end{aligned} \tag{3}$$

where ω is a field limited (of reference) contents in $\mathbb{R}^2$, to see Figure 2.

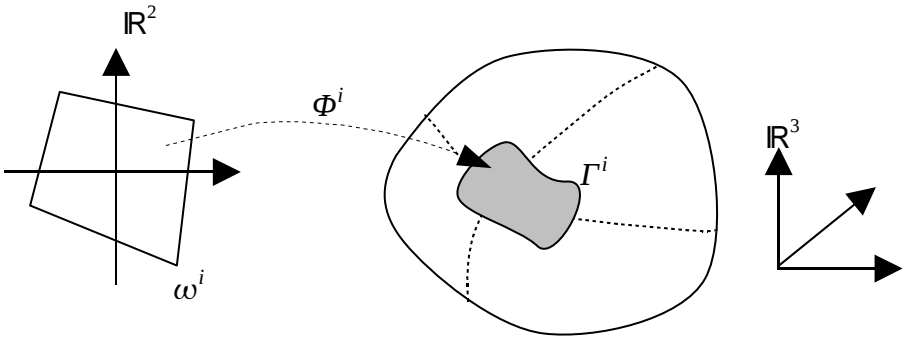

Figure 2: Description of surfaces

the coordinates of contact surfaces Γ_c^i are transported by application of the strain, at time t. They are thus built by composition of the two functions:

$$\begin{aligned}\boldsymbol{\Phi}_t^i(\boldsymbol{\zeta},t):\quad & \Gamma_c^i\times[0,T]\rightarrow\gamma_c^i\\ & (\boldsymbol{\zeta},t)\rightarrow\boldsymbol{\Phi}_t^i(\boldsymbol{\zeta})=\boldsymbol{\varphi}_t^i\circ\boldsymbol{\Phi}^i(\boldsymbol{\zeta})\end{aligned} \tag{4}$$

One defines the tensor gradient of the transformation $\underline{\underline{\boldsymbol{F}}}_t^i$ by:

$$\underline{\underline{\boldsymbol{F}}}_t^i=\frac{\partial\boldsymbol{x}_t^i}{\partial\boldsymbol{p}^i} \tag{5}$$

If one indicates by $\boldsymbol{u}^i$ the field of displacements of solid B^i :

$$\boldsymbol{x}_t^i=\boldsymbol{p}^i+\boldsymbol{u}^i \tag{6}$$

Then the tensor gradient of the transformation is rewritten:

$$\underline{\underline{\boldsymbol{F}}}_t^i=\nabla\otimes\boldsymbol{u}^i+\underline{\underline{\boldsymbol{I}}} \tag{7}$$

By means of the function $\boldsymbol{\varphi}_t^i$, one can write:

$$\underline{\underline{\boldsymbol{F}}}_t^i=\frac{\partial\boldsymbol{\varphi}_t^i(\boldsymbol{p}^i,t)}{\partial\boldsymbol{p}^i}=\nabla_{\boldsymbol{p}^i}\otimes\boldsymbol{\varphi}_t^i \tag{8}$$

Where $\otimes$ is the tensor product and $\nabla_{\boldsymbol{p}^i}$ the operator gradient (in space). The jacobian of the transformation is introduced:

$$J_t^i=det\left(\underline{\underline{\boldsymbol{F}}}_t^i\right) \tag{9}$$

One will note $\overline{\Omega^i}$ the strict interior of Ω^i and Γ^i his border:

$$\Omega^i=\overline{\Omega^i}\cup\Gamma^i \tag{10}$$

2.1.1 Construction of the local bases

the two applications $\boldsymbol{\Phi}^i$ and $\boldsymbol{\varphi}_t^i$ make it possible to define the local coordinate system on solid i. This local coordinate system is established by construction of the tangent plane at the point considered and thus, by the data of the two tangent vectors $\boldsymbol{t}_{\alpha=1,2}^i$ on the deformed configuration of surface γ^i :

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$$\boldsymbol{t}_{\alpha}^{i}\left(\boldsymbol{\zeta}_{c}^{i}\right)=\frac{\partial \boldsymbol{\Phi}_{t}^{i}}{\partial \zeta_{\alpha}}=\frac{\partial \boldsymbol{\varphi}_{t}^{i}\left(\boldsymbol{p}^{i},t\right)}{\partial \boldsymbol{p}^{i}}.\frac{\partial \boldsymbol{\Phi}^{i}\left(\boldsymbol{\zeta}\right)}{\partial \zeta_{\alpha}} \text{ for } \alpha=1,2 \tag{11}$$

One can in the same way define the basic vectors of initial surface Γ_{c}^{i} :

$$\boldsymbol{T}_{\alpha}^{i}\left(\boldsymbol{\zeta}\right)=\frac{\partial \boldsymbol{\Phi}^{i}\left(\boldsymbol{\zeta}\right)}{\partial \zeta_{\alpha}} \text{ for } \alpha=1,2 \tag{12}$$

By means of the definition of the tensor gradient of the transformation (8), one expresses the relation explicitly enters $\boldsymbol{T}_{\alpha}^{i}$ and $\boldsymbol{t}_{\alpha}^{i}$:

$$\boldsymbol{t}_{\alpha}^{i}=\underline{\underline{\boldsymbol{F}}}^{i}\boldsymbol{T}_{\alpha}^{i} \tag{13}$$

There is no reason which $\boldsymbol{t}_{\alpha}^{i}$ are unit and which the reference $\left(\boldsymbol{t}_{1}^{i},\boldsymbol{t}_{2}^{i}\right)$ is orthogonal. We obtain the norms by simple cross product:

$$\boldsymbol{n}^{i}=\frac{\boldsymbol{t}_{1}^{i}\times\boldsymbol{t}_{2}^{i}}{\left\|\boldsymbol{t}_{1}^{i}\times\boldsymbol{t}_{2}^{i}\right\|} \text{ and } \boldsymbol{N}^{i}=\frac{\boldsymbol{T}_{1}^{i}\times\boldsymbol{T}_{2}^{i}}{\left\|\boldsymbol{T}_{1}^{i}\times\boldsymbol{T}_{2}^{i}\right\|} \tag{14}$$

In *Code_Aster*, the norm is always obtained according to the algorithm below:

- Computation of the tangent vectors on the present configuration ;
- Modification of the tangent vectors (according to options VECT_MAIT, VECT_ESCL);
- Computation of the normal vector by cross product at the point considered.

By construction, the tangent vectors $\boldsymbol{t}_{\alpha}^{i}$ are orthogonal with the norm $\boldsymbol{n}^{i}$.

2.1.2 Regularization of the local bases

It is sometimes necessary to use a field of norms “smoothed” to minimize the risks of instabilities of the problem of contact when surface is described by polynomial fields (typically the case of the finite elements). For that, a very simple strategy was implemented in the code (activable via option LISSAGE). The algorithm is described below:

•For each element e of surface, for each node s of this element divided by $k(s)$ elements, one calculates the unit $\boldsymbol{n}^{e}(s)$ norm in this node s ;

•For each node s , one makes the average $\boldsymbol{n}_{L}(s)$ between the norms of each element sharing this common node:

$$\boldsymbol{n}_{L}(s)=\frac{\sum_{e=1}^{k(s)}\boldsymbol{n}^{e}(s)}{\left\|\sum_{e=1}^{k(s)}\boldsymbol{n}^{e}(s)\right\|} \tag{15}$$

•One smoothes the norm obtained by an interpolation in conformity with the degree of surface (linear or quadratic). If one indicates by $N_{e}^{s}\ \ nno(e)$ the shape functions of the element e , then the norm smoothed $\boldsymbol{n}_{\text{lis}}(\overline{\boldsymbol{\zeta}})$ in a point $\overline{\boldsymbol{\zeta}}$ of the element is written simply:

$$\boldsymbol{n}_{\text{lis}}(\overline{\boldsymbol{\zeta}})=\sum_{s=1}^{nno(e)}N_{e}^{s}(\overline{\boldsymbol{\zeta}}).\boldsymbol{n}_{L}(s) \tag{16}$$

Note:: given that one makes an arithmetic mean NON-balanced between the norms, if two adjacent elements are serious and of very different norms, the lissage obtained can be of poor quality.

2.1.3 Some remarkable identities

In this paragraph, we will present a certain number of remarkable identities relating to the properties of the base $(\boldsymbol{t}_{1},\boldsymbol{t}_{2},\boldsymbol{n})$. We point out initially the definition of the tangent vectors:

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$$\boldsymbol{t}_\alpha = \frac{\partial \overline{\boldsymbol{x}}}{\partial \zeta_\alpha} \tag{17}$$

L are vectors $\boldsymbol{t}_\alpha$ and $\boldsymbol{n}$ are ortho gona ux between them , therefore their scalar product is null :

$$\boldsymbol{t}_\alpha \cdot \boldsymbol{n} = 0$$ L1818

) E vector $\boldsymbol{n}$ is unit by definition, its product by itself is a constant (which is worth 1) :

$$\boldsymbol{n} \cdot \boldsymbol{n} = 1 \tag{19}$$

One defines L has matrix of the first form fundamental or tensor metric $\underline{\underline{\boldsymbol{m}}}$ whose components are :

$$m_{\alpha\beta} = \boldsymbol{t}_\alpha \cdot \boldsymbol{t}_\beta \tag{20}$$

If the base were orthonormal, this tensor would be reduced to the tensor identity.
The reverse of the metric tensor is also the first fundamental form of dual space, its components are worth :

$$\frac{1}{m_{\alpha\beta}} = m^{\alpha\beta} = \boldsymbol{t}^\alpha \cdot \boldsymbol{t}^\beta = \frac{\partial \overline{\boldsymbol{x}}}{\partial \zeta^\alpha} \cdot \frac{\partial \overline{\boldsymbol{x}}}{\partial \zeta^\beta} \tag{21}$$

to pass from the tangent vectors of the base covariante at the base contravariante, one uses the metric tensor [1] :

$$\boldsymbol{t}^\alpha = m^{\alpha\beta} \boldsymbol{t}_\beta \tag{22}$$

With:

$$m_{\alpha\beta} m^{\alpha\beta} = 1 \tag{23}$$

One defines also the tensor of the second fundamental form, related to the curvature, whose components are the following ones:

$$\kappa_{\alpha\beta} = \frac{\partial^2 \overline{\boldsymbol{x}}}{\partial \zeta_\alpha \partial \zeta_\beta} \cdot \boldsymbol{n} \tag{24}$$

2.2 Kinematical condition of NON-interpenetration

to translate the NON-interpenetration, one proceeds as follows:

- One couples the points of contact surfaces two to two: it is *pairing;*
- One imposes between the two points paired a condition of NON-interpenetration according to a given direction (generally the norm with the one of two surfaces at the point concerned).

2.2.1 Problem of pairing

Per convention, one will call surface slave the contact surface defined on solid 1 and surfaces main surface paired definite on solid 2 . One carries out pairing while searching, for any point $\boldsymbol{x}$ of the border γ_c^1 , the point $\overline{\boldsymbol{x}}$ of γ_c^2 nearest. That amounts solving the problem of minimization according to:

$$\begin{gathered} \forall\, \boldsymbol{x} \in \gamma_c \ et\ \forall\, t \geqslant 0 \ \text{ with} \\ \boldsymbol{x}_t = \boldsymbol{\varphi}_t^1\left(\boldsymbol{p}^1, t\right) \ \text{ and } \ \boldsymbol{p}^1 = \boldsymbol{\Phi}^1(\boldsymbol{\zeta}) \quad \boldsymbol{\zeta}\left(\zeta_1, \zeta_2\right) \in \omega \\ \text{to find } \overline{\boldsymbol{\zeta}} \in \omega \times [0, t] \text{ such as:} \\ \overline{\boldsymbol{\zeta}} = \underset{\boldsymbol{\zeta} \in \omega}{\operatorname{ArgMin}} \left\{ \frac{1}{2} . \left\| \boldsymbol{\varphi}_t^1\left(\boldsymbol{\Phi}^1(\boldsymbol{\zeta}), t\right) - \boldsymbol{\varphi}_t^2\left(\boldsymbol{\Phi}^2(\boldsymbol{\zeta}), t\right) \right\|^2 \right\} \end{gathered} \tag{25}$$

the solution $\overline{\boldsymbol{\zeta}}$ is the position within the space of parametric reference of the projection M of the slave node P on the master mesh.

1 tensor metric is often described “as elevator of index”

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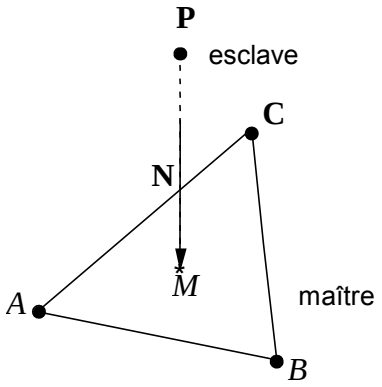

Figure 3: Projection of the slave node on the master mesh

the details of the implementation of the algorithm used for the problem of pairing are available in the document [R5.03.50] (the method of pairing is strictly equivalent between the discrete methods and the continuous method).

2.2.2 Definition of clearance

pairing makes it possible to define the point $\overline{\boldsymbol{p}}$ of surface Γ_c^2, paired with $\boldsymbol{p}$ at time t in the configuration of origin, and the point $\overline{\boldsymbol{x}}$ paired with $\boldsymbol{x}$ at the moment T in the present configuration. The vector outdistances $\boldsymbol{d}$ between the paired points is quite simply:

$$\boldsymbol{d}=\boldsymbol{x}-\overline{\boldsymbol{x}}=\boldsymbol{\Phi}_t^1(\boldsymbol{\zeta})-\boldsymbol{\Phi}_t^2(\overline{\boldsymbol{\zeta}}) \tag{26}$$

It is result of the minimization described in the system (25):

$$\overline{\boldsymbol{d}}=\min_{\boldsymbol{\zeta}\in\omega}\left\{\frac{1}{2}.\left\|\boldsymbol{\varphi}_t^1\left(\boldsymbol{\Phi}^1(\boldsymbol{\zeta}),t\right)-\boldsymbol{\varphi}_t^2\left(\boldsymbol{\Phi}^2(\boldsymbol{\zeta}),t\right)\right\|^2\right\} \tag{27}$$

the first condition of optimality is written:

$$\left[\boldsymbol{\Phi}_t^1(\boldsymbol{\zeta})-\boldsymbol{\Phi}_t^2(\overline{\boldsymbol{\zeta}})\right]\cdot\boldsymbol{t}_\alpha(\overline{\boldsymbol{\zeta}})=0 \text{ for } \alpha=1,2 \tag{28}$$

With the definition of the tangent vectors on deformed surface γ_c^2 :

$$\boldsymbol{t}_\alpha(\overline{\boldsymbol{\zeta}})=\frac{\partial\boldsymbol{\Phi}_t^2}{\partial\zeta_\alpha}=\frac{\partial\boldsymbol{\varphi}_t^2}{\partial\overline{\boldsymbol{p}}}.\frac{\partial\boldsymbol{\Phi}^2(\overline{\boldsymbol{\zeta}})}{\partial\zeta_\alpha} \text{ for } \alpha=1,2 \tag{29}$$

One can go back to the basic vectors of initial surface Γ_c^2 :

$$\boldsymbol{T}_\alpha(\overline{\boldsymbol{\zeta}}_t)=\frac{\partial\boldsymbol{\Phi}^2(\overline{\boldsymbol{\zeta}}_t)}{\partial\zeta_\alpha} \text{ for } \alpha=1,2 \tag{30}$$

By means of the definition of the tensor gradient of the transformation, one expresses the relation explicitly enters $\boldsymbol{T}_\alpha$ and $\boldsymbol{t}_\alpha$:

$$\boldsymbol{t}_\alpha=\underline{\underline{\boldsymbol{F}}}^2\boldsymbol{T}_\alpha \tag{31}$$

We obtain the norm by simple cross product:

$$\boldsymbol{n}(\overline{\boldsymbol{\zeta}}_t)=\frac{\boldsymbol{t}_1\times\boldsymbol{t}_2}{\|\boldsymbol{t}_1\times\boldsymbol{t}_2\|} \tag{32}$$

the base built by the three vectors $(\boldsymbol{t}_1,\boldsymbol{t}_2,\boldsymbol{n})$ does not have any reason *a priori* to be orthonormal. On the other hand, by construction, the norm $\boldsymbol{n}$ is unit and normal with the plane defined by the vectors $(\boldsymbol{t}_1,\boldsymbol{t}_2)$ which define the tangent plane in deformed surface γ_c^2. By taking again the equality (28) one necessarily has:

$$\left[\boldsymbol{\Phi}_t^1(\boldsymbol{\zeta})-\boldsymbol{\Phi}_t^2(\overline{\boldsymbol{\zeta}})\right]=d_n\boldsymbol{n} \tag{33}$$

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d_n is a signed algebraic quantity. Like $x=\Phi_t^1(\zeta)$, O N obtains:

$$x=\Phi_t^2(\overline{\zeta})+d_n n \tag{34}$$

d_n clearance **between** two solids at the point is called x . What gives us the relation between the base (t_1^1,t_2^1,n^1) defined on the tangent plane in the point slave and the basis (t_1,t_2,n) built by the projection of the point x on deformed surface (main) γ_c^2 :

$$t_\alpha^1(\overline{\zeta})=t_\alpha(\overline{\zeta})+d_n\frac{\partial n}{\partial \zeta_\alpha}(\overline{\zeta}) \text{ for } \alpha=1,2$$
$$n^1=n \tag{35}$$

the two references are coïncidents if the points are in contact (if $d_n=0$). The distance breaks up into a normaleet part d_n a tangential part d_τ :

$$d=d_n n+d_\tau \tag{36}$$

One obtains for the statement of d_τ :

$$d_\tau=\underline{\underline{P}}^\tau d=\underline{\underline{P}}^\tau(x-\overline{x}) \tag{37}$$

With $\underline{\underline{P}}^\tau$ the matrix of projection on the tangent level:

$$\underline{\underline{P}}^\tau=\underline{\underline{I}}-n\otimes n \tag{38}$$

One can write the condition of NON-interpenetration enters $x\in\gamma_c^1$ and $\overline{x}\in\gamma_c^2$ according to the direction of search $n=-n^2(\overline{x})$, norm entering unit at the point $\overline{x}$ (directed towards the interior of Ω_t^2):

$$d_n=(x-\overline{x})\cdot n\le 0 \tag{39}$$

This quantity is objective.

Note:
• Taking into account the choice of the norm, clearance “is reversed” compared to the case of the contact in discrete formulation in *Code_Aster* : there is interpenetration if $d_n\ge 0$;

2.2.3 Relative velocity

to write the friction law of Coulomb, we will need the notion relative velocity of sliding between two solids. We take again for that the definition of normal clearance (33):

$$\left[\Phi_t^1(\zeta)-\Phi_t^2(\overline{\zeta})\right]=d_n n \tag{40}$$

One starts by deriving the term from right compared to time:

$$\frac{d}{dt}(d_n n)=\dot{d}_n n+d_n\left(\frac{\partial n}{\partial t}+\frac{\partial n}{\partial\overline{\zeta^\alpha}}\frac{\partial\overline{\zeta^\alpha}}{\partial t}\right) \tag{41}$$

With the usual convention of summation on the dumb indices α . Then one derives the term from left:

$$\frac{d}{dt}\left(\Phi_t^1(\zeta)-\Phi_t^2(\overline{\zeta})\right)=\frac{\partial\Phi_t^1}{\partial t}-\frac{\partial\Phi_t^2}{\partial t}-\frac{\partial\Phi_t^2}{\partial\overline{\zeta^\alpha}}\frac{\partial\overline{\zeta^\alpha}}{\partial t} \tag{42}$$

One recognizes the definition of t_α in the term $\frac{\partial\Phi_t^2}{\partial\zeta_\alpha}$ (see (29)). One equalizes the two terms (41) and (42):

$$\dot{d}_n.n+d_n.\left(\frac{\partial n}{\partial t}+\frac{\partial n}{\partial\overline{\zeta^\alpha}}.\frac{\partial\overline{\zeta}}{\partial t}\right)=\frac{\partial\Phi_t^1}{\partial t}-\frac{\partial\Phi_t^2}{\partial t}-\frac{\partial\overline{\zeta_\alpha}}{\partial t}t_\alpha \tag{43}$$

By posing $v(x,t)$ the velocity of the point x and $\overline{v}(\overline{x},t)$ the velocity of the point $\overline{x}$ paired:

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$$\boldsymbol{v}(\boldsymbol{x},t)=\frac{\partial \boldsymbol{\Phi}_t^1}{\partial t} \text{ and } \bar{\boldsymbol{v}}(\bar{\boldsymbol{x}},t)=\frac{\partial \boldsymbol{\Phi}_t^2}{\partial t} \tag{44}$$

One obtains finally by means of (35):

$$\dot{d}_n\boldsymbol{n}+d_n\frac{\partial \boldsymbol{n}}{\partial t}+\frac{\partial \overline{\zeta_\alpha}}{\partial t}\boldsymbol{t}_\alpha^1=\boldsymbol{v}^1-\bar{\boldsymbol{v}} \tag{45}$$

One notes $\boldsymbol{v}_{gliss}$ the velocity of sliding:

$$\boldsymbol{v}_{gliss}=\boldsymbol{v}-\left(\bar{\boldsymbol{v}}+d_n\frac{\partial \boldsymbol{n}}{\partial t}\right) \tag{46}$$

This quantity is objective because it is invariant by change of reference. The relative velocity of sliding is identified unlike velocity of two solids, only when they are in contact ($d_n=0$).

2.2.4 Operator of discontinuity

One notes $[\![\boldsymbol{g}(\boldsymbol{p})]\!]$ the discontinuity of the quantity $\boldsymbol{g}$ between solid 1 and the solid 2. This discontinuity is written at the point $\boldsymbol{p}$ on 1 which is projected in $\bar{\boldsymbol{p}}$ on 2 according to the application of pairing:

$$[\![\boldsymbol{g}(\boldsymbol{p})]\!]=\boldsymbol{g}^1(\boldsymbol{p})-\boldsymbol{g}^2(\bar{\boldsymbol{p}}) \tag{47}$$

Thereafter one will give up the notation $[\![\boldsymbol{g}(\boldsymbol{p})]\!]$ with the profit of $[\![\boldsymbol{g}]\!]$ by making the implicit assumption that discontinuity is written on the couple $(\boldsymbol{p},\bar{\boldsymbol{p}})$ paired points. One can break up this discontinuity into a normal part $[\![\boldsymbol{g}]\!]_n$ and a tangential part $[\![\boldsymbol{g}]\!]_\tau$:

$$[\![\boldsymbol{g}]\!]=[\![g]\!]_n\boldsymbol{n}+[\![\boldsymbol{g}]\!]_\tau \tag{48}$$

With:

$$[\![g]\!]_n=[\![\boldsymbol{g}]\!]\cdot\boldsymbol{n} \tag{49}$$

Where $\boldsymbol{n}$ is the norm on the surface Master, directed towards the interior of solid 2.
The tangential part:

$$[\![\boldsymbol{g}]\!]_\tau=\underline{\underline{\boldsymbol{P}}}^\tau[\![\boldsymbol{g}]\!] \tag{50}$$

With the introduced notations, clearance d_n can be written:

$$d_n=[\![x]\!]_n=[\![p+u]\!]_n \tag{51}$$

It is appropriate to notice that the quantities projected on the norm are necessarily scalar, from where the notation in $[\![g]\!]_n$ and not in $[\![\boldsymbol{g}]\!]_n$.

2.3 Variations of the kinematical quantities

contact surfaces are described by a set of four parameters $(\zeta_1^1,\zeta_2^1,\zeta_1^2,\zeta_2^2)$, within the space of reference (parametric) of the finite elements. The approach Master/slave makes it possible to reduce this number by describing the kinematical variables $\boldsymbol{\zeta}$ compared to a body taken as reference (in fact, the body defined as "Master"). The adequate choice of the surface of reference is not immediate and remains an assumption of modelization to be taken into account, especially in the cases of a great relative sliding of two surfaces. It is necessary to distinguish Lagrangian derivatives from eulerian derivatives. This paragraph aims at evaluating the various variations. One defines the variation $\delta f(x_0)$ of a function $f(x)$ in a variable in a point given x_0 by the quantity:

$$\delta f(x_O)=\lim_{\delta x\to 0}\left(f(x_0+\delta_x)-f(x_0)\right)=\left.\frac{\partial f}{\partial x}\right|_{x_0}\delta x \tag{formulate5 252}$$

) a function with several variables, one distinguishes the total (compared to all its variables), noted variation $\tilde{\delta} f$ and the partial *variation* compared to a variable x, noted $\delta_x f$. The total variation is the sum of all the partial

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variations. Let us note $\tilde{\delta}$ the total variation of a kinematical quantity. If t variable time is noted, one will have, in the case of an eulerian quantity $\bar{a}$:

$$\tilde{\delta}\bar{a}=\frac{\partial\bar{a}}{\partial t}\delta t+\frac{\partial\bar{a}}{\partial\zeta_\alpha}\delta\zeta^\alpha \tag{53}$$

It is thus the sum of a variation δ_t compared to time t and variation δ_ζ compared to the parametric coordinates:

$$\tilde{\delta}\bar{a}=\delta_t\bar{a}+\delta_{\zeta_\alpha}\bar{a} \tag{54}$$

If a Lagrangian quantity now is considered a , its variation compared to the parametric coordinates ζ is null:

$$\tilde{\delta}a=\delta_t a \text{ car } \delta_{\zeta_\alpha}a=0 \tag{55}$$

the quantities related to surface slave are Lagrangian quantities, those related to surface Master are eulerian quantities.

2.3.1 The formulas of Weingarten

One uses initially the fact that the variation of a constant is null some is the parameter which varies. However according to (19) the product $(\boldsymbol{n}\cdot\boldsymbol{n})$ is constant, therefore:

$$\frac{1}{2}\delta(\boldsymbol{n}\cdot\boldsymbol{n})=0 \tag{56}$$

While developing (56) one obtains:

$$\delta\boldsymbol{n}\cdot\boldsymbol{n}=0 \tag{57}$$

the product is null if, and only if, $\delta\boldsymbol{n}$ is orthogonal with $\boldsymbol{n}$, and thus $\delta\boldsymbol{n}$ belongs to the tangent plane:

$$\delta\boldsymbol{n}=a\boldsymbol{t}_1+b\boldsymbol{t}_2 \quad \forall a,b\in\mathbb{R}\times\mathbb{R} \tag{58}$$

One by means of notices that the variation of the norm belongs to the tangent plane, the constants c_β :

$$\delta\boldsymbol{n}=c_\beta\boldsymbol{t}_\beta \quad \forall c_\beta\in\mathbb{R} \tag{59}$$

We now will establish very practical formulas which one will need later. The first thing to be made is to give the formula establishing the value of the coefficients c_β . One starts by establishing the following scalar product:

$$\delta\boldsymbol{n}\cdot\boldsymbol{t}_\alpha=c_\beta\boldsymbol{t}_\beta\cdot\boldsymbol{t}_\alpha=c_\beta m_{\alpha\beta} \tag{60}$$

Then, one leaves the property (18):

$$\frac{\partial\bar{\boldsymbol{x}}}{\partial\zeta_\alpha}\cdot\boldsymbol{n}=0 \tag{61}$$

One calculates its first variation, the variation of a constant is null some is the parameter which varies:

$$\delta\left(\frac{\partial\bar{\boldsymbol{x}}}{\partial\zeta_\alpha}\right)\cdot\boldsymbol{n}+\frac{\partial\bar{\boldsymbol{x}}}{\partial\zeta_\alpha}\cdot\delta\boldsymbol{n}=0\rightarrow\frac{\partial\bar{\boldsymbol{x}}}{\partial\zeta_\alpha}\cdot\delta\boldsymbol{n}=-\delta\left(\frac{\partial\bar{\boldsymbol{x}}}{\partial\zeta_\alpha}\right)\cdot\boldsymbol{n}=-\kappa_{\alpha\beta} \tag{62}$$

One finds the components of the second fundamental form, except for the sign. By identification with (60), there is the following relation:

$$c_\beta m_{\alpha\beta}=-\kappa_{\alpha\beta}\rightarrow c_\beta=-m^{\alpha\beta}\kappa_{\alpha\beta} \tag{63}$$

the variation of the norm is thus worth:

$$\delta\boldsymbol{n}=-m^{\alpha\beta}\kappa_{\alpha\beta}\boldsymbol{t}_\beta \tag{formulate6 464}$$

) rear application of (22), one a:

$$\delta\boldsymbol{n}=-\kappa_{\alpha\beta}\boldsymbol{t}^\alpha \tag{65}$$

By means of (62) , one a:

$$\delta\boldsymbol{n}=-\left(\delta\left(\frac{\partial\bar{\boldsymbol{x}}}{\partial\zeta_\alpha}\right)\cdot\boldsymbol{n}\right)\frac{\partial\bar{\boldsymbol{x}}}{\partial\zeta^\alpha} \tag{66}$$

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It is the formula of Weingarten.

2.3.2 Variation of normal clearance

From the definition of normal clearance (33):

$$\boldsymbol{x}-\overline{\boldsymbol{x}}=d_n\boldsymbol{n} \tag{67}$$

One applies the first total variation :

$$\tilde{\delta}\,\boldsymbol{x}=\tilde{\delta}\,\overline{\boldsymbol{x}}+\tilde{\delta}\,d_n\boldsymbol{n}+d_n\tilde{\delta}\,\boldsymbol{n} \tag{68}$$

By definition $\overline{\boldsymbol{x}}$ is an eulerian quantity, its variation thus expresses itself by the formula (53):

$$\tilde{\delta}\,\overline{\boldsymbol{x}}=\delta_t\overline{\boldsymbol{x}}+\frac{\partial\overline{\boldsymbol{x}}}{\partial\zeta_\alpha}\delta\zeta^\alpha \tag{69}$$

In the same way for the variation of the norm:

$$\tilde{\delta}\,\boldsymbol{n}=\delta_t\boldsymbol{n}+\frac{\partial\boldsymbol{n}}{\partial\zeta_\alpha}\delta\zeta^\alpha \tag{70}$$

As $\boldsymbol{x}$ is a Lagrangian quantity, $\tilde{\delta}\,\boldsymbol{x}=\delta_t\boldsymbol{x}$. Finally:

$$\delta_t\boldsymbol{x}=\delta_t\overline{\boldsymbol{x}}+\frac{\partial\overline{\boldsymbol{x}}}{\partial\zeta_\alpha}\delta\zeta^\alpha+\tilde{\delta}\,d_n\boldsymbol{n}+d_n\left(\delta_t\boldsymbol{n}+\frac{\partial\boldsymbol{n}}{\partial\zeta_\alpha}\delta\zeta^\alpha\right) \tag{71}$$

If one makes the product of this identity with the norm $\boldsymbol{n}$:

$$\delta_t\boldsymbol{x}\cdot\boldsymbol{n}=\delta_t\overline{\boldsymbol{x}}\cdot\boldsymbol{n}+\delta\zeta^\alpha\frac{\partial\overline{\boldsymbol{x}}}{\partial\zeta_\alpha}\cdot\boldsymbol{n}+\tilde{\delta}\,d_n\boldsymbol{n}\cdot\boldsymbol{n}+d_n\left(\delta_t\boldsymbol{n}+\delta\zeta^\alpha\frac{\partial\boldsymbol{n}}{\partial\zeta_\alpha}\right)\cdot\boldsymbol{n} \tag{72}$$

One considers the properties (18), (19) and (57), i.e.:

$$\frac{\partial\overline{\boldsymbol{x}}}{\partial\zeta_\alpha}\cdot\boldsymbol{n}=0\ ,\ \boldsymbol{n}\cdot\boldsymbol{n}=1\ \text{ and }\ \delta\boldsymbol{n}\cdot\boldsymbol{n}=0 \tag{73}$$

What simplifies the statement:

$$\tilde{\delta}\,d_n=\boldsymbol{n}\cdot\left(\delta_t\boldsymbol{x}-\delta_t\overline{\boldsymbol{x}}\right)=\delta_t d_n \tag{74}$$

the first variation of clearance does not involve the appearance of terms of transport (as in equation 55). d_n is thus a purely Lagrangian quantity.

