Contact in small slidings with X-FEM

Summarized:

This document presents a new approach to deal with the problems of contact in small slidings with the eXtended Finite Element Method (X-FEM) [bib1]. One considers the continuous hybrid formulation of problems of contact between solids [bib2] and the strategy of resolution is similar to that already implemented in Code_Aster for the frame conventional finite elements [bib3]. A new type of mixed element of contact is introduced, specific with the frame X-FEM.

The approach is implemented in *Code_Aster* in 2D and 3D, and treats at the same time interfaces completely cut by a crack as well as interfaces with crack tip. It is usable with command STAT_NON_LINE [U4.51.03]. The friction of the Coulomb type is taken into account.

Contents

1 Introduction

To take into account the contact rubbing on the lips of crack with X-FEM, we chose the frame of the continuous method [biberon142], [biberon242].

Contrary to the discrete approaches where the problem of contact is taken into account by assembly of nodal forces, the equations are discretized here by the finite element method and the problem of contact is taken into account by an assembly of elementary contributions. This approach was developed by many authors, like Alart *and al.* [biberon342], Laursen *and al.* [biberon842], Wriggers [biberon942], Curnier *and al.* [biberon1042], Pietrzak [biberon1142].

In this “continuous” approach, the conditions of contact are seen like a model of interface and not like boundary conditions. With the notion of model of interface between deformable bodies, one can during associate the transition of the continuous formalism with the discrete model, the notion of element of contact. The resolution exact, and thus rigorous, of the models of contact (nondifferentiable) can be carried out via a hybrid element contact including the forces of contact in the unknowns of the problem. In this frame, the hybrid formulation led to an asymmetric matrix tangent, nondefinite positive, badly conditioned, with zeros on the diagonal. The difficulty of the problem lies in the non-differentiability of the system to solve [biberon1242].

The method suggested by Ben Dia is very similar, but one chooses of the contact to eliminate the non-differentiability by an algorithm from active stresses, and that of friction by a fixed problem of point on friction so as to obtain a succession of regular problems having methods of resolution whose convergence is established. Many alternatives, according to the adopted algorithms and their fitting exist, but the total convergence of the diagram is not assured

One points out only one one starts from formal Lagrangian of the problem of contact between two deformable solids, which one introduces into the principle of the virtual wors. A variational formulation mixed displacement-pressure is deduced by incorporating weak formulations from the models from contact. The equations are discretized by the finite element method. The choice of spaces finite elements of discretization as well as the diagrams of integration (terms of contact) is clarified.

With X-FEM, the lips of crack are treated as only one geometrical surface of discontinuity which can be interior with the finite elements. The integration of the terms of contact on this surface (nonwith a grid) then called on the quantities carried by the nodes of the elements crossed by this one. In small displacements, no pairing is not necessary because the points of the surface of crack in opposite are intrinsically dependant (they correspond to the same geometrical entity). The jump of displacement is expressed according to the discontinuous degrees of freedom of enrichment introduced by X-FEM.

This document is articulated around 6 principal sections, of which this introduction which holds place of section 1. The problem of rubbing contact as well as the put equations concerned are introduced in the paragraph [§22]. On the basis of a Lagrangian approach of the contact introduced into the principle of the virtual wors, the paragraph [§33] ends in a statement of the variational formulation mixed displacement-pressure. The paragraph [§44] evokes the choice of the discretizations finite elements of the sphere of activities of contact. One specifies the strategy of resolution in the paragraph [§55]. The statements of the elementary terms of contact and friction resulting from the approach X-FEM are detailed in the paragraph [§5.628]. The paragraph [§66] is interested in a particular condition of compatibility of the fields of displacements and pressure. An algorithm aiming at determining a space of the intensifiers of adequate contact (i.e. respecting the LBB condition) is clarified.

2 Strong formulation of the problem of contact rubbing

2.1 Formulation of the equations of the general problem

One will indicate by r^1 and the r^2 densities of the forces due to interactions of contact rubbing possible between two surfaces. One will indicate by t_c^1 and the t_c^2 forces1Force[1] due to the possible interactions of cohesion between two surfaces in the case of the opening of an interface.

Constitutive law	$\sigma = C : \varepsilon \quad \text{dans } \Omega$
Balances	$\nabla \cdot \sigma = f \quad \text{dans } \Omega$
imposed surface Forces	$\sigma \cdot n_{ext} = t \quad \text{sur } \Gamma_t$
Density of the forces of contact	$\sigma \cdot n^i = r^i \quad \text{sur } \Gamma^i \quad i = 1,2$
Density of the forces of cohesion	$\sigma \cdot n^i = t_c^i \quad \text{sur } \Gamma^i \quad i = 1,2$
imposed Displacements	$u = 0 \quad \text{sur } \Gamma_u$

Table 2.1-1 : Equations of the general problem

Note:

> *In spite of appearances, the relations on the densities of force of contact and the densities of force of cohesion are not incompatible. Indeed, the first is valid only when the solids are in contact whereas the second is applicable only if the solids are separated.*