2.3.3 Variation of the parametric coordinates

One sets out again of L has first variation D E the statement (71) (by taking account owing to the fact that d_n is a Lagrangian quantity) :

$$\delta_t\boldsymbol{x}=\delta_t\overline{\boldsymbol{x}}+\frac{\partial\overline{\boldsymbol{x}}}{\partial\zeta_\alpha}\delta\zeta^\alpha+\delta\,d_n\boldsymbol{n}+d_n\left(\delta_t\boldsymbol{n}+\frac{\partial\boldsymbol{n}}{\partial\zeta_\alpha}\delta\zeta^\alpha\right) \tag{formulate7575}$$

) ue one multiplies this time by $\frac{\partial\overline{\boldsymbol{x}}}{\partial\zeta_\beta}$:

$$\delta_t\boldsymbol{x}\cdot\frac{\partial\overline{\boldsymbol{x}}}{\partial\zeta_\beta}=\delta_t\overline{\boldsymbol{x}}\cdot\frac{\partial\overline{\boldsymbol{x}}}{\partial\zeta_\beta}+\left(\delta\zeta^\alpha\frac{\partial\overline{\boldsymbol{x}}}{\partial\zeta_\alpha}\right)\cdot\frac{\partial\overline{\boldsymbol{x}}}{\partial\zeta_\beta}+\delta\,d_n\boldsymbol{n}\cdot\frac{\partial\overline{\boldsymbol{x}}}{\partial\zeta_\beta}+d_n\left(\delta_t\boldsymbol{n}\cdot\frac{\partial\overline{\boldsymbol{x}}}{\partial\zeta_\beta}+\left(\delta\zeta^\alpha\frac{\partial\boldsymbol{n}}{\partial\zeta_\alpha}\right)\cdot\frac{\partial\overline{\boldsymbol{x}}}{\partial\zeta_\beta}\right) \tag{76}$$

One finds L E tensor metric $\underline{\underline{\boldsymbol{m}}}$ whose components covariantes are worth :

$$m_{\alpha\beta}=\boldsymbol{t}_\alpha\cdot\boldsymbol{t}_\beta \tag{77}$$

If O N C one S id è R E L has property (18) , i.e.:

$$\frac{\partial\overline{\boldsymbol{x}}}{\partial\zeta_\alpha}\cdot\boldsymbol{n}=0 \tag{78}$$

the statement is simplified:

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$$\frac{\partial \overline{\boldsymbol{x}}}{\partial \zeta_\beta}\cdot\left(\delta_t \boldsymbol{x}-\delta_t \overline{\boldsymbol{x}}\right)-d_n \delta_t \boldsymbol{n}\cdot\frac{\partial \overline{\boldsymbol{x}}}{\partial \zeta_\beta}=\delta\zeta^\alpha\left(m_{\alpha\beta}+d_n\frac{\partial \boldsymbol{n}}{\partial \zeta_\alpha}\cdot\frac{\partial \overline{\boldsymbol{x}}}{\partial \zeta_\beta}\right) \tag{79}$$

One sets out again of the statement (78) which one will differentiate:

$$\frac{\partial \overline{\boldsymbol{x}}}{\partial \zeta_\alpha}\cdot\boldsymbol{n}=0\rightarrow\frac{\partial}{\partial \zeta_\beta}\left(\frac{\partial \overline{\boldsymbol{x}}}{\partial \zeta_\alpha}\cdot\boldsymbol{n}\right)=0 \tag{80}$$

While developing:

$$\frac{\partial}{\partial \zeta_\beta}\left(\frac{\partial \overline{\boldsymbol{x}}}{\partial \zeta_\alpha}\cdot\boldsymbol{n}\right)=\frac{\partial^2 \overline{\boldsymbol{x}}}{\partial \zeta_\alpha \partial \zeta_\beta}\cdot\boldsymbol{n}+\frac{\partial \overline{\boldsymbol{x}}}{\partial \zeta_\alpha}\cdot\frac{\partial \boldsymbol{n}}{\partial \zeta_\beta}=0\rightarrow\frac{\partial \overline{\boldsymbol{x}}}{\partial \zeta_\alpha}\cdot\frac{\partial \boldsymbol{n}}{\partial \zeta_\beta}=-\frac{\partial^2 \overline{\boldsymbol{x}}}{\partial \zeta_\alpha \partial \zeta_\beta}\cdot\boldsymbol{n} \tag{81}$$

It is the tensor of curvature (24) except for the sign. To find $\delta\zeta^\alpha$, should be solved the following system:

$$C_{\alpha\beta}\delta\zeta^\alpha=B_\beta \tag{82}$$

With:

$$C_{\alpha\beta}=\left(m_{\alpha\beta}-d_n\kappa_{\alpha\beta}\right)\text{ and }B_\beta=\frac{\partial \overline{\boldsymbol{x}}}{\partial \zeta_\beta}\cdot\left(\delta_t \boldsymbol{x}-\delta_t \overline{\boldsymbol{x}}\right)-d_n\delta_t \boldsymbol{n}\cdot\frac{\partial \overline{\boldsymbol{x}}}{\partial \zeta_\beta} \tag{83}$$

2.3.4 Variation of the tangents

As $\boldsymbol{t}_\alpha$ is an eulerian quantity, one appliqu E (53) out of (17):

$$\tilde{\delta}\boldsymbol{t}_\alpha=\tilde{\delta}\left(\frac{\partial \overline{\boldsymbol{x}}}{\partial \zeta_\alpha}\right)=\frac{\partial\left(\delta_t \overline{\boldsymbol{x}}\right)}{\partial \zeta_\alpha}+\frac{\partial^2 \overline{\boldsymbol{x}}}{\partial \zeta_\alpha \partial \zeta_\beta}\delta\zeta^\beta \tag{84}$$

2.3.5 Second variation of normal clearance

the second variation will be noted $\Delta\delta$ in order to preserve the distinction between derivatives Lagrangian and eulerian (and the notation $\tilde{}$). To compute: the second variation of normal clearance, one does not leave the statement (74) but directly (33) :

$$d_n\boldsymbol{n}=\left(\boldsymbol{x}-\overline{\boldsymbol{x}}\right) \tag{85}$$

the first variation of the term of right of (85) is worth:

$$\tilde{\delta}\left(\boldsymbol{x}-\overline{\boldsymbol{x}}\right)=\left(\delta_t \boldsymbol{x}-\delta_t \overline{\boldsymbol{x}}-\frac{\partial \overline{\boldsymbol{x}}}{\partial \zeta_\alpha}\delta\zeta^\alpha\right) \tag{86}$$

One now will calculate the second variation (total) by applying the operator $\tilde{\Delta}$ to (86). Initially, the fact is used that $\delta_t \boldsymbol{x}$ is a Lagrangian quantity, one thus has:

$$\tilde{\Delta}\,\delta_t \boldsymbol{x}=\Delta_t\delta_t \boldsymbol{x} \tag{87}$$

On the other hand, $\delta_t \overline{\boldsymbol{x}}$ is an eulerian quantity:

$$\tilde{\Delta}\,\delta_t \overline{\boldsymbol{x}}=\Delta_t\delta_t \overline{\boldsymbol{x}}+\frac{\partial\left(\delta_t \overline{\boldsymbol{x}}\right)}{\partial \zeta_\alpha}\Delta\zeta^\alpha \tag{88}$$

Second variation of the convective term of (86):

$$\tilde{\Delta}\left(\frac{\partial \overline{\boldsymbol{x}}}{\partial \zeta_\alpha}\delta\zeta^\alpha\right)=\tilde{\Delta}\left(\frac{\partial \overline{\boldsymbol{x}}}{\partial \zeta_\alpha}\right)\delta\zeta^\alpha+\frac{\partial \overline{\boldsymbol{x}}}{\partial \zeta_\alpha}\tilde{\Delta}\,\delta\zeta^\alpha \tag{formulate8989}$$

), the second variation of (86) is worth:

$$\tilde{\Delta}\,\tilde{\delta}\left(\boldsymbol{x}-\overline{\boldsymbol{x}}\right)=\left(\Delta_t\delta_t \boldsymbol{x}-\Delta_t\delta_t \overline{\boldsymbol{x}}-\frac{\partial\left(\delta_t \overline{\boldsymbol{x}}\right)}{\partial \zeta_\alpha}\Delta\zeta^\alpha-\tilde{\Delta}\left(\frac{\partial \overline{\boldsymbol{x}}}{\partial \zeta_\alpha}\right)\delta\zeta^\alpha-\frac{\partial \overline{\boldsymbol{x}}}{\partial \zeta_\alpha}\tilde{\Delta}\,\delta\zeta^\alpha\right) \tag{90}$$

the first variation of the term of left of (85) is worth:

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$$\tilde{\delta}\left(d_n \boldsymbol{n}\right)=\tilde{\delta}\, d_n \boldsymbol{n}+d_n \tilde{\delta}\, \boldsymbol{n} \tag{91}$$

One now will calculate the second variation by applying the operator $\tilde{\Delta}$ on (91). She is written:

$$\tilde{\Delta}\,\tilde{\delta}\left(d_n \boldsymbol{n}\right)=\tilde{\Delta}\,\tilde{\delta}\, d_n \boldsymbol{n}+\tilde{\delta}\, d_n \tilde{\Delta}\, \boldsymbol{n}+\tilde{\Delta}\, d_n \tilde{\delta}\, \boldsymbol{n}+d_n \tilde{\Delta}\,\tilde{\delta}\, \boldsymbol{n} \tag{92}$$

By equalizing the two equations (90) and (92), one obtains:

$$\begin{aligned}&\left(\Delta_t \delta_t \boldsymbol{x}-\Delta_t \delta_t \overline{\boldsymbol{x}}-\frac{\partial\left(\delta_t \overline{\boldsymbol{x}}\right)}{\partial \zeta_\alpha}\Delta\zeta^\alpha-\tilde{\Delta}\left(\frac{\partial \overline{\boldsymbol{x}}}{\partial \zeta_\alpha}\right)\delta\zeta^\alpha-\frac{\partial \overline{\boldsymbol{x}}}{\partial \zeta_\alpha}\tilde{\Delta}\,\delta\zeta^\alpha\right)= \\ &\left(\tilde{\Delta}\,\tilde{\delta}\, d_n \boldsymbol{n}+\tilde{\delta}\, d_n \tilde{\Delta}\, \boldsymbol{n}+\tilde{\Delta}\, d_n \tilde{\delta}\, \boldsymbol{n}+d_n \tilde{\Delta}\,\tilde{\delta}\, \boldsymbol{n}\right)\end{aligned} \tag{93}$$

One will multiply (93) by the norm $\boldsymbol{n}$. One considers the properties (18) and (19), i.e.:

$$\frac{\partial \overline{\boldsymbol{x}}}{\partial \zeta_\alpha}\cdot\boldsymbol{n}=0 \text{ and } \boldsymbol{n}\cdot\boldsymbol{n}=1 \tag{94}$$

What gives us, after simplifications:

$$\left(\Delta_t \delta_t \boldsymbol{x}-\Delta_t \delta_t \overline{\boldsymbol{x}}-\frac{\partial\left(\delta_t \overline{\boldsymbol{x}}\right)}{\partial \zeta_\alpha}\Delta\zeta^\alpha-\tilde{\Delta}\left(\frac{\partial \overline{\boldsymbol{x}}}{\partial \zeta_\alpha}\right)\delta\zeta^\alpha\right)\cdot\boldsymbol{n}=\tilde{\Delta}\,\tilde{\delta}\, d_n+\left(\tilde{\delta}\, d_n \tilde{\Delta}\, \boldsymbol{n}\right)\cdot\boldsymbol{n}+\left(d_n \tilde{\Delta}\,\tilde{\delta}\, \boldsymbol{n}\right)\cdot\boldsymbol{n} \tag{95}$$

One knows that d_n is a purely Lagrangian quantity (§15), therefore:

$$\tilde{\delta}\, d_n=\delta_t d_n \text{ and } \tilde{\Delta}\,\tilde{\delta}\, d_n=\tilde{\Delta}\,\delta_t d_n=\Delta_t \delta_t d_n \tag{96}$$

One supposes that the variations $\Delta_t \delta_t \boldsymbol{x}$ and $\Delta_t \delta_t \overline{\boldsymbol{x}}$ are null. Moreover, by analogy with (57), one a:

$$\tilde{\Delta}\,\boldsymbol{n}\cdot\boldsymbol{n}=0 \tag{97}$$

What gives us:

$$\Delta_t \delta_t d_n=\left(-\frac{\partial\left(\delta_t \overline{\boldsymbol{x}}\right)}{\partial \zeta_\alpha}\Delta\zeta^\alpha-\tilde{\Delta}\left(\frac{\partial \overline{\boldsymbol{x}}}{\partial \zeta_\alpha}\right)\delta\zeta^\alpha\right)\cdot\boldsymbol{n}-\left(d_n \tilde{\Delta}\,\tilde{\delta}\, \boldsymbol{n}\right)\cdot\boldsymbol{n} \tag{98}$$

One uses (84):

$$\tilde{\Delta}\left(\frac{\partial \overline{\boldsymbol{x}}}{\partial \zeta_\alpha}\right)=\frac{\partial\left(\Delta_t \overline{\boldsymbol{x}}\right)}{\partial \zeta_\alpha}+\frac{\partial^2 \overline{\boldsymbol{x}}}{\partial \zeta_\alpha \partial \zeta_\beta}\Delta\zeta^\beta \tag{formulate9999}$$

):

$$\Delta_t \delta_t d_n=\left(-\frac{\partial\left(\delta_t \overline{\boldsymbol{x}}\right)}{\partial \zeta_\alpha}\Delta\zeta^\alpha-\delta\zeta^\alpha\frac{\partial^2 \overline{\boldsymbol{x}}}{\partial \zeta_\alpha \partial \zeta_\beta}\Delta\zeta^\beta-\frac{\partial\left(\Delta_t \overline{\boldsymbol{x}}\right)}{\partial \zeta_\alpha}\delta\zeta^\alpha\right)\cdot\boldsymbol{n}-\left(d_n \tilde{\Delta}\,\tilde{\delta}\, \boldsymbol{n}\right)\cdot\boldsymbol{n} \tag{100}$$

One recognizes the tensor of curvature (24):

$$\begin{aligned}\Delta_t \delta_t d_n= \quad &-\boldsymbol{n}\cdot\left(\Delta\zeta_\alpha\frac{\partial\left(\delta_t \overline{\boldsymbol{x}}\right)}{\partial \zeta_\alpha}+\frac{\partial\left(\Delta_t \overline{\boldsymbol{x}}\right)}{\partial \zeta_\alpha}\delta\zeta_\alpha\right) \\ &-\delta\zeta_\alpha \kappa_{\alpha\beta}\Delta\zeta_\beta \\ &-\left(d_n \tilde{\Delta}\,\tilde{\delta}\, \boldsymbol{n}\right)\cdot\boldsymbol{n}\end{aligned} \tag{101}$$

It remains to specify the statement of $\tilde{\Delta}\,\tilde{\delta}\, \boldsymbol{n}\cdot\boldsymbol{n}$. One a:

$$\tilde{\delta}\,\boldsymbol{n}\cdot\boldsymbol{n}=0\rightarrow\tilde{\Delta}\left(\tilde{\delta}\,\boldsymbol{n}\cdot\boldsymbol{n}\right)=0\rightarrow\tilde{\Delta}\,\tilde{\delta}\,\boldsymbol{n}\cdot\boldsymbol{n}=-\tilde{\delta}\,\boldsymbol{n}\cdot\tilde{\Delta}\,\boldsymbol{n} \tag{102}$$

the formula of Weingarten (66) gives us:

$$\tilde{\Delta}\,\tilde{\delta}\,\boldsymbol{n}\cdot\boldsymbol{n}=-\left(\tilde{\delta}\left(\frac{\partial \overline{\boldsymbol{x}}}{\partial \zeta_\alpha}\right)\cdot\boldsymbol{n}\right)\cdot\frac{\partial \overline{\boldsymbol{x}}}{\partial \zeta^\alpha}\cdot\left(\tilde{\Delta}\left(\frac{\partial \overline{\boldsymbol{x}}}{\partial \zeta_\beta}\right)\cdot\boldsymbol{n}\right)\cdot\frac{\partial \overline{\boldsymbol{x}}}{\partial \zeta^\beta} \tag{103}$$

the scalar product being commutative, by means of the metric tensor contravariant, one obtains:

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$$\tilde{\Delta}\,\tilde{\delta}\,\boldsymbol{n}\cdot\boldsymbol{n}=-m^{\alpha\beta}\left(\tilde{\delta}\left(\frac{\partial\overline{\boldsymbol{x}}}{\partial\zeta_\alpha}\right)\cdot\boldsymbol{n}\right)\left(\tilde{\Delta}\left(\frac{\partial\overline{\boldsymbol{x}}}{\partial\zeta_\beta}\right)\cdot\boldsymbol{n}\right) \tag{104}$$

It is necessary to develop the variations by application of the statements of (84):

$$\tilde{\Delta}\,\tilde{\delta}\,\boldsymbol{n}\cdot\boldsymbol{n}=-m^{\alpha\beta}\boldsymbol{n}\cdot\left(\frac{\partial\left(\delta_t\overline{\boldsymbol{x}}\right)}{\partial\zeta_\alpha}+\frac{\partial^2\overline{\boldsymbol{x}}}{\partial\zeta_\alpha\partial\zeta_\gamma}\delta\zeta^\gamma\right)\left(\frac{\partial\left(\delta_t\overline{\boldsymbol{x}}\right)}{\partial\zeta_\beta}+\frac{\partial^2\overline{\boldsymbol{x}}}{\partial\zeta_\beta\partial\zeta_\sigma}\delta\zeta^\sigma\right)\cdot\boldsymbol{n} \tag{105}$$

Once more, one finds the tensor of curvature:

$$\tilde{\Delta}\,\tilde{\delta}\,\boldsymbol{n}\cdot\boldsymbol{n}=-m^{\alpha\beta}\left(\boldsymbol{n}\cdot\frac{\partial\left(\delta_t\overline{\boldsymbol{x}}\right)}{\partial\zeta_\alpha}+\kappa_{\alpha\gamma}\delta\zeta^\gamma\right)\left(\boldsymbol{n}\cdot\frac{\partial\left(\delta_t\overline{\boldsymbol{x}}\right)}{\partial\zeta_\beta}+\kappa_{\beta\sigma}\delta\zeta^\sigma\right) \tag{106}$$

Finally:

$$\begin{aligned}\Delta_t\delta_t d_n= \quad & -\boldsymbol{n}\cdot\left(\Delta\zeta_\alpha\frac{\partial\left(\delta_t\overline{\boldsymbol{x}}\right)}{\partial\zeta_\alpha}+\frac{\partial\left(\Delta_t\overline{\boldsymbol{x}}\right)}{\partial\zeta_\alpha}\delta\zeta_\alpha\right)\\ & -\delta\zeta_\alpha\kappa_{\alpha\beta}\Delta\zeta_\beta\\ & -d_n m^{\alpha\beta}\left(\boldsymbol{n}\cdot\frac{\partial\left(\delta_t\overline{\boldsymbol{x}}\right)}{\partial\zeta_\alpha}+\kappa_{\alpha\gamma}\delta\zeta^\gamma\right)\left(\boldsymbol{n}\cdot\frac{\partial\left(\delta_t\overline{\boldsymbol{x}}\right)}{\partial\zeta_\beta}+\kappa_{\beta\sigma}\delta\zeta^\sigma\right)\end{aligned} \tag{107}$$

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3 Models of contact and from friction

In this chapter, we will write the various forms equivalent to the models of standard contact-friction (model of Signorini and model of Coulomb). The demonstration of equivalence between the various forms (Signorini, Coulomb, Lagrangian increased, Lagrangian stabilized) is available in [9].
In *Code_Aster* , in fact the forms “Lagrangian stabilized” are used.

3.1 Model of contact

3.1.1 Model of Signorini

the model of Signorini for the unilateral contact is written:

$$\left(P^C\right)\equiv\begin{cases}\lambda_n\leqslant 0 & (a)\\ d_n\leqslant 0 & (b)\\ \lambda_n d_n=0 & (c)\end{cases} \tag{108}$$

where d_n is the algebraic clearance defined by the equation (39) and λ_n is the contact pressure. The graph of the model of Signorini is represented on Figure 4. The first condition retranscribes the condition of NON-dependancy and the second condition translates impenetrability.

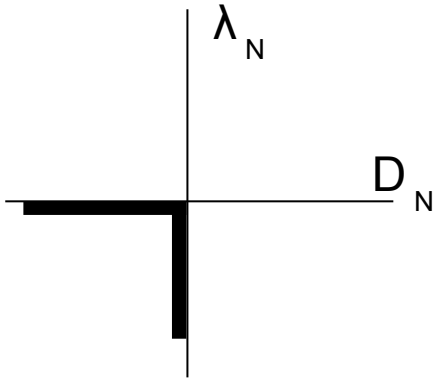

Figure 4: Graph of the model of unilateral contact of Signorini

the problem of contact thus posed introduced a relation *NON-univocal* (λ_n is not a function of d_n), *semi-definite positive* and *NON-differentiable* in $\lambda_n=d_n=0$. The model of contact of Signorini is equivalent to a constitutive law. It is similar to the case of perfect rigid plasticity. It is thus advisable to note that clearance d_n is equivalent worthy of a strain and that he must thus be objective.

3.1.2 Model of contact – convex Analysis

By means of the tools of the convex analysis, one can write the model of Signorini in the following form:

$$\left(P^C_{cvxe}\right)\equiv\begin{cases}\lambda_n\in\partial I_{\mathbb{R}^-}\left(d_n\right) & (a)\\ d_n\in\partial I_{\mathbb{R}^-}\left(\lambda_n\right) & (b)\end{cases} \tag{109}$$

∂ indicating the under-differential and I_K the indicating function of the convex one K :

$$I_K(x)=\begin{cases}0 & \text{si } x\in K\\ \infty & \text{si } x\notin K\end{cases} \tag{110}$$

the model of contact of Signorini consists of inequalities (search for solution in convex given), one uses this writing when one solves the problem of contact by methods of optimization under stresses: it is the privileged methodology of the discrete *formulations* in *Code_Aster*. In the continuous method, one uses a form of semi-regularization by means of operators of projection in convex given.

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3.1.3 Model of contact – Projection

the method of Lagrangian increased applied to the contact consists in using projection on the convex one $\mathbb{R}^-$ of a quantity called multiplier of contact *increased* g_n , defined by:

$$g_n=\lambda_n-\rho_n d_n \tag{111}$$

where ρ_n is a strictly positive reality, homogeneous with a force on a unit of volume:

$$[\rho_n]=\frac{[F]}{[L]^3} \tag{112}$$

the problem of Signorini P^C rewrites itself then:

$$\left\{P^C_{Proj_{\mathbb{R}^-}(g_n)}\right\}\equiv\begin{cases}\lambda_n=Proj_{\mathbb{R}^-}(g_n) & (a)\\ g_n=\lambda_n-\rho_n d_n & (b)\end{cases} \tag{113}$$

the idea of this formulation is of the contact to replace the energy part NON-differentiable, by the square of the projection of g_n in convex suitable, which returns the functional energy of Lagrangian increased differentiable. The operator of projection is worth explicitly:

$$Proj_{\mathbb{R}^-}(g_n)=\begin{cases}g_n \text{ si } g_n\le 0\\ 0 \text{ si } g_n>0\end{cases} \tag{114}$$

the multiplier of contact *increased* g_n is a line in space (d_n,λ_n) . With the operator of projection $Proj_{\mathbb{R}^-}(x)$, one brings back itself on the graph of Signorini. $Proj_{\mathbb{R}^-}(x)$ is continuous on $\mathbb{R}^-$ and almost everywhere differentiable.

3.1.4 Model of contact – Lagrangian stabilized

the idea of the formulation by field of sign (also called stabilized *formulation*) is to dissociate the statute of a point of contact of the value of Lagrangian increased. The same multiplier as g_n in the case of the Lagrangian one increased is introduced (equation 111). One defines the field of sign (“level-set”) noted S_u^x by the following statement:

$$S_u^{g_n}=\begin{cases}1 & \text{si } g_n\le 0\\ 0 & \text{si } g_n>0\end{cases} \tag{115}$$

This operator is the function characteristic of the variable g_n on the convex $\mathbb{R}^-$. He is not differentiable. The problem of Signorini P^C is rewritten then:

$$\left\{P^C_{stab}\right\}\equiv\begin{cases}\lambda_n-S_u^{g_n}(\lambda_n-\rho_n d_n)=0 & (a)\\ g_n=\lambda_n-\rho_n d_n & (b)\\ S_u^{g_n}=\begin{cases}1 & \text{si } g_n\le 0\\ 0 & \text{si } g_n>0\end{cases} & (c)\end{cases} \tag{116}$$

the multiplier of increased *contact* g_n is a line. The non-regular character of the model of contact is the NON-differentiable character coming from the field of sign. S_u^x is almost everywhere continuous (except in $x=0$) and almost everywhere differentiable. The two operators S_u^x and $Proj_{\mathbb{R}^-}(x)$ are bound by a simple relation:

$$x\,S_u^x=Proj_{\mathbb{R}^-}(x) \tag{117}$$

It is this form which is used in *Code_Aster* .

3.2 Friction law

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3.2.1 Model of Coulomb

to represent the phenomena of friction, one uses *the model of Coulomb* which is written as follows:

$$\left\{P^F\right\}\equiv \begin{cases} \text{Si } \|\boldsymbol{\lambda}_\tau\|<\mu|\lambda_n| \text{ alors } \boldsymbol{v}_\tau=\boldsymbol{0} & (a) \\ \text{Si } \|\boldsymbol{\lambda}_\tau\|=\mu|\lambda_n| \text{ alors } \dfrac{\boldsymbol{v}_\tau}{\|\boldsymbol{v}_\tau\|}=-\dfrac{\boldsymbol{\lambda}_\tau}{\|\boldsymbol{\lambda}_\tau\|} & (b) \\ \boldsymbol{v}_\tau\left(\|\boldsymbol{\lambda}_\tau\|-\mu|\lambda_n|\right)=\boldsymbol{0} & (c) \end{cases} \quad (118)$$

where μ is called *coefficient of kinetic friction of Coulomb* (presumedly constant and isotropic), and $\boldsymbol{v}_\tau$ is *tangent relative velocity*, definite like projection in the tangent plane on the surface at the point of contact of the jump of velocity of solid 1 compared to solid 2. When the contact is slipping, as λ_n is always negative by assumption, one can write:

$$\boldsymbol{\lambda}_\tau=\mu\,\lambda_n\frac{\boldsymbol{v}_\tau}{\|\boldsymbol{v}_\tau\|} \quad (119)$$

the formula $\frac{\boldsymbol{v}_\tau}{\|\boldsymbol{v}_\tau\|}$ corresponds to the direction of sliding, unknown, but which one knows that it takes place in the tangent plane made up by both vectors $\left(\boldsymbol{t}_1,\boldsymbol{t}_2\right)$. The third model, similar to that of the equation (108) introduced for the contact, imposes the nullity of the frictional forces when there is not contact. The graph of the friction law of Coulomb is represented on Figure 5.

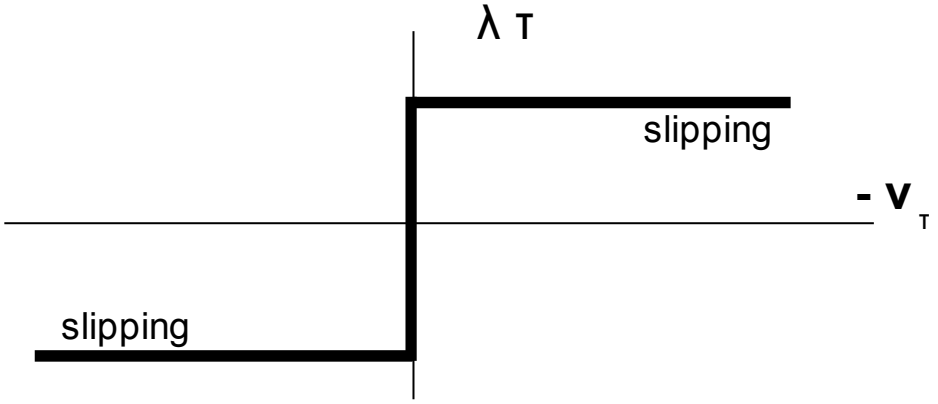

Figure 5: Graph of the friction law of Coulomb

friction induces the notion of threshold. The relation introduced by the friction of Coulomb is *NON-univocal* and *NON-differentiable*. The statement of $\boldsymbol{v}_\tau$ is given by:

$$\boldsymbol{v}_\tau\left(\boldsymbol{p}^1,t\right)=[\![\boldsymbol{v}]\!]_\tau=[\![\boldsymbol{v}]\!]-[\![v]\!]_n\boldsymbol{n} \quad (120)$$

the jump tangential velocity $\boldsymbol{v}_\tau$ **is not objective** (see [12]). In any rigor, formula would have $\boldsymbol{v}_{gliss}$ (see p.1212). However, these two definitions are identical when there is contact (formula $d_n=0$. The friction law of Coulomb is written of *velocity* and not in *displacement*, contrary to the model of Signorini. This constitutes one of the difficulties in the case of considering a quasi-static resolution of the problem of contact-friction. One cannot directly replace the velocity by displacement without making the assumption that the loading in the contact zone varies in a monotonous way, which, in practice, is an assumption impossible to satisfy for an unspecified loading with structure.
Nevertheless, one can write the model of Signorini in incremental form, by considering *the displacement increment*, this way make is correct as regards respect of the model of Coulomb. But the difficulty is that by not knowing *a priori* the value of the total quantity of the contact pressure, the problem of Coulomb must be solved in two times.

3.2.2 Friction law – convex Analysis

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By means of the tools of the convex analysis, one can write the model of Coulomb in the following form:

$$\left\{P^F_{cvxe}\right\}\equiv\begin{cases}\boldsymbol{\lambda}_\tau\in\partial I_{C(\lambda_n)}(\boldsymbol{v}_\tau) & (a)\\ \boldsymbol{v}_\tau\in\partial I_{C^*(\lambda_n)}(\boldsymbol{\lambda}_\tau) & (b)\end{cases} \tag{121}$$

With the convex $C(\lambda_n)$ which represents the cone of Coulomb:

$$C(\lambda_n)=\left\{\forall\,\boldsymbol{\lambda}_\tau \text{ tel que } \|\boldsymbol{\lambda}_\tau\|<\mu|\lambda_n|\right\} \tag{122}$$

3.2.3 Friction law – Projection

the method of Lagrangian increased applied to friction is similar to the case of the contact, except that the convex one considered changes. Let us consider the multiplier of friction *increased* $\boldsymbol{g}_\tau$, defined by:

$$\boldsymbol{g}_\tau=\boldsymbol{\lambda}_\tau+\rho_\tau\boldsymbol{v}_\tau \tag{123}$$

where ρ_τ is a strictly positive reality, homogeneous with a force on a unit of volume:

$$[\rho_\tau]=\frac{[F]}{[L]^3} \tag{124}$$

the problem of Coulomb P^F rewrites itself then:

$$\left\{P^F_{Proj_{C(\lambda_n)}}\right\}\equiv\begin{cases}\boldsymbol{\lambda}_\tau=Proj_{C(\lambda_n)}(\boldsymbol{g}_\tau) & (a)\\ \boldsymbol{g}_\tau=\boldsymbol{\lambda}_\tau+\rho_\tau\boldsymbol{v}_\tau & (b)\end{cases} \tag{125}$$

the operator of projection is not orthogonal:

$$Proj_{C(\lambda_n)}(\boldsymbol{g}_\tau)=\begin{cases}\boldsymbol{g}_\tau \text{ si } \boldsymbol{g}_\tau\in C(\lambda_n)\\ \dfrac{\mu\left|\lambda_n-\rho_n d_n\right|\boldsymbol{g}_\tau}{\|\boldsymbol{g}_\tau\|} \text{ si } \boldsymbol{g}_\tau\notin C(\lambda_n)\end{cases} \tag{126}$$

3.2.4 Friction law – Projection standardized

In the contact, we used the same definition of the multiplier increased between the increased Lagrangian formulation and the stabilized Lagrangian formulation. On the other hand for friction, we will proceed differently. Initially, one introduces a quantity $\boldsymbol{\Lambda}$ called **semi-multiplier of Lagrange** :

$$\boldsymbol{\lambda}_\tau=\mu\,\lambda_n\boldsymbol{\Lambda} \tag{127}$$

This standardized quantity belongs to the ball unit:

$$\boldsymbol{\Lambda}\in B(0,1)\subset\mathbb{R}^2 \tag{128}$$

One calls it **semi-multiplier** of Lagrange because this quantity has two meanings:
1.In the case of adherent friction, $\boldsymbol{\Lambda}$ represents the density of force of dependancy. It is thus formally the multiplier of Lagrange associated with the condition with dependancy (tangential relative velocity null);

2.In the case of the sliding, $\boldsymbol{\Lambda}$ indicates the direction of sliding (standardized) and one has $\boldsymbol{\Lambda}=\dfrac{-\boldsymbol{\lambda}_\tau}{\|\boldsymbol{\lambda}_\tau\|}=\dfrac{\boldsymbol{v}_\tau}{\|\boldsymbol{v}_\tau\|}$

; what is not the definition of a multiplier of Lagrange to the “classical” meaning (see p.22);
It is noticed that λ_n being always negative or null (convention of usual pressure), $\boldsymbol{\lambda}_\tau$ is well of sign opposed to $\boldsymbol{\Lambda}$, which indicates the direction of sliding. This change of variable makes it possible to pass from the unknown in tangent force towards the direction of this force. Consequently, one introduces the semi-multiplier increased of friction $\boldsymbol{h}_\tau$ (to distinguish it from $\boldsymbol{g}_\tau$ the “classic”, to see (123) such as:

$$\boldsymbol{h}_\tau=\boldsymbol{\Lambda}+\rho_t\boldsymbol{v}_\tau \tag{129}$$

It should be noted that the coefficient of increase ρ_t has a unit different from ρ_τ in the case of the Lagrangian one increased. Indeed, he is written like the ratio of a time over a length:

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$$\left[\rho_t\right]=\frac{\left[T\right]}{\left[L\right]} \tag{130}$$

In the increased Lagrangian formulation, the coefficient of increase ρ_τ is same unit as the coefficient of increase in contact (see 112). The problem of Coulomb P^F is rewritten :

$$\left\{P^F_{Proj_{B(0,1)}}\right\}\equiv\begin{cases}\boldsymbol{\Lambda}=Proj_{B(0,1)}\left(\boldsymbol{h}_\tau\right) & (a)\\ \boldsymbol{h}_\tau=\boldsymbol{\Lambda}+\rho_t\boldsymbol{v}_\tau & (b)\end{cases} \tag{131}$$

This operator of projection is orthogonal:

$$Proj_{B(0,1)}\left(\boldsymbol{h}_\tau\right)=\begin{cases}\boldsymbol{h}_\tau \text{ si } \boldsymbol{h}_\tau\in B(0,1)\\ \boldsymbol{\tau}=\dfrac{\boldsymbol{h}_\tau}{\|\boldsymbol{h}_\tau\|} \text{ si } \boldsymbol{h}_\tau\notin B(0,1)\end{cases} \tag{132}$$

One will note:

$$\boldsymbol{\tau}=\frac{\boldsymbol{h}_\tau}{\|\boldsymbol{h}_\tau\|} \tag{133}$$

One will proceed to the calculus of the variation compared to the variable $\boldsymbol{h}_\tau$:

$$\delta_{\boldsymbol{h}_\tau}Proj_{B(0,1)}\left(\boldsymbol{h}_\tau\right)=\begin{cases}\underline{\underline{\mathbf{1}}} \text{ si } \boldsymbol{h}_\tau\in B(0,1)\\ \underline{\underline{\boldsymbol{P}}}^{B(0,1)} \text{ si } \boldsymbol{h}_\tau\notin B(0,1)\end{cases} \tag{134}$$

One will note the matrix of projection on the ball unit:

$$\underline{\underline{\boldsymbol{P}}}^{B(0,1)}=\frac{1}{\|\boldsymbol{h}_\tau\|}\left(\underline{\underline{\mathbf{1}}}-\frac{\boldsymbol{h}_\tau\otimes\boldsymbol{h}_\tau}{\|\boldsymbol{h}_\tau\|^2}\right) \tag{135}$$

3.2.5 Friction law – Lagrangian stabilized

One defines the field of sign (“level-set”) noted $S_f^{\boldsymbol{h}_\tau}$ by the following statement:

$$S_f^{\boldsymbol{h}_\tau}=\begin{cases}1 & \text{si } \boldsymbol{h}_\tau\in B(0,1)\\ 0 & \text{sinon}\end{cases} \tag{136}$$

This operator is the function characteristic of the variable $\boldsymbol{h}_\tau$ on the convex $B(0,1)$. He is not differentiable. The problem of Coulomb P^F is rewritten then:

$$\left\{P^F_{stab}\right\}\equiv\begin{cases}\left(1-S_u^{g_n}\right)\boldsymbol{\Lambda}+S_u^{g_n}\left(\left(1-S_f^{\boldsymbol{h}_\tau}\right)\boldsymbol{\Lambda}-\rho_t\boldsymbol{v}_\tau S_f^{\boldsymbol{h}_\tau}-\left(1-S_f^{\boldsymbol{h}_\tau}\right)\boldsymbol{\tau}\right)=0 & (a)\\ \boldsymbol{h}_\tau=\boldsymbol{\Lambda}+\rho_t\boldsymbol{v}_\tau & (b)\\ S_f=\begin{cases}1 & \text{si } \boldsymbol{h}_\tau\in B(0,1)\\ 0 & \text{sinon}\end{cases} & (c)\end{cases} \tag{137}$$

the stabilized Lagrangian form is an exact ***formulation*** of the problem of friction. It is this form which is used in *Code_Aster*.

3.3 Model of Augmented Lagrangian

3.3.1 contact-friction

We write the Lagrangian one increased problem of contact-friction. Let us consider the convex one K such as:

$$K=\left\{\boldsymbol{\lambda}\,;\boldsymbol{\lambda}=\boldsymbol{\lambda}_\tau+\lambda_n\boldsymbol{n} \quad \text{avec} \quad \lambda_n\in\mathbb{R}^- \quad \text{et} \quad \boldsymbol{\lambda}_\tau\in C\left(\lambda_n\right)\right\} \tag{138}$$

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Then the Lagrangian one increased problem of contact/friction is written:

$$\left\{P_{augm}^{CF}\right\}\equiv\begin{cases}\boldsymbol{\lambda}-Proj_K\left(\hat{\boldsymbol{\lambda}}\right)=0 & (a)\\ \hat{\boldsymbol{\lambda}}=\boldsymbol{g}_\tau+g_n\boldsymbol{n} & (b)\\ g_n=\lambda_n-\rho_n d_n & (c)\\ \boldsymbol{g}_\tau=\boldsymbol{\lambda}_\tau+\rho_t\boldsymbol{v}_\tau & (d)\end{cases} \quad (139)$$

projection on the convex one K is worth explicitly:

$$Proj_K\left(\hat{\boldsymbol{\lambda}}\right)=\begin{cases}\hat{\boldsymbol{\lambda}}\ si\ \hat{\boldsymbol{\lambda}}\in K & (a)\\ g_n\boldsymbol{n}+\mu\lambda_n\dfrac{\boldsymbol{g}_\tau}{\|\boldsymbol{g}_\tau\|}\ si\ g_n\le 0\ et\ \|\boldsymbol{g}_\tau\|>\mu\lambda_n & (b)\\ 0\ si\ \lambda_n>0 & (c)\end{cases} \quad (140)$$

the case (140) corresponds to the adherent contact, the case (140) is the slipping contact and the case (140) is the case without contact (nor friction).

3.3.2 Lagrangian stabilized

We write the Lagrangian one stabilized problem of contact/friction:

$$\left\{P_{stab}^{CF}\right\}\equiv\begin{cases}\lambda_n-S_u^{g_n}\left(\lambda_n-\rho_n d_n\right)=0 & (a)\\ \left(1-S_u^{g_n}\right)\boldsymbol{\Lambda}+S_u^{g_n}\left(\left(1-S_f^{\boldsymbol{h}_\tau}\right)\boldsymbol{\Lambda}-\rho_t\boldsymbol{v}_\tau S_f^{\boldsymbol{h}_\tau}-\left(1-S_f^{\boldsymbol{h}_\tau}\right)\boldsymbol{\tau}\right)=0 & (b)\\ g_n=\lambda_n-\rho_n d_n & (c)\\ \boldsymbol{h}_\tau=\boldsymbol{\Lambda}+\rho_t\boldsymbol{v}_\tau & (d)\\ \boldsymbol{\lambda}_\tau=\mu\lambda_n\boldsymbol{\Lambda} & (e)\\ \boldsymbol{\tau}=\dfrac{\boldsymbol{h}_\tau}{\|\boldsymbol{h}_\tau\|} & (f)\end{cases} \quad (141)$$

With the two fields of sign:

$$S_u^{g_n}=\begin{cases}1 & \text{si } g_n\le 0\\ 0 & \text{si } g_n>0\end{cases}\ \text{and}\ S_f^{\boldsymbol{h}_\tau}=\begin{cases}1 & \text{si } \boldsymbol{h}_\tau\in B(0,1)\\ 0 & \text{sinon}\end{cases} \quad (142)$$

It is this form which is used in *Code_Aster* .

3.3.3 Differences between Augmented Lagrangian and Lagrangian stabilized

the two writings belong to the category of the methods of optimization without stresses. The principal differences come from the use of a “field of sign” (or level-set), and, for friction, of a change of variable reducing the disc of Coulomb to a disc of radius unit.

- By dissociating the writing of Lagrangian increased that of the statute (contact/separation and member/sliding), this enables us to introduce the unknowns of statutes *explicitly* in the case of the Lagrangian one stabilized;
- In the Lagrangian one stabilized, one uses a semi-multiplier of friction, which has for principal effect to have a different unit between the Lagrangian one increased friction ρ_t and the Lagrangian one increased contact ρ_n ; It is this form which east uses in *Code_Aster*.

One will find the demonstration of equivalence between these two writings in [9].

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4 Hybrid formulation continues of the contact/friction

to express the equilibrium of structure, in taking into account contact-friction, we have two ways proceed:
1.One expresses the classical form of the principle of the virtual wors by regarding the contact as a force of interaction of surface (one integrates it in the statement of the work of the internal forces); To supplement the system, one adds to it the weak statement of the models of contact-friction;
2.One directly writes the energy of the system in the form of variational inequality mixed (without forcing the space of the solutions by the conditions of contact/friction). The statement of the criteria of optimality by the use of Lagrangian increased (problem of POINT-saddles) makes it possible to find the system established starting from the PTV;

4.1 Spaces

the space of Hilbert $H^1(\Omega)$, complete vector space, usually the space of Sobolev provided with a suitable norm (the scalar product), is the space of the measurable functions of square L^2 and whose derivative with the weak meaning also belongs to L^2. The dual space of $H^1(\Omega)$ will be noted H^{-1}. One indicates by CA^i space containing the kinematically admissible functions such as:

$$CA^i=\left\{\boldsymbol{u}^i\in H^1(\Omega^i);\boldsymbol{u}^i=\mathbf{0}\ \text{sur}\ \Gamma_u^i\right\} \tag{143}$$

$H^{1/2}(\Gamma)$ is the space of the traces of functions (pertaining to H^1) on the border Γ and $H^{-1/2}(\Gamma)$ is its dual space. To simplify, one will note $H=H^{-1/2}(\Gamma)$ and $\boldsymbol{H}=\left(H^{-1/2}(\Gamma)\right)^2$ his counterpart with two dimensions (for friction).

4.2 Principle of the virtual wors

the equations for the equilibrium of the body B^i are:

$$\begin{cases}\mathbf{div}\,\underline{\underline{\boldsymbol{\Pi}}}^i+\boldsymbol{f}_v^i=0 & \text{dans}\ \Omega_0^i \quad (a)\\ \underline{\underline{\boldsymbol{\Pi}}}^i\boldsymbol{N}^i=\boldsymbol{f}_s^i & \text{sur}\ \Gamma_g^i \quad (b)\\ \boldsymbol{u}^i=\mathbf{0} & \text{sur}\ \Gamma_u^i \quad (c)\\ \underline{\underline{\boldsymbol{\Pi}}}^i\boldsymbol{N}^i=\boldsymbol{\lambda}^i & \text{sur}\ \Gamma_c^i \quad (d)\end{cases} \tag{144}$$

Where $\underline{\underline{\boldsymbol{\Pi}}}^i$ indicates the first *tensor of the Piola-Kirchhoff stresses* (NON-symmetric), $\boldsymbol{u}^i$ the field of displacements and $\boldsymbol{\lambda}^i$ density of force due to the interactions of contact rubbing between two solids.

4.2.1 Virtual wor of the internal forces

the principle of virtual wors (PTV) makes it possible to express the bilinear form of the internal forces :

$$G_{\text{int}}^i=\int_{\Omega_0^i}\left(\underline{\underline{\boldsymbol{E}}}^i\,\underline{\underline{\boldsymbol{S}}}^i:\underline{\underline{\boldsymbol{grad}}}\,\tilde{\boldsymbol{u}}^i\right)d\Omega \tag{145}$$

Where $\tilde{\boldsymbol{u}}^i$ indicates the field of virtual displacements. $\underline{\underline{\boldsymbol{S}}}^i$ is the second tensor of the Piola-Kirchhoff stresses, symmetric and purely Lagrangian, related to the first tensor $\underline{\underline{\boldsymbol{\Pi}}}^i$ by the relation:

$$\underline{\underline{\boldsymbol{S}}}^i=\left(\underline{\underline{\boldsymbol{E}}}^i\right)^{-1}\underline{\underline{\boldsymbol{\Pi}}}^i \tag{146}$$

4.2.2 Virtual wor of the external forces

the density of the volume forces $\boldsymbol{f}_v^i$ and surface $\boldsymbol{f}_s^i$ applied to two solids produce a work G_{ext}^i :

$$G_{\text{ext}}^i=\int_{\Omega_0^i}\boldsymbol{f}_v^i(\boldsymbol{u}^i)\tilde{\boldsymbol{u}}^i\,d\Omega+\int_{\Gamma_g^i}\boldsymbol{f}_s^i(\boldsymbol{u}^i)\tilde{\boldsymbol{u}}^i\,d\Gamma \tag{147}$$

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4.2.3 Virtual wor of the forces of contact-friction

G_{cf}^{i} is the work of the forces of contact-friction on solid B^{i} :

$$G_{\text{cf}}^{i}=\int_{\Gamma_c^i}\boldsymbol{\lambda}^{i}\left(\boldsymbol{p}^{i},t\right)\tilde{\boldsymbol{u}}^{i}d\Gamma \tag{148}$$

$\boldsymbol{\lambda}^{i}$ is the density of the forces of contact-friction undergone by solid B^{i} . By means of the procedure of pairing, described with the subparagraph §2.2.12.2.1, *the principle of the action and the reaction* is written locally, on the initial configurations in the following form:

$$\boldsymbol{\lambda}^{1}\left(\boldsymbol{p}^{1},t\right)d\Gamma_c^1+\boldsymbol{\lambda}^{2}\left(\bar{\boldsymbol{p}},t\right)d\Gamma_c^2=0 \tag{149}$$

One poses $\Gamma_c=\Gamma_c^1$ and $\boldsymbol{\lambda}^{1}=\boldsymbol{\lambda}$ and one can thus take:

$$\boldsymbol{\lambda}^{1}d\Gamma_c^1=-\boldsymbol{\lambda}^{2}d\Gamma_c^2=\boldsymbol{\lambda}\,d\Gamma_c \tag{150}$$

What of the contact gives us a new form of the term of reaction:

$$G_{\text{cf}}^{i}\rightarrow G_{\text{cf}}=\int_{\Gamma_c}\boldsymbol{\lambda}\left(\tilde{\boldsymbol{u}}^{1}-\tilde{\boldsymbol{u}}^{2}\right)d\Gamma_c=\int_{\Gamma_c}\boldsymbol{\lambda}\left[\!\left[\tilde{\boldsymbol{u}}\right]\!\right]d\Gamma_c \tag{151}$$

$\tilde{\boldsymbol{u}}^{i}$ are the fields of virtual displacements, kinematically admissible on each solid. Moreover, one posed $\Gamma_c=\Gamma_c^1$ and $\boldsymbol{\lambda}^{1}=\boldsymbol{\lambda}$. It is announced that the density of forces $\boldsymbol{\lambda}^{2}$ is prolonged by zero at the points of Γ_c^2 without opposite on Γ_c^1 . One breaks up the density of force of contact $\boldsymbol{\lambda}$ into a normal part $\lambda_n.\boldsymbol{n}$ and a tangential part $\boldsymbol{\lambda}_{\tau}$ such as:

$$\boldsymbol{\lambda}=\boldsymbol{\lambda}_{\tau}+\lambda_n\boldsymbol{n} \tag{152}$$

While projecting on the ball unit (see (127) in the §22), one a:

$$\boldsymbol{\lambda}_{\tau}=\mu\,\lambda_n\boldsymbol{\Lambda} \tag{153}$$

the reaction is discontinuous and takes two values (contact or not). One uses the field of sign $S_u^{g_n}$ introduced with the §20:

$$\boldsymbol{\lambda}=S_u^{g_n}\left(\lambda_n\boldsymbol{n}+\mu\,\lambda_n\boldsymbol{\Lambda}\right) \tag{154}$$

the semi-multiplier of friction is also discontinuous and takes two values (slipping or adherent). By means of the field of sign $S_f^{\boldsymbol{h}_{\tau}}$ introduced with the §23 :

$$\boldsymbol{\Lambda}=S_f^{\boldsymbol{h}_{\tau}}\boldsymbol{\Lambda}+\left(1-S_f^{\boldsymbol{h}_{\tau}}\right)\boldsymbol{\tau} \tag{155}$$

By reinjecting equation 155 in equation 154 :

$$\boldsymbol{\lambda}=S_u^{g_n}\left(\lambda_n\boldsymbol{n}+\mu\,\lambda_n\left(S_f^{\boldsymbol{h}_{\tau}}\boldsymbol{\Lambda}+\left(1-S_f^{\boldsymbol{h}_{\tau}}\right)\boldsymbol{\tau}\right)\right) \tag{156}$$

We will add here the term known as “of stabilization”, which will make it possible to find the behavior of Lagrangian increased standard. The idea is the following one, in the event of contact with dependancy ($S_u^{g_n}=1$ and $S_f^{\boldsymbol{h}_{\tau}}=1$), the reaction of contact friction (156) is written:

$$\boldsymbol{\lambda}=\lambda_n\boldsymbol{n}+\mu\,\lambda_n\boldsymbol{\Lambda} \tag{157}$$

One can modify the writing of the reaction of contact/friction:

$$\boldsymbol{\lambda}=S_u^{g_n}\left(\left(\lambda_n-\rho_n d_n\right)\boldsymbol{n}+\mu\,\lambda_n\left(S_f^{\boldsymbol{h}_{\tau}}\boldsymbol{h}_{\tau}+\left(1-S_f^{\boldsymbol{h}_{\tau}}\right)\boldsymbol{\tau}\right)\right) \tag{158}$$

This addition does not change anything in the case of the adherent contact, because, in this case, $d_n=0$ and $\boldsymbol{v}_{\tau}=0$, which makes it possible to find (157). On the other hand, in the other cases (not of contact or slipping contact), the parameters ρ_n and ρ_t make it possible to avoid the zero on the diagonal of the matrix. Finally, the term of reaction of contact-friction in the PTV can break up into a component for the contact:

$$G_c=\int_{\Gamma_c}S_u^{g_n}g_n\left[\!\left[\tilde{u}\right]\!\right]_n d\Gamma_c \tag{159}$$

And a component for friction:

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$$\begin{aligned} G_{\mathrm{f}}= \quad & \int_{\Gamma_c} \mu \lambda_n S_u^{g_n} S_f^{\boldsymbol{h}_\tau} \boldsymbol{h}_\tau \cdot [\![\tilde{\boldsymbol{u}}]\!]_\tau d\Gamma_c + \\ & \int_{\Gamma_c} \mu \lambda_n S_u^{g_n} \left(1 - S_f^{\boldsymbol{h}_\tau}\right) \boldsymbol{\tau} \cdot [\![\tilde{\boldsymbol{u}}]\!]_\tau d\Gamma_c \end{aligned} \tag{160}$$

4.2.4 weak Formulation of the model of contact

We introduce the weak form of the model of contact. It is enough for that to take again the equation (141) by multiplying it by the field-test of the Lagrange of contact:

$$\tilde{G}_{\mathrm{c}} = -\frac{1}{\rho_n} \int_{\Gamma_c} \left\{ \lambda_n - S_u^{g_n} g_n \right\} \tilde{\lambda}_n d\Gamma_c \tag{161}$$

4.2.5 weak Formulation of the friction law

We introduce the weak form of the friction law. It is enough for that to take again the equation (141) by multiplying it by the field-test of the Lagrange of friction:

$$\begin{aligned} \tilde{G}_{\mathrm{f}} = \quad & \frac{1}{\rho_t} \int_{\Gamma_c} \mu \lambda_n S_u^{g_n} \boldsymbol{\Lambda} \cdot \tilde{\boldsymbol{\Lambda}} d\Gamma_c - \\ & \frac{1}{\rho_t} \int_{\Gamma_c} \mu \lambda_n S_u^{g_n} S_f^{\boldsymbol{h}_\tau} \boldsymbol{h}_\tau \cdot \tilde{\boldsymbol{\Lambda}} d\Gamma_c - \\ & \frac{1}{\rho_t} \int_{\Gamma_c} \mu \lambda_n S_u^{g_n} (1 - S_f^{\boldsymbol{h}_\tau}) \boldsymbol{\tau} \cdot \tilde{\boldsymbol{\Lambda}} d\Gamma_c + \\ & \int_{\Gamma_c} \left(1 - S_u^{g_n}\right) \boldsymbol{\Lambda} \cdot \tilde{\boldsymbol{\Lambda}} d\Gamma_c \end{aligned} \tag{162}$$

4.2.6 Application of the principle of the virtual wors

If the quasi-static problem is considered, one seeks to solve the equilibrium in weak form by application of the principle of the virtual wors:

To find the fields $\boldsymbol{W} = \left(\boldsymbol{u}^1, \boldsymbol{u}^2, \lambda_n, \boldsymbol{\Lambda}\right) \in \mathrm{CA}^1 \times \mathrm{CA}^2 \times H \times \boldsymbol{H}$ such as

$$\begin{cases} \sum_{i=1}^{2} \left[G_{\mathrm{ext}}^i(\boldsymbol{W}, \tilde{\boldsymbol{W}}) - G_{\mathrm{int}}^i(\boldsymbol{W}, \tilde{\boldsymbol{W}}) \right] - G_{\mathrm{c}}(\boldsymbol{W}, \tilde{\boldsymbol{W}}) - G_{\mathrm{f}}(\boldsymbol{W}, \tilde{\boldsymbol{W}}) = 0 & (a) \\ \tilde{G}_{\mathrm{c}}(\boldsymbol{W}, \tilde{\boldsymbol{W}}) = 0 & (b) \\ \tilde{G}_{\mathrm{f}}(\boldsymbol{W}, \tilde{\boldsymbol{W}}) = 0 & (b) \end{cases} \tag{163}$$

$$\forall \tilde{\boldsymbol{W}} = \left(\tilde{\boldsymbol{u}}^1, \tilde{\boldsymbol{u}}^2, \tilde{\lambda}_n, \tilde{\boldsymbol{\Lambda}}\right) \in \mathrm{CA}^1 \times \mathrm{CA}^2 \times H \times \boldsymbol{H}$$

4.3 mixed variational Inequality

All in all, it acts to find the solution of the following problem (mixed variational inequality):

To find the fields $\boldsymbol{W} = \left(\boldsymbol{u}^1, \boldsymbol{u}^2, \lambda_n, \boldsymbol{\Lambda}\right) \in \mathrm{CA}^1 \times \mathrm{CA}^2 \times H \times \boldsymbol{H}$ such as

$$\left(\boldsymbol{u}^1, \boldsymbol{u}^2, \lambda_n, \boldsymbol{\Lambda}\right) = \underset{(\tilde{\boldsymbol{u}}^1, \tilde{\boldsymbol{u}}^2)}{\operatorname{argmin}} \ \underset{(\tilde{\lambda}_n, \tilde{\boldsymbol{\Lambda}})}{\operatorname{argmax}} \left\{ \sum_{i=1}^{2} \left[W_{\mathrm{int}}^i - W_{\mathrm{ext}}^i \right] - W_{\mathrm{c}} - W_{\mathrm{f}} \right\} \tag{164}$$

W_{int}^i and W_{ext}^i correspond to internal energy and external energy. We will write the statement of the energy of contact and friction. That is to say l_n the density continues and differentiable energy of contact:

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$$W_{\mathrm{c}}=\int_{\Gamma_c} l_n d\Gamma_c \tag{165}$$

And l_t the density continues and differentiable energy of friction:

$$W_{\mathrm{f}}=\int_{\Gamma_c} \mu\lambda_n l_t d\Gamma_c \tag{166}$$

Compared to a classical increased Lagrangian formulation, one carried out the change of variable $\boldsymbol{\lambda}_\tau=\mu\lambda_n\boldsymbol{\Lambda}$ (see §22). For the density of energy of contact:

$$l_n=\frac{1}{2\rho_n}\left(S_u^{g_n}g_n^2-\lambda_n^2\right) \tag{167}$$

One recalls that L E normal clearance d_n is written:

$$d_n=[\![x]\!]_n=(\boldsymbol{x}-\bar{\boldsymbol{x}})\cdot\boldsymbol{n} \tag{168}$$

In the same way, the field of sign depends on the fields of displacements and Lagrangian of contact:

$$S_u^{g_n}=\begin{cases}1 & \text{si } g_n\leq 0\\ 0 & \text{si } g_n>0\end{cases} \tag{169}$$

For the density of energy of friction:

$$\begin{aligned} l_t= \quad & \frac{1}{2\rho_\tau}\left(1-S_u^{g_n}\right)\boldsymbol{\Lambda}\cdot\boldsymbol{\Lambda}+\\ & \frac{1}{2\rho_\tau}S_u^{g_n}S_f^{\boldsymbol{h}_\tau}\left[\boldsymbol{h}_\tau\cdot\boldsymbol{h}_\tau-\boldsymbol{\Lambda}\cdot\boldsymbol{\Lambda}\right]+\\ & \frac{1}{2\rho_\tau}S_u^{g_n}\left(1-S_f^{\boldsymbol{h}_\tau}\right)\boldsymbol{\tau}\cdot\boldsymbol{\tau}+\\ & \frac{1}{2\rho_\tau}S_u^{g_n}\left(1-S_f^{\boldsymbol{h}_\tau}\right)\left(-2\|\boldsymbol{h}_\tau\|+\boldsymbol{\Lambda}\cdot\boldsymbol{\Lambda}\right)\end{aligned} \tag{170}$$

to find the statement of the equilibrium to the weak meaning, it is enough to apply the condition of optimality (164) out of (165) and (166). By variation $\delta_{\boldsymbol{u}^i}$ on the energy of contact W_{c}, one obtains:

$$\delta_{\boldsymbol{u}^i}W_{\mathrm{c}}=\delta_{\boldsymbol{u}^i}\left(\int_{\Gamma_c} l_n d\Gamma_c\right)=\int_{\Gamma_c} S_u^{g_n}g_n\delta d_n d\Gamma_c \tag{171}$$

One finds the statement of the virtual wor of the forces of contact (159). In the same way, by variation $\delta_{\boldsymbol{u}^i}$ on the energy of contact W_{f}, one obtains the statement of the virtual wor of the forces of friction (160):

$$\begin{aligned}\delta_{\boldsymbol{u}^i}W_{\mathrm{f}} \quad &=\delta_{\boldsymbol{u}^i}\left(\int_{\Gamma_c}\mu\lambda_n l_t d\Gamma_c\right)\\ &=\int_{\Gamma_c} S_u^{g_n}\mu\lambda_n S_f^{\boldsymbol{h}_\tau}\boldsymbol{h}_\tau\cdot\delta\boldsymbol{v}_\tau d\Gamma_c+\\ &\quad \int_{\Gamma_c} S_u^{g_n}\mu\lambda_n\left(1-S_f^{\boldsymbol{h}_\tau}\right)\boldsymbol{\tau}\cdot\delta\boldsymbol{v}_\tau d\Gamma_c\end{aligned} \tag{172}$$

to find the statement of the virtual wor of the model of contact (equation 161), one calculates the variation from W_c the Lagrangian one of contact λ_n and for the virtual wor of the friction law (equation 162), one expresses the variation from W_f the Lagrangian one of standardized friction $\boldsymbol{\Lambda}$.

4.4 Variational formulation penalized

to write the penalized variational formulation, one regularizes the models of contact and friction while writing:

$$\begin{aligned}\lambda_n&=-(\kappa_n d_n)^+\\ \boldsymbol{\Lambda}&=P^{B(0,1)}(\kappa_\tau\boldsymbol{v}_\tau)\end{aligned} \tag{173}$$

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the two parameters κ_n and κ_τ are strictly positive parameters of penalization. $(.)^+$ the positive part and $Proj_{B(0,1)}$ projection on the ball unit (see definition with the §22). The weak form of the reaction of contact is written:

$$G_c = \int_{\Gamma_c} \kappa_n d_n S_u^{g_n} [\![\tilde{u}]\!]_n \, d\Gamma_c \tag{174}$$

For the reaction of friction:

$$\begin{aligned} G_f = \quad & -\int_{\Gamma_c} \mu \lambda_n \kappa_\tau \tilde{S}_u^{g_n} \tilde{S}_f^{\boldsymbol{h}_\tau} \boldsymbol{v}_\tau \cdot [\![\tilde{\boldsymbol{u}}]\!]_\tau \, d\Gamma_c \\ & -\int_{\Gamma_c} \mu \lambda_n \tilde{S}_u^{g_n} \left(1 - \tilde{S}_f^{\boldsymbol{h}_\tau}\right) \frac{\kappa_t \boldsymbol{v}_\tau}{\|\kappa_t \boldsymbol{v}_\tau\|} \cdot [\![\tilde{\boldsymbol{u}}]\!]_\tau \, d\Gamma_c \end{aligned} \tag{175}$$

the fields of signs are modified:

$$\begin{aligned} \tilde{S}_u^{g_n} &= \begin{cases} 1 & \text{si } -d_n \leq 0 \text{ (contact)} \\ 0 & \text{si } -d_n > 0 \text{ (décollement)} \end{cases} \\ \tilde{S}_f^{\boldsymbol{h}_\tau} &= \begin{cases} 1 & \text{si } \|\kappa_t \boldsymbol{v}_\tau\| \leq 1 \text{ (adhérence)} \\ 0 & \text{si } \|\kappa_t \boldsymbol{v}_\tau\| > 1 \text{ (décollement)} \end{cases} \end{aligned} \tag{176}$$

In the penalized formulation, the multipliers of contact λ_n and the semi-multipliers of friction $\boldsymbol{\Lambda}$ do not have theoretically more raison d'être but one preserves them for reasons of data-processing architecture. One thus writes a weak form corresponding to the model of contact/friction (173) : one thus has access to contact pressures in penalized formulation:

$$\tilde{G}_c = -\frac{1}{\kappa_n} \int_{\Gamma_c} \left(\lambda_n + \tilde{S}_u^{g_n} \kappa_n d_n\right) \tilde{\lambda}_n \, d\Gamma_c \tag{177}$$

Like with the density of force of dependancy thanks to the weak form of the model of Coulomb:

$$\begin{aligned} \tilde{G}_f = \quad & \frac{1}{\kappa_t} \int_{\Gamma_c} \mu \lambda_n \tilde{S}_u^{g_n} \boldsymbol{\Lambda} \cdot \tilde{\boldsymbol{\Lambda}} \, d\Gamma_c - \\ & \frac{1}{\kappa_t} \int_{\Gamma_c} \mu \lambda_n \tilde{S}_u^{g_n} \tilde{S}_f^{\boldsymbol{h}_\tau} \kappa_t \boldsymbol{v}_\tau \cdot \tilde{\boldsymbol{\Lambda}} \, d\Gamma_c - \\ & \frac{1}{\kappa_t} \int_{\Gamma_c} \mu \lambda_n \tilde{S}_u^{g_n} (1 - \tilde{S}_f^{\boldsymbol{h}_\tau}) \frac{\kappa_t \boldsymbol{v}_\tau}{\|\kappa_t \boldsymbol{v}_\tau\|} \cdot \tilde{\boldsymbol{\Lambda}} \, d\Gamma_c + \\ & \int_{\Gamma_c} \left(1 - \tilde{S}_u^{g_n}\right) \boldsymbol{\Lambda} \cdot \tilde{\boldsymbol{\Lambda}} \, d\Gamma_c \end{aligned} \tag{178}$$

4.5 Dynamic – Writing of velocity/impulse

The model of Signorini is unsuited to the problems of contact-impact, i.e. with the problems of contact in dynamics. As the fields of displacements are irregular (field of sign), their temporal integration by classical diagrams of finite differences of Newmark type causes parasitic oscillations (NON-physics) at the time of the shocks. These oscillations are all the more important as the order of the diagram is high.

4.5.1 Model of Signorini-Moreau for the Moreau

contact rewrote the model of Signorini to adapt it to the case of the dynamics:

$$\begin{cases} \lambda_n = 0 & \text{si } d_n < 0 \quad (a) \\ [\![v]\!]_n \leqslant 0 \text{ et } \lambda_n \leqslant 0 \text{ et } \lambda_n \boldsymbol{v} = 0 & \text{si } d_n \geqslant 0 \quad (b) \end{cases} \tag{179}$$

With the normal velocity defined by:

$$[\![v]\!]_n = [\![\boldsymbol{v}]\!] \cdot \boldsymbol{n} \tag{180}$$

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The model of Moreau justifies itself by writing that so the model of Signorini is true at initial time t_0, then the respect of the conditions of Moreau amounts solving the problem of Signorini for any $t>t_0$. For $t=t_0$, the models of Signorini-Moreau are strictly equivalent to those introduced into the §19 :

$$\lambda_n=S_u^{g_n}g_n \text{ formulate } S_u^{g_n}=\begin{cases}1 & \text{si } g_n\leq 0\\ 0 & \text{si } g_n>0\end{cases} \quad \text{formula18 1181}$$

) with the multiplier of contact increased g_n defined by:

$$g_n=\lambda_n-\rho_n d_n \quad (182)$$

For $t>t_0$, the models of Signorini are written:

$$\lambda_n=S_u^{d_n}S_v^{\dot{g}_n}\dot{g}_n \quad (183)$$

with the two fields of following signs:

$$S_u^{d_n}=\begin{cases}1 & \text{si } -d_n\leq 0\\ 0 & \text{si } -d_n>0\end{cases} \text{ and } S_v^{\dot{g}_n}=\begin{cases}1 & \text{si } \dot{g}_n\leq 0\\ 0 & \text{si } \dot{g}_n>0\end{cases} \quad (184)$$

And the multiplier of contact increased *of velocity* $\dot{g}_n$, definite by:

$$\dot{g}_n=\lambda_n-\rho_n[\![v]\!]_n \quad (185)$$

One will notice that the stress on clearance moved partly on the normal velocity $[\![v]\!]_n$. It is also necessary to note the change of nature of the parameter of increase ρ_n which has from now on the dimension of a force on a velocity (and either of a force on a displacement).

4.5.2 Model of Coulomb for the friction

The model of Coulomb for friction is already naturally written of velocity:

$$\left(1-S_u^{d_n}\right)\boldsymbol{\Lambda}+S_u^{d_n}\left\{\left(1-S_f^{\boldsymbol{h}_\tau}\right)\boldsymbol{\Lambda}-\rho_\tau \boldsymbol{v}_\tau S_f^{\boldsymbol{h}_\tau}-(1-S_f^{\boldsymbol{h}_\tau})\frac{\boldsymbol{h}_\tau}{\|\boldsymbol{h}_\tau\|}\right\}=\mathbf{0} \text{ sur } \Gamma_c \quad (186)$$

4.5.3 Application of the principle of the virtual wors

For the problem in dynamics, one adds the term of inertia G_{dyn}^i with the balance equation. The system to be solved is thus:

To find the fields $\boldsymbol{W}=\left(\boldsymbol{u}^1,\boldsymbol{u}^2,\lambda_n,\boldsymbol{\Lambda}\right)\in \text{CA}^1\times\text{CA}^2\times H\times\boldsymbol{H}$ such as

$$\begin{cases}\sum_{i=1}^{2}\left[G_{\text{dyn}}^i(\boldsymbol{W},\tilde{\boldsymbol{W}})+G_{\text{int}}^i(\boldsymbol{W},\tilde{\boldsymbol{W}})-G_{\text{ext}}^i(\boldsymbol{W},\tilde{\boldsymbol{W}})\right]-G_{\text{cf}}(\boldsymbol{W},\tilde{\boldsymbol{W}})=0 & (a)\\ \tilde{G}_{\text{cf}}(\boldsymbol{W},\tilde{\boldsymbol{W}})=0 & (b)\end{cases} \quad (187)$$

$$\forall\,\tilde{\boldsymbol{W}}=\left(\tilde{\boldsymbol{u}}^1,\tilde{\boldsymbol{u}}^2,\tilde{\lambda}_n,\tilde{\boldsymbol{\Lambda}}\right)\in \text{CA}^1\times\text{CA}^2\times H\times\boldsymbol{H}$$

G_{dyn}^i is the work of the forces of inertia:

$$G_{\text{dyn}}^i=\int_{\Omega_0^i}\left(\rho_p^i\frac{\partial^2\boldsymbol{u}^i}{\partial t^2}\cdot\tilde{\boldsymbol{u}}^i\right)d\Omega^i \quad (188)$$

It is appropriate to add to the system of the initial conditions:

$$\begin{aligned}\boldsymbol{u}_t^i&=\boldsymbol{u}_0^i \text{ dans } \Omega_0^i\\ \boldsymbol{v}_t^i&=\boldsymbol{v}_0^i \text{ dans } \Omega_0^i\end{aligned} \quad (189)$$

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to supplement the system, it is also necessary to describe the diagram of temporal integration, which connects displacements at the speeds:

$$\boldsymbol{u}_t^i=\boldsymbol{u}_0^i+\int_{t_0}^t \boldsymbol{v}_s^i d\tau \tag{190}$$

By leaning on works of Jean [10], we will use a diagram theta of order one:

$$\boldsymbol{u}_{k+1}^i=\boldsymbol{u}_k^i+\Delta t_k\left[(1-\theta)\boldsymbol{v}_k^i+\theta\,\boldsymbol{v}_{k+1}^i\right] \text{ and } \dot{\boldsymbol{v}}_{k+1}^i=\frac{\boldsymbol{v}_{k+1}^i-\boldsymbol{v}_k^i}{\Delta t_k} \tag{191}$$

If $\theta=0$, one finds a purely explicit diagram. If $\theta=1$, one finds a purely implicit diagram of Eulerian type (one indeed finds the approximation velocity produced into quasi-static):

$$\boldsymbol{v}_{k+1}=\frac{1}{\Delta t_k}\left(\boldsymbol{u}_{k+1}-\boldsymbol{u}_k\right) \tag{192}$$

the diagram is stable if:

$$\theta\geqslant 0 \text{ et } 1-\theta\leqslant\frac{2}{\omega_m\Delta t_k} \tag{193}$$

where ω_m is the maximum pulsation of the dynamic system. In practice, one recommends to choose θ between 0.5 and 1 .

4.5.4 Virtual wor of the forces of contact-friction

We now will write the virtual wor of the forces of contact-friction in strong form. There is a component for the contact:

$$G_c=-\int_{\Gamma_c}\lambda_n S_u^{d_n} S_v^{\dot{g}_n}[\![\tilde{u}]\!]_n d\Gamma_c \tag{194}$$

And a component for friction:

$$\begin{aligned} G_f= \quad & \int_{\Gamma_c}\mu\lambda_n S_u^{d_n} S_f^{\boldsymbol{h}_\tau}\boldsymbol{\Lambda}\cdot[\![\tilde{\boldsymbol{u}}]\!]_\tau d\Gamma_c+ \\ & \int_{\Gamma_c}\mu\lambda_n S_u^{d_n}\left(1-S_f^{\boldsymbol{h}_\tau}\right)\boldsymbol{\tau}\cdot[\![\tilde{\boldsymbol{u}}]\!]_\tau d\Gamma_c \end{aligned} \tag{195}$$

One does not see the field of sign velocities $S_v^{\dot{g}_n}$ in the statement of the strong form of the reaction due to friction, but it is present implicitly in the statement of the multiplier of contact λ_n .