2.2 Models of the contact

Is P a point of Γ_c. One notes P^1 points P^2 and the coinciding on Γ^1 and Γ^2 respectively. The condition of NON-interpenetration enters P^1 and P^2 is written in the direction n, the norm with Γ^1 :

$$d_n = \left(x\left(P^1\right) - x\left(P^2\right) \right) \cdot n \leq 0$$

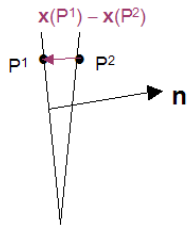


Figure 2.2-1 : Definition of clearance

One breaks up the density of force of contact r into a normal part λ which indicates the normal pressure of contact and another tangential r_τ. Thus, the density of force of contact is written:

$$r = \lambda n + r_\tau$$

1 per unit of area, homogeneous with a stress

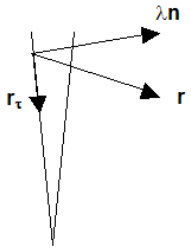


Figure 2.2-2 : Definition of the density of force of contact

With these notations, the models of contact (models of Signorini) are written in the following form:

$$d_n \leq 0, \quad \lambda \leq 0, \quad \lambda d_n = 0$$

These models utilize inequations, but those do not lend themselves easily to a weak formulation. For that, one rewrites these models in another form, by transforming them into only one equivalent equation biberon142 :

$$\lambda - \chi(g_n) g_n = 0$$

In this statement, χ is the indicating function of $\Re^-$ defined by

$$\chi(x) = \begin{cases} 1 & \text{si } x < 0 \\ 0 & \text{si } x \geq 0 \end{cases}$$

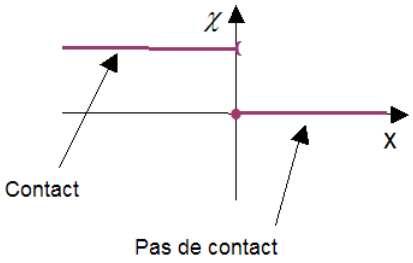


Figure 2.2-3 : Definition of the indicating function of $\Re^-$

and g_n the multiplier (known as of contact increased biberon342) defined by:

$$g_n = \lambda - \rho_n d_n$$

where ρ_n is a strictly positive reality.
The problem of contact posed by the models of Signorini introduces a NON-univocal relation (λ is not a function of d_n), semi-definite positive and nondifferentiable in $\lambda = d_n = 0$ like illustrates it [Figure 2.2-4].

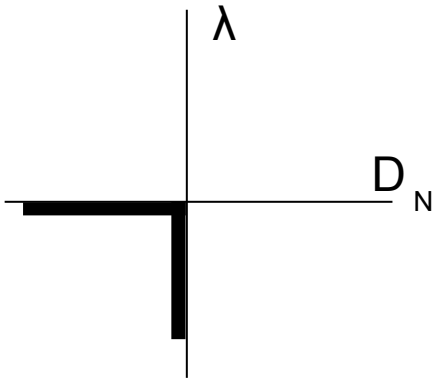


Figure 2.2-4 : Graph of the model of unilateral contact of Signorini

a regularization by penalization of the interpenetration makes it possible to make this relation univocal, to see [Figure 2.2-5].

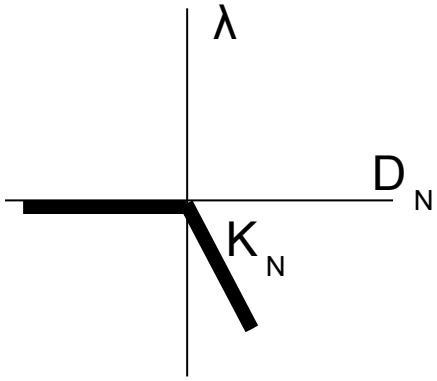


Figure 2.2-5 : Graph of the model of contact regularized

Physically, one authorizes solids to be interpenetrated at the cost of a very κ_n high stiffness. The contact pressure is then given by $\lambda=-\kappa_n d_n$ and becomes equal to the multiplier of contact g_n. It is noted that the more the coefficient of penalization is increased κ_n, the more one approaches the classical model of contact.

2.3 Models of friction

For the phenomena of friction, one uses the models of Coulomb which are written as follows:

$$\|r_\tau\| \leq \mu|\lambda|$$
$$\text{Si } \|r_\tau\| < \mu|\lambda| \text{ alors } \quad v_\tau = 0$$
$$\text{Si } \|r_\tau\| = \mu|\lambda| \text{ alors } \exists \alpha \geq 0 \ ; \ v_\tau = -\alpha r_\tau$$

where μ is the coefficient of kinetic friction of Coulomb and v_τ the tangent relative velocity.

One notes thereafter x_τ the projection of x on the tangent level at the contact surface, defined by $x_\tau=(\mathbf{Id}-n\otimes n)x$, where the symbol $\otimes$ indicates the tensor product.

Just as for the models of contact, one can write the friction laws as follows in an equivalent way:

$$r_\tau=\mu\lambda\Lambda$$
$$\Lambda-P_{B(0,1)}(g_\tau)=0$$
$$g_\tau=\Lambda+\rho_\tau v_\tau$$

In these statements, Λ is a semi-multiplier (vectorial) of friction, g_τ is the semi-multiplier (vectorial) of increased friction, $P_{B(0,1)}$ is projection on the ball unit and ρ_τ a strictly positive parameter. The semi-multiplier of friction Λ , whose modulus is always lower or equal to 1, corresponds to the direction of sliding when its modulus is worth 1, and corresponds to the direction of dependancy when its modulus is strictly lower than 1. [Figure 2.3-1] presents in 3D a case of sliding and a case of dependancy.

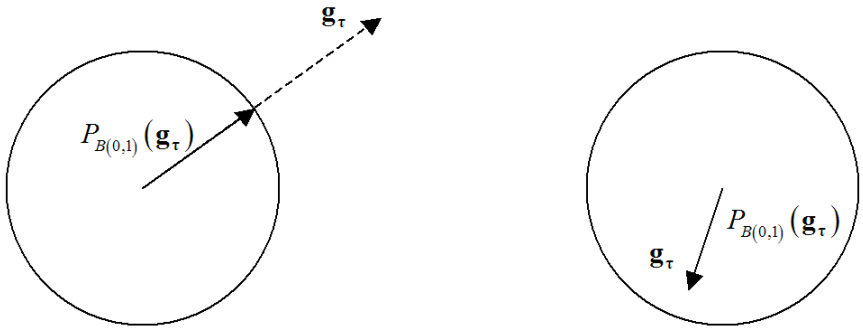


Figure 2.3-1 : Projections on the ball unit in 2D: sliding (on the left) and dependancy (on the right)

the friction laws are supplemented by the equation (of standard exclusion) following:

$$d_n\Lambda=0 \quad \text{ou} \quad (1-\chi(g_n))\Lambda=0$$

The problem of friction expressed via the models of Coulomb introduces a NON-univocal relation (R_τ is not a function of v_τ), and nondifferentiable (see [Figure 2.3-2]).

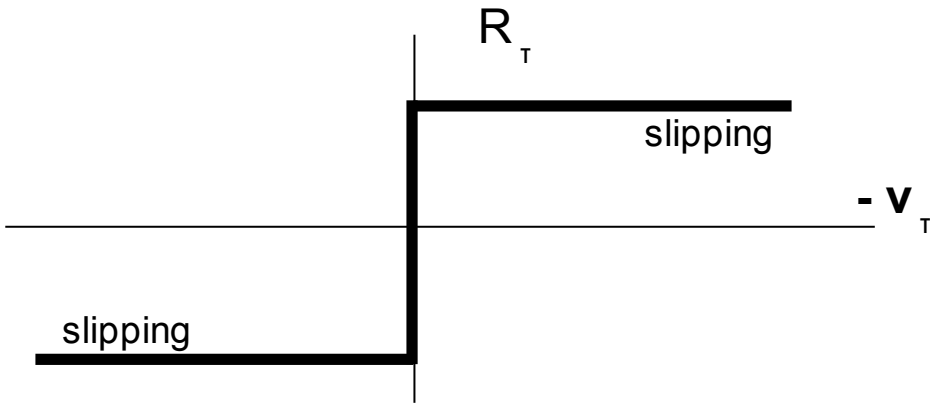


Figure 2.3-2 : Graph of the friction law of Coulomb

the penalized method makes it possible to make this relation univocal (see [Figure 2.3-3]).

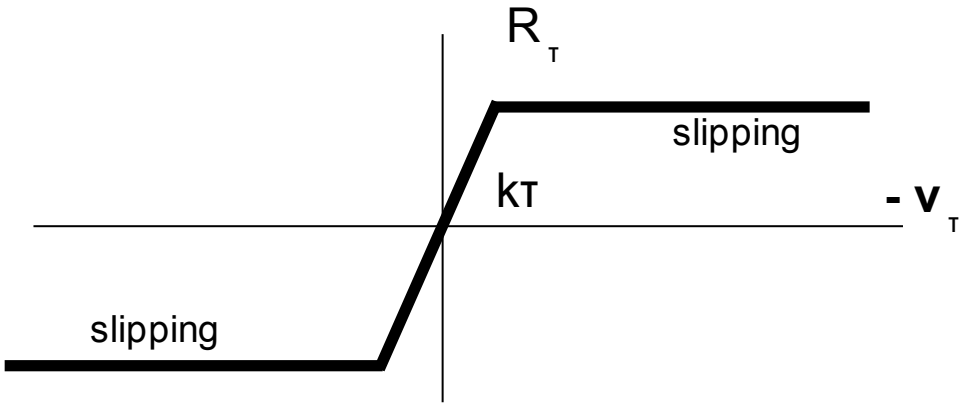


Figure 2.3-3 : Graph of the regularized friction law

the semi-multiplier of vectorial friction is then given by the relation $\Lambda = P_{B(0,1)}(\kappa_\tau \nu_\tau)$.
It is noted that the more the coefficient of penalization is increased κ_τ, the more one approaches the classical model of contact.

2.4 Cohesive models

2.4.1 regularized cohesive Models

the first type of models which we can consider is a model in which the initial dependancy is not perfect: the initial slope is finished. Two cohesive models of this kind are available in Code_Aster, models CZM_EXP_REG and CZM_LIN_REG whose characteristics are detailed in [R7.02.11]. We detail here the extension to XFEM for model CZM_EXP_REG, while basing ourselves on the standard commodity [biberon1742]. The extension to model CZM_LIN_REG is made while following the same paradigm exactly.

We write the jump of displacement such as definite for the contact algorithm, with the notation defined by figure 2.2-1:

$$[\![\boldsymbol{u}]\!](P^1) = \boldsymbol{u}(P^1) - \boldsymbol{u}(P^2)$$

We have then $[\![u_n]\!] = [\![\boldsymbol{u}]\!] \cdot \boldsymbol{n}$ negative in opening and positive in interpenetration. The notations of [R7.02.11] pose a jump of displacement $\boldsymbol{\delta}$ such as $\boldsymbol{\delta} \cdot \boldsymbol{n}$ is positive in opening and negative in interpenetration. To bring back to us to our notations, we re-use the results of [R7.02.11] while posing $\boldsymbol{\delta} = -[\![\boldsymbol{u}]\!]$. We are satisfied here to recall the principal points of them. The reader can refer to it for a thorough comprehension.