4.5.5 Weak formulation of the model of contact

We now write the weak form of the model of contact established by the equation (183):

$$\tilde{G}_c=-\frac{1}{\rho_n}\int_{\Gamma_c}\lambda_n-S_u^{d_n}S_v^{\dot{g}_n}\left(\lambda_n-\rho_n[\![\boldsymbol{v}]\!]_n\right)\tilde{\lambda}_n d\Gamma_c \tag{196}$$

4.5.6 weak Formulation of the friction law

We now write the weak form of the friction law drawn up by the equation (186):

$$\begin{aligned} \tilde{G}_f= \quad & \frac{1}{\rho_t}\int_{\Gamma_c}\mu\lambda_n S_u^{d_n}\boldsymbol{\Lambda}\cdot\tilde{\boldsymbol{\Lambda}}\,d\Gamma_c- \\ & \frac{1}{\rho_t}\int_{\Gamma_c}\mu\lambda_n S_u^{d_n} S_f^{\boldsymbol{h}_\tau}\boldsymbol{h}_\tau\cdot\tilde{\boldsymbol{\Lambda}}\,d\Gamma_c- \\ & \frac{1}{\rho_t}\int_{\Gamma_c}\mu\lambda_n S_u^{d_n}(1-S_f^{\boldsymbol{h}_\tau})\frac{\boldsymbol{h}_\tau}{\|\boldsymbol{h}_\tau\|}\cdot\tilde{\boldsymbol{\Lambda}}\,d\Gamma_c+ \\ & \int_{\Gamma_c}\left(1-S_u^{d_n}\right)\boldsymbol{\Lambda}\,\tilde{\boldsymbol{\Lambda}}\,d\Gamma_c \end{aligned} \tag{197}$$

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4.5.7 Conservation of energy

the Lagrangian formulation stabilized for the contact preserves the total energy of the system, the linear linear momentum and the angular linear momentum. In the case of friction, energy is preserved if friction is adherent ($S_u^{d_n} S_v^{\dot{g}_n} = 1$ and $S_f^{\boldsymbol{h}_\tau} = 1$), on the other hand it in the case of the reduction (dissipation) slipping friction ($S_u^{d_n} S_v^{\dot{g}_n} = 1$ and $S_f^{\boldsymbol{h}_\tau} = 0$).

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5 Algorithms of resolution of the nonlinear system

5.1 incremental Formulation

We will transform the continuous mixed variational formulations into an incremental form, because the problem is always solved in this form, whether it is into quasi-static or dynamics. One gives oneself a discretization of the time interval $I=[0,T]$ with calculating:

$$0=t_0<t_1<\cdots<t_{nT}=T \tag{198}$$

With a stage k loading given, corresponding to fictitious time t_k, for $k\in[1,n_T]$, one supposes known the fields at time t_{k-1} noted $\boldsymbol{W}_{k-1}=\left(\boldsymbol{u}_{k-1}^1,\boldsymbol{u}_{k-1}^2,\lambda_{n,k-1},\boldsymbol{\Lambda}_{k-1}\right)$. One seeks the new fields solutions at time t_k noted $\boldsymbol{W}_k=\left(\boldsymbol{u}_k^1,\boldsymbol{u}_k^2,\lambda_{n,k},\boldsymbol{\Lambda}_k\right)$. The hybrid formulation is written then in the following generic form:

To find the fields $\boldsymbol{W}_k\in \mathrm{CA}^1\times\mathrm{CA}^2\times H\times \boldsymbol{H}$ such as:

$$\begin{cases}\displaystyle\sum_{i=1}^{2}\left[G_{\mathrm{int},k}^i\left(\boldsymbol{W}_k,\tilde{\boldsymbol{W}}_k\right)-G_{\mathrm{ext},k}^i\left(\boldsymbol{W}_k,\tilde{\boldsymbol{W}}_k\right)\right]-G_{\mathrm{c},k}\left(\boldsymbol{W}_k,\tilde{\boldsymbol{W}}_k\right)-G_{\mathrm{f},k}\left(\boldsymbol{W}_k,\tilde{\boldsymbol{W}}_k\right)=0 & (a)\\ \tilde{G}_{\mathrm{c},k}\left(\boldsymbol{W}_k,\tilde{\boldsymbol{W}}_k\right)=0 & (b)\\ \tilde{G}_{\mathrm{f},k}\left(\boldsymbol{W}_k,\tilde{\boldsymbol{W}}_k\right)=0 & (c)\end{cases} \tag{199}$$

$$\forall\,\tilde{\boldsymbol{W}}_k\in \mathrm{CA}^1\times\mathrm{CA}^2\times H\times \boldsymbol{H}$$

the variational formulation given by the system (199) is strongly nonlinear. Indeed, in addition to “classical” non-linearities due to the large deformations or the nonlinear behaviors of the materials, several levels of non-linearity inherent in contact-friction can be in addition distinguished:

- A geometrical non-linearity due to the ignorance *a priori* of the couples of points in contact between two surfaces;
- A non-linearity due to the ignorance of the state of contact of the paired points;
- A non-linearity due to the dependence in λ_n threshold of friction $\lambda_s=\mu\lambda_n$.

The group is written in the form of a nonlinear system:

$$\boldsymbol{F}\left(\boldsymbol{W}_k,p(\boldsymbol{W})\right)=0 \tag{200}$$

With $\boldsymbol{W}_k$ which represents the unknowns of the system and $p(\boldsymbol{W})$ the parameters of the problem. The general idea is to solve this nonlinear system by fixing certain parameters of contact-friction then to linearize the system by a method of Newton:

1.To fix the parameters of computation $p_0(\boldsymbol{W}_0)$;

2.To solve the system by the method of Newton $\boldsymbol{F}\left(\boldsymbol{W},p_0(\boldsymbol{W}_0)\right)=0$;

3.To buckle into 1 as long as $p_0\neq p(\boldsymbol{W})$;

L method of Newton has consists in applying a development of Taylor to the first order:

$$\boldsymbol{F}\left(\boldsymbol{W}_{k-1}+\delta\boldsymbol{W}\right)\approx\boldsymbol{F}\left(\boldsymbol{W}_{k-1}\right)+\frac{\partial\boldsymbol{F}\left(\boldsymbol{W}_{k-1}\right)}{\partial\boldsymbol{W}_{k-1}}\delta\boldsymbol{W} \tag{201}$$

According to the parameters which will be fixed, the method of resolution declines itself in several alternatives:

- Method of the fixed points;
- Partial method of Newton;
- Method of Newton generalized;

5.2 Temporal discretization for the model of Coulomb

It is appropriate to notice that the friction law of Coulomb introduces the notion relative velocity of sliding $\boldsymbol{v}_\tau$ between two solids. Into quasi-static, an implicit diagram of Eulerian is used. On the other hand, in dynamics, two situations arise:

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- One uses the formulation of velocity/impulse suggested in p.29 ;
- One uses a diagram of standard Newmark in displacement;

In this last case, one will not use the velocity given by the diagram, but the incremental form of implicit Eulerian type, as in static. This velocity is written in incremental form:

$$\boldsymbol{v}_\tau = \frac{1}{\Delta t} [\![\Delta \boldsymbol{u}]\!]_\tau \quad (202)$$

where $[\![\Delta \boldsymbol{u}]\!]_\tau$ represents the increment of the projection of the jump of displacement $\boldsymbol{u}$ on the tangent level at contact surface at the point of contact considered. This incremental writing in displacement of the problem of Coulomb is equivalent to the original formulation of velocity. In practice, it in Δt the parameter is integrated ρ_τ , i.e. one asks to the user to choose them in taking into account that one must integrate into it Δt "typical" problem. One must thus take care so that the parameter ρ_t which one introduces *via* COEF_FROT is worth $\bar{\rho}_t = \frac{\rho_t}{\Delta t}$ and not ρ_t .

5.3 Parameters of computation

the parameters of computation come from the way treat non-linearities of the problems of contact-friction. According to the assumptions of resolution, there exist several algorithms of resolution. In the initial algorithm (see [5], [6] and [7]), three non-linearities of the contact/friction are solved by a series of overlapping fixed points. The algorithm uses five overlapping loops:

Buckle <K> on time step
the Loop <J> on the geometry
Buckles <L> on the threshold of friction
Buckles <I> on the statute of contact
Buckles <N> on Fine
Newton buckles <N>
End buckles <I>
End buckles <L>
End buckles <J>
End buckles <K>

It remains on the level <N> only the nonlinear terms not having been able to be treated. There are thus only the internal forces and dependant non-linearity at the end NON-differentiable of projection on the ball unit of the model of Coulomb.
There are three parameters which one can fix or not by key keys ALGO_RESO_*, corresponding to the loops J, L and I, therefore eight algorithms. They do not have the role to be all used. In practice only three are useful.

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	Code	Geometry (<J> buckles) ALGO_RESO_GEOM	Threshold of friction (<L> buckles) ALGO_RESO_FROT	Statutes of contact (<I> buckles) ALGO_RESO_CONT
Newton generalized	GNM	NEWTON	NEWTON	NEWTON
partial Newton in geometry (Mode by default)	PNM	POINT_FIXE	NEWTON	NEWTON
Points fixed	FPM	POINT_FIXE	POINT_FIXE	POINT_FIXE
partial Newton	Mode not recommended	to POINT_FIXE	POINT_FIXE	NEWTON
partial Newton	Mode not recommended	to NEWTON	NEWTON	POINT_FIXE
partial Newton	Mode not recommended	to NEWTON	POINT_FIXE	NEWTON
partial Newton	Mode not recommended	to POINT_FIXE	NEWTON	POINT_FIXE
Newton partial	Mode not recommended	POINT_FIXE	POINT_FIXE	NEWTON

In the algorithm of generalized Newton (see [9]), no parameter is fixed and the entirety of nonthe linearities is treated within the same iteration of Newton.

Buckle <K> on time step
the Loop <N> on Fine
Newton buckles <N>
End buckles <K>

In practice, only three versions are interesting:

•Newton generalized GNM is the algorithm which gives the results most quickly, but it misses sometimes robustness (and thus causes failures of convergence). Moreover, it is very depend on the choice of the parameters of increase and requires a control of the cycles to be powerful (see p.56);

•Algorithm FPM is most robust, but it is very expensive (many iterations of Newton). Moreover of pathologies of cycling can appear in certain situations. On the other hand, it is relatively insensitive with the choice of the parameters of increase;

•Intermediate algorithm PNM proposes to keep the geometrical parameter in the fixed loop of point and to solve other non-linearities by Newton. It is an excellent compromise between the two preceding algorithms.

5.3.1 Buckle on the geometry

the problem of geometrical non-linearity comes owing to the fact that one does not know *a priori* which will be the couples of points in contact. This ignorance also involves that of the local coordinate system (normal and plane tangent) in each potential point of contact.

This non-linearity is particularly marked for the problems with the left contact surfaces (i.e. nonplane) and/or potentially subjected to *great transformations (great relative slidings)*. In practice, only one correction is often sufficient if contact surfaces undergo little geometrical variation (small displacements).

It is important to understand what implies the geometrical reactualization: it is, with each iteration, a completely new problem of contact-friction which one solves, and where it is necessary to set out again of the determination of the couples of paired points. This is why the loop of geometry is most external.

The non-linearity of pairing involves also the NON-knowledge of the local coordinate system. However the local coordinate system running (the norm $\boldsymbol{n}$ and the two tangents $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$ in 3D, in the deformed configuration) is present several time in the writing of the equations of Signorini-Coulomb. If the case of the norm is taken, that means that she is written, during displacement $\boldsymbol{u}^1$:

$$\boldsymbol{n}(\bar{\boldsymbol{x}}) = \boldsymbol{n}\left(\bar{\boldsymbol{p}} + \boldsymbol{u}^1(\bar{\boldsymbol{p}})\right) \tag{203}$$

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the norm is thus nonlinear in displacement. This non-linearity can be derived in the equations or one can make the assumption that the norm remains constant during the loop <J> on the geometry:

$$\boldsymbol{n}(\bar{\boldsymbol{x}}) \approx \boldsymbol{n}\left(\bar{\boldsymbol{p}} + \boldsymbol{u}_j^1(\bar{\boldsymbol{p}})\right) \tag{204}$$

In our case, one makes the assumption that the norm (and the tangent) remains constant during the loop on the geometry. The nombre of iterations of the loop is automatically given by the code but one can fix it by NB_ITER_GEOM. It should be noted that there is always at least a pairing at the beginning of time step, even if NB_ITER_GEOM=0.

5.3.2 Buckle on the threshold of friction

the non-linearity of threshold (friction of Coulomb) comes owing to the fact that the threshold of friction depends on the normal density of the force of contact λ_n, which is itself an unknown of the problem. The loop of POINT-fixes <L> on the threshold of friction λ_s comes down to dealing with the problem of contact-friction with a threshold of fixed *friction*, which returns to each iteration to dealing *with a problem of Tresca*. This strategy is very current to solve the friction of Coulomb, and is justified theoretically: by fixing the threshold of friction, one can derive *a convex potential associated* with the model with Coulomb, which makes it possible to connect the unknowns of displacement to their associated dual variables, the frictional forces.

5.3.3 Buckle on the statutes of contact

last non-linearity relates to the ignorance of the effective state of contact. We adopted a method derived from the method of the active stresses implemented in *Code_Aster* for the discrete contact (see [R5.03.50]). By default, the initial statute of the nodes of contact is lack of contact in *Code_Aster*. In addition λ_s is initialized to zero (except if one uses SEUIL_INIT). That returns starting with solving a problem without contact with the first iteration, then with contact but without friction and finally to activate friction with a threshold of fixed λ_s friction (problem of Tresca). One can nevertheless start from a state initially contacting by means of option CONTACT_INIT. In this case, it is possible to solve geometrically unstable problems with the formulation continues. It is an important advantage of the continuous formulation of the contact compared to the discrete formulations which do not support a mechanical problem which “is held” only by the contact.
Moreover, the method “slide” makes it possible to simulate of the contact bilateral by supposing that the nodes making contact will remain it throughout the transient (functionality activable by the keyword SLIDE).

5.4 Linearizations of the terms

the nonlinear algebraic system (199) will be solved by linearization of the Newton type. We take again the statements of the equilibrium (PTV, to see p.25) and of the model of Signorini and Coulomb in weak form. The virtual quantities are then infinitesimal variations, therefore the quantities expressed in term of brought up to date coordinates (of the kind $\delta \boldsymbol{x}$) are transformed into variation on displacements $\delta \boldsymbol{u}$:

$$\boldsymbol{x} = \boldsymbol{p} + \boldsymbol{u} \rightarrow \delta \boldsymbol{x} = \delta \boldsymbol{u} \text{ and } \boldsymbol{x} = \boldsymbol{p} + \boldsymbol{u} \rightarrow \Delta \boldsymbol{x} = \Delta \boldsymbol{u} \tag{205}$$

important Note: the linearization of the terms in the following paragraphs implies the use of a diagram of Newmark in dynamics or the quasi-static case, which uses the incremental writing in displacement of the friction law.

The PTV understands in particular the reaction of contact, which will be written with the variation of normal clearance:

$$G_c = \int_{\Gamma_c} S_u^{g_n} g_n \tilde{\delta} d_n \, d\Gamma_c \tag{206}$$

For the reaction of friction in the PTV:

$$G_f = \int_{\Gamma_c} \mu \lambda_n S_u^{g_n} S_f^{\boldsymbol{h}_\tau} \boldsymbol{h}_\tau \cdot \tilde{\delta} \boldsymbol{d}_\tau \, d\Gamma_c + \int_{\Gamma_c} \mu \lambda_n S_u^{g_n} \left(1 - S_f^{\boldsymbol{h}_\tau}\right) \boldsymbol{\tau} \cdot \tilde{\delta} \boldsymbol{d}_\tau \, d\Gamma_c \tag{207}$$

It is important to understand the meaning of the notations. In the two preceding cases, $\tilde{\delta}$ gets along like the total variation compared to displacements (statements drawn up with p.13).

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For the model of Signorini in weak form one considers:

$$\tilde{G}_c=-\frac{1}{\rho_n}\int_{\Gamma_c}\left(\lambda_n-S_u^{g_n}g_n\right)\delta\lambda_n d\Gamma_c \tag{208}$$

In this case, δ compared to the means like the variation contact pressure λ_n.

Finally for the model of Coulomb in weak form:

$$\begin{aligned}\tilde{G}_f=\quad & \frac{1}{\rho_t}\int_{\Gamma_c}\mu\lambda_n S_u^{g_n}\boldsymbol{\Lambda}\cdot\delta\boldsymbol{\Lambda}\,d\Gamma_c-\\ & \frac{1}{\rho_t}\int_{\Gamma_c}\mu\lambda_n S_u^{g_n}S_f^{\boldsymbol{h}_\tau}\boldsymbol{h}_\tau\cdot\delta\boldsymbol{\Lambda}\,d\Gamma_c-\\ & \frac{1}{\rho_t}\int_{\Gamma_c}\mu\lambda_n S_u^{g_n}(1-S_f^{\boldsymbol{h}_\tau})\boldsymbol{\tau}\cdot\delta\boldsymbol{\Lambda}\,d\Gamma_c+\\ & \int_{\Gamma_c}\left(1-S_u^{g_n}\right)\boldsymbol{\Lambda}\cdot\delta\boldsymbol{\Lambda}\,d\Gamma_c\end{aligned} \tag{209}$$

δ gets along like the variation compared to the Lagrange of friction $\boldsymbol{\Lambda}$.

By adding the terms corresponding to the equilibrium of structure (G_{int}^i and G_{ext}^i), there is the following nonlinear system to solve:

$$\begin{cases}L_{\boldsymbol{u}}\left(\boldsymbol{W}_k,\delta\boldsymbol{W}_k\right)=L_{\boldsymbol{u},k}=0 & (a)\\ L_c\left(\boldsymbol{W}_k,\delta\boldsymbol{W}_k\right)=L_{c,k}=0 & (b)\\ L_f\left(\boldsymbol{W}_k,\delta\boldsymbol{W}_k\right)=L_{f,k}=0 & (c)\end{cases} \tag{210}$$

With $\boldsymbol{W}_k$ which represents the unknowns of the system and $\delta\boldsymbol{W}_k$ the virtual quantities. The first equation represents the equilibrium of structure with contact-friction, as the variation thus gets along compared to displacements, one a:

$$L_{\boldsymbol{u},k}=G_{\text{ext},k}^i-G_{\text{int},k}^i-G_{\text{c},k}-G_{\text{f},k}\quad\forall\,\delta\boldsymbol{u}_k^i\in\text{CA}_h^i \tag{211}$$

L `following equation is the form of the model of contact of Signorini. The variation gets along compared to Lagrangian contact :

$$L_{c,k}=\tilde{G}_{\text{c},k}\quad\forall\,\delta\lambda_{n,k}\in H_h\left(\Gamma_c\right) \tag{212}$$

For the friction law of Coulomb. The variation gets along compared to Lagrangian friction:

$$L_{f,k}=\tilde{G}_{\text{f},k}\quad\forall\,\delta\boldsymbol{\Lambda}_k\in\boldsymbol{H}_h\left(\Gamma_c\right) \tag{213}$$

We already made an assumption in these two last equations: the Lagrange multipliers of the contact λ_n and the semi-multipliers of friction $\boldsymbol{\Lambda}$ were written only on one of two solids. It was thus supposed that $\Gamma_c=\Gamma_c^1$ or $\Gamma_c=\Gamma_c^2$. Privileged surface is called surface slave. One will evaluate the integrals on this surface. The linearization introduces in particular terms corresponding to the generalized jacobian of contact-friction. For the equilibrium:

$$\begin{aligned}L_{\boldsymbol{u},k}\approx\quad & G_{\text{ext},k-1}^i-G_{\text{int},k-1}^i-G_{\text{c},k-1}-G_{\text{f},k-1}+\\ & \left(\frac{\partial G_{\text{ext}}^i}{\partial\boldsymbol{u}^i}\bigg|_{X_{k-1}}-\frac{\partial G_{\text{int}}^i}{\partial\boldsymbol{u}^i}\bigg|_{X_{k-1}}-\frac{\partial G_{\text{c}}}{\partial\boldsymbol{u}^i}\bigg|_{X_{k-1}}-\frac{\partial G_{\text{f}}}{\partial\boldsymbol{u}^i}\bigg|_{X_{k-1}}\right)\Delta\boldsymbol{u}^i\\ & \left(-\frac{\partial G_{\text{c}}}{\partial\lambda_n}\bigg|_{X_{k-1}}-\frac{\partial G_{\text{f}}}{\partial\lambda_n}\bigg|_{X_{k-1}}\right)\Delta\lambda_n\\ & \left(-\frac{\partial G_{\text{c}}}{\partial\boldsymbol{\Lambda}}\bigg|_{X_{k-1}}-\frac{\partial G_{\text{f}}}{\partial\boldsymbol{\Lambda}}\bigg|_{X_{k-1}}\right)\Delta\boldsymbol{\Lambda}\end{aligned} \tag{214}$$

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For the model of contact of Signorini:

$$L_{c,k} \approx \tilde{G}_{\mathrm{c},k-1} + \left.\frac{\partial \tilde{G}_{\mathrm{c}}}{\partial \boldsymbol{u}^i}\right|_{\boldsymbol{X}_{k-1}} \Delta \boldsymbol{u}^i + \left.\frac{\partial \tilde{G}_{\mathrm{c}}}{\partial \lambda_n}\right|_{\boldsymbol{X}_{k-1}} \Delta \lambda_n + \left.\frac{\partial \tilde{G}_{\mathrm{c}}}{\partial \boldsymbol{\Lambda}}\right|_{\boldsymbol{X}_{k-1}} \Delta \boldsymbol{\Lambda} \tag{215}$$

For the friction law of Coulomb:

$$L_{f,k} \approx \tilde{G}_{\mathrm{f},k-1} + \left.\frac{\partial \tilde{G}_{\mathrm{f}}}{\partial \boldsymbol{u}^i}\right|_{\boldsymbol{X}_{k-1}} \Delta \boldsymbol{u}^i + \left.\frac{\partial \tilde{G}_{\mathrm{f}}}{\partial \lambda_n}\right|_{\boldsymbol{X}_{k-1}} \Delta \lambda_n + \left.\frac{\partial \tilde{G}_{\mathrm{f}}}{\partial \boldsymbol{\Lambda}}\right|_{\boldsymbol{X}_{k-1}} \Delta \boldsymbol{\Lambda} \tag{216}$$

With final, we will obtain the following linearized system:

$$\begin{Bmatrix} L_{\boldsymbol{u},k} \\ L_{c,k} \\ L_{f,k} \end{Bmatrix} \approx \begin{Bmatrix} L_{\boldsymbol{u},k-1} \\ L_{c,k-1} \\ L_{f,k-1} \end{Bmatrix} + \begin{bmatrix} J_{\boldsymbol{uu}} & J_{\boldsymbol{u}c} & J_{\boldsymbol{u}f} \\ J_{c\boldsymbol{u}} & J_{cc} & J_{cf} \\ J_{f\boldsymbol{u}} & J_{fc} & J_{ff} \end{bmatrix} \begin{Bmatrix} \Delta \boldsymbol{u}^i \\ \Delta \lambda_n \\ \Delta \boldsymbol{\Lambda} \end{Bmatrix} = \begin{Bmatrix} L_{\boldsymbol{u},k-1} \\ L_{c,k-1} \\ L_{f,k-1} \end{Bmatrix} + \left[J\right] \left\{\Delta \boldsymbol{W}_k\right\} \tag{217}$$

We must evaluate all the terms of the jacobian matrix $\left[J\right]$.

5.4.1 The alternatives for the linearization of the model of Coulomb

the model of Coulomb connects the contact pressure to two quantities:
- Intensity of the shear stresses (frictional forces);
- Direction of friction $\boldsymbol{\tau}$;

The dependence of the direction of friction compared to the intensity of the contact pressure is an important point and which shows the difficulty of taking into account of the model of Coulomb. This dependence is paramount in 3D, on curved surfaces.
If one takes again the statement of the equilibrium, with integration of the reaction of friction G_f, one thus has, in any rigor, a dependence with the contact pressure λ_n $\|\boldsymbol{\lambda}_\tau\|$ and of $\boldsymbol{\tau}$, but also, by the form of the weak model of friction, a dependence of $\tilde{G}_f$ to the contact pressure λ_n. Three quantities are thus variable:

$$\Delta_{\lambda_n}\left(\|\boldsymbol{\lambda}_\tau\|\right) \neq 0 \quad \Delta_{\lambda_n}\left(\boldsymbol{\tau}\right) \neq 0 \text{ , } \Delta_{\lambda_n}\left(\tilde{G}_f\right) \neq 0 \tag{218}$$

It is the most general case, however, one can distinguish in any rigor 8 possible configurations:

Alternative	$\Delta_{\lambda_n}\left(\|\boldsymbol{\lambda}_\tau\|\right)=0$	$\Delta_{\lambda_n}\left(\boldsymbol{\tau}\right)$	$\Delta_{\lambda_n}\left(\tilde{G}_f\right)$	Stamps
1	0	0	0	Symmetric
2	0	0	$\neq 0$	Asymmetric
3	0	$\neq 0$	0	Asymmetric
4	0	$\neq 0$	$\neq 0$	Asymmetric
5	$\neq 0$	0	0	Asymmetric
6	$\neq 0$	0	$\neq 0$	Asymmetric
7	$\neq 0$	$\neq 0$	0	Asymmetric
8	$\neq 0$	$\neq 0$	$\neq 0$	Asymmetric

All these alternatives are numerically acceptable (they are alternatives of Newton) but only some have a true physical meaning. The alternative 1 corresponds if the contact pressure is constant: it is the model of Tresca or the alternative known as “fixed point on the threshold of friction”, it is the only one which produces a symmetric tangent matrix. Alternative 8 is the most general case (method of Newton generalized). Except for these two cases, only two other alternatives have a physical meaning:
- Alternative 4 supposes that one compared to the neglects the dependence of the intensity of friction contact pressure but which one can have of the variations of direction of the tangential directions “abrupt”. The typical case is that of a curved surface in 3D with low intensities of contact pressure (shaving contact);

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•Alternative 6 supposes that one can compared to the neglect the dependence of the direction of friction contact pressure (case 2D or 3D plane) but not the dependence of the frictional force. The typical case is that of a plane surface 3D or 2D, with a strong intensity of contact pressure;
In practice, alternative 8 introduced of the terms numerically badly conditioned which slow down convergence. The dependence with the direction of friction couples geometrical terms with terms in pressure. We thus introduced the alternative 5 which seems to be a good compromise. The computation terms of the jacobienne is deferred in appendix (see p.5959)
On the assumptions of simplification, one neglected the second variation of tangent clearance (equation 313) because the equations which result from this are very complex and do not seem to bring a significant gain on the velocity of convergence. Of course, one thus loses the theoretical quadratic convergence of the algorithm of Newton.

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6 Discretization of the mixed variational formulation

One describes in this section the discretization of the problem defined by the variational formulations data p.25. The variational formulations used have the following common points:

- The writing of PTV for the equilibrium of structure, by means of a weak formulation of the balance equation;
- A vector field of unknowns: that of the primal variables representative either displacements, or velocities (in dynamics, in the case of the theta-diagram);
- The geometry is represented by a regular triangulation (classical polynomial finite elements);
- An unknown scalar field representing the multiplier of Lagrange of contact λ ;
- U N unknown vector field representing the semi-multiplier of Lagrange of friction $\boldsymbol{\Lambda}$;
- The model of contact of Signorini is applied in a weak way, with an increased Lagrangian formulation;
- The model of friction of Coulomb is applied in a weak way, with an increased Lagrangian formulation;
- Fields of sign, discontinuous, for the contact and friction were introduced into the formulation.

In this chapter, we will introduce all the ingredients necessary to the fabrication of the discretized model, then, in the following chapter, we will see the way solve the problem.

Important: from now on, we will note λ and either λ_n the contact pressure to reduce the notations. The pressure of friction is always noted Λ .

6.1 Theoretical elements

6.1.1 Paving of the field

the formula Ω^i are approached by fields Ω_h^i polygonal. With each formula Ω^i a family with triangulations is associated T_{h^i} with the parameter h^i which is the step of the mesh of the field Ω^i . By noting K_j^i NE^i elements of each triangulation, we can write:

$$\Omega_h^i = \bigcup_{j=1}^{NE^i} K_j^i \tag{219}$$

For the discretization of the border of contact Γ_c^i , we are vis-a-vis a first important assumption. Indeed, this border can be discretized of two ways:

- One takes the geometrical trace of the mesh of the field Ω^i subjacent;
- One creates a new independent mesh, with his own step of discretization.

In practice, the mesh of the border is almost always built on the first assumption, because that would imply if not a work of mesh difficult to implement, particularly in 3D. One will thus use always the geometrical trace of the mesh of the field Ω^i like mesh for the border of contact. The trace of T_{h^i} forms a mesh $T_{h^i}^{\Gamma}$ of Γ_c^i . The discretized border $\Gamma_{c,h}^i$ is made up of NE_c^i elements k_j^i :

$$\Gamma_{c,h}^i = \bigcup_{j=1}^{NE_c^i} k_j^i \tag{220}$$

6.1.2 discrete Spaces of approximations

the discrete version of space $H^1\left(\Omega^i\right)$, on the field Ω^i , which one will note $H_h^{i,q}$, is such as:

$$H_h^{i,q} = \left\{ \boldsymbol{u}^i \in \left(C\left(\bar{\Omega}^i\right)\right)^d ; \boldsymbol{u}_{|K_j^i}^i \in \left(P_q\left(K_j^i\right)\right)^d \text{ pour } 1 \leqslant j \leqslant NE^i \right\} \tag{221}$$

with NE^i the number of elements coming from the triangulation (219), d the dimension of the problem, $C\left(\bar{\Omega}^i\right)$ the space of the continuous functions on the dependancy of Ω^i and $P_q\left(K_j^i\right)$ the space of polynomials of degree lower or equal to q on the field. One indicated by CA^i space containing the kinematically admissible functions. The discrete version of this space, noted $CA_h^{i,q}$ is built like (221):

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$$CA_h^{i,q}=\left\{\boldsymbol{u}^i\in\left(C\left(\bar{\Omega}^i\right)\right)^d;\boldsymbol{u}^i_{|K^i_j}\in\left(P_q\left(K^i_j\right)\right)^d \text{ pour } 1\leqslant j\leqslant NE^i \text{ et } \boldsymbol{u}^i_{|\Gamma^i_u}=0\right\} \quad (222)$$

We will write the discrete version as of the space of the traces of functions on the border:

$$CA_h^{\gamma,i}=\left\{trace\left(\boldsymbol{u}^i\right) \text{ sur } \Gamma^i_c;\boldsymbol{u}^i\in CA_h^{i,q}\right\} \quad (223)$$

From now on, one will forget the notation q the order of the polynomials to indicate.

6.1.3 The Inf-Sup *condition* or condition LBB

the mixed character of the formulation imposes *a dependence* between spaces finite elements of the multipliers of contact and friction and space finite elements of displacements. This choice concerns the compatibility condition (or Inf-Sup condition or condition LBB of Ladyzenskaya-Babuska-Brezzi). A mechanical problem is in general brought back to solve *a variational equation* of the type:

$$\begin{array}{l}\text{Trouver } \boldsymbol{u}\in U \text{ tel que } \forall\,\boldsymbol{u}^*\in U\\ \begin{cases}a\left(\boldsymbol{u},\boldsymbol{u}^*\right)=\langle l,\boldsymbol{u}^*\rangle & \text{dans } U\\ u=g & \text{sur } \partial U\end{cases}\end{array} \quad (224)$$

where U is a space of Hilbert. He is often more convenient to rewrite this problem in a form known as *mixed*, with two unknowns $\boldsymbol{u}\in U$ etformule $\lambda\in M$:

$$\begin{array}{l}\text{Trouver } \boldsymbol{u}\in U \text{ et } \lambda\in M \text{ tel que } \forall\,\boldsymbol{u}^*\in U \text{ et } \forall\,\lambda^*\in M\\ \begin{cases}a\left(\boldsymbol{u},\boldsymbol{u}^*\right)+b\left(\boldsymbol{u}^*,\lambda\right)=\langle l,\boldsymbol{u}^*\rangle & \text{dans } U\\ b\left(\boldsymbol{u}^*,\lambda\right)=\langle g,\lambda^*\rangle & \text{sur } \partial U\end{cases}\end{array} \quad (225)$$

where M is also a space of Hilbert, dual of U by the bilinear form b . The problem is that the formulation given by (224) is not equivalent to the formulation given by (225). Thus, in this very general frame, *the equivalence* of the two formulations, as well as *a requirement and sufficient of existence and unicity* (moreover, of course, classical assumptions on *the ellipticity* and *the coercity* of the bilinear form a) solution of (225), is given by the condition known as of *Ladyzenskaya*, *Babuska* and *Brezzi* (LBB):

$$\exists\alpha\in\mathbb{R}>0,\ \inf_{\lambda^*\in M}\sup_{\boldsymbol{u}^*\in U}\frac{\left|b\left(\boldsymbol{u}^*,\lambda^*\right)\right|}{\left\|\lambda^*\right\|_M.\left\|\boldsymbol{u}^*\right\|_U}\geqslant\alpha>0 \quad (226)$$

This condition indicates *how* space should be chosen M , so that the formulations are equivalent. However, it is necessary to acknowledge that from an operational point of view, one is not so much more advanced, because this condition, very abstract, is also very technical to implement. The more so as this condition is expressed in the continuous frame and that its transition in the discrete frame is not guaranteed! One will be satisfied to give implementing rules resulting from the literature.

The application of condition LBB to our problem of contact depends on the nature of the interfaces:

- If the interface is left, obtaining a Inf-Sup *condition* with a discontinuous discretization of the multipliers of contact requires a very fine discretization of these multipliers;
- If the interface is plane, the space $\mathrm{CA}_h^{\gamma,i}$ of the traces of the fields of displacements on the interface of contact as spaces discretization of the field of multiplier of contact meets condition LBB;

The continuous method uses the space of the traces of the fields of displacements, knowing that stability (with meaning LBB) will not be guaranteed for the nonplane interfaces.

One takes fields of multipliers in space $\mathrm{CA}_h^{\gamma,i}$ on all the interface of contact2Sauf[2]. For left *surfaces* (Figure 6), this choice is convenient (since we discretize the field of multipliers in a way identical to the field of displacement), but it does not have no proof *of stability there*. Indeed, the problem comes from what the choice of one of two surfaces as surfaces integration constitutes a more or less rough approximation, like illustrates it Figure 6. The experiment shows however that this difficulty can be circumvented by the use of fields of approximation as smooth as possible (lissage of the norms, continuity of the fields of multipliers). We will see later that the respect of the Inf-Sup condition is also strongly related to the numerical diagram of squaring used to evaluate the integrals.