The opening of a crack in mixed mode is characterized by a criterion of damage defined by means of the jump of equivalent displacement and local variable α. The material remains in the elastic domain as long as the inequality is checked:

$$f([\![u]\!]_{eq}, \alpha) = [\![u]\!]_{eq} - \alpha \leq 0$$

- $[\![u]\!]_{éq} = \sqrt{\langle [\![u_n]\!] \rangle_-^2 + [\![\boldsymbol{u}_\tau]\!]^2}$ is the jump of displacement are equivalent,
- $[\![\boldsymbol{u}_\tau]\!] = [\![\boldsymbol{u}]\!] - [\![u_n]\!]\boldsymbol{n}$ is the jump of tangent displacement,
- $\alpha(t) = max\{\alpha_0, \underset{v \in [0,t]}{max} [\![u(v)]\!]_{éq}\}$ is the local variable of the cohesive model,

- α_0 is the initial value of α. This value is given by the user via material parameter PENA_ADHERENCE so that $\alpha_0 = \frac{G_c}{\sigma_c}$ PENA_ADHERENCE .

The cohesive stress is written then like adds of an elastic stress, a dissipative stress and a stress of penalization which gives an account of the contact:

$$\boldsymbol{t_c} = H\left(\llbracket u \rrbracket_{\text{èq}} - \alpha\right) \boldsymbol{\sigma_{lin}} + \left(1 - H\left(\llbracket u \rrbracket_{\text{èq}} - \alpha\right)\right) \boldsymbol{\sigma_{dis}} + \boldsymbol{\sigma_{pen}}$$

where H is the indicating function of $\Re^+$

- $\boldsymbol{\sigma}_{pen} = -C \langle \llbracket u_n \rrbracket \rangle_+ \boldsymbol{n}$ is the stress of penalization.

where C Is a coefficient of penalization clarified in [R7.02.11] given starting from material parameter PENA_CONTACT.

- $\boldsymbol{\sigma}_{lin} = \boldsymbol{\sigma}_{dis} = -\frac{\sigma_c}{\alpha} \exp\left(-\frac{\sigma_c}{G_c} \alpha\right) \llbracket \boldsymbol{u} \rrbracket$ is the statement common to the linear stresses and dissipative, with $\alpha = \llbracket u \rrbracket_{èq}$ for $\boldsymbol{\sigma_{dis}}$.

where σ_c is the critical stress with the fracture

and G_c is the energy tenacity of the material. It corresponds indeed to energy necessary to the complete opening of the interface over a unit length. A fast computation with the preceding statements makes it possible to confirm:

$$\int_{-\infty}^{0} \boldsymbol{t}_{c,\tau} \cdot d\llbracket \boldsymbol{u}_\tau \rrbracket + \int_{-\infty}^{0} \left(\boldsymbol{t}_{c,n} \cdot \boldsymbol{n}\right) d\llbracket u_n \rrbracket = G_c$$

One represents on figure 2.4.1-1 the cohesive stress for a loading in pure I mode according to the jump of normal displacement.

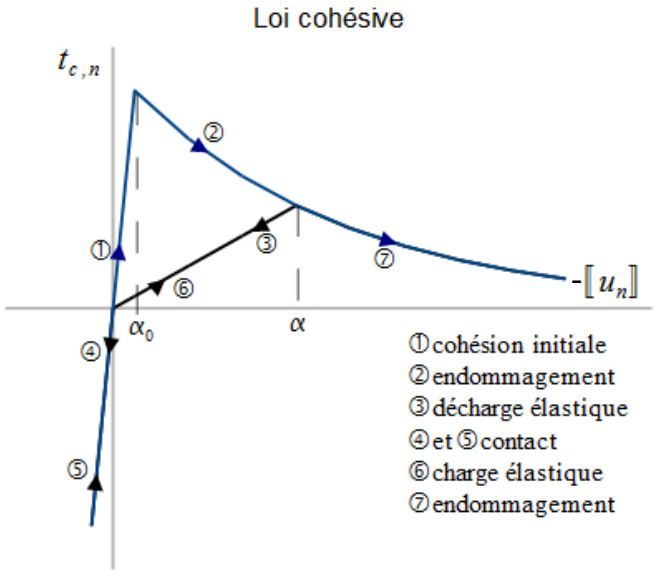


Figure 2.4.1-1: Evolution of the force of normal cohesion according to the jump of displacement

For $[\![u_n]\!]<0$, it is also usual to define a force of equivalent cohesion $t_{c,eq}$ thanks to the energy condition of equivalence:

$$t_{c,eq}[\![\dot{u}]\!]_{eq}=\boldsymbol{t}_{c,n}\cdot\boldsymbol{n}[\![\dot{u}_n]\!]+\boldsymbol{t}_{c,\tau}\cdot[\![\dot{\boldsymbol{u}}_\tau]\!]$$

To find his value, one derives $[\![u]\!]_{èq}$ compared to time.

$$[\![\dot{u}]\!]_{èq}=\frac{[\![\dot{u}_n]\!][\![u_n]\!]+[\![\dot{\boldsymbol{u}}_\tau]\!]\cdot[\![\boldsymbol{u}_\tau]\!]}{[\![u]\!]_{èq}}$$

From where one identifies: $t_{c,èq}=\boldsymbol{t}_{c,n}\cdot\boldsymbol{n}\dfrac{[\![u]\!]_{èq}}{[\![u_n]\!]}=\|\boldsymbol{t}_{c,\tau}\|\dfrac{[\![u]\!]_{èq}}{\|[\![\boldsymbol{u}_\tau]\!]\|}=-\dfrac{\sigma_c}{\alpha}\exp\left(-\dfrac{\sigma_c}{G_c}\alpha\right)[\![u]\!]_{èq}$

Figure 2.4.1-2 represents the evolution of the force of cohesion equivalent according to the jump of equivalent displacement according to this constitutive law.

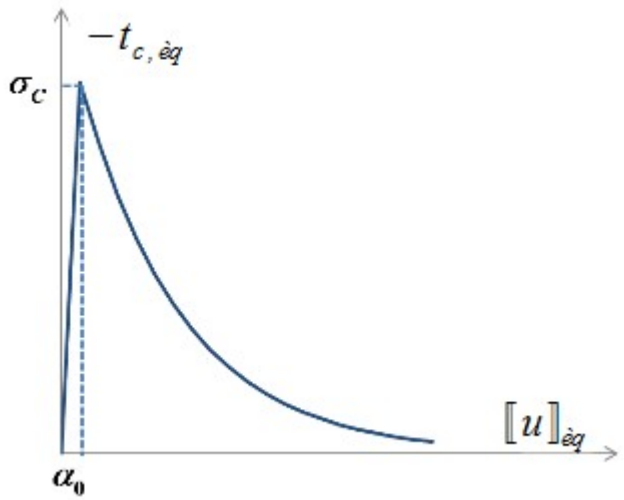


Figure 2.4.1-2: Evolution of the force of cohesion equivalent according to the jump of equivalent displacement

Note:

One defines sometimes a jump of equivalent displacement $[\![u]\!]_{èq}=\sqrt{\langle[\![u_n]\!]\rangle_-^2+\beta^2[\![\boldsymbol{u}_\tau]\!]^2}$ *where* β *is an experimental coefficient which represents the ratio of intensity of the forces of opening in mode* I *and mode* II*. By taking again the preceding reasoning then with:*

$$t_{c,èq}=-\frac{\sigma_c}{\alpha}\exp\left(-\frac{\sigma_c}{G_c}\alpha\right)[\![u]\!]_{èq}$$

One deduces the statements of the components:

$$\boldsymbol{t}_{c,n}=\frac{t_{c,eq}[\![u_n]\!]}{[\![u]\!]_{èq}}\boldsymbol{n}=-\frac{\sigma_c}{\alpha}\exp\left(-\frac{\sigma_c}{G_c}\alpha\right)[\![u_n]\!]\boldsymbol{n}\ ,\ \boldsymbol{t}_{c,\tau}=\beta^2\frac{t_{c,eq}[\![\boldsymbol{u}_\tau]\!]}{[\![u]\!]_{èq}}=-\beta^2\frac{\sigma_c}{\alpha}\exp\left(-\frac{\sigma_c}{G_c}\alpha\right)[\![\boldsymbol{u}_\tau]\!]$$

2.4.2 Mixed cohesive Models

the second type of models which we can consider is, contrary, a model in which the initial dependancy is perfect: the initial slope is infinite. Two cohesive models of this kind are available with XFEM in

Code_Aster, models CZM_OUV_MIX and CZM_TAC_MIX whose characteristics are detailed in [R7.02.11]. We point out model CZM_OUV_MIX here, represented on figure 2.4.2-1.

As partly 8, we introduce the jump of displacement $[\![\boldsymbol{u}]\!]$ such as $[\![u_n]\!]$ is negative in interpenetration and positive in opening, and we are reduced to the notations [R7.02.11] by posing $\boldsymbol{\delta}=-[\![\boldsymbol{u}]\!]$: one returns to this documentation for the form of the model and his derivative. Let us note that in the same way that for the regularized models, the material remains in the elastic domain as long as:

$$f([\![u]\!]_n,\alpha)=-[\![u]\!]_n-\alpha\leq 0$$

On the other hand, this time, the local variable has an initial value rigorously null, so that the stress is not explicit any more according to displacement, as shown in the figure 2.4.2-1.

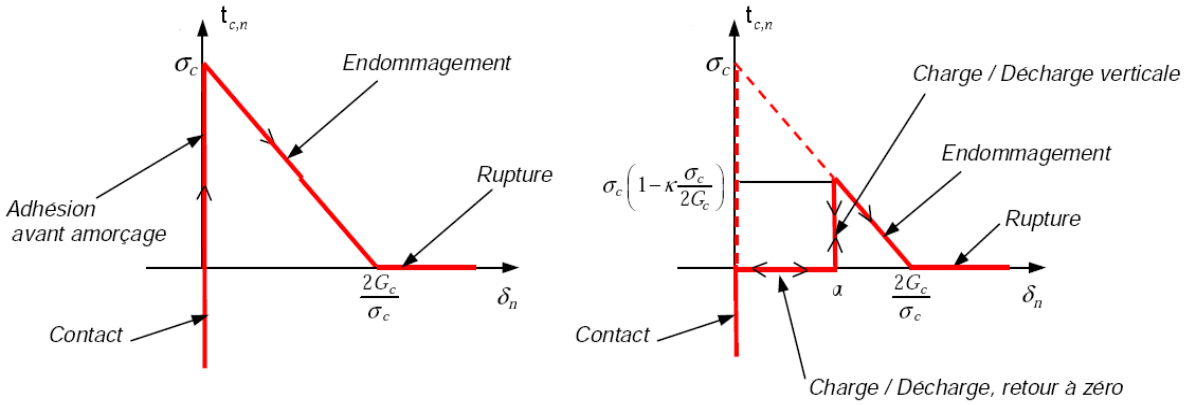


Figure 2.4.2-1: Normal component of the vector forced according to the normal jump for model CZM_OUV_MIX (threshold α no one on the left and positive on the right).