2 if one wants to treat certain incompatibilities between contact-friction and the limiting conditions

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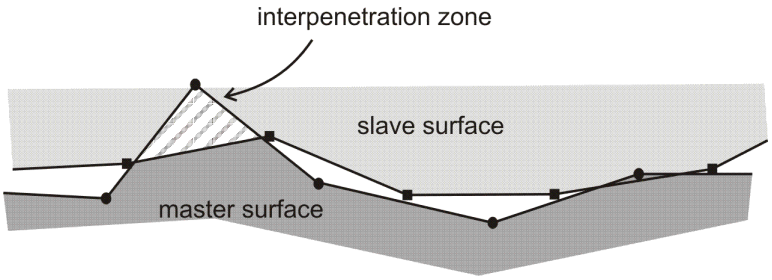

Figure 6: Incompatible left surfaces

For the user of the contact in *Code_Aster*, this choice means that it is likely to have difficulties of convergence since the interfaces of contact are left, that the relative sliding is significant or that the difference in discretization is important between two surfaces.

6.1.4 Fields of sign

the choice of spaces of approximations (of polynomial finite elements type) for the field of displacement, the field of multiplier of Lagrange of contact and the field of the semi-multipliers of friction are not sufficient to build the discrete problem. There remains the case of the fields of signs (for the contact, friction, the velocity field, etc) which are naturally in infinite spaces. One uses a collocation method to approximate these fields of sign. One thus chooses a finished collection of points of discretized contact surface $\Gamma_{c,h}^{i}$:

$$\left(p_j^i\right)_{1\leqslant j\leqslant n_c^i}\in\Gamma_{c,h}^{i} \tag{227}$$

a field of sign S will thus be approached by S_h , which is the following discrete sum:

$$S_h=\sum_{j=1}^{n_c}\omega_j S\left(p_j\right) \tag{228}$$

Attention! The number of collocation points n_c^i is not necessarily the same one as the number of points resulting from the discretization finite elements from the multipliers from contact/friction (which rests, one recalls it, on the geometrical trace of discretized volume).

6.1.5 Numerical integration

to numerically evaluate the integrals, it is appropriate to choose a diagram of numerical squaring suitable. One is based for that on the integration of the discontinuous terms (fields of sign) which “guide” the best diagram to be adopted. Question thus moves towards the choice of the optimal collocation points to obtain forces of precise contact.
The respect of the Inf-Sup condition was approached in the frame of the choice of the functions of approximation of the fields of multipliers of contact and friction. In optics to always observe this condition, it was known as that it was strongly related to the accuracy of the numerical integration which one is able to obtain. The phenomenon is understood when one makes the analogy with the methods of numerical under-integration sometimes employed to solve problems of numerical blockings due to a too great wealth of the approximation, while then making “lower” the degree of interpolation of the fields into under-integral. The reasoning can thus be reversed easily: if numerical integration “distorts” (while enriching or while impoverishing) the polynomial interpolation of the fields of multipliers, one is likely to leave the field of stability of condition LBB.

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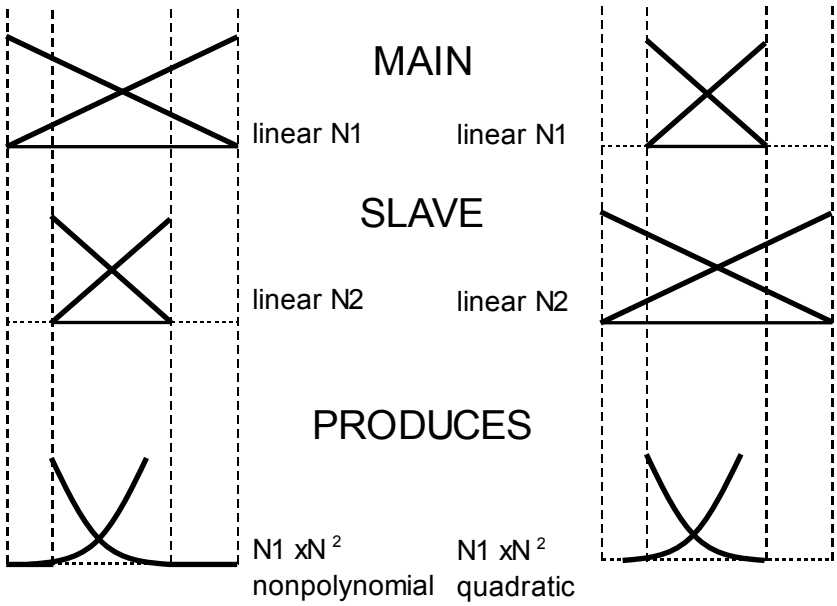

Figure 7: Numerical integration and compatibility

It is the case, in particular in the event of incompatibility. Let us imagine that the elementary area of integration on surface slave recovers several elements of main surface. As the function-test have as a support NON-no one a finite element, the fact of integrating numerically, by a squaring of Gauss, a nonpolynomial function, but identically null on part of the area of integration, is clearly not exact, because squaring is completely inappropriate (see Figure 7). However it is what one does when one tries to integrate a functional field of a main element on *an element slave much wider* (not compatible). By reversing the choice of surfaces, result is clearly better (Figure 7 on the right), but still without guarantee to have a satisfactory integration. Thereafter, we will describe the method adopted for *Code_Aster*, based on a subdivision of the element *which is used only for integration*. This method, which consists in refining the surface of integration, limits the error of numerical integration, and has especially the advantage of being extremely simple to implement.
Several ways were explored to try to minimize these problems of incompatibilities between surfaces. That which was retained in Code_Aster is to propose various diagrams of numerical integration, usable according to the needs. There are several types of diagrams:

- Numerical integration with the nodes;
- Numerical integration with Gauss points ;
- Numerical integration by the method of *Simpson* with 3,5 or 9 points of integration;
- Integration by the method of *Newton-Dimensions* with 4,5 or 10 points of integration;

The integration of Simpson makes it possible to improve the results in the event of incompatibility of mesh between two contact surfaces, thanks to the subdivision of the elements of integration which we already evoked above.
The integration of Newton-Dimensions makes it possible to integrate exactly polynomials of a high nature (higher than three). It proves very useful when the interpolation functions used on one of two contact surfaces are of order higher than one ($P1-P2$, $P2-P2$ …). Moreover, squarings of Newton-Dimensions also allow (like those of Simpson) a subdivision of the element of integration, and thus apply in the event of incompatibility of meshes.
However, one will take care well not to believe that a systematic use of a richer squaring (Simpson, Newton-Dimensions) is always a good choice. Initially because, in most case, squaring with the nodes proves in practice sufficient ($P1-P1$ with sufficiently fine meshes). Then because the richer one squaring is (of many points of integration), the more important the computational loads are, and one will realize well quickly that, for example, the use of the squaring of Newton-Dimensions is quickly crippling. There is thus no systematic rule a priori as for the choice of a squaring, except the few indications which we gave: it is necessary to take account of the specificity of the model and the mechanical problem studied, and to make a compromise between the accuracy wanted for the solution and the time computing necessary.

6.2 Elements of contact

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After having discretized the mixed variational formulation and after having given the diagrams of numerical integration, it any more but does not remain us of the finite elements to set up for contact-friction.

The idea to use of the finite elements is not immediate. For example, in the discrete methods of contact friction [R5.03.50], the kinematical conditions of contact are imposed without using finite elements and while intervening directly on the algebraic system to solve. Such a strategy has the disadvantage of returning the programming and architecture completely dependant on the formulation, which prohibits in contact with to profit from improvements (architecture, performance) which would be carried out in the rest of the code finite elements. A contrario, we will see that to develop contact-friction of the finite elements implies to leave the standard diagram of the finite elements and that it is a sub-optimal strategy from the point of view of the performances.

To determine the stiffness matrix (and the second member) of the discretized system, one must of the finite elements calculate the elementary terms before carrying out the assembly of it. This elementary level (classical) is not enough for the processing to our problem of contact-friction. Several incompatibilities come to return the problem NON-standard within the meaning of the finite elements.

6.2.1 Geometrical incompatibility

One notes already a geometrical incompatibility (Figure 8) between the discretizations of two contact surfaces in opposite, an element slave, being able for example to overlap several main elements.

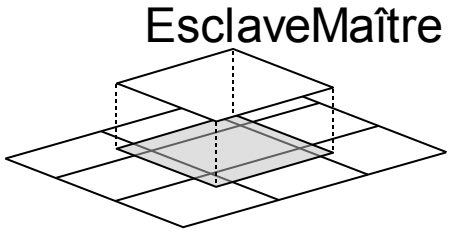

Figure 8: Geometrical incompatibility

6.2.2 Incompatibility of the statutes

the discontinuous character of the states of contact/friction involves an incompatibility of the statutes (Figure 9). Within the same element of contact, can coexist several states of contact-friction;

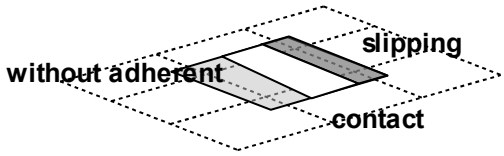

Figure 9: Incompatibility of the statutes

This incompatibility is regulated easily because the assumption is made that the field of sign (thus indicating the statute of contact and friction) is discretized by collocation and that the collocation points were selected to be those of numerical integration. What wants to say that the statute of contact is not carried by an element of contact, but by a point of integration of an element of contact.

6.2.3 Construction of the elements of contact

It remains to build these elements of contact. We 2D will illustrate the principle on a case.

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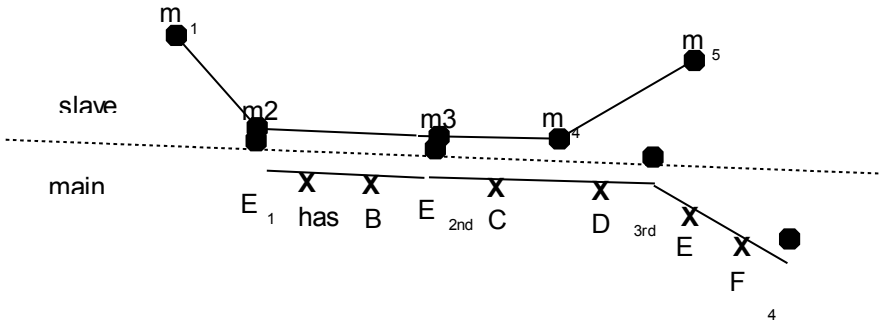

Figure 10: Example 2D of pairing between two incompatible surfaces

On figure 10, we represent an example 2D surfaces geometrically incompatible. The systems represent the points of integration and here pairings and the statutes of contact:

Not integration	Element slave	main Element paired	Statute
a	e_1-e_2	m_2-m_3	Adherent
b	e_1-e_2	m_2-m_3	Slipping
c	e_2-e_3	m_3-m_4	Slipping
d	e_2-e_3	m_4-m_5	Not contact
e	e_3-e_4	m_4-m_5	a contact
f	e_3-e_4	no	No contact

Table 6.2.3-1

There are thus three elements of contact which one represents on figure 11. Each element is geometrically defined by the nodes slaves and Masters. The statute is represented by the collocation point which acts like a point of integration for each element of contact.

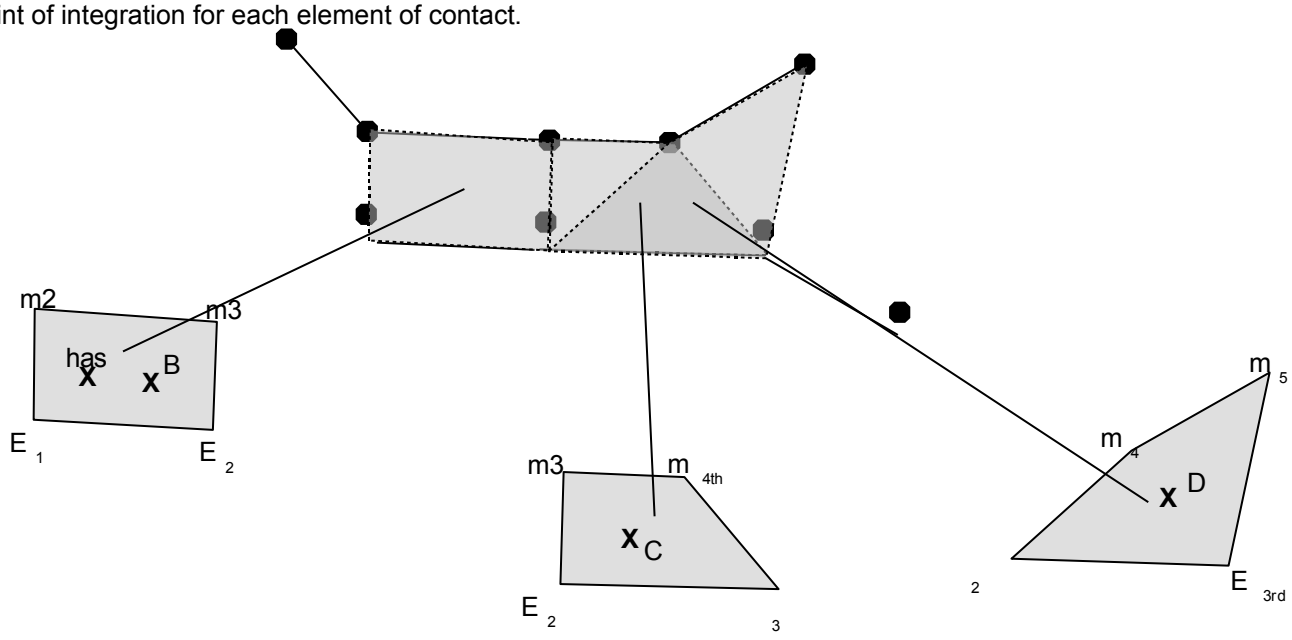

Figure 11: Creation of the three elements of contact

We put the collocation points (a,b,c,d) like Gauss points, but their site of course will depend on the diagram of integration. Each element of contact thus represents a matrix or an elementary vector on the degrees of freedom carried by the nodes slaves and the main nodes. In 3D, the largest mesh of surface is `the QUAD9` (quadrangle with nine nodes), which makes $2\times9\times3=54$ degrees of freedom for displacements. In the case

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of the contact with friction in 3D, one adds on surface slave three additional degrees of freedom by the node is outside the field of definition with a right profile of the EXCLU type node: the Lagrange of the multiplier of contact and the two Lagrange of the semi-multiplier of friction, which gives $9\times3=27$ additional degrees of freedom. Therefore, in 3D, for the element of contact with friction, the maximum size of the elementary matrix is of 81 by 81 degrees of freedom.

In practice the method is the following one:
- Creation of the element of contact on the main nodes and slaves;
- For each element of contact, buckles on the collocation points;
- For each collocation point, computation of the matrix (or vector) elementary.

It is because of last point that the creation of the finite element is sub-optimal. Indeed, one calls the computation of the elementary matrix for each collocation point. However, for the conventional finite elements, the loop on the points of integration is inside elementary computation.

We thus see that the use of a diagram of collocation makes it possible to make coexist different states of contact within the same element. However, there remains a problem of compatibility between the approximation EF of the shape functions and the diagram of integration. Let us take the example of a segment (figure 12).

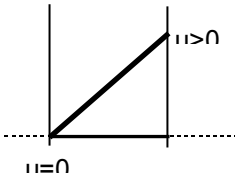

Figure 12: EF/collocation incompatibility on a segment

On this segment, displacement is approached by a linear shape function and it is worth 0 with the node of left and it is higher than zero with the node of right. From the point of view of the contact, that means that point located in the middle of the segment necessarily will fall apart. However the diagram of integration per collocation does not indicate this state inevitably. There is a decoupling between the polynomial approximation of displacements and the discretization by collocation of the fields of sign. What means that the collocation method is perfect for zones with constant states of contact on the element but which it is much less relevant if the state of contact varies. Such a situation generates oscillations in the zones of transition. These oscillations cannot be reduced by means of more collocation points but by refining the mesh in this zone.

Note:
- The elements of contact-friction not having a geometrical meaning, they are not in conformity, without any impact on the quality of result: they are infinitely flattened when there is contact, they can be very distorted and to have “strange” connectivities in the case of connection between surfaces of different degrees;
- The fact of using the diagram of integration to the nodes does not make the elements of contact more powerful from the point of view of their use, since the principle describes above is generic for all the diagrams of integration;
- There is no procedure dedicated to the perfectly compatible meshes. Two meshes chosen to be compatible will not give results faster than two incompatible meshes;
- The fact of having to build of the finite elements late results in a refactorisation of the total matrix to each iteration of Newton, since the topology of the matrix will be modified.

6.3 Preparation of the discretization

6.3.1 Discretization in time

the discretization (pseudonym) - temporal will be indicated by the index k. For example, displacement $\boldsymbol{u}_k$ with time step running is written as the sum of the displacement cumulated $\boldsymbol{u}_k^{cum}$ since the beginning of the process of Newton of time step running and of the displacement increment of the current iteration of Newton $\Delta\boldsymbol{u}$ which one seeks:

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$$\boldsymbol{u}_k=\boldsymbol{u}_{k-1}+\boldsymbol{u}_k^{cum}+\Delta\boldsymbol{u} \tag{229}$$

This iterative writing is also valid for the other nodal unknowns (field of the Lagrange multipliers):

$$\lambda_k=\lambda_{k-1}+\lambda_k^{cum}+\Delta\lambda \tag{230}$$

And:

$$\boldsymbol{\Lambda}_k=\boldsymbol{\Lambda}_{k-1}+\boldsymbol{\Lambda}_k^{cum}+\Delta\boldsymbol{\Lambda} \tag{231}$$

In *Code_Aster*, one made the choice to work in quantity total (and nonincremental) for the Lagrange multipliers, they are thus initialized to zero at each beginning of time step and one thus has:

$$\lambda_{k-1}=0\rightarrow\lambda_k=\lambda_k^{cum}+\Delta\lambda \tag{232}$$

And:

$$\boldsymbol{\Lambda}_{k-1}=0\rightarrow\boldsymbol{\Lambda}_k=\boldsymbol{\Lambda}_k^{cum}+\Delta\boldsymbol{\Lambda} \tag{233}$$

One will use a notation which separates with time step running k the quantities *known* $\hat{\boldsymbol{u}}_k$ and the unknown *quantities* $\Delta\boldsymbol{u}$:

$$\begin{aligned}\boldsymbol{u}_k&=\hat{\boldsymbol{u}}_k+\Delta\boldsymbol{u}\ \text{ with }\ \hat{\boldsymbol{u}}_k=\boldsymbol{u}_{k-1}+\boldsymbol{u}_k^{cum}\\ \bar{\boldsymbol{u}}_k&=\hat{\bar{\boldsymbol{u}}}_k+\Delta\bar{\boldsymbol{u}}\ \text{ with }\ \hat{\bar{\boldsymbol{u}}}_k=\bar{\boldsymbol{u}}_{k-1}+\bar{\boldsymbol{u}}_k^{cum}\end{aligned} \tag{234}$$

For the brought up to date coordinates, as one has $\boldsymbol{x}=\boldsymbol{p}+\boldsymbol{u}$:

$$\begin{aligned}\boldsymbol{x}_k&=\hat{\boldsymbol{x}}_k+\Delta\boldsymbol{x}\ \text{ with }\ \hat{\boldsymbol{x}}_k=\boldsymbol{p}+\boldsymbol{u}_{k-1}+\boldsymbol{u}_k^{cum}=\boldsymbol{p}+\hat{\boldsymbol{u}}_k\\ \bar{\boldsymbol{x}}_k&=\hat{\bar{\boldsymbol{x}}}_k+\Delta\bar{\boldsymbol{x}}\ \text{ with }\ \hat{\bar{\boldsymbol{x}}}_k=\bar{\boldsymbol{p}}+\bar{\boldsymbol{u}}_{k-1}+\bar{\boldsymbol{u}}_k^{cum}=\bar{\boldsymbol{p}}+\hat{\bar{\boldsymbol{u}}}_k\end{aligned} \tag{235}$$

For the field of contact pressure:

$$\lambda_k=\hat{\lambda}_k+\Delta\lambda\ \text{ with }\ \hat{\lambda}_k=\lambda_k^{cum} \tag{236}$$

For the field of pressure of friction:

$$\boldsymbol{\Lambda}_k=\hat{\boldsymbol{\Lambda}}_k+\Delta\boldsymbol{\Lambda}\ \text{ with }\ \hat{\boldsymbol{\Lambda}}_k=\boldsymbol{\Lambda}_k^{cum} \tag{237}$$

This notation remains valid for other quantities like the increased multiplier of contact:

$$g_{n,k}=\hat{g}_{n,k}+\Delta g_n\ \text{ with }\ \hat{g}_{n,k}=\hat{\lambda}_k-\rho_n\hat{d}_{n,k} \tag{238}$$

And the increased semi-multiplier of friction:

$$\boldsymbol{h}_{\tau,k}=\hat{\boldsymbol{h}}_{\tau,k}+\Delta\boldsymbol{h}_\tau\ \text{ with }\ \hat{\boldsymbol{h}}_{\tau,k}=\hat{\boldsymbol{\Lambda}}_k+\bar{\rho}_t[\![\hat{\boldsymbol{u}}]\!]_{\tau,k} \tag{239}$$

For the operator of projection on the ball unit:

$$\hat{\boldsymbol{\tau}}_k=\frac{\hat{\boldsymbol{h}}_{\tau,k}}{\left\|\hat{\boldsymbol{h}}_{\tau,k}\right\|} \tag{240}$$

6.3.2 Discretization of the local coordinate system

One does not specify the way in which are interpolated the norms and the tangent, because all will depend on the options specified in DEFI_CONTACT (activation of the lissage for example). It should simply be known that one uses the same shape functions as the description of the geometry. It will be written directly that the norm or the tangents is quantities expressed at the current point of integration:

$$\begin{aligned}\boldsymbol{n}&\overset{Discrétisation}{\rightarrow}\left\{n_h\right\}\\ \boldsymbol{t}^1&\overset{Discrétisation}{\rightarrow}\left\{t_h^1\right\}\\ \boldsymbol{t}^2&\overset{Discrétisation}{\rightarrow}\left\{t_h^2\right\}\end{aligned} \tag{241}$$

One will need to transform unknowns of space 3D usual towards the tangent plane. One uses for that the rectangular matrix $\left[T_h\right]$ such as:

$$\left[T_h\right]=\left[\left\{t_h^1\right\}\quad\left\{t_h^2\right\}\right] \tag{242}$$

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the discretization of the matrix of normal projection notes itself:

$$\underline{\underline{\boldsymbol{P}}}^{n} \overset{\text{Discrétisation}}{\rightarrow} \left[P_h^n\right] \tag{243}$$

By developing the terms of the norm $\boldsymbol{n}$ (discretized) and its three components in the total reference (x, y, z), one a:

$$\left[P_h^n\right]=\begin{bmatrix} n_x^2 & n_x n_y & n_x n_z \\ n_x n_y & n_y^2 & n_y n_z \\ n_x n_z & n_y n_z & n_z^2 \end{bmatrix} \tag{244}$$

Lastly, the discretization of the matrix of tangent projection is noted:

$$\underline{\underline{\boldsymbol{P}}}^{\tau} \overset{\text{Discrétisation}}{\rightarrow} \left[P_h^\tau\right] \tag{245}$$

Attention not to confuse $\left[P_h^\tau\right]$ with the matrix $\left[T_h\right]$ (242) on the tangent level. $\left[P_h^\tau\right]$ is worth:

$$\left[P_h^\tau\right]=\begin{bmatrix} 1-n_x^2 & -n_x n_y & -n_x n_z \\ -n_x n_y & 1-n_y^2 & -n_y n_z \\ -n_x n_z & -n_y n_z & 1-n_z^2 \end{bmatrix} \text{ or } \left[P_h^\tau\right]=[\mathbf{1}]-\left[P_h^n\right] \tag{246}$$

the matrixes $\left[P_h^\tau\right]$ and $\left[P_h^n\right]$ are symmetric:

$$\left[P_h^\tau\right]=\left[P_h^\tau\right]^T \text{ and } \left[P_h^n\right]=\left[P_h^n\right]^T \tag{247}$$

6.3.3 Discretization of the fields solutions

Consider the vector field describing the geometry (coordinated nodes) on the element slave $\boldsymbol{p}^e$. This field has two or three components according to the dimension of space, we will note it in the following form:

$$\boldsymbol{p}^e=\begin{Bmatrix} p_x^e \\ p_y^e \\ p_z^e \end{Bmatrix} \tag{248}$$

the component p_x^e is approximated by:

$$p_x^e=\sum_{j=1}^{NN_e} p_{x,j}^e N_j^e \tag{249}$$

With NN_e the number of nodes on the element slave, $p_{x,j}^e$ J-ième nodal value of the field following dimension x and N_j^e the shape functions. While projecting on the bases finite elements, the shape functions being identical for three dimensions of space:

$$\boldsymbol{p}^e=\begin{Bmatrix} p_x^e \\ p_y^e \\ p_z^e \end{Bmatrix}=\sum_{j=1}^{NN_e}\begin{bmatrix} N_j^e & 0 & 0 \\ 0 & N_j^e & 0 \\ 0 & 0 & N_j^e \end{bmatrix}\begin{Bmatrix} p_{x,j}^e \\ p_{y,j}^e \\ p_{z,j}^e \end{Bmatrix} \tag{250}$$

One will write in more compact form the nodal values $\left\{p_h^e\right\}$:

$$\boldsymbol{p}^e=\left[N^e\right]\left\{p_h^e\right\} \tag{251}$$

In a similar way for the main element:

$$\boldsymbol{p}^m=\left[N^m\right]\left\{p_h^m\right\} \tag{252}$$

For the fields of displacements:

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$$\boldsymbol{u}^e=\left[N^e\right]\left\{u_h^e\right\} \text{ and } \boldsymbol{u}^m=\left[N^m\right]\left\{u_h^m\right\} \tag{253}$$

And for the current coordinate:

$$\boldsymbol{x}^e=\left[N^e\right]\left\{x_h^e\right\} \text{ and } \boldsymbol{x}^m=\left[N^m\right]\left\{x_h^m\right\} \tag{254}$$

the field of multiplier of contact is scalar:

$$\lambda=\sum_{j=1}^{NN_e}\lambda_j\psi_j \tag{255}$$

With ψ_j the shape functions used for the Lagrange multipliers. One will write in more compact form:

$$\lambda=\langle\psi\rangle\left\{\lambda_h\right\}=\langle\lambda_h\rangle\left\{\psi\right\} \tag{256}$$

the norm and the tangent must also be discretized. One leaves the definition of the tangent reference on the master mesh, and one interpolates the norms and the tangent thanks to the same shape functions. One will note $\left\{n_h\right\}$, $\left\{t_h^1\right\}$ and $\left\{t_h^2\right\}$ these quantities discretized. At the time of the writing of the various relations, we supposed that the density of normal force of friction was a total quantity, defined in the total reference:

$$\boldsymbol{\Lambda}=\begin{Bmatrix}\Lambda_x\\ \Lambda_y\\ \Lambda_z\end{Bmatrix} \tag{257}$$

the component Λ_x on an edge element, is written as follows:

$$\Lambda_x=\sum_{j=1}^{NN_c}\Lambda_{x,j}\phi_j \tag{258}$$

With NN_c the number of nodes on the edge element, $\Lambda_{x,j}$ J-ième nodal value of the component x field of pressure of friction and ϕ_j the shape functions. In fact, as there exists naturally a close connection between the contact and friction (that of the definition of the local coordinate system), the elementary functions ψ_j and ϕ_j are not independent, and one will break up the semi-multiplier of Lagrange of total friction $\boldsymbol{\Lambda}$ in two components (in 3D) on the tangent level defined in the point of contact per $\left(\boldsymbol{t}^1,\boldsymbol{t}^2\right)$:

$$\boldsymbol{\Lambda}=\begin{Bmatrix}\Lambda_x\\ \Lambda_y\\ \Lambda_z\end{Bmatrix}=\sum_{j=1}^{NN_e}\begin{bmatrix}\left\{t_h^1\right\} & \left\{t_h^2\right\}\end{bmatrix}\begin{bmatrix}\psi_j & 0\\ 0 & \psi_j\end{bmatrix}\begin{Bmatrix}\Lambda_j^1\\ \Lambda_j^2\end{Bmatrix} \tag{259}$$

That one will rewrite in more compact form:

$$\boldsymbol{\Lambda}=\left[T_h\right]\left[\psi\right]\left\{\Lambda_h\right\} \tag{260}$$

If one considers 3D case, the field of the semi-multiplier of friction $\boldsymbol{\Lambda}$ is thus of dimension three times the number of nodes slaves. Its discrete version $\left\{\Lambda_h\right\}$, projection on the tangent level is of dimension twice the number of nodes slaves. We thus have nothing any more but three bases finite elements a priori independent:

- The base describing the geometry of structure;
- The base describing the primal field of unknowns (displacement or velocity);
- The base describing the multiplier of Lagrange of the contact and the projection of semi-multiplier of friction on the tangent level at the point of contact.

6.3.4 Numerical integration

the transformation of the integral continues of a scalar quantity f in discrete integral is written:

$$\int_{\Omega_h} f\,d\,\Omega\approx\sum_{c=1}^{NP}f\left(\boldsymbol{\zeta}_c\right)\omega_c J_c \tag{261}$$

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With NP the number of points of integrations, $\boldsymbol{\zeta}_c$ the parametric coordinate of the point of integration on the slave mesh, ω_c the weight of integration of this point and J_c the jacobian of the transformation of the field of integration. This point is project of surface slave towards surface Master by the application of pairing A_t. The evaluating on surface Master of the "paired" quantities noted $\bar{f}$, is thus written simply:

$$\int_{\Omega_h} \bar{f}\, d\Omega \approx \sum_{c=1}^{NP} \bar{f}\left(A_t(\boldsymbol{\zeta}_c)\right)\omega_c J_c$$

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) the contact, it is appropriate to notice that each point of contact is also a point of integration (method of discretization of the field of sign by collocation). All the continuous integrals will thus be discretized by a numerical diagram confused with the collocation method of the fields of sign.