3 Mixed variational formulation

Let us transform the strong form of the problem into a weak formulation, adapted better to the finite elements. The field u must belong to all V_0 the kinematically admissible fields of displacements:

$$V_0=\left\{v\in H^1, v \text{ discontinu à travers } \Gamma_c, v=0 \text{ sur } \Gamma_u\right\}$$

Let us start then by giving a unified formulation common to the cases of contact-friction and the regularized cohesive **models**. The mixed cohesive models obey when them with an energy formulation according to a different logic, explained partly 13.

3.1 Formalism common to cohesive regularized and contact-friction

With this intention we will note $\boldsymbol{r}=\boldsymbol{t}_c$ in the case as of regularized models. We note $H=H^{-1/2}(\Gamma)$ for the cohesive models, and we in the case of indicate H formula the subspace $H^{-1/2}(\Gamma)$ formula of the sphere of activities of contact contact-friction. The weak formulation of the problem of rubbing contact is written as follows:
To find $(u, r^1, r^2)\in V_0\times H\times H$ such as:

$$\int_\Omega \sigma(u):\varepsilon(u^*)\,d\Omega=\int_\Omega f\cdot u^*\,d\Omega+\int_{\Gamma_t} t\cdot u^*\,d\Gamma+\int_{\Gamma^1} r^1\cdot u^{*1}\,d\Gamma^1+\int_{\Gamma^2} r^2\cdot u^{*2}\,d\Gamma^2 \quad \forall u^*\in V_0$$

By writing the jump with the following notation

$$[[x]](P^1)=x(P^1)-x(P^2)\ ,$$

and while noting $r=r^1$ and thereafter, the weak formulation of the problem of rubbing contact is written in an equivalent way as follows:

$$\int_\Omega \sigma(u):\varepsilon(u^*)\,d\Omega=\int_\Omega f\cdot u^*\,d\Omega+\int_{\Gamma_t} t\cdot u^*\,d\Gamma+\int_{\Gamma_c} r\cdot[[u^*]]\,d\Gamma_c\quad\forall u^*\in V_0$$

Spaces of the unknowns of contact are the following:

$$H=\left\{\lambda\in H^{-1/2}(\Gamma_c),\lambda\le 0\ \text{ sur }\ \Gamma_c\right\}$$
$$H=\left\{r_\tau\in H^{-1/2}(\Gamma_c),\|r_\tau\|\le\mu\lambda\ \text{ sur }\ \Gamma_c\right\}$$

3.2 Method of Augmented Lagrangian

the weak formulation at three fields is written finally, in the case of an Augmented Lagrangian training:
To find $(u,\lambda,\Lambda)\in V_0\times H\times H$
$\forall(u^*,\lambda^*,\Lambda^*)\in V_0\times H\times H$

Balance equation
$$\int_\Omega \sigma(u):\varepsilon(u^*)\,d\Omega-\int_\Omega f\cdot u^*\,d\Omega-\int_{\Gamma_t} t\cdot u^*\,d\Gamma$$
$$-\int_{\Gamma_c}\chi(g_n)\,g_n n\cdot[[u^*]]\,d\Gamma_c-\int_{\Gamma_c}\chi(g_n)\,\mu\lambda_s P_{B(0,1)}(g_\tau)\cdot[[u_\tau^*]]\,d\Gamma_c=0$$

Model of contact
$$\int_{\Gamma c}\frac{-1}{\rho_n}\left(\lambda-\chi(g_n)g_n\right)\lambda^*\,d\Gamma_c=0$$

Friction law
$$\int_{\Gamma c}\frac{\mu\chi(g_n)\lambda_s\Delta t}{\rho_\tau}\left(\Lambda-P_{B(0,1)}(g_\tau)\right)\Lambda^*\,d\Gamma_c$$
$$+\int_{\Gamma c}\left(1-\chi(g_n)\right)\Lambda\Lambda^*\,d\Gamma_c=0$$

3.3 penalized Method

the weak formulation at three fields is finally written, in the case of a purely penalized training:
To find $(u,\lambda,\Lambda)\in V_0\times H\times H$
$\forall(u^*,\lambda^*,\Lambda^*)\in V_0\times H\times H$

Balance equation
$$\int_\Omega \sigma(u):\varepsilon(u^*)\,d\Omega-\int_\Omega f\cdot u^*\,d\Omega-\int_{\Gamma_t} t\cdot u^*\,d\Gamma$$
$$-\int_{\Gamma_c}\chi(g_n)\lambda\cdot[[u^*]]\cdot n\,d\Gamma_c$$
$$-\int_{\Gamma_c}\chi(g_n)\mu\lambda_s P_{B(0,1)}(\kappa_\tau\nu_\tau)\cdot[[u_\tau^*]]\,d\Gamma_c=0$$

Model of contact
$$\int_{\Gamma c}\frac{-1}{\kappa_n}\left(\lambda+\chi(g_n)\kappa_n d_n\right)\lambda^*\,d\Gamma_c=0$$

Friction law
$$\int_{\Gamma c}\frac{\mu\chi(g_n)\lambda_s}{\kappa_\tau}\left(\Lambda-P_{B(0,1)}(\kappa_\tau\nu_\tau)\right)\Lambda^*\,d\Gamma_c$$
$$+\int_{\Gamma c}\left(1-\chi(g_n)\right)\Lambda\Lambda^*\,d\Gamma_c=0$$

One will notice that the value of λ obtained by the model of contact is an average when the state is contacting forces of interpenetration for the penalized model. The use of the statements λ or $-\kappa_n d_n$ in the balance equation should thus lead to same result if it is not that condition LBB applies only to λ . To have equivalence, it would thus be necessary to defer the processing of the LBB on the

fields of displacement for the penalized method. As that is not done here, one uses the term λ for which processing LBB is made and one reinjects it in the balance equation which thus takes into account this processing. One could try the same processing for the friction law but Λ is obtained like an average for slipping or adherent situations. To reinject this state realised on the level of the balance equation involves nona convergence of the algorithm of Newton. The difference in behavior for λ and Λ comes owing to the fact that one or not integrates discontinuous quantities according to the state contacting contacting for λ and following the state slipping or adherent for Λ , but that for the state not contacting the contributions on λ are null in the model of contact. One can deduce from it that if there exists a formulation penalized satisfying fully conditions LBB for the contact, there does not exist for the moment not formulation penalized satisfying fully conditions LBB for friction. The choice of coeffcient κ_τ in will be all the more important as a result (of strong values being likely to display blockings for the adherent part).

3.4 Formulation for a regularized cohesive model

When a cohesive model is used , the contact is managed by the coefficient of penalization defined in the cohesive model. Knowing the form of the cohesive model according to $[\![u]\!]$, the formulation is written:

To find $(u,\lambda,\Lambda)\in V_0\times H\times H$ such as:

$\forall(u^*,\lambda^*,\Lambda^*)\in V_0\times H\times H$

Normal
$$\int_\Omega \sigma(u):\varepsilon(u^*)\,d\Omega-\int_\Omega f\cdot u^*\,d\Omega-\int_{\Gamma_t} t\cdot u^*\,d\Gamma$$
$$-\int_{\Gamma_c} t_{c,n}\cdot[\![u^*]\!]_n\,d\Gamma_c-\int_{\Gamma_c} t_{c,\tau}\cdot[\![u^*]\!]_\tau\,d\Gamma_c=0$$

balance equation Postprocessing left
$$\int_{\Gamma c}\left(\lambda-t_{c,n}\cdot n\right)\lambda^*\,d\Gamma_c=0$$

Postprocessing left tangential
$$\int_{\Gamma c}\left(\Lambda-t_{c,\tau}\right)\Lambda^*\,d\Gamma_c=0$$

One notices that the multipliers λ and Λ do not intervene in the resolution. They are only used to store the cohesive stresses in an explicit way.

3.5 Formulation for a mixed cohesive model

P rear opposition to the preceding formulation, the processing of such a model will require a true formulation at several fields, in the meaning where a vectorial dual field $\boldsymbol{\lambda}$ indeed will enter the formulation, instead of being an artifice of postprocessing as into 13 . This formulation follows an energy reasoning, explained in detail in documentation [R3.06.13]. Let us summarize in the principal points:

It is written that the opening of crack costs an energy proportional to surface to be opened, that is to say :

$$E_{\text{fr}}(\boldsymbol{\delta})=\int_\Gamma \Pi(\boldsymbol{\delta})\,dS$$

where $\Pi(\boldsymbol{\delta})$ is the density of cohesive energy. For model CZM_OUV_MIX , we have for example $\Pi(\boldsymbol{\delta})=\int_0^{\delta_n} t_{c,n}(\delta')\,d\delta'$.

The field of discontinuity $\boldsymbol{\delta}$ appearing in the preceding statements is then defined like a field except for whole, integrated in the formulation as a new unknown . Total energy is written then:

$$E(\boldsymbol{u},\boldsymbol{\delta})=\int_{\Omega\setminus\Gamma}\Phi(\boldsymbol{\varepsilon}(\boldsymbol{u}))\,d\Omega-W_{ext}(\boldsymbol{u})+\int_\Gamma \Pi(\boldsymbol{\delta})\,d\Gamma$$

the solution of the problem consists then of the minimization of this total energy under the stress which δ corresponds, with our sign conventions, contrary to the jump of displacement. One seeks:

$$\min_{\substack{u,\delta \\ [\![\boldsymbol{u}]\!]=-\boldsymbol{\delta}}} E(\boldsymbol{u},\boldsymbol{\delta})$$

In order to solve this one, we introduce the Lagrangian associated one with the problem, to which we add a term of increase whose utility will appear thereafter:

$$L_r(\boldsymbol{u},\boldsymbol{\delta},\boldsymbol{\lambda}) \underset{déf.}{=} E(\boldsymbol{u},\boldsymbol{\delta}) + \int_\Gamma \boldsymbol{\lambda}\cdot(-[\![\boldsymbol{u}]\!]-\boldsymbol{\delta})\,d\Gamma + \int_\Gamma \frac{r}{2}([\![\boldsymbol{u}]\!]+\boldsymbol{\delta})^2\,d\Gamma$$

We can then write the first condition of optimality of this Lagrangian:

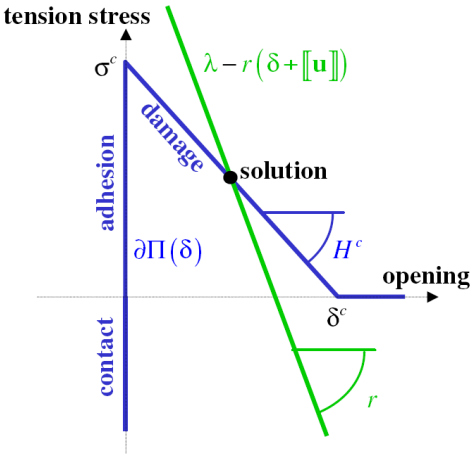
$$\forall\ \delta^* \quad \int_\Gamma \left[t-\lambda+r\,([\![u]\!]+\delta)\right]\cdot\delta^* = 0 \quad \text{avec} \quad t\in\partial\Pi(\delta)$$

This equation utilized the cohesive stress t_c . However we have like statement of t_c only one local constitutive law. This first equation thus must, to have meaning, to be discretized in a way which allows s.e to bring back to a local statement . This is possible if δ is discretized by collocation with Gauss points interface, coordinates x_g .