6.3.5 Discretization of the jumps

the various quantities related to the "jump" are functions of a quantity on surface slave with his projection on their surface Master. Let us consider initially the projection of the jump on the norm:

$$[\![u]\!]_n = [\![\boldsymbol{u}]\!]\cdot\boldsymbol{n} = (\boldsymbol{u}-\bar{\boldsymbol{u}})\cdot\boldsymbol{n} \quad (263)$$

By recalling that $\boldsymbol{u}$ is a quantity defined on surface slave and $\bar{\boldsymbol{u}}$ his projection on surface Master. We will need also to develop terms of the tangential type:

$$[\![\boldsymbol{u}]\!]_{\boldsymbol{\tau}} = \left[P^{\tau}\right][\![\boldsymbol{u}]\!] = \left[P^{\tau}\right](\boldsymbol{u}-\bar{\boldsymbol{u}}) \quad (264)$$

One points out the discretization defined on the unknown $\boldsymbol{u}$ by (253) uses the shape functions defined in the current point of integration $\boldsymbol{\zeta}_c$ on surface slave. The projectionformule $\bar{\boldsymbol{u}}$ of this quantity on surface Master will thus use the point of integration transferred $A_t(\boldsymbol{\zeta}_c)$ onto surface Master:

$$\boldsymbol{u} = \left[N^e(\boldsymbol{\zeta}_c)\right]\left\{u_h^e\right\} \text{ and } \bar{\boldsymbol{u}} = \left[N^m\left(A_t(\boldsymbol{\zeta}_c)\right)\right]\left\{u_h^m\right\} \quad (265)$$

That one will note in a more compact way by:

$$\boldsymbol{u} = \left[N^e\right]\left\{u_h^e\right\} \text{ and } \bar{\boldsymbol{u}} = \left[\bar{N}^m\right]\left\{u_h^m\right\} \quad (266)$$

One has finally:

$$[\![u]\!]_n = \langle n_h\rangle\left(\left[N^e\right]\{u_h^e\} - \left[\bar{N}^m\right]\{u_h^m\}\right) = \left(\langle u_h^e\rangle\left[N^e\right]^T - \langle u_h^m\rangle\left[\bar{N}^m\right]^T\right)\{n_h\} \quad (267)$$

It is a scalar. For the tangential jump:

$$[\![\boldsymbol{u}]\!]_{\boldsymbol{\tau}} = \left[P^{\tau}\right]\left(\left[N^e\right]\{u_h^e\} - \left[\bar{N}^m\right]\{u_h^m\}\right) \quad (268)$$

If one now considers the normal jump of displacement d_n :

$$d_n = [\![x]\!]_n = [\![\boldsymbol{x}]\!]\cdot\boldsymbol{n}$$

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):

$$d_n = \langle n_h\rangle\left(\left[N^e\right]\{x_h^e\} - \left[\bar{N}^m\right]\{x_h^m\}\right) \quad (270)$$

In the same way:

$$[\![\boldsymbol{x}]\!]_{\boldsymbol{\tau}} = \left[P^{\tau}\right]\left(\left[N^e\right]\{x_h^e\} - \left[\bar{N}^m\right]\{x_h^m\}\right) \quad (271)$$

From where:

$$\boldsymbol{d}_{\tau} = \left[P^{\tau}\right]\left(\left[N^e\right]\{x_h^e\} - \left[\bar{N}^m\right]\{x_h^m\}\right) \quad (272)$$

the derived quantities compared to the parametric coordinates:

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$$\frac{\partial \bar{\boldsymbol{x}}}{\partial \zeta^{\alpha}}=\sum_{j=1}^{NN_m}\begin{bmatrix}\frac{\partial \bar{N}_j^m}{\partial \zeta^{\alpha}} & 0 & 0\\ 0 & \frac{\partial \bar{N}_j^m}{\partial \zeta^{\alpha}} & 0\\ 0 & 0 & \frac{\partial \bar{N}_j^m}{\partial \zeta^{\alpha}}\end{bmatrix}\begin{Bmatrix}x_j^m\\ y_j^m\\ z_j^m\end{Bmatrix} \tag{273}$$

By noting $\left[\bar{B}_{\alpha}^m\right]$ the matrix of derivatives of the shape functions compared to the coordinated parametric ζ^{α}. That is to say:

$$\frac{\partial \bar{\boldsymbol{x}}}{\partial \zeta^{\alpha}}=\left[\bar{B}_{\alpha}^m\right]\{x_h^m\} \tag{274}$$

6.3.6 Discretization of (semi) - multiplying

Let us consider the increased multiplier of contact g_n :

$$g_n=\lambda-\rho_n d_n \tag{275}$$

Its discretization gives:

$$g_n=\langle\psi\rangle\left[\lambda_h\right]-\rho_n\langle n_h\rangle\left(\left[N^e\right]\{x_h^e\}-\left[\bar{N}^m\right]\{x_h^m\}\right) \tag{276}$$

now Consider the increased semi-multiplier of friction $\boldsymbol{h}_{\tau}$:

$$\boldsymbol{h}_{\tau}=\boldsymbol{\Lambda}+\rho_t\boldsymbol{v}_{\tau} \tag{277}$$

We consider the incremental writing in displacement (see §33):

$$\boldsymbol{h}_{\tau}=\boldsymbol{\Lambda}+\bar{\rho}_t[\![\boldsymbol{u}]\!]_{\tau} \tag{278}$$

the discretization of the semi-multiplier gives:

$$\boldsymbol{h}_{\tau}=[T]\left[\psi\right]\left[\Lambda_h\right]+\bar{\rho}_{\tau}\left[P^{\tau}\right]\left(\left[N^e\right]\{u_h^e\}-\left[\bar{N}^m\right]\{u_h^m\}\right) \tag{279}$$

6.3.7 Statutes of contact and of friction

One defined the field of sign of contact noted $S_u^{g_n}$ by the following statement:

$$S_u^{g_n}=\begin{cases}1 & \text{si } g_n\leq 0\\ 0 & \text{si } g_n>0\end{cases} \tag{280}$$

This operator is the function characteristic of the increased multiplier of contact g_n on the convex $\mathbb{R}^-$. One also defined the field of sign of friction noted $S_f^{\boldsymbol{h}_{\tau}}$ by the following statement:

$$S_f^{\boldsymbol{h}_{\tau}}=\begin{cases}1 & \text{si } \boldsymbol{h}_{\tau}\in B(0,1)\\ 0 & \text{sinon}\end{cases} \tag{281}$$

This operator is the function characteristic of the semi-multiplier increased of friction $\boldsymbol{h}_{\tau}$ on the convex one $B(0,1)$. The discretization of the fields of the signs is commonplace because one uses collocation on these discontinuous fields of nature (see §42). The discretized value of a multiplier (of contact or friction) is equal to its value at the point of integration. These operators being discontinuous, their linearization does not have a meaning. One thus writes directly the value of these fields at time running k but right before the current iteration of Newton:

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$$\hat{S}_{u,k}^{g_n}=\begin{cases}1 & \text{si } \hat{g}_{n,k}\leq 0\\ 0 & \text{si } \hat{g}_{n,k}>0\end{cases} \text{ and } \hat{S}_{f,k}^{\boldsymbol{h}_\tau}=\begin{cases}1 & \text{si } \hat{\boldsymbol{h}}_{\tau,k}\in B(0,1)\\ 0 & \text{sinon}\end{cases} \tag{282}$$

6.3.8 Discretization of the virtual quantities

We now will consider the writing of the discretized forms of the virtual quantities. For δd_n :

$$\delta d_n=\langle n_h\rangle\left(\left[N^e\right]\{\delta u_h^e\}-\left[\bar{N}^m\right]\{\delta u_h^m\}\right)=\left(\langle\delta u_h^e\rangle\left[N^e\right]^T-\langle\delta u_h^m\rangle\left[\bar{N}^m\right]^T\right)\{n_h\} \tag{283}$$

For $\delta \boldsymbol{d}_\tau$:

$$\delta \boldsymbol{d}_\tau=\left[P^\tau\right]\left(\left[N^e\right]\{\delta u_h^e\}-\left[\bar{N}^m\right]\{\delta u_h^m\}\right) \tag{284}$$

the variation of the increased semi-multiplier of friction is worth:

$$\delta \boldsymbol{h}_\tau=[T][\psi]\{\delta\Lambda_h\}+\bar{\rho}_\tau\left[P^\tau\right]\left(\left[N^e\right]\{\delta u_h^e\}-\left[\bar{N}^m\right]\{\delta u_h^m\}\right) \tag{285}$$

6.4 resulting discretized System

One sets out again of the system (199), which represents the equilibrium of system (PTV) in incremental form:

To find the fields $\boldsymbol{W}_k\in \mathrm{CA}^1\times\mathrm{CA}^2\times H\times \boldsymbol{H}$ such as:

$$\begin{cases}\sum_{i=1}^{2}\left[G_{\text{int},k}^i\left(\boldsymbol{W}_k,\delta\boldsymbol{W}_k\right)-G_{\text{ext},k}^i\left(\boldsymbol{W}_k,\delta\boldsymbol{W}_k\right)\right]-G_{\text{c},k}\left(\boldsymbol{W}_k,\delta\boldsymbol{W}_k\right)-G_{\text{f},k}\left(\boldsymbol{W}_k,\delta\boldsymbol{W}_k\right)=0 & (a)\\ \tilde{G}_{\text{c},k}\left(\boldsymbol{W}_k,\delta\boldsymbol{W}_k\right)=0 & (b)\\ \tilde{G}_{\text{f},k}\left(\boldsymbol{W}_k,\delta\boldsymbol{W}_k\right)=0 & (c)\end{cases} \tag{286}$$

$$\forall\,\delta\boldsymbol{W}_k\in \mathrm{CA}^1\times\mathrm{CA}^2\times H\times \boldsymbol{H}$$

After discretization, we obtain the following linearized system:

$$\begin{cases}\sum_{i=1}^{2}\left[G_{\text{int},k}^i-G_{\text{ext},k}^i\right]-G_{\text{c},k}-G_{\text{f},k}=0 \overset{\text{Discrétisation}}{\rightarrow}\left\{L_{\boldsymbol{u},k}\right\}=0\\ \tilde{G}_{\text{c},k}=0 \overset{\text{Discrétisation}}{\rightarrow}\left\{L_{c,k}\right\}=0\\ \tilde{G}_{\text{f},k}=0 \overset{\text{Discrétisation}}{\rightarrow}\left\{L_{f,k}\right\}=0\end{cases} \tag{287}$$

$$\forall\,\delta\boldsymbol{W}_k\in \mathrm{CA}^1\times\mathrm{CA}^2\times H\times \boldsymbol{H}$$

This system is linearized (see §36):

$$\begin{Bmatrix}L_{\boldsymbol{u},k}\\ L_{c,k}\\ L_{f,k}\end{Bmatrix}\approx\begin{Bmatrix}L_{\boldsymbol{u},k-1}\\ L_{c,k-1}\\ L_{f,k-1}\end{Bmatrix}+\begin{bmatrix}J_{\boldsymbol{uu}} & J_{\boldsymbol{u}c} & J_{\boldsymbol{u}f}\\ J_{c\boldsymbol{u}} & J_{cc} & J_{cf}\\ J_{f\boldsymbol{u}} & J_{fc} & J_{ff}\end{bmatrix}\begin{Bmatrix}\Delta\boldsymbol{u}^i\\ \Delta\lambda_n\\ \Delta\boldsymbol{\Lambda}\end{Bmatrix}=\begin{Bmatrix}L_{\boldsymbol{u},k-1}\\ L_{c,k-1}\\ L_{f,k-1}\end{Bmatrix}+[J]\{\Delta\boldsymbol{W}_k\} \tag{288}$$

$$\forall\,\delta\boldsymbol{W}\in \mathrm{CA}^1\times\mathrm{CA}^2\times H\times \boldsymbol{H}$$

After linearization, one thus finds itself with a system of the following form:

$$\left[\boldsymbol{J}(\delta\boldsymbol{W})\right]\{\Delta\boldsymbol{W}\}+\left\{\boldsymbol{L}_{k-1}(\delta\boldsymbol{W})\right\}\overset{\text{Discrétisation}}{\rightarrow}\langle\delta\boldsymbol{W}\rangle\left(\left[\boldsymbol{K}\right]\{\Delta\boldsymbol{W}\}+\{\boldsymbol{L}\}\right)=0 \tag{289}$$

With the vector of the nodal unknowns (real or virtual) $\langle\delta\boldsymbol{W}\rangle$:

$$\langle\delta\boldsymbol{W}\rangle=\langle\delta\boldsymbol{u}^e\quad\delta\boldsymbol{u}^m\quad\delta\lambda\quad\delta\boldsymbol{\Lambda}\rangle \tag{290}$$

the matrix $[K]$:

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$$[\boldsymbol{K}]=\begin{bmatrix} [K_{ee}^t]+[K_{ee}] & [K_{em}] & [K_{ec}] & [K_{ef}] \\ [K_{me}] & [K_{mm}^t]+[K_{mm}] & [K_{mc}] & [K_{mf}] \\ [K_{ce}] & [K_{cm}] & [K_{cc}] & [K_{cf}] \\ [K_{fe}] & [K_{fm}] & [K_{fc}] & [K_{ff}] \end{bmatrix} \quad (291)$$

the vector $\{\boldsymbol{L}\}$:

$$\{\boldsymbol{L}\}=\begin{Bmatrix} L_{\text{int}}^e+L_c^e+L_f^e \\ L_{\text{int}}^m+L_c^m+L_f^m \\ \tilde{L}_c^c \\ \tilde{L}_f^f \end{Bmatrix} \quad (292)$$

the matrixes $[K_{ee}^t]$ and $[K_{mm}^t]$ are the tangent matrixes coming from the linearization of the internal forces (discretization of the jacobian matrixes $J_{\boldsymbol{uu}}^{\boldsymbol{u}}$, to see p.59). The other matrixes come from the linearization of the terms of contact and friction (see §36). The system to be solved will be thus:

$$[\boldsymbol{K}]\{\Delta \boldsymbol{W}\}=-\{\boldsymbol{L}\} \quad (293)$$

In the developed statements, one will always consider the case of generalized Newton, while basing itself on the "incrémentalo-iterative" discretization specified in the §46. The discretized quantities **will be written in only one point of integration** of coordinate $\boldsymbol{\zeta}_c$, in such a way that one will introduce the weight of integration ω_c of this point and the jacobian J_c of the transformation of the field of integration:

$$\int_{\Omega_k} f\, d\Omega \overset{\text{Discrétisation}}{\rightarrow} f(\boldsymbol{\zeta}_c)\,\omega_c J_c \quad (294)$$

important Remark: It is advisable to notice that with the notations above it is in fact $-[L]$ that one finds on the right of equal sign. However as the forces of contact-friction are added in the code to the internal forces which themselves are cut off with the second member, one does not need to take account of this change of sign. The statements which are written in the elementary terms of the code are thus quite identical to those presented.

The statement of the second member vector is found in appendix with the p. 65 and that of the tangent matrix to p. 68 .

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7 Management of the incompatibilities in the processing of the contact

the presence of contact and boundary conditions of Dirichlet imposed on points of a common surface can generate problems of redundancy leading to the failure of computation (in general, the tangent stiffness matrix becomes singular). In what follows, we present some typical cases of redundancy which we identified, and the specific processing suggested in [7].

7.1 Redundancy between the boundary conditions and the conditions of contact-friction

7.1.1 Example

the case more flow is a redundancy of the conditions of contact with the boundary conditions. Let us take the example of a block in contact rubbing on a rigid level (ssnv128 benchmark, to see Figure 13): one models half of this block, at the end of which the conditions of symmetry are applied: $u_x=0$. The loading is a pressure imposed F on the sides of dimensioned and top. The face of bottom is, as one said, in contact rubbing with his support. On the segment $[AC]$, one will have redundancy between the boundary condition and the condition of sliding.

When one discretized the mixed formulation variational, one wrote that there was coupling between displacements and the semi-multipliers of friction (matrixes $\left[A_u^\Lambda\right]$ and $\left[A_\Lambda^u\right]$), which wants to say that the condition of NON-sliding will give $u_x=0$, whereas this last condition is already imposed in the total system (the matrix $[B]$ of the boundary conditions). One thus imposes twice the relation $u_x=0$. This situation leads to an error of null pivot during the inversion of the tangent stiffness matrix.

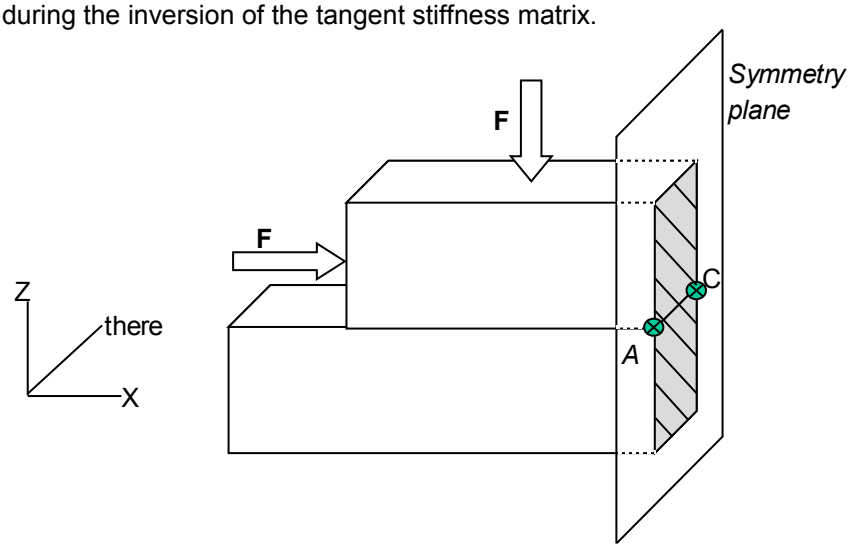

Figure 13: Plate in contact rubbing on a rigid level

7.1.2 Choice of the space of approximation

the adopted solution is always to give precedence to the boundary conditions of the conditions of dependancy-sliding. For that, it is enough and to uncouple the multipliers from friction displacements for the component to the node concerned with the redundancy (to take again our example, one will uncouple u_x and Λ_x with the nodes A and C). That amounts by means of modifying the space of approximation of the field of the multiplier of friction a function which is worth zero for the suitable degree of freedom. This operation is illustrated on figure 14 in the case 2D where the condition of redundancy is at the point A .
One takes $\boldsymbol{H}_h$ for space of approximation of the semi-multiplier of friction μ_h :

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$$\boldsymbol{H}_h=\left\{\mu_h\in \mathrm{CA}_h ; \mu_h(A)=0\right\} \quad (295)$$

This space is in conformity from the point of view of condition LBB. In practice, one puts simply of the zeros at the suitable sites of the matrixes $\left[A_e^{\Lambda}\right]$ and $\left[A_{\Lambda}^{e}\right]$ [3].

Option SANS_GROUP_NO_FR/SANS_NOEUD_FR of *Code_Aster* makes it possible to remove the contribution to the matrix of friction of a set of nodes, while keeping the boundary conditions and of unilateral contact in these nodes.

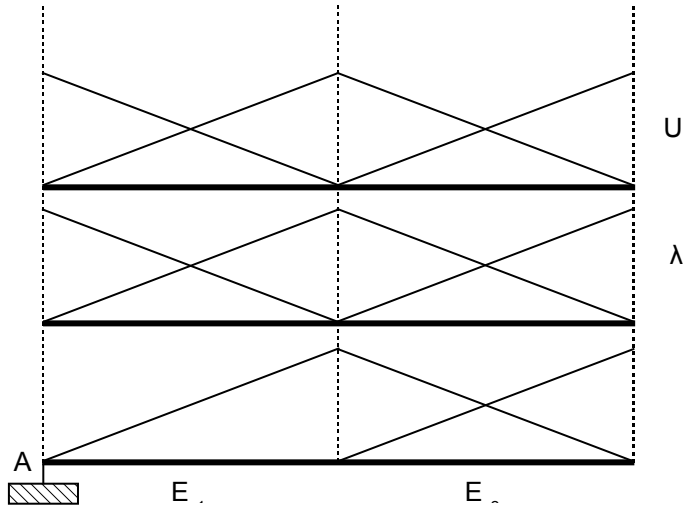

Figure 14: Modification of spaces of approximation for the redundancy friction/condition limits

In 2D, one and only one direction of sliding exists. The suppression of the condition of dependancy for the problematic points is thus sufficient to eliminate any redundancy. It is done by uncoupling the semi-multiplier from friction associated with a connection with the unknowns with displacement.
In 3D on the other hand, the sliding or the dependancy occurring in the tangent plane, one can want to privilege a particular direction of sliding (for example perpendicular to a blocking). For that, the user provides under key word DIRE_EXCL_FROT the direction to be excluded. This direction allows the construction of a local coordinate system which facilitates the decoupling of only one semi-multiplier of friction. If no direction is provided then that amounts not treating friction on the nodes concerned (decoupling of the two multipliers).

3 acts only on the coupling displacement multiplying slave/of friction because the risk of redundancy will appear only if the boundary conditions are imposed on surface slave, owing to the fact that the Lagrange multipliers (contact and friction) are discretized on surface slave

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8 Sometimes cycling during the process of

8.1 Newton Problématiqu

E the use of an algorithm of Newton on nondifferentiable quantities reveals phenomena of numerical cycling. In a rigorous way , one can use the algorithm of Newton on quasi-differentiable quantities (with the classical meaning), in fact, one will speak here about *B-différentiabilité* [11], i.e. the points of non-differentiability belong to a space of measurement null. In practice, with the numerical approximations introduced by the discretization and the transition with an algebra in floating point makes that it “probable very little” that one meets these points of non-differentiability, but problems of cycling are not excluded, particularly if the contact/friction is coupled with other non-linearities.
A cycling corresponds on the way of a point of non-differentiability during the process of Newton (i.e. of an iteration of Newton to the other).
In [11], O N distinguishes from three kinds (see figure 15) :
-Cycling on the statute of contact: a point of the mesh passes alternatively from a case of contact to a case of lack of contact;
-Cycling on the statute of friction: a point of the mesh passes alternatively from a case of sliding to a case of dependancy;
-Cycling on the threshold of sliding: a point of the mesh passes alternatively from a case of front sliding to a case of back sliding;
From a more heuristic point of view, one also shows that only cycles of order 2 with more are likely to occur the iterations of Newton during (what corresponds to at least three iterations of Newton) .

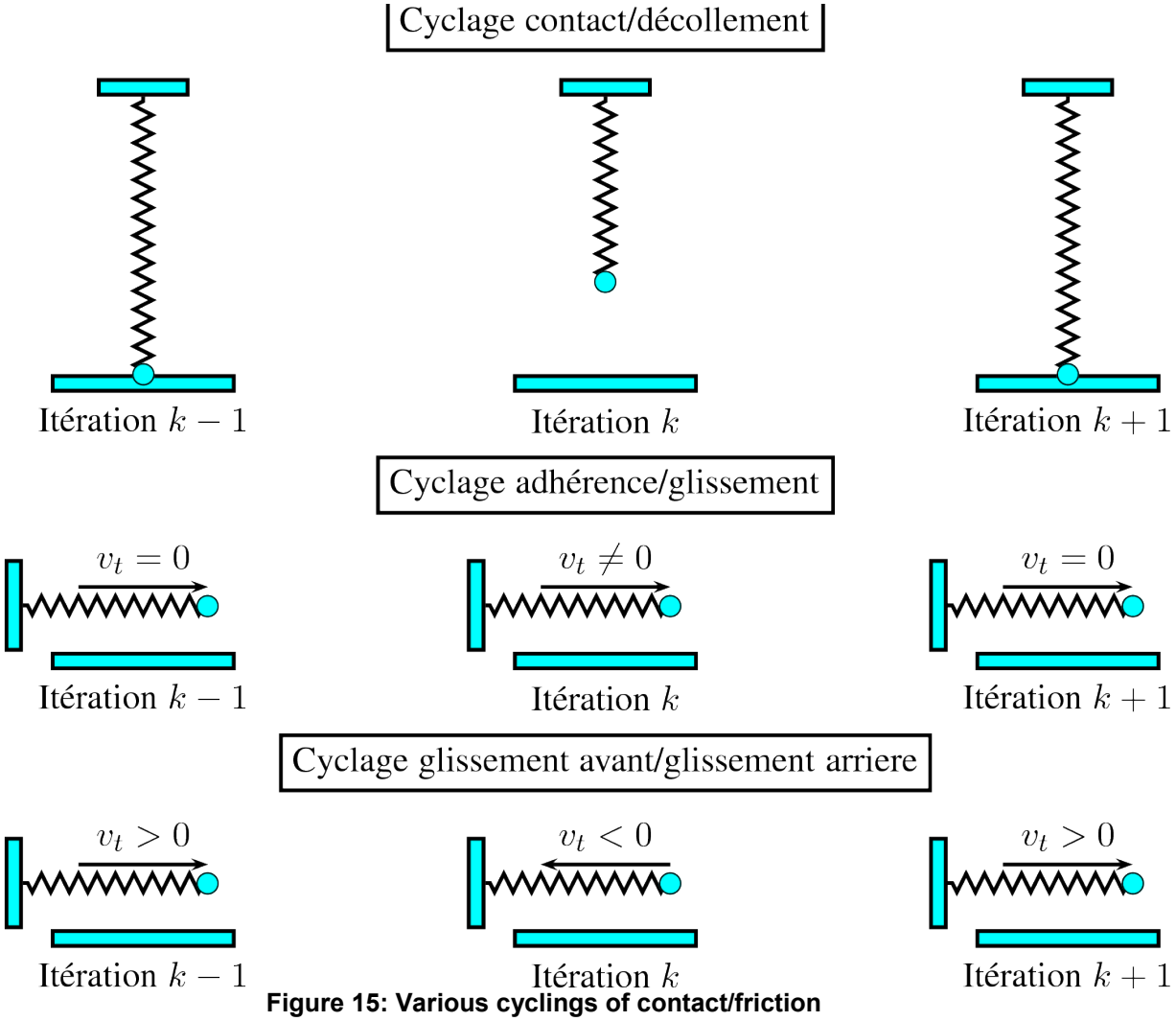

Figure 15: Various cyclings of contact/friction

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8.2 Detection

the three kinds of cycle are detected automatically by Code_Aster. An additional column appears then in the table of convergence with the mention CYCLING CONT , CYCLING AV/AR and CYCLING ADH/GLIS for, respectively, the cycling of contact, the cycling of sliding before/back and the cycling of friction dependancy/sliding.
It is important to note that there is a priority of detection between the cycles: cycling sliding before/back is priority in front of the cycling dependancy/sliding which itself is priority in front of the cycling of contact. In other words, one will display only the highest cycle of priority in the column, and this, on all the points of contact.

8.3 Processing

It does not have there, for time, automatic processing of these cyclings. For the user, the appearance of these cyclings is the sign of a certain difficulty of converging and one can deal with this problem of three ways:
−To use a more robust algorithm like partial Newton or incomplete Newton. In this last case, another type of cycling appear: it is the flip-flop, a cycling of type contact but of order 15 (!);
−To modify the parameters of increase. Here, there are no rules pre-established on the meaning of the modification (to decrease or increase the value) and on its amplitude. Nevertheless, it is advisable to observe that the parameter of increase ρ_t is very sensitive.
− To modify its model. Maybe by refining/déraffinant the mesh (attention! refinement can exacerbate the phenomena of cycling), that is to say while degrading the model (transition with an elastic material, even rigid).

9 Features and checking

This documentation describes the modelization of contact-friction by the method known as “continuous”, for its use, one can refer to the documentation of use of the command DEFI_CONTACT and the handbook of U2.04.04 modelization.
There exist about sixty benchmarks validating the various features related to the rubbing contact , among which:
•The benchmark of the contact of Hertz: to see ssnv104;
•The benchmark of the rubbing shoe: to see ssnv128;
•Benchmarks of benchmark NAFEMS on the contact/friction: to see ssna122, ssnp154, ssnp155, ssnp156 and ssnp157;
•The case test “shallow ironing”, to see ssnp153;

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11 Appendix a: computation of the terms of the jacobienne

11.1 Balance equation

For the balance equation the virtual quantities are total variations compared to displacements. The notation $\tilde{\delta}$ will thus be employed. The first line of the jacobian matrix $[J]$ consists of three terms:

$$\begin{aligned} J_{uu}&=J_{uu}^{\text{ext}}+J_{uu}^{\text{int}}+J_{uu}^{c}+J_{uu}^{f} \\ J_{uc}&=J_{uc}^{c}+J_{uc}^{f} \\ J_{uf}&=J_{uf}^{c}+J_{uf}^{f} \end{aligned} \tag{296}$$

One knows already that the internal forces and external bodies will not depend on Lagrangian on contact, nor of Lagrangian of friction:

$$J_{uc}^{\text{int}}=J_{uc}^{\text{ext}}=0 \text{ and } J_{uf}^{\text{int}}=J_{uf}^{\text{ext}}=0 \tag{297}$$

But also that the contact pressure does not depend on the pressure of friction:

$$J_{uf}^{\text{c}}=0 \tag{298}$$

Whereas the pressure of friction depends on the contact pressure (see §38):

$$J_{uc}^{\text{f}}\neq 0 \tag{299}$$

It does not act of assumptions of simplification, but of the logical consequence of the writing of the physical model.

11.1.1 Second variation compared to displacements

In this paragraph, the second variation will get along like total *variation* compared to displacements $\boldsymbol{u}^i$. It will be noted $\Delta_{\boldsymbol{u}^i}=\tilde{\Delta}$. We will not reconsider in detail the computation of the nonlinear terms J_{uu}^{int} and J_{uu}^{ext} with the internal forces G_{int}^i and external G_{ext}^i since they relate to the behavior and the kinematics of solids without contact-friction. For example, the variation of the work of the internal forces introduces the tangent matrixes $\underline{\underline{\boldsymbol{K}}}_t^i$ such as :

$$J_{uu}^{\text{int}}=\tilde{\Delta}\left(G_{\text{int}}^i\right)=\delta\boldsymbol{u}^i\,\underline{\underline{\boldsymbol{K}}}_t^i\,\Delta\boldsymbol{u}^i \tag{300}$$

We start by evaluating J_{uu}^{c} written with the variation of normal clearance:

$$J_{uu}^{\text{c}}=-\tilde{\Delta}\left(G_{\text{c}}\right)=-\tilde{\Delta}\left(\int_{\Gamma_c} S_u^{g_n} g_n\,\tilde{\delta} d_n\, d\Gamma_c\right) \text{ with } g_n=\left(\lambda_n-\rho_n d_n\right) \tag{301}$$

Crudely:

$$\tilde{\Delta} g_n=-\rho_n\tilde{\Delta} d_n \tag{302}$$

By developing the statement:

$$J_{uu}^{\text{c}}=-\int_{\Gamma_c} S_u^{g_n} g_n\,\tilde{\Delta}\tilde{\delta} d_n\, d\Gamma_c+\int_{\Gamma_c} S_u^{g_n}\rho_n\,\tilde{\Delta} d_n\tilde{\delta} d_n\, d\Gamma_c \tag{303}$$

Take again the statements drawn up in the § 13 . The first total variation of normal clearance is worth (see (74)) :

$$\tilde{\delta} d_n=\left(\delta_t\boldsymbol{x}-\delta_t\bar{\boldsymbol{x}}\right)\cdot\boldsymbol{n} \text{ and } \tilde{\Delta} d_n=\left(\Delta_t\boldsymbol{x}-\Delta_t\bar{\boldsymbol{x}}\right)\cdot\boldsymbol{n} \tag{304}$$

And thus, by application of (205):

$$\tilde{\delta} d_n=\left(\delta_t\boldsymbol{u}-\delta_t\bar{\boldsymbol{u}}\right)\cdot\boldsymbol{n} \text{ and } \tilde{\Delta} d_n=\left(\Delta_t\boldsymbol{u}-\Delta_t\bar{\boldsymbol{u}}\right)\cdot\boldsymbol{n} \tag{305}$$

the second variation is much more compl E Xe (see (107)) :

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$$\begin{aligned}\tilde{\Delta}\tilde{\delta}d_n=\Delta_t\delta_t d_n= & -\boldsymbol{n}\cdot\left(\Delta\zeta_\alpha\frac{\partial\left(\delta_t\overline{\boldsymbol{x}}\right)}{\partial\zeta_\alpha}+\frac{\partial\left(\Delta_t\overline{\boldsymbol{x}}\right)}{\partial\zeta_\alpha}\delta\zeta_\alpha\right)\\ & -\delta\zeta_\alpha\kappa_{\alpha\beta}\Delta\zeta_\beta\\ & -d_n\left(\boldsymbol{n}\cdot\frac{\partial\left(\delta_t\overline{\boldsymbol{x}}\right)}{\partial\zeta_\alpha}+\kappa_{\alpha\gamma}\delta\zeta^\gamma\right)m^{\alpha\beta}\left(\boldsymbol{n}\cdot\frac{\partial\left(\delta_t\overline{\boldsymbol{x}}\right)}{\partial\zeta_\beta}+\kappa_{\beta\sigma}\delta\zeta^\sigma\right)\end{aligned}\tag{306}$$

One uses the metric tensor $\underline{\underline{\boldsymbol{m}}}$ and the tensor of the second fundamental form $\underline{\underline{\boldsymbol{\kappa}}}$. All these statements intervene only in the event of contact ($S_u^{g_n}=1$) and thus when $d_n=0$. This assumption simplifies much the statement. For (306), by removing the term in d_n and with (205):

$$\tilde{\Delta}\tilde{\delta}d_n=\Delta_t\delta_t d_n=-\boldsymbol{n}.\left(\Delta\zeta_\alpha\frac{\partial\left(\delta_t\overline{\boldsymbol{u}}\right)}{\partial\zeta_\alpha}+\frac{\partial\left(\Delta_t\overline{\boldsymbol{u}}\right)}{\partial\zeta_\alpha}\delta\zeta_\alpha\right)-\delta\zeta_\alpha\kappa_{\alpha\beta}\Delta\zeta_\beta\tag{307}$$

to find $\delta\zeta_\alpha$ and $\Delta\zeta_\alpha$, it is necessary to solve the following system (version contravariante of (82)):

$$\begin{aligned}m^{\alpha\beta}\delta\zeta_\alpha=\frac{\partial\overline{\boldsymbol{x}}}{\partial\zeta^\beta}\cdot\left(\delta_t\boldsymbol{x}-\delta_t\overline{\boldsymbol{x}}\right)\rightarrow\delta\zeta_\alpha=m_{\alpha\beta}\left(\delta_t\boldsymbol{u}-\delta_t\overline{\boldsymbol{u}}\right)\cdot\frac{\partial\overline{\boldsymbol{x}}}{\partial\zeta^\beta}\\ m^{\alpha\beta}\Delta\zeta_\alpha=\frac{\partial\overline{\boldsymbol{x}}}{\partial\zeta^\beta}\cdot\left(\Delta_t\boldsymbol{x}-\Delta_t\overline{\boldsymbol{x}}\right)\rightarrow\Delta\zeta_\alpha=m_{\alpha\beta}\left(\Delta_t\boldsymbol{u}-\Delta_t\overline{\boldsymbol{u}}\right)\cdot\frac{\partial\overline{\boldsymbol{x}}}{\partial\zeta^\beta}\end{aligned}\tag{308}$$

O N has Pr is $d_n=0$, which simplifies the statements of (83). Finally, one a:

$$J_{\boldsymbol{uu}}^{\mathrm{c}}=\underbrace{-\int_{\Gamma_c}S_u^{g_n}g_n\tilde{\Delta}\tilde{\delta}d_n d\Gamma_c}_{J_{\boldsymbol{uu}}^{\mathrm{c},1}}+\underbrace{\int_{\Gamma_c}S_u^{g_n}\rho_n\tilde{\Delta}d_n\tilde{\delta}d_n d\Gamma_c}_{J_{\boldsymbol{uu}}^{\mathrm{c},2}}\tag{309}$$

With:

$$\begin{aligned}J_{\boldsymbol{uu}}^{\mathrm{c},1}= & \int_{\Gamma_c}S_u^{g_n}g_n\boldsymbol{n}\cdot\left(\left(\delta_t\boldsymbol{u}-\delta_t\overline{\boldsymbol{u}}\right)\cdot\frac{\partial\overline{\boldsymbol{x}}}{\partial\zeta^\beta}m_{\alpha\beta}\frac{\partial\left(\Delta_t\overline{\boldsymbol{u}}\right)}{\partial\zeta_\alpha}\right)d\Gamma_c\\ & \int_{\Gamma_c}S_u^{g_n}g_n\boldsymbol{n}\cdot\left(\frac{\partial\left(\delta_t\overline{\boldsymbol{u}}\right)}{\partial\zeta_\alpha}\frac{\partial\overline{\boldsymbol{x}}}{\partial\zeta^\beta}m_{\alpha\beta}\cdot\left(\Delta_t\boldsymbol{u}-\Delta_t\overline{\boldsymbol{u}}\right)\right)d\Gamma_c\\ & \int_{\Gamma_c}S_u^{g_n}g_n\left(\delta_t\boldsymbol{u}-\delta_t\overline{\boldsymbol{u}}\right)\cdot\frac{\partial\overline{\boldsymbol{x}}}{\partial\zeta^\beta}m_{\alpha\beta}\kappa_{\alpha\beta}m_{\alpha\beta}\frac{\partial\overline{\boldsymbol{x}}}{\partial\zeta^\alpha}\cdot\left(\Delta_t\boldsymbol{u}-\Delta_t\overline{\boldsymbol{u}}\right)d\Gamma_c\end{aligned}\tag{310}$$

And:

$$J_{\boldsymbol{uu}}^{\mathrm{c},2}=\int_{\Gamma_c}S_u^{g_n}\rho_n\left(\left(\delta_t\boldsymbol{u}-\delta_t\overline{\boldsymbol{u}}\right)\cdot\boldsymbol{n}\right)\left(\boldsymbol{n}\cdot\left(\Delta_t\boldsymbol{u}-\Delta_t\overline{\boldsymbol{u}}\right)\right)d\Gamma_c\tag{311}$$

We now will consider the term of equilibrium $J_{\boldsymbol{uu}}^{\mathrm{f}}$ corresponds to the reaction for friction:

$$J_{\boldsymbol{uu}}^{\mathrm{f}}=-\tilde{\Delta}_{\boldsymbol{u}^i}\left(G_{\mathrm{f}}\right)=-\tilde{\Delta}_{\boldsymbol{u}^i}\left(\int_{\Gamma_c}\mu\lambda_n S_u^{g_n}S_f^{\boldsymbol{h}_\tau}\boldsymbol{h}_\tau\cdot\tilde{\delta}\boldsymbol{d}_\tau d\Gamma_c+\int_{\Gamma_c}\mu\lambda_n S_u^{g_n}\left(1-S_f^{\boldsymbol{h}_\tau}\right)\boldsymbol{\tau}\cdot\tilde{\delta}\boldsymbol{d}_\tau d\Gamma_c\right)\tag{312}$$

the equation (312) uses the total variation of tangent clearance $\tilde{\delta}\boldsymbol{d}_\tau$ compared to displacements. We will make assumptions of simplification. One will neglect the second variation of tangent clearance, i.e.:

$$\tilde{\Delta}\tilde{\delta}\boldsymbol{d}_\tau=0\tag{313}$$

What gives us:

$$J_{\boldsymbol{uu}}^{\mathrm{f}}=-\int_{\Gamma_c}\mu\lambda_n S_u^{g_n}S_f^{\boldsymbol{h}_\tau}\tilde{\Delta}\boldsymbol{h}_\tau\cdot\tilde{\delta}\boldsymbol{d}_\tau d\Gamma_c-\int_{\Gamma_c}\mu\lambda_n S_u^{g_n}\left(1-S_f^{\boldsymbol{h}_\tau}\right)\tilde{\Delta}\boldsymbol{\tau}\cdot\tilde{\delta}\boldsymbol{d}_\tau d\Gamma_c\tag{314}$$

One writes the variation of the semi-multiplier of Lagrange of friction $\boldsymbol{h}_\tau$. By means of the kinematical assumption (202):

$$\boldsymbol{h}_\tau=\boldsymbol{\Lambda}+\rho_t\boldsymbol{v}_\tau=\boldsymbol{\Lambda}+\bar{\rho}_t[\![\boldsymbol{u}]\!]_\tau\tag{315}$$

And thus:

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$$\tilde{\delta}\boldsymbol{h}_\tau=\tilde{\delta}\boldsymbol{\Lambda}+\bar{\rho}_t[\![\tilde{\delta}\boldsymbol{u}]\!]_\tau=\bar{\rho}_t[\![\tilde{\delta}\boldsymbol{u}]\!]_\tau \text{ because } \tilde{\delta}\boldsymbol{\Lambda}=0 \tag{316}$$

As follows:

$$\tilde{\delta}\boldsymbol{h}_\tau=\bar{\rho}_t\underline{\underline{\boldsymbol{P}}}^\tau\left(\delta_t\boldsymbol{u}-\delta_t\overline{\boldsymbol{u}}\right) \tag{317}$$

In a similar way:

$$\tilde{\delta}\boldsymbol{d}_\tau=\underline{\underline{\boldsymbol{P}}}^\tau\left(\delta_t\boldsymbol{u}-\delta_t\overline{\boldsymbol{u}}\right) \tag{318}$$

For the variation of the semi-multiplier normalized $\tilde{\Delta}\boldsymbol{\tau}$, which corresponds to projection on the ball unit, one considers only the variation in sliding ($S_f^{\boldsymbol{h}_\tau}=0$) since the term is cancelled in the adherent case. One has (see (134)):

$$\Delta_{\boldsymbol{h}_\tau}(\boldsymbol{\tau})=\underline{\underline{\boldsymbol{P}}}^{B(0,1)} \tag{319}$$

By application of the variation of a made up function:

$$\tilde{\Delta}\boldsymbol{\tau}=\Delta_{\boldsymbol{h}_\tau}(\boldsymbol{\tau})\tilde{\Delta}\boldsymbol{h}_\tau=\bar{\rho}_t\underline{\underline{\boldsymbol{P}}}^\tau\,\underline{\underline{\boldsymbol{P}}}^{B(0,1)}\left(\Delta_t\boldsymbol{u}-\Delta_t\overline{\boldsymbol{u}}\right) \tag{320}$$

What gives us:

$$J^{\mathrm{f}}_{\boldsymbol{uu}}=\underbrace{-\int_{\Gamma_c}\mu\lambda_n S_u^{g_n}S_f^{\boldsymbol{h}_\tau}\tilde{\Delta}\boldsymbol{h}_\tau\cdot\tilde{\delta}\boldsymbol{d}_\tau d\Gamma_c}_{J^{\mathrm{f},1}_{\boldsymbol{uu}}}\underbrace{-\int_{\Gamma_c}\mu\lambda_n S_u^{g_n}\left(1-S_f^{\boldsymbol{h}_\tau}\right)\tilde{\Delta}\boldsymbol{\tau}\cdot\tilde{\delta}\boldsymbol{d}_\tau d\Gamma_c}_{J^{\mathrm{f},2}_{\boldsymbol{uu}}} \tag{321}$$

With:

$$J^{\mathrm{f},1}_{\boldsymbol{uu}}=-\int_{\Gamma_c}\mu\lambda_n\bar{\rho}_t S_u^{g_n}S_f^{\boldsymbol{h}_\tau}\left[\underline{\underline{\boldsymbol{P}}}^\tau\left(\delta_t\boldsymbol{u}-\delta_t\overline{\boldsymbol{u}}\right)\right]\cdot\left[\underline{\underline{\boldsymbol{P}}}^\tau\left(\Delta_t\boldsymbol{u}-\Delta_t\overline{\boldsymbol{u}}\right)\right]d\Gamma_c \tag{322}$$

And:

$$J^{\mathrm{f},2}_{\boldsymbol{uu}}=-\int_{\Gamma_c}\mu\lambda_n\bar{\rho}_t S_u^{g_n}\left(1-S_f^{\boldsymbol{h}_\tau}\right)\left[\underline{\underline{\boldsymbol{P}}}^\tau\left(\delta_t\boldsymbol{u}-\delta_t\overline{\boldsymbol{u}}\right)\right]\cdot\left[\underline{\underline{\boldsymbol{P}}}^\tau\,\underline{\underline{\boldsymbol{P}}}^{B(0,1)}\left(\Delta_t\boldsymbol{u}-\Delta_t\overline{\boldsymbol{u}}\right)\right]d\Gamma_c \tag{323}$$

11.1.2 Second variation compared to the contact pressure

Now, the second variation gets along like variation compared to λ_n. We start by evaluating $J^{\mathrm{c}}_{\boldsymbol{u}c}$ such as:

$$J^{\mathrm{c}}_{\boldsymbol{u}c}=-\Delta_{\lambda_n}(G_{\mathrm{c}})=-\Delta_{\lambda_n}\left(\int_{\Gamma_c}S_u^{g_n}g_n\tilde{\delta}d_n d\Gamma_c\right) \tag{324}$$

One recalls that L has variation of normal clearance is worth (305) :

$$\tilde{\delta}d_n=\left(\delta_t\boldsymbol{u}-\delta_t\overline{\boldsymbol{u}}\right)\cdot\boldsymbol{n} \tag{325}$$

And:

$$\Delta_{\lambda_n}(\lambda_n)=\Delta\lambda_n \text{ and } \Delta_{\lambda_n}(\tilde{\delta}d_n)=0 \tag{326}$$

It remains:

$$J^{\mathrm{c}}_{\boldsymbol{u}c}=-\int_{\Gamma_c}S_u^{g_n}\left(\delta_t\boldsymbol{u}-\delta_t\overline{\boldsymbol{u}}\right)\cdot\boldsymbol{n}\,\Delta\lambda_n d\Gamma_c \tag{327}$$

We now consider to evaluate $J^{\mathrm{f}}_{\boldsymbol{u}c}$ such as:

$$J^{\mathrm{f}}_{\boldsymbol{u}c}=-\Delta_{\lambda_n}(G_{\mathrm{f}})=-\Delta_{\lambda_n}\left(\int_{\Gamma_c}\mu\lambda_n S_u^{g_n}S_f^{\boldsymbol{h}_\tau}\boldsymbol{h}_\tau\cdot\tilde{\delta}\boldsymbol{d}_\tau d\Gamma_c+\int_{\Gamma_c}\mu\lambda_n S_u^{g_n}\left(1-S_f^{\boldsymbol{h}_\tau}\right)\boldsymbol{\tau}\cdot\tilde{\delta}\boldsymbol{d}_\tau d\Gamma_c\right) \tag{328}$$

We supposed (alternative 5, to see § 38):

$$\Delta_{\lambda_n}(\boldsymbol{\tau})=0 \tag{329}$$

And we have:

$$\Delta_{\lambda_n}\tilde{\delta}\boldsymbol{d}_\tau=0 \text{ and } \Delta_{\lambda_n}\boldsymbol{h}_\tau=0 \text{ and } \Delta_{\lambda_n}(\lambda_n)=\Delta\lambda_n \tag{330}$$

It remains :

$$J^{\mathrm{f}}_{\boldsymbol{u}c}=\underbrace{-\int_{\Gamma_c}\mu S_u^{g_n}S_f^{\boldsymbol{h}_\tau}\boldsymbol{h}_\tau\cdot\tilde{\delta}\boldsymbol{d}_\tau\Delta\lambda_n d\Gamma_c}_{J^{\mathrm{f},1}_{\boldsymbol{u}c}}\underbrace{-\int_{\Gamma_c}\mu S_u^{g_n}\left(1-S_f^{\boldsymbol{h}_\tau}\right)\boldsymbol{\tau}\cdot\tilde{\delta}\boldsymbol{d}_\tau\Delta\lambda_n d\Gamma_c}_{J^{\mathrm{f},2}_{\boldsymbol{u}c}} \tag{331}$$

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With:

$$J_{uc}^{f,1}=-\int_{\Gamma_c}\mu\, S_u^{g_n} S_f^{h_\tau}\left[\underline{\underline{P}}^{\tau}\left(\delta_t \boldsymbol{u}-\delta_t \overline{\boldsymbol{u}}\right)\right]\cdot \boldsymbol{h}_\tau \Delta\lambda_n d\Gamma_c \tag{332}$$

And:

$$J_{uc}^{f,2}=-\int_{\Gamma_c}\mu\, S_u^{g_n}\left(1-S_f^{h_\tau}\right)\left[\underline{\underline{P}}^{\tau}\left(\delta_t \boldsymbol{u}-\delta_t \overline{\boldsymbol{u}}\right)\right]\cdot \boldsymbol{\tau}\, \Delta\lambda_n d\Gamma_c \tag{333}$$

11.1.3 Second variation compared to the pressure of friction

Finally, the second variation will get along like variation compared to Λ. The only term NON-no one is J_{uf}^{f} such as:

$$J_{uf}^{f}=-\Delta_{\Lambda}\left(G_f\right)=-\Delta_{\Lambda}\left(\int_{\Gamma_c}\mu\,\lambda_n S_u^{g_n} S_f^{h_\tau}\boldsymbol{h}_\tau\cdot\tilde{\delta}\boldsymbol{d}_\tau d\Gamma_c+\int_{\Gamma_c}\mu\,\lambda_n S_u^{g_n}\left(1-S_f^{h_\tau}\right)\boldsymbol{\tau}\cdot\tilde{\delta}\boldsymbol{d}_\tau d\Gamma_c\right) \tag{334}$$

the variations of $\boldsymbol{h}_\tau$ compared to Λ are worth :

$$\Delta_{\Lambda}\left(\boldsymbol{h}_\tau\right)=\Delta_{\Lambda}\left(\Lambda+\bar{\rho}_t[\![\boldsymbol{u}]\!]_\tau\right)=\Delta\Lambda \text{ because } \Delta_{\Lambda}\left([\![\boldsymbol{u}]\!]_\tau\right)=0 \tag{335}$$

One a:

$$\Delta_{\Lambda}\boldsymbol{\tau}=\Delta_{\boldsymbol{h}_\tau}\left(\boldsymbol{\tau}\right)\Delta_{\Lambda}\left(\boldsymbol{h}_\tau\right)=\underline{\underline{\boldsymbol{P}}}^{B(0,1)}\Delta\boldsymbol{\Lambda} \tag{336}$$

In the same way, crudely:

$$\Delta_{\Lambda}\tilde{\delta}\boldsymbol{d}_\tau=0 \text{ because } \tilde{\delta}\boldsymbol{d}_\tau=\underline{\underline{\boldsymbol{P}}}^{\tau}\left(\delta_t \boldsymbol{u}-\delta_t \overline{\boldsymbol{u}}\right) \tag{337}$$

What gives us:

$$J_{uf}^{f}=\underbrace{-\int_{\Gamma_c}\mu\,\lambda_n S_u^{g_n} S_f^{h_\tau}\Delta\boldsymbol{\Lambda}\cdot\tilde{\delta}\boldsymbol{d}_\tau d\Gamma_c}_{J_{uf}^{f,1}}\underbrace{-\int_{\Gamma_c}\mu\,\lambda_n S_u^{g_n}\left(1-S_f^{h_\tau}\right)\underline{\underline{\boldsymbol{P}}}^{B(0,1)}\Delta\boldsymbol{\Lambda}\cdot\tilde{\delta}\boldsymbol{d}_\tau d\Gamma_c}_{J_{uf}^{f,2}} \tag{338}$$

With:

$$J_{uf}^{f,1}=-\int_{\Gamma_c}\mu\,\lambda_n S_u^{g_n} S_f^{h_\tau}\Delta\boldsymbol{\Lambda}\cdot\left[\underline{\underline{\boldsymbol{P}}}^{\tau}\left(\delta_t \boldsymbol{u}-\delta_t \overline{\boldsymbol{u}}\right)\right]d\Gamma_c \tag{339}$$

And:

$$J_{uf}^{f,2}=-\int_{\Gamma_c}\mu\,\lambda_n S_u^{g_n}\left(1-S_f^{h_\tau}\right)\left[\underline{\underline{\boldsymbol{P}}}^{\tau}\left(\delta_t \boldsymbol{u}-\delta_t \overline{\boldsymbol{u}}\right)\right]\cdot\left[\underline{\underline{\boldsymbol{P}}}^{B(0,1)}\Delta\boldsymbol{\Lambda}\right]d\Gamma_c \tag{340}$$

11.2 Model of contact

For the model of contact (in weak form) the virtual quantities are variations compared to the contact pressure λ_n. The notation δ will thus be employed in this meaning. The second line of the jacobian matrix $[J]$ consists of three terms:

$$J_{c\boldsymbol{u}}\quad J_{cc}\quad J_{cf} \tag{341}$$

As the contact pressure does not depend on the pressure of friction, one knows already that:

$$J_{cf}=0 \tag{342}$$

11.2.1 Second variation compared to displacements

In this paragraph, the second variation will always get along like total *variation* compared to displacements $\boldsymbol{u}^i$. It will be noted $\Delta_{\boldsymbol{u}^i}=\tilde{\Delta}$. We start by evaluating $J_{c\boldsymbol{u}}$ such as:

$$J_{c\boldsymbol{u}}=\Delta_{\boldsymbol{u}^i}\left(\tilde{G}_c\right)=\tilde{\Delta}\left(-\frac{1}{\rho_n}\int_{\Gamma_c}\left\{\lambda_n-S_u^{g_n}g_n\right\}\delta\lambda_n d\Gamma_c\right) \tag{343}$$

One has $\tilde{\Delta}\,\delta\lambda_n=0$. There thus remains only the term $\tilde{\Delta}\,d_n$ whose statement is given by (74):

$$J_{c\boldsymbol{u}}=-\int_{\Gamma_c}S_u^{g_n}\delta\lambda_n\boldsymbol{n}\cdot\left(\Delta_t \boldsymbol{u}-\Delta_t \overline{\boldsymbol{u}}\right)d\Gamma_c \tag{344}$$

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One observes immediately that $J_{cu} \neq J_{uc}$ because of (331). In friction, the matrix is not symmetric.

11.2.2 Second variation compared to the contact pressure

the second variation will get along like variation compared to λ_n. We calculate J_{cc} such as:

$$J_{cc} = \Delta_{\lambda_n}\left(\tilde{G}_c\right) = \Delta_{\lambda_n}\left(-\frac{1}{\rho_n}\int_{\Gamma_c}\left\{\lambda_n - S_u^{g_n}\left(\lambda_n - \rho_n d_n\right)\right\}\delta\lambda_n d\Gamma_c\right) \tag{345}$$

the second variation $\Delta_{\lambda_n}\delta\lambda_n = 0$ is null and $\Delta_{\lambda_n}\left(\lambda_n\right) = \Delta\lambda_n$, therefore:

$$J_{cc} = -\frac{1}{\rho_n}\int_{\Gamma_c}\Delta\lambda_n\left(1 - S_u^{g_n}\right)\delta\lambda_n d\Gamma_c \tag{346}$$

One notices ruffle which C E term intervenes only when there is no contact ($S_u^{g_n} = 0$) .

11.2.3 Second variation compared to the pressure of friction

L has second variation will get along like variation compared to $\boldsymbol{\Lambda}$. L E formula J_{cf} is null :

$$J_{cf} = \Delta_{\boldsymbol{\Lambda}}\left(\tilde{G}_c\right) = 0 \tag{347}$$

11.3 Friction law

For the friction law (in weak form) the virtual quantities are variations compared to the pressure of friction $\boldsymbol{\Lambda}$. The notation δ will thus be employed in this meaning. The back-row forward of the jacobian matrix $[J]$ consists of three terms:

$$J_{f\boldsymbol{u}} \quad J_{fc} \quad J_{ff} \tag{348}$$

11.3.1 Second variation compared to displacements

In this paragraph, the second variation will always get along like total *variation* compared to displacements $\boldsymbol{u}^i$. It will be noted $\Delta_{\boldsymbol{u}^i} = \tilde{\Delta}$. We start by evaluating $J_{f\boldsymbol{u}}$ such as:

$$\begin{aligned} J_{f\boldsymbol{u}} = \Delta_{\boldsymbol{u}^i}\left(\tilde{G}_f\right) = \quad & +\tilde{\Delta}\left(\frac{1}{\bar{\rho}_t}\int_{\Gamma_c}\mu\lambda_n S_u^{g_n}\boldsymbol{\Lambda}\cdot\delta\boldsymbol{\Lambda}\, d\Gamma_c\right) \\ & -\tilde{\Delta}\left(\frac{1}{\bar{\rho}_t}\int_{\Gamma_c}\mu\lambda_n S_u^{g_n} S_f^{\boldsymbol{h}_\tau}\boldsymbol{h}_\tau\cdot\delta\boldsymbol{\Lambda}\, d\Gamma_c\right) \\ & -\tilde{\Delta}\left(\frac{1}{\bar{\rho}_t}\int_{\Gamma_c}\mu\lambda_n S_u^{g_n}\left(1 - S_f^{\boldsymbol{h}_\tau}\right)\boldsymbol{\tau}\cdot\delta\boldsymbol{\Lambda}\, d\Gamma_c\right) \\ & -\tilde{\Delta}\left(\int_{\Gamma_c}\left(1 - S_u^{g_n}\right)\boldsymbol{\Lambda}\cdot\delta\boldsymbol{\Lambda}\, d\Gamma_c\right) \end{aligned} \tag{349}$$

One re-uses (320):

$$\tilde{\Delta}\boldsymbol{\tau} = \bar{\rho}_t\underline{\boldsymbol{P}}^{\tau}\underline{\boldsymbol{P}}^{B(0,1)}\left(\Delta_t\boldsymbol{u} - \Delta_t\bar{\boldsymbol{u}}\right) \tag{350}$$

One has also $\tilde{\Delta}\delta\boldsymbol{\Lambda} = 0$ and $\tilde{\Delta}\boldsymbol{\Lambda} = 0$. As follows:

$$J_{f\boldsymbol{u}} = \underbrace{-\frac{1}{\bar{\rho}_t}\int_{\Gamma_c}\mu\lambda_n S_u^{g_n} S_f^{\boldsymbol{h}_\tau}\delta\boldsymbol{\Lambda}\cdot\tilde{\Delta}\boldsymbol{h}_\tau\, d\Gamma_c}_{J_{f\boldsymbol{u}}^1} \underbrace{-\frac{1}{\bar{\rho}_t}\int_{\Gamma_c}\mu\lambda_n S_u^{g_n}\left(1 - S_f^{\boldsymbol{h}_\tau}\right)\delta\boldsymbol{\Lambda}\cdot\tilde{\Delta}\left(\frac{\boldsymbol{h}_\tau}{\|\boldsymbol{h}_\tau\|}\right)d\Gamma_c}_{J_{f\boldsymbol{u}}^2} \tag{351}$$

With:

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$$J_{f\boldsymbol{u}}^{1}=-\int_{\Gamma_c}\mu\lambda_n S_u^{g_n} S_f^{\boldsymbol{h}_\tau}\delta\boldsymbol{\Lambda}\cdot\left[\underline{\underline{\boldsymbol{P}}}^{\tau}\left(\Delta_t\boldsymbol{u}-\Delta_t\bar{\boldsymbol{u}}\right)\right]d\Gamma_c \tag{352}$$

And:

$$J_{f\boldsymbol{u}}^{2}=-\int_{\Gamma_c}\mu\lambda_n S_u^{g_n}(1-S_f^{\boldsymbol{h}_\tau})\delta\boldsymbol{\Lambda}\cdot\left[\underline{\underline{\boldsymbol{P}}}^{\tau}\,\underline{\underline{\boldsymbol{P}}}^{B(0,1)}\left(\Delta_t\boldsymbol{u}-\Delta_t\bar{\boldsymbol{u}}\right)\right]d\Gamma_c \tag{353}$$

11.3.2 Second variation compared to the contact pressure

L has second variation will get along like variation compared to λ_n . We start by evaluating J_{fc} such as:

$$J_{fc}=\Delta_{\lambda_n}\left(\tilde{G}_{\mathrm{f}}\right)$$ formulate3 54354

) alternative 5 (see § 38), one has $\Delta_{\lambda_n}\left(\tilde{G}_f\right)=0$, which implies $J_{fc}=0$.

11.3.3 Second variation compared to the pressure of friction

L has second variation will get along like variation compared to $\boldsymbol{\Lambda}$. We start by evaluating J_{ff} such as:

$$\begin{aligned}J_{ff}=\Delta_{\boldsymbol{\Lambda}}\left(\tilde{G}_{\mathrm{f}}\right)= \quad &+\Delta_{\boldsymbol{\Lambda}}\left(\frac{1}{\bar{\rho}_t}\int_{\Gamma_c}\mu\lambda_n S_u^{g_n}\boldsymbol{\Lambda}\cdot\delta\boldsymbol{\Lambda}\,d\Gamma_c\right)\\ &-\Delta_{\boldsymbol{\Lambda}}\left(\frac{1}{\bar{\rho}_t}\int_{\Gamma_c}\mu\lambda_n S_u^{g_n} S_f^{\boldsymbol{h}_\tau}\boldsymbol{h}_\tau\cdot\delta\boldsymbol{\Lambda}\,d\Gamma_c\right)\\ &-\Delta_{\boldsymbol{\Lambda}}\left(\frac{1}{\bar{\rho}_t}\int_{\Gamma_c}\mu\lambda_n S_u^{g_n}(1-S_f^{\boldsymbol{h}_\tau})\boldsymbol{\tau}\cdot\delta\boldsymbol{\Lambda}\,d\Gamma_c\right)\\ &-\Delta_{\boldsymbol{\Lambda}}\left(\int_{\Gamma_c}\left(1-S_u^{g_n}\right)\boldsymbol{\Lambda}\cdot\delta\boldsymbol{\Lambda}\,d\Gamma_c\right)\end{aligned} \tag{355}$$

the variations of $\boldsymbol{h}_\tau$ compared to $\boldsymbol{\Lambda}$ are worth (335):

$$\Delta_{\boldsymbol{\Lambda}}\left(\boldsymbol{h}_\tau\right)=\Delta_{\boldsymbol{\Lambda}}\left(\boldsymbol{\Lambda}+\bar{\rho}_t[\![\boldsymbol{u}]\!]_\tau\right)=\Delta\boldsymbol{\Lambda}\ \text{ because }\ \Delta_{\boldsymbol{\Lambda}}\left([\![\boldsymbol{u}]\!]_\tau\right)=0 \tag{356}$$

the contact pressure does not depend on the pressure of friction, therefore:

$$\Delta_{\boldsymbol{\Lambda}}\left(\lambda_n\right)=0\ \text{ and }\ \Delta_{\boldsymbol{\Lambda}}\left(\boldsymbol{\Lambda}\right)=\Delta\boldsymbol{\Lambda} \tag{357}$$

One also has (336):

$$\Delta_{\boldsymbol{\Lambda}}\boldsymbol{\tau}=\underline{\underline{\boldsymbol{P}}}^{B(0,1)}\Delta\boldsymbol{\Lambda} \tag{358}$$

Finally:

$$\begin{aligned}J_{ff}= \quad &\frac{1}{\bar{\rho}_t}\int_{\Gamma_c}\mu\lambda_n S_u^{g_n}\left(1-S_f^{\boldsymbol{h}_\tau}\right)\left(\underline{\underline{\mathbf{1}}}-\underline{\underline{\boldsymbol{P}}}^{B(0,1)}\right)\Delta\boldsymbol{\Lambda}\cdot\delta\boldsymbol{\Lambda}\,d\Gamma_c\\ &-\int_{\Gamma_c}\left(1-S_u^{g_n}\right)\Delta\boldsymbol{\Lambda}\cdot\delta\boldsymbol{\Lambda}\,d\Gamma_c\end{aligned} \tag{359}$$

It has there a contribution only in slipping contact and when the contact is inactive.

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12 Appendix b: second member vector

the statements considered here relate to only the terms related to the contact friction. It is thus a question of discretizing the statement of the virtual wors given by the equations (206) to (209). It is pointed out that the system to be solved is:

$$[\boldsymbol{K}]\{\Delta \boldsymbol{W}\}=-\{\boldsymbol{L}\} \tag{360}$$

12.1 Terms of equilibrium

One starts with the statement of the reaction of contact in the balance equation, from G_c (206) which one time step writes on the configuration known with k, since the beginning of the process of Newton:

$$G_c=\int_{\Gamma_c}\hat{S}_{u,k}^{g_n}\hat{g}_{n,k}\tilde{\delta}\, d_n d\Gamma_c \tag{361}$$

After discretization:

$$-G_c \overset{\text{Discrétisation}}{\rightarrow} \{L_c^e\},\{L_c^m\} \tag{362}$$

the sign $-$ comes from the statement of the equilibrium in (287). One obtains, by means of (283) and by separating the contributions slave and Master:

$$\begin{aligned}\{L_c^e\}&=-\omega_c J_c \hat{S}_{u,k}^{g_n}\hat{g}_{n,k}[N^e]^T\{n_h\}\\ \{L_c^m\}&=+\omega_c J_c \hat{S}_{u,k}^{g_n}\hat{g}_{n,k}[\bar{N}^m]^T\{n_h\}\end{aligned} \tag{363}$$

These terms are active only when there is contact (formula $\hat{S}_{u,k}^{g_n}=1$.

One considers then the reaction of friction (207):

$$G_f=\int_{\Gamma_c}\mu\hat{\lambda}_k\hat{S}_{u,k}^{g_n}\hat{S}_{f,k}^{\boldsymbol{h}_\tau}\boldsymbol{h}_{\tau,k}\cdot\tilde{\delta}\,\boldsymbol{d}_\tau d\Gamma_c+\int_{\Gamma_c}\mu\hat{\lambda}_k\hat{S}_{u,k}^{g_n}\left(1-\hat{S}_{f,k}^{\boldsymbol{h}_\tau}\right)\hat{\boldsymbol{\tau}}_k\cdot\tilde{\delta}\,\boldsymbol{d}_\tau d\Gamma_c \tag{formulate3 64364}$$

) the operator of projection (240). After discretization:

$$-G_f \overset{\text{Discrétisation}}{\rightarrow} \{L_f^e\},\{L_f^m\} \tag{365}$$

One obtains:

$$\begin{aligned}\{L_f^e\}&=-\mu\hat{\lambda}_k\hat{S}_{u,k}^{g_n}\omega_c J_c[N^e]^T[P^\tau]\left(\hat{S}_{f,k}^{\boldsymbol{h}_\tau}\{\hat{h}_{\tau,k}\}+\left(1-\hat{S}_{f,k}^{\boldsymbol{h}_\tau}\right)\{\hat{\tau}_k\}\right)\\ \{L_f^m\}&=+\mu\hat{\lambda}_k\hat{S}_{u,k}^{g_n}\omega_c J_c[\bar{N}^m]^T[P^\tau]\left(\hat{S}_{f,k}^{\boldsymbol{h}_\tau}\{\hat{h}_{\tau,k}\}+\left(1-\hat{S}_{f,k}^{\boldsymbol{h}_\tau}\right)\{\hat{\tau}_k\}\right)\end{aligned} \tag{366}$$

One observes an active term in slipping contact ($\hat{S}_{f,k}^{\boldsymbol{h}_\tau}=0$):

$$\begin{aligned}\{L_f^e\}_{\text{glissant}}&=-\mu\hat{\lambda}_k\omega_c J_c[N^e]^T[P^\tau]\{\hat{\tau}_k\}\\ \{L_f^m\}_{\text{glissant}}&=+\mu\hat{\lambda}_k\omega_c J_c[\bar{N}^m]^T[P^\tau]\{\hat{\tau}_k\}\end{aligned} \tag{367}$$

And an active term in adherent contact ($\hat{S}_{f,k}^{\boldsymbol{h}_\tau}=1$:

$$\begin{aligned}\{L_f^e\}_{\text{adhérent}}&=-\mu\hat{\lambda}_k\omega_c J_c[N^e]^T[P^\tau]\{\hat{h}_{\tau,k}\}\\ \{L_f^m\}_{\text{adhérent}}&=+\mu\hat{\lambda}_k\omega_c J_c[\bar{N}^m]^T[P^\tau]\{\hat{h}_{\tau,k}\}\end{aligned} \tag{368}$$

12.2 Terms of the model of Signorini

We now will consider the term corresponding to the unknown in contact pressure, that is to say from (208):

$$\tilde{G}_c=-\frac{1}{\rho_n}\int_{\Gamma_c}\left\{\hat{\lambda}_k-\hat{S}_{u,k}^{g_n}\hat{g}_{n,k}\right\}\delta\lambda\, d\Gamma_c \tag{369}$$

After discretization:

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$$\tilde{G}_c \overset{\text{Discrétisation}}{\rightarrow} \left\{L_c^c\right\} \tag{370}$$

One obtains:

$$\left\{L_c^c\right\} = -\frac{1}{\rho_n}\left(\hat{\lambda}_k - \hat{S}_{u,k}^{g_n}\,\hat{g}_{n,k}\right)\omega_c J_c\left\{\psi\right\} \tag{371}$$

This statement can break up into two parts. If the contact is inactive ($\hat{S}_{u,k}^{g_n}=0$), one a:

$$\left\{L_c^c\right\}_{\text{sans contact}} = -\frac{1}{\rho_n}\hat{\lambda}_k\,\omega_c J_c\left\{\psi\right\} \tag{372}$$

If the contact is active ($\hat{S}_{u,k}^{g_n}=1$), one has terms being simplified between $\hat{\lambda}_k$ and $\hat{g}_{n,k}$, which gives:

$$\left\{L_c^c\right\}_{\text{avec contact}} = -\hat{d}_{n,k}\,\omega_c J_c\left\{\psi\right\} \tag{formulate3 73373}$$

12.3) Terms of the model of

Coulomb the term corresponds to the unknown in pressure of friction, from the weak statement of the model of Coulomb (209):

$$\begin{aligned}\tilde{G}_f = \quad & \frac{1}{\bar{\rho}_t}\int_{\Gamma_c}\mu\,\hat{\lambda}_k\,\hat{S}_{u,k}^{g_n}\hat{\boldsymbol{\Lambda}}_k\cdot\delta\boldsymbol{\Lambda}\,d\Gamma_c - \\ & \frac{1}{\bar{\rho}_t}\int_{\Gamma_c}\mu\,\hat{\lambda}_k\,\hat{S}_{u,k}^{g_n}\hat{S}_{f,k}^{\boldsymbol{h}_\tau}\hat{\boldsymbol{h}}_{\tau,k}\cdot\delta\boldsymbol{\Lambda}\,d\Gamma_c - \\ & \frac{1}{\bar{\rho}_t}\int_{\Gamma_c}\mu\,\hat{\lambda}_k\,\hat{S}_{u,k}^{g_n}(1-\hat{S}_{f,k}^{\boldsymbol{h}_\tau})\hat{\boldsymbol{\tau}}_k\cdot\delta\boldsymbol{\Lambda}\,d\Gamma_c + \\ & \int_{\Gamma_c}\left(1-\hat{S}_{u,k}^{g_n}\right)\hat{\boldsymbol{\Lambda}}_k\cdot\delta\boldsymbol{\Lambda}\,d\Gamma_c\end{aligned} \tag{374}$$

After discretization:

$$\tilde{G}_f \overset{\text{Discrétisation}}{\rightarrow} \left\{L_f^f\right\} \tag{375}$$

One obtains:

$$\begin{aligned}\left\{L_f^f\right\} = \quad & \frac{1}{\bar{\rho}_t}\mu\,\omega_c J_c\,\hat{\lambda}_k\,\hat{S}_{u,k}^{g_n}\left[\psi\right]^T\left[T\right]^T\left\{\hat{\Lambda}_k\right\} - \\ & \frac{1}{\bar{\rho}_t}\mu\,\omega_c J_c\,\hat{\lambda}_k\,\hat{S}_{u,k}^{g_n}\hat{S}_{f,k}^{\boldsymbol{h}_\tau}\left[\psi\right]^T\left[T\right]^T\left\{\hat{h}_{\tau,k}\right\} - \\ & \frac{1}{\bar{\rho}_t}\mu\,\omega_c J_c\,\hat{\lambda}_k\,\hat{S}_{u,k}^{g_n}(1-\hat{S}_{f,k}^{\boldsymbol{h}_\tau})\left[\psi\right]^T\left[T\right]^T\left\{\hat{\tau}_k\right\} + \\ & \left(1-\hat{S}_{u,k}^{g_n}\right)\omega_c J_c\left[\psi\right]^T\left[T\right]^T\left\{\hat{\Lambda}_k\right\}\end{aligned} \tag{376}$$

This complex statement can break up into three parts. If the contact is inactive ($S_{u,k}^{g_n}=0$), one a:

$$\left\{L_f^f\right\}_{\text{sans contact}} = \omega_c J_c\left[\psi\right]^T\left[T\right]^T\left\{\hat{\Lambda}_k\right\} \tag{377}$$

If one is in slipping contact ($S_{u,k}^{g_n}=1$ and $S_{f,k}^{\boldsymbol{h}_\tau}=0$), one a:

$$\left\{L_f^f\right\}_{\text{glissant}} = \frac{1}{\bar{\rho}_t}\mu\,\hat{\lambda}_k\,\omega_c J_c\left[\psi\right]^T\left[T\right]^T\left(\left\{\hat{\Lambda}_k\right\} - \left\{\hat{\tau}_k\right\}\right) \tag{378}$$

If one is in adherent contact ($\hat{S}_{u,k}^{g_n}=1$ and $\hat{S}_{f,k}^{\boldsymbol{h}_\tau}=1$), one has, certain terms are simplified between the first and the second-row forward of (376), in particular, one a:

$$\left\{\hat{\Lambda}_k\right\} - \left\{\hat{h}_{\tau,k}\right\} = -\bar{\rho}_\tau\left\{\left[\!\left[\hat{x}\right]\!\right]_{\tau,k}\right\} \tag{379}$$

With:

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$$\left\{\hat{\Lambda}_k\right\}-\left\{\hat{h}_{\tau,k}\right\}=-\bar{\rho}_\tau\left\{[\![\hat{x}]\!]_{\tau,k}\right\}=-\bar{\rho}_\tau\left[P^\tau\right]\left\{[\![\hat{x}]\!]_k\right\}$$ formulate3 80380

):

$$\left\{L_f^f\right\}_{\text{adhérent}}=-\mu\,\hat{\lambda}_k\,\omega_c J_c\left[\psi\right]^T\left[T\right]^T\left[P^\tau\right]\left\{[\![\hat{x}]\!]_k\right\}\quad(381)$$

In short:

Without contact	$\left\{L_c^c\right\}_{\text{sans contact}}=-\frac{1}{\rho_n}\hat{\lambda}_k\,\omega_c J_c\left\{\psi\right\}$ $\left\{L_f^f\right\}_{\text{sans contact}}=\omega_c J_c\left[\psi\right]^T\left[T\right]^T\left\{\hat{\Lambda}_k\right\}$
slipping formulates	$\left\{L_c^e\right\}=-\left[N^e\right]^T\left\{n_h\right\}\hat{g}_{n,k}\,\omega_c J_c$ $\left\{L_c^m\right\}_{\text{glissant}}=+\left[N^m\right]^T\left\{n_h\right\}\hat{g}_{n,k}\,\omega_c J_c$ $\left\{L_f^e\right\}_{\text{glissant}}=-\mu\,\hat{\lambda}_k\,\omega_c J_c\left[N^e\right]^T\left[P^\tau\right]\left\{\hat{\tau}_k\right\}$ $\left\{L_f^m\right\}_{\text{glissant}}=+\mu\,\hat{\lambda}_k\,\omega_c J_c\left[\bar{N}^m\right]^T\left[P^\tau\right]\left\{\hat{\tau}_k\right\}$ $\left\{L_c^c\right\}_{\text{glissant}}=-\hat{d}_{n,k}\,\omega_c J_c\left\{\psi\right\}$ $\left\{L_f^f\right\}_{\text{glissant}}=\frac{1}{\bar{\rho}_t}\mu\,\hat{\lambda}_k\,\omega_c J_c\left[\psi\right]^T\left[T\right]^T\left(\left\{\hat{\Lambda}_k\right\}-\left\{\hat{\tau}_k\right\}\right)$¿
adherent Contact	$\left\{L_c^e\right\}_{\text{adhérent}}=-\left[N^e\right]^T\left\{n_h\right\}\hat{g}_{n,k}\,\omega_c J_c$ $\left\{L_c^m\right\}_{\text{adhérent}}=+\left[N^m\right]^T\left\{n_h\right\}\hat{g}_{n,k}\,\omega_c J_c$ $\left\{L_f^e\right\}_{\text{adhérent}}=-\mu\,\hat{\lambda}_k\,\omega_c J_c\left[N^e\right]^T\left[P^\tau\right]\left\{\hat{h}_{\tau,k}\right\}$ $\left\{L_f^m\right\}_{\text{adhérent}}=+\mu\,\hat{\lambda}_k\,\omega_c J_c\left[\bar{N}^m\right]^T\left[P^\tau\right]\left\{\hat{h}_{\tau,k}\right\}$ $\left\{L_c^c\right\}_{\text{adhérent}}=-\hat{d}_{n,k}\,\omega_c J_c\left\{\psi\right\}$ $\left\{L_f^f\right\}_{\text{adhérent}}=-\mu\,\hat{\lambda}_k\,\omega_c J_c\left[\psi\right]^T\left[T\right]^T\left[P^\tau\right]\left\{[\![\hat{x}]\!]_k\right\}$

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13 Annexes C : stamp tangent

the statements considered relate to always only the terms related to contact-friction. It is thus a question of discretizing the linearized statements. The process of linearization reveals matric terms which we will discretize here.

13.1 Preliminary matric quantities

One starts by describing the discretized form of a certain number of quantities. The discretization of the direction $\hat{\boldsymbol{\tau}}_k$ will give us:

$$\hat{\boldsymbol{\tau}}_k = \frac{\hat{\boldsymbol{h}}_{\tau,k}}{\|\hat{\boldsymbol{h}}_{\tau,k}\|} \tag{382}$$

Discretization of the operator of projection on the ball unit:

$$\underline{\underline{\hat{\boldsymbol{P}}}}_k^{B(0,1)} \overset{\text{Discrétisation}}{\rightarrow} \left[\hat{P}_k^B\right] \tag{383}$$

the process of linearization of the quantity relating to the ball unit (§22, to see (135)) we gives:

$$\underline{\underline{\hat{\boldsymbol{P}}}}_k^{B(0,1)} = \frac{1}{\|\hat{\boldsymbol{h}}_{\tau,k}\|}\left(\underline{\underline{\mathbf{1}}} - \frac{\hat{\boldsymbol{h}}_{\tau,k}\otimes\hat{\boldsymbol{h}}_{\tau,k}}{\|\hat{\boldsymbol{h}}_{\tau,k}\|^2}\right) \tag{384}$$

One will note the identity on the way:

$$\underline{\underline{\hat{\boldsymbol{P}}}}_k^{B(0,1)} = \frac{1}{\|\hat{\boldsymbol{h}}_{\tau,k}\|}\left(\underline{\underline{\mathbf{1}}} - \hat{\boldsymbol{\tau}}_k\otimes\hat{\boldsymbol{\tau}}_k\right) \tag{385}$$

the discretization uses the definition (279), with (238) one can thus write:

$$\underline{\underline{\hat{\boldsymbol{P}}}}_k^{B(0,1)} \overset{\text{Discrétisation}}{\rightarrow} \left[\hat{P}_k^B\right] = \frac{1}{\|\hat{\boldsymbol{h}}_{\tau,k}\|}\left([1] - \{\hat{\tau}_k\}\langle\hat{\tau}_k\rangle\right) \tag{386}$$

13.2 Quantities for the equilibrium

One starts by considering the matric terms resulting from the linearization of the balance equation. Discretization of the jacobiennes (310) and (311):

$$\begin{aligned} J_{\boldsymbol{uu}}^{c,1} &\overset{\text{Discrétisation}}{\rightarrow} \left[K_{ee}^{c,1}\right], \left[K_{em}^{c,1}\right], \left[K_{me}^{c,1}\right], \left[K_{mm}^{c,1}\right] \\ J_{\boldsymbol{uu}}^{c,2} &\overset{\text{Discrétisation}}{\rightarrow} \left[K_{ee}^{c,2}\right], \left[K_{em}^{c,2}\right], \left[K_{me}^{c,2}\right], \left[K_{mm}^{c,2}\right] \end{aligned} \tag{387}$$

One starts with (311) :

$$J_{\boldsymbol{uu}}^{c,2} = \int_{\Gamma_c} \rho_n \hat{S}_{u,k}^{g_n}\left(\left(\delta_t \hat{\boldsymbol{u}}_k^e - \delta_t \hat{\boldsymbol{u}}_k^m\right)\cdot\hat{\boldsymbol{n}}_k\right)\left(\hat{\boldsymbol{n}}_k\cdot\left(\Delta_t \boldsymbol{u}^e - \Delta_t \boldsymbol{u}^m\right)\right) d\Gamma_c$$

formulate3 88388

):

$$\left[K_{uu}^{c,2}\right] = \omega_c J_c \rho_n \hat{S}_{u,k}^{g_n}\left(\langle\delta_t u_h^e\rangle\left[N^e\right]^T - \langle\delta_t u_h^m\rangle\left[\bar{N}^m\right]^T\right)\{n_h\}\langle n_h\rangle\left(\left[N^e\right]\{\Delta u_h^e\} - \left[\bar{N}^m\right]\{\Delta u_h^m\}\right) \tag{389}$$

One uses $\left[P^n\right] = \{n_h\}\langle n_h\rangle$. Finally:

$$\begin{aligned} \left[K_{ee}^{c,2}\right] &= +\omega_c J_c \rho_n \hat{S}_{u,k}^{g_n}\left[N^e\right]^T\left[P^n\right]\left[N^e\right] \\ \left[K_{mm}^{c,2}\right] &= +\omega_c J_c \rho_n \hat{S}_{u,k}^{g_n}\left[\bar{N}^m\right]^T\left[P^n\right]\left[\bar{N}^m\right] \\ \left[K_{em}^{c,2}\right] &= -\omega_c J_c \rho_n \hat{S}_{u,k}^{g_n}\left[N^e\right]^T\left[P^n\right]\left[\bar{N}^m\right] \\ \left[K_{me}^{c,2}\right] &= -\omega_c J_c \rho_n \hat{S}_{u,k}^{g_n}\left[\bar{N}^m\right]^T\left[P^n\right]\left[N^e\right] \end{aligned} \tag{390}$$

From (310):

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$$J_{\boldsymbol{uu}}^{\mathrm{c},1}= \int_{\Gamma_c} \hat{S}_{u,k}^{g_n} \hat{g}_{n,k} \hat{\boldsymbol{n}}_k \cdot \left\{ \left(\delta_t \hat{\boldsymbol{u}}_k^e - \delta_t \hat{\boldsymbol{u}}_k^m \right) \cdot \frac{\partial \overline{\boldsymbol{x}}}{\partial \zeta^\beta} m_{\alpha\beta} \frac{\partial \left(\Delta_t \overline{\boldsymbol{u}} \right)}{\partial \zeta_\alpha} \right\} d\Gamma_c$$
$$\int_{\Gamma_c} \hat{S}_{u,k}^{g_n} \hat{g}_{n,k} \hat{\boldsymbol{n}}_k \cdot \left\{ \frac{\partial \left(\delta_t \overline{\boldsymbol{u}} \right)}{\partial \zeta_\alpha} m_{\alpha\beta} \frac{\partial \overline{\boldsymbol{x}}}{\partial \zeta^\beta} \cdot \left(\Delta_t \boldsymbol{u}^e - \Delta_t \boldsymbol{u}^m \right) \right\} d\Gamma_c \quad (391)$$
$$\int_{\Gamma_c} \hat{S}_{u,k}^{g_n} \hat{g}_{n,k} \left(\delta_t \hat{\boldsymbol{u}}_k^e - \delta_t \hat{\boldsymbol{u}}_k^m \right) \cdot \frac{\partial \overline{\boldsymbol{x}}}{\partial \zeta^\beta} m_{\alpha\beta} \kappa_{\alpha\beta} m_{\alpha\beta} \frac{\partial \overline{\boldsymbol{x}}}{\partial \zeta^\alpha} \cdot \left(\Delta_t \boldsymbol{u}^e - \Delta_t \boldsymbol{u}^m \right) d\Gamma_c$$

Consider the first integral, while discretizing, one obtains:

$$J_{\boldsymbol{uu}}^{\mathrm{c},1a} \overset{\text{Discrétisation}}{\rightarrow} \left[K_{em}^{\mathrm{c},1a} \right], \left[K_{mm}^{\mathrm{c},1a} \right] \quad (392)$$

With the statements:

$$\left[K_{em}^{\mathrm{c},1a} \right] = +\omega_c J_c \hat{S}_{u,k}^{g_n} \hat{g}_{n,k} \left[N^e \right]^T [G] \left[\bar{N}^m \right]$$
$$\left[K_{mm}^{\mathrm{c},1a} \right] = -\omega_c J_c \hat{S}_{u,k}^{g_n} \hat{g}_{n,k} \left[\bar{N}^m \right]^T [G] \left[\bar{N}^m \right] \quad (393)$$

With the matrix $[G]$ such as:

$$[G] = m_{11} \left[\bar{B}_1^m \right] \{ x_h^m \} \langle n_h \rangle \left[\bar{B}_1^m \right] + m_{21} \left[\bar{B}_2^m \right] \{ x_h^m \} \langle n_h \rangle \left[\bar{B}_1^m \right]$$
$$+ m_{12} \left[\bar{B}_1^m \right] \{ x_h^m \} \langle n_h \rangle \left[\bar{B}_2^m \right] + m_{22} \left[\bar{B}_2^m \right] \{ x_h^m \} \langle n_h \rangle \left[\bar{B}_2^m \right] \quad (394)$$

Consider the second integral, while discretizing, one obtains:

$$J_{\boldsymbol{uu}}^{\mathrm{c},2a} \overset{\text{Discrétisation}}{\rightarrow} \left[K_{me}^{\mathrm{c},2a} \right], \left[K_{mm}^{\mathrm{c},2a} \right] \quad (395)$$

With the statements:

$$\left[K_{me}^{\mathrm{c},2a} \right] = +\omega_c J_c \hat{S}_{u,k}^{g_n} \hat{g}_{n,k} \left[\bar{N}^m \right]^T [H] \left[N^e \right]$$
$$\left[K_{mm}^{\mathrm{c},2a} \right] = -\omega_c J_c \hat{S}_{u,k}^{g_n} \hat{g}_{n,k} \left[\bar{N}^m \right]^T [H] \left[\bar{N}^m \right] \quad (396)$$

With the matrix $[H]$ such as:

$$[H] = m_{11} \left[\bar{B}_1^m \right]^T \{ n_h \} \langle x_h^m \rangle \left[\bar{B}_1^m \right]^T + m_{21} \left[\bar{B}_2^m \right]^T \{ n_h \} \langle x_h^m \rangle \left[\bar{B}_1^m \right]^T + m_{12} \left[\bar{B}_1^m \right]^T \{ n_h \} \langle x_h^m \rangle \left[\bar{B}_2^m \right]^T + m_{22} \left[\bar{B}_2^m \right]^T \{ \quad (397)$$

Lastly, the last contribution:

$$J_{\boldsymbol{uu}}^{\mathrm{c},3a} \overset{\text{Discrétisation}}{\rightarrow} \left[K_{me}^{\mathrm{c},3a} \right], \left[K_{mm}^{\mathrm{c},3a} \right], \left[K_{ee}^{\mathrm{c},3a} \right], \left[K_{em}^{\mathrm{c},3a} \right] \quad (398)$$

With the statements:

$$\left[K_{ee}^{\mathrm{c},3a} \right] = +\omega_c J_c \hat{S}_{u,k}^{g_n} \hat{g}_{n,k} \left[N^e \right]^T [L] \left[N^e \right]$$
$$\left[K_{mm}^{\mathrm{c},3a} \right] = +\omega_c J_c \hat{S}_{u,k}^{g_n} \hat{g}_{n,k} \left[\bar{N}^m \right]^T [L] \left[\bar{N}^m \right]$$
$$\left[K_{em}^{\mathrm{c},2a} \right] = -\omega_c J_c \hat{S}_{u,k}^{g_n} \hat{g}_{n,k} \left[N^e \right]^T [L] \left[\bar{N}^m \right] \quad (399)$$
$$\left[K_{me}^{\mathrm{c},2a} \right] = -\omega_c J_c \hat{S}_{u,k}^{g_n} \hat{g}_{n,k} \left[\bar{N}^m \right]^T [L] \left[N^e \right]$$

With the matrix $[L]$ such as:

$$[H] = m_{11} \kappa_{11} m_{11} \left[\bar{B}_1^m \right]^T \left[\bar{B}_1^m \right] + m_{21} \kappa_{21} m_{21} \left[\bar{B}_2^m \right]^T \{ n_h \} \langle x_h^m \rangle \left[\bar{B}_1^m \right] + m_{12} \kappa_{12} m_{12} \left[\bar{B}_1^m \right]^T \{ n_h \} \langle x_h^m \rangle \left[\bar{B}_2^m \right] + m \quad (400)$$

One considers the discretization of the jacobiennes now (322) and (323):

$$J_{\boldsymbol{uu}}^{\mathrm{f},1} \overset{\text{Discrétisation}}{\rightarrow} \left[K_{ee}^{\mathrm{f},1} \right], \left[K_{em}^{\mathrm{f},1} \right], \left[K_{me}^{\mathrm{f},1} \right], \left[K_{mm}^{\mathrm{f},1} \right]$$
$$J_{\boldsymbol{uu}}^{\mathrm{f},2} \overset{\text{Discrétisation}}{\rightarrow} \left[K_{ee}^{\mathrm{f},2} \right], \left[K_{em}^{\mathrm{f},2} \right], \left[K_{me}^{\mathrm{f},2} \right], \left[K_{mm}^{\mathrm{f},2} \right] \quad (401)$$

the first part relates to the adherent case ($\hat{S}_{u,k}^{g_n} = 1$ and $\hat{S}_{f,k}^{\boldsymbol{h}_\tau} = 1$):

$$J_{\boldsymbol{uu}}^{\mathrm{f},1} = -\int_{\Gamma_c} \mu \bar{p}_t \hat{\lambda}_k \left[\underline{\underline{\boldsymbol{P}}}^\tau \left(\delta_t \hat{\boldsymbol{u}}_k^e - \delta_t \hat{\boldsymbol{u}}_k^m \right) \right] \cdot \left[\underline{\underline{\boldsymbol{P}}}^\tau \left(\Delta_t \boldsymbol{u}^e - \Delta_t \boldsymbol{u}^m \right) \right] d\Gamma_c \quad (402)$$

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the discretization of $\underline{\underline{\boldsymbol{P}}}^{\tau}$ and displacements gives us:

$$\begin{aligned}\left[K_{ee}^{\mathrm{f},1}\right]&=-\omega_c J_c \mu\, \bar{\rho}_t \hat{\lambda}_k \left[N^e\right]^T \left[P^{\tau}\right]\left[P^{\tau}\right]\left[N^e\right]\\ \left[K_{mm}^{\mathrm{f},1}\right]&=-\omega_c J_c \mu\, \bar{\rho}_t \hat{\lambda}_k \left[N^m\right]^T \left[P^{\tau}\right]\left[P^{\tau}\right]\left[N^m\right]\\ \left[K_{em}^{\mathrm{f},1}\right]&=+\omega_c J_c \mu\, \bar{\rho}_t \hat{\lambda}_k \left[N^e\right]^T \left[P^{\tau}\right]\left[P^{\tau}\right]\left[N^m\right]\\ \left[K_{me}^{\mathrm{f},2}\right]&=+\omega_c J_c \mu\, \bar{\rho}_t \hat{\lambda}_k \left[N^m\right]^T \left[P^{\tau}\right]\left[P^{\tau}\right]\left[N^e\right]\end{aligned} \tag{403}$$

the second part relates to the slipping case ($\hat{S}_{u,k}^{g_n}=1$ and $\hat{S}_{f,k}^{\boldsymbol{h}_{\tau}}=0$):

$$J_{\boldsymbol{u}\boldsymbol{u}}^{\mathrm{f},2}=-\int_{\Gamma_c} \mu\, \bar{\rho}_t \hat{\lambda}_k \left[\underline{\underline{\boldsymbol{P}}}^{\tau}\left(\delta_t \hat{\boldsymbol{u}}_k^e-\delta_t \hat{\boldsymbol{u}}_k^m\right)\right]\cdot\left[\underline{\underline{\boldsymbol{P}}}^{\tau}\,\hat{\underline{\underline{\boldsymbol{P}}}}_k^{B(0,1)}\left(\Delta_t \boldsymbol{u}^e-\Delta_t \boldsymbol{u}^m\right)\right] d\Gamma_c \tag{404}$$

What is written, by means of in particular discretization of projection on the ball unit (386):

$$\begin{aligned}\left[K_{ee}^{\mathrm{f},2}\right]&=-\omega_c J_c \mu\, \bar{\rho}_t \hat{\lambda}_k \left[N^e\right]^T \left[P^{\tau}\right]\left[\hat{P}_k^B\right]\left[P^{\tau}\right]\left[N^e\right]\\ \left[K_{mm}^{\mathrm{f},2}\right]&=-\omega_c J_c \mu\, \bar{\rho}_t \hat{\lambda}_k \left[N^m\right]^T \left[P^{\tau}\right]\left[\hat{P}_k^B\right]\left[P^{\tau}\right]\left[N^m\right]\\ \left[K_{em}^{\mathrm{f},2}\right]&=+\omega_c J_c \mu\, \bar{\rho}_t \hat{\lambda}_k \left[N^e\right]^T \left[P^{\tau}\right]\left[\hat{P}_k^B\right]\left[P^{\tau}\right]\left[N^m\right]\\ \left[K_{me}^{\mathrm{f},2}\right]&=+\omega_c J_c \mu\, \bar{\rho}_t \hat{\lambda}_k \left[N^m\right]^T \left[P^{\tau}\right]\left[\hat{P}_k^B\right]\left[P^{\tau}\right]\left[N^e\right]\end{aligned} \tag{405}$$

We now will discretize the quantities concerning the second column, relating to the contact pressure. From the reaction of contact and jacobienne (327):

$$J_{\boldsymbol{u}c}^{\mathrm{c}}=-\int_{\Gamma_c} \hat{S}_{u,k}^{g_n}\left(\delta_t \hat{\boldsymbol{u}}_k^e-\delta_t \hat{\boldsymbol{u}}_k^m\right)\cdot \boldsymbol{n}_k \Delta\lambda\, d\Gamma_c \tag{406}$$

One must discretize:

$$J_{\boldsymbol{u}c}^{\mathrm{c}} \overset{\text{Discrétisation}}{\rightarrow} \left[K_{ec}^{\mathrm{c}}\right],\left[K_{mc}^{\mathrm{c}}\right] \tag{407}$$

What gives us:

$$\begin{aligned}\left[K_{ec}^{\mathrm{c}}\right]&=-\omega_c J_c \hat{S}_{u,k}^{g_n}\left[N^e\right]^T\{n_h\}\langle\psi\rangle\\ \left[K_{mc}^{\mathrm{c}}\right]&=+\omega_c J_c \hat{S}_{u,k}^{g_n}\left[N^m\right]^T\{n_h\}\langle\psi\rangle\end{aligned} \tag{408}$$

From the reaction of friction, i.e. jacobiennes (332) and (333):

$$\begin{aligned}J_{\boldsymbol{u}c}^{\mathrm{f},1} &\overset{\text{Discrétisation}}{\rightarrow} \left[K_{ec}^{\mathrm{f},1}\right],\left[K_{mc}^{\mathrm{f},1}\right]\\ J_{\boldsymbol{u}c}^{\mathrm{f},2} &\overset{\text{Discrétisation}}{\rightarrow} \left[K_{ec}^{\mathrm{f},2}\right],\left[K_{mc}^{\mathrm{f},2}\right]\end{aligned} \tag{409}$$

the first part relates to the adherent case ($\hat{S}_{u,k}^{g_n}=1$ and $\hat{S}_{f,k}^{\boldsymbol{h}_{\tau}}=1$) :

(410)

the discretization of $\underline{\underline{\boldsymbol{P}}}^{\tau}$, of displacements $\boldsymbol{u}$ and of the contact pressure λ gives us:

$$\begin{aligned}\left[K_{ec}^{\mathrm{f},1}\right]&=-\omega_c J_c \mu\left[N^e\right]^T\left[P^{\tau}\right]\{\psi\}\\ \left[K_{mc}^{\mathrm{f},1}\right]&=+\omega_c J_c \mu\left[N^m\right]^T\left[P^{\tau}\right]\{\psi\}\end{aligned} \tag{411}$$

the second part relates to the slipping case ($\hat{S}_{u,k}^{g_n}=1$ and $\hat{S}_{f,k}^{\boldsymbol{h}_{\tau}}=0$):

$$J_{\boldsymbol{u}c}^{\mathrm{f},2}=-\int_{\Gamma_c} \mu\left[\underline{\underline{\boldsymbol{P}}}^{\tau}\left(\delta_t \hat{\boldsymbol{u}}_k^e-\delta_t \hat{\boldsymbol{u}}_k^m\right)\right]\cdot\hat{\boldsymbol{\tau}}_k \Delta\lambda\, d\Gamma_c \tag{412}$$

What is written, by means of in particular discretization of projection on the ball unit (386):

$$\begin{aligned}\left[K_{ec}^{\mathrm{f},2}\right]&=-\omega_c J_c \mu\left[N^e\right]^T\left[P^{\tau}\right]\left\{\hat{\tau}_k\right\}\{\psi\}\\ \left[K_{mc}^{\mathrm{f},2}\right]&=+\omega_c J_c \mu\left[N^m\right]^T\left[P^{\tau}\right]\left\{\hat{\tau}_k\right\}\{\psi\}\end{aligned} \tag{413}$$

We now will discretize the quantities concerning the third column, relating to the pressure of friction. From the reaction of friction and jacobiennes (339) and (337):

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$$\begin{aligned} J_{\boldsymbol{u}f}^{\mathrm{f},1} &\overset{\text{Discrétisation}}{\rightarrow} \left[K_{ef}^{\mathrm{f},1}\right],\left[K_{mf}^{\mathrm{f},1}\right] \\ J_{\boldsymbol{u}f}^{\mathrm{f},2} &\overset{\text{Discrétisation}}{\rightarrow} \left[K_{ef}^{\mathrm{f},2}\right],\left[K_{mf}^{\mathrm{f},2}\right] \end{aligned} \tag{414}$$

the first part relates to the adherent case ($\hat{S}_{u,k}^{g_n}=1$ and $\hat{S}_{f,k}^{\boldsymbol{h}_\tau}=1$):

$$J_{\boldsymbol{u}f}^{\mathrm{f},1}=-\int_{\Gamma_c}\mu\hat{\lambda}_k\left[\underline{\underline{\boldsymbol{P}}}^{\tau}\left(\delta_t\hat{\boldsymbol{u}}_k^e-\delta_t\hat{\boldsymbol{u}}_k^m\right)\right]\cdot\Delta\boldsymbol{\Lambda}\,d\Gamma_c \tag{415}$$

the discretization of $\underline{\underline{\boldsymbol{P}}}^{\tau}$, displacements $\boldsymbol{u}$ and of the pressure of friction $\boldsymbol{\Lambda}$ gives us:

$$\begin{aligned} \left[K_{ef}^{\mathrm{f},1}\right] &= -\omega_c J_c \mu \hat{\lambda}_k \left[N^e\right]^T \left[P^{\tau}\right][T][\psi] \\ \left[K_{mf}^{\mathrm{f},1}\right] &= +\omega_c J_c \mu \hat{\lambda}_k \left[N^m\right]^T \left[P^{\tau}\right][T][\psi] \end{aligned} \tag{416}$$

the second part relates to the slipping case ($\hat{S}_{u,k}^{g_n}=1$ and $\hat{S}_{f,k}^{\boldsymbol{h}_\tau}=0$):

$$J_{\boldsymbol{u}f}^{\mathrm{f},2}=-\int_{\Gamma_c}\mu\hat{\lambda}_k\left[\underline{\underline{\boldsymbol{P}}}^{\tau}\left(\delta_t\hat{\boldsymbol{u}}_k^e-\delta_t\hat{\boldsymbol{u}}_k^m\right)\right]\cdot\left[\hat{\underline{\underline{\boldsymbol{P}}}}_k^{B(0,1)}\Delta\boldsymbol{\Lambda}\right]d\Gamma_c \tag{417}$$

What is written, by means of in particular discretization of projection on the ball unit (386):

$$\begin{aligned} \left[K_{ef}^{\mathrm{f},2}\right] &= -\omega_c J_c \mu \hat{\lambda}_k \left[N^e\right]^T \left[P^{\tau}\right]\left[\hat{P}_k^B\right][T][\psi] \\ \left[K_{mf}^{\mathrm{f},2}\right] &= +\omega_c J_c \mu \hat{\lambda}_k \left[N^m\right]^T \left[P^{\tau}\right]\left[\hat{P}_k^B\right][T][\psi] \end{aligned} \tag{418}$$

13.3 Quantities for the model of contact

One considers the matric terms resulting from the linearization of the model of contact and thus with the second line from the total matrix. Discretization of the jacobienne (344) corresponding to the first column of the total matrix:

$$J_{c\boldsymbol{u}} \overset{\text{Discrétisation}}{\rightarrow} \left[K_{ce}\right],\left[K_{cm}\right] \tag{419}$$

From the jacobienne:

$$J_{c\boldsymbol{u}}=-\int_{\Gamma_c}\hat{S}_{u,k}^{g_n}\delta\lambda\left(\Delta_t\boldsymbol{u}^e-\Delta_t\boldsymbol{u}^m\right)\cdot\boldsymbol{n}_k\,d\Gamma_c \tag{420}$$

One obtains two quantities:

$$\begin{aligned} \left[K_{ce}\right] &= -\omega_c J_c \hat{S}_{u,k}^{g_n}[\psi]\langle n_h\rangle\left[N^e\right] \\ \left[K_{cm}\right] &= +\omega_c J_c \hat{S}_{u,k}^{g_n}[\psi]\langle n_h\rangle\left[N^m\right] \end{aligned} \tag{421}$$

By the discretization of the jacobienne (346), one obtains the statement of the quantities corresponding to the second column:

$$J_{cc} \overset{\text{Discrétisation}}{\rightarrow} \left[K_{cc}\right] \tag{422}$$

From the jacobienne:

$$J_{cc}=-\frac{1}{\rho_n}\int_{\Gamma_c}\Delta\lambda\left(1-\hat{S}_{u,k}^{g_n}\right)\delta\lambda\,d\Gamma_c \tag{423}$$

One obtains:

$$\left[K_{cc}\right]=-\omega_c J_c\frac{1}{\rho_n}\left(1-\hat{S}_{u,k}^{g_n}\right)[\psi]\langle\psi\rangle \tag{424}$$

13.4 Quantities for the friction law

One considers the matric terms resulting from the linearization of the friction law and thus with the back-row forward from the total matrix. Discretization of the jacobiennes (352) and (353) corresponding to the first column of the total matrix:

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$$\begin{aligned} J^1_{f\,\boldsymbol{u}} &\overset{\text{Discrétisation}}{\rightarrow} \left[K^1_{fe}\right],\left[K^1_{fm}\right] \\ J^2_{f\,\boldsymbol{u}} &\overset{\text{Discrétisation}}{\rightarrow} \left[K^2_{fe}\right],\left[K^2_{fm}\right] \end{aligned} \tag{425}$$

the first part relates to the adherent case ($\hat{S}^{g_n}_{u,k}=1$ and $\hat{S}^{\boldsymbol{h}_\tau}_{f,k}=1$):

$$J^1_{f\,\boldsymbol{u}}=-\int_{\Gamma_c}\mu\,\hat{\lambda}_k\,\delta\boldsymbol{\Lambda}\cdot\left[\underline{\underline{\boldsymbol{P}}}^{\tau}\left(\Delta_t\boldsymbol{u}^e-\Delta_t\boldsymbol{u}^m\right)\right]d\Gamma_c \tag{426}$$

the discretization of $\underline{\underline{\boldsymbol{P}}}^{\tau}$, displacements $\boldsymbol{u}$ and of the pressure of friction $\boldsymbol{\Lambda}$ gives us:

$$\begin{aligned} \left[K^1_{fe}\right] &= -\omega_c J_c\,\mu\,\hat{\lambda}_k\left[\psi\right]^T\left[T\right]^T\left[P^{\tau}\right]\left[N^e\right] \\ \left[K^1_{fm}\right] &= +\omega_c J_c\,\mu\,\hat{\lambda}_k\left[\psi\right]^T\left[T\right]^T\left[P^{\tau}\right]\left[N^m\right] \end{aligned} \tag{427}$$

the second part relates to the slipping case ($\hat{S}^{g_n}_{u,k}=1$ and $\hat{S}^{\boldsymbol{h}_\tau}_{f,k}=0$):

$$J^2_{f\,\boldsymbol{u}}=-\int_{\Gamma_c}\mu\,\hat{\lambda}_k\,\delta\boldsymbol{\Lambda}\cdot\left[\underline{\underline{\boldsymbol{P}}}^{\tau}\,\underline{\underline{\hat{\boldsymbol{P}}}}^{B(0,1)}_k\left(\Delta_t\boldsymbol{u}^e-\Delta_t\boldsymbol{u}^m\right)\right]d\Gamma_c \tag{428}$$

What is written, by means of in particular discretization of projection on the ball unit (386):

$$\begin{aligned} \left[K^2_{fe}\right] &= -\omega_c J_c\,\mu\,\hat{\lambda}_k\left[\psi\right]^T\left[T\right]^T\left[P^{\tau}\right]\left[\hat{P}^B_k\right]\left[N^e\right] \\ \left[K^2_{fe}\right] &= +\omega_c J_c\,\mu\,\hat{\lambda}_k\left[\psi\right]^T\left[T\right]^T\left[P^{\tau}\right]\left[\hat{P}^B_k\right]\left[N^m\right] \end{aligned} \tag{429}$$

the discretization of the jacobienne (359) corresponding to the third column of the total matrix:

$$J_{f\,f}\overset{\text{Discrétisation}}{\rightarrow}\left[K_{ff}\right] \tag{430}$$

the jacobienne being written:

$$\begin{aligned} J_{f\,f}= \quad & \frac{1}{\bar{\rho}_t}\int_{\Gamma_c}\mu\,\hat{\lambda}_k\hat{S}^{g_n}_{u,k}\left(1-\hat{S}^{\boldsymbol{h}_\tau}_{f,k}\right)\left(\underline{\underline{\boldsymbol{1}}}-\underline{\underline{\hat{\boldsymbol{P}}}}^{B(0,1)}_k\right)\Delta\boldsymbol{\Lambda}\cdot\delta\boldsymbol{\Lambda}\,d\Gamma_c \\ & -\int_{\Gamma_c}\left(1-\hat{S}^{g_n}_{u,k}\right)\Delta\boldsymbol{\Lambda}\cdot\delta\boldsymbol{\Lambda}\,d\Gamma_c \end{aligned} \tag{431}$$

One prefers to divide it into two. A contribution in the case of slipping contact ($\hat{S}^{g_n}_{u,k}=1$ and $\hat{S}^{\boldsymbol{h}_\tau}_{f,k}=0$):

$$J^1_{f\,f}=\frac{1}{\bar{\rho}_t}\int_{\Gamma_c}\mu\,\hat{\lambda}_k\left(\underline{\underline{\boldsymbol{1}}}-\underline{\underline{\hat{\boldsymbol{P}}}}^{B(0,1)}_k\right)\Delta\boldsymbol{\Lambda}\cdot\delta\boldsymbol{\Lambda}\,d\Gamma_c \tag{432}$$

And a contribution for the case without contact:

$$J^2_{f\,f}=-\int_{\Gamma_c}\Delta\boldsymbol{\Lambda}\cdot\delta\boldsymbol{\Lambda}\,d\Gamma_c \tag{433}$$

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Historical versions of the document

Index Doc.	VersionAster	Author (S) or contributor (S), organization	Description of the modifications
A	6.4	**P. Massin**, H. Ben Dhia, Mr. Zarroug, S. Lamarche, C. Zammali, Mr. Torkhani	Description of L" algorithm of the continuous method of contact in *Code_Aster.*
B	8.4	**Mr. Kham** , Mr. Torkhani, P. Massin, H. Ben Dhia, C.Zammali, Mr. Zarroug	Additions: model of contact in dynamics (Moreau) and diagram D" integration, management of the redundancies, model of multilevel compliance
C	9.4	**Mr. Abbas,** M.Torkhani, P. Massin, H. Ben Dhia	Additions: Setting with the norms of the style of documentation Addition of L" wear Addition of the formulation generalized
D	10.2	**Mr. Abbas,** P. Massin, T. De Soza	Additions: Resumption of the styles and classification Corrections of the elementary matrixes/vectors Addition of the formulation penalized
E	11.1	**Mr. Abbas,** T. De Soza	Suppressions: Compliance, linear-quadratic connection, processing of the crack tip
F	11.3	**Mr. Abbas,**	Suppressions: Wear Archard Additions: method of Newton partial, method of Newton generalized, terms derived, cycles (D.Kudawoo thesis)

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