Indeed, has vec such a discretization , the resolution of the first conditions of optimality amounts satisfying the cohesive model in each collocation point:

$$t_c(\delta_g,\alpha_g) = \lambda_g - r([\![u_g]\!]+\delta_g)$$

where we noted $\lambda_g=\lambda(x_g)$, for example, the values of a field to Gauss points, and where $t_c(\delta_g,\alpha_g)$ follows model 2.4.2-1 . The graphic translation of this constitutive law is the following one: the solution corresponds to the intersection of the linear function $\delta\to\lambda_g-r[\![u_g]\!]-r\,\delta$ (with a negative slope given by the coefficient of penalty r) with the graph $t_c(\delta,\alpha)$. We see whereas for r rather large, the solution is single, from where the interest to have increased the Lagrangian one.



Appear 3.5-a : Solution of the integration of the behavior.

The field δ is thus written locally like a function of $\lambda-r[\![u]\!]$, which we will call increased multiplier and will note p . Consequently, it disappears from the unknown fields of the problem. The formulation is then given by the two conditions of optimality of Lagrangian remaining:

To find $(u,\lambda)\in V_0\times H$ such as:

$\forall(u^*,\lambda^*)\in V_0\times H$

Balance equation
$$\int_\Omega \sigma(u):\varepsilon(u^*)\,d\Omega-\int_\Omega f\cdot u^*\,d\Omega-\int_{\Gamma_t} t\cdot u^*\,d\Gamma + \int_{\Gamma_c}\left[-\lambda+r\left(\llbracket u\rrbracket+\delta(p)\right)\right]\cdot\llbracket u^*\rrbracket\,d\Gamma=0 \quad \text{avec} \quad p=\lambda-r\llbracket u\rrbracket$$

Model of interface
$$-\int_{\Gamma_c}\left[\llbracket u\rrbracket+\delta(p)\right]\cdot\lambda^*\,d\Gamma=0$$

With regard to the discretization of these two fields of unknowns , a simple observing stability condition inf-sup , and consistent discretization with the discretization of δ per collocation, consists in discretizing displacement with elements P2 and the multiplier in a way P1 adapted to X-FEM, of which we give the detail in part 15 to follow .

4 Discretizations EF

One details in this part the discretization of the unknowns of contact-friction, which is done with the nodes tops of the element relative.

4.1 Multipliers of contact

the unknowns for the contact pressure λ and the semi-multiplier of friction Λ are carried to the nodes tops of the element relative. The approximation of the contact pressure utilizes them ϕ_i , shape function of the linear element relative and is written:

$$\lambda^h(x)=\sum_{i=1}^{nno}\lambda_i\phi_i(x)$$

where nno is the number of nodes of the linear element relative. The pressure is then defined as the trace on the approximated interface of this field λ^h .

To clarify this notion of approximated interface, let us suppose that one cuts out, if need, the element relative in simpliciaux subelements (i.e. triangles in 2D, tetrahedrons in 3D). By approximated interface, we understand in 2D broken line connecting between them the points of intersection of the edges of such subelements with line of crack. In 3D, the points of intersection of the edges of such subelements with the surface of crack define a polygon, not necessarily plane within this subelement: there can for example be 4 points of intersection, not inevitably coplanar. The adopted method was that which consists in cutting out this polygon in triangular facets whose tops are these points of intersection ([fig. 4.3.3-1]). The group of the process describes in this paragraph is partly detailed 16.

4.2 Semi-multipliers of friction

Just as for the multipliers of contact, the semi-multipliers of friction are interpolated with ϕ_i the shape function of the element relative. In 3D, Λ is a vector of the tangent plane on the surface of crack. The gradients of the level sets make it possible to define a base covariante in the surface of the crack, in which Λ will be expressed. One defines the 2 vectors of the base covariante by:

$$\left(n^{ls},\tau^1,\tau^2\right)=\left(\nabla lsn,\nabla lst,\nabla lsn\times\nabla lst\right)$$

where n^{ls} is the local norm resulting from the gradient of the level set norm. The vectors τ^1,τ^2 resulting from the gradients (nodal) of the level sets, they can be interpolated within the elements so as to obtain vectors at the tops i of the facets of contacts, that is to say τ_i^1 and τ_i^2 $i=1,3$. The approximation of the semi-multipliers of friction on a facet of contact is written then:

$$\Lambda^h(x)=\sum_{i=1}^{nno}\left(\Lambda_i^1\tau_i^1+\Lambda_i^2\tau_i^2\right)\phi_i(x)\ ,$$

where the vectors τ_i^1,τ_i^2 for a node I correspond to those of the point of intersection of the interface with the edge of the element associated with node I. The associated point of intersection is this node (if it is about a node by where the interface of contact passes), that is to say the intersected edge

containing this node (if it is about a cut edge). If the node belongs to several intersected edges, one associates it with the point of intersection of the vital edge.

4.3 Finite element of contact

4.3.1 general Case

the degrees of freedom of contact are carried exclusively by the nodes tops.

By way of an example, figure 4.3.1 shows the bearing nodes the degrees of freedom of contact. It is noticed that one distinguishes on the one hand the elements X-FEM crossed by crack which will carry degrees of freedom of contact and on the other hand the elements X-FEM not crossed which do not need degrees of freedom of contact. One introduces degrees of freedom of contact only on the nodes tops is on the whole 4 degrees of freedom. Two relations of equalities connect respectively nodes 1 and 3 as well as nodes 2 and 4 in order to satisfy condition LBB (see [§66]). That makes a total of 6 introduced variables.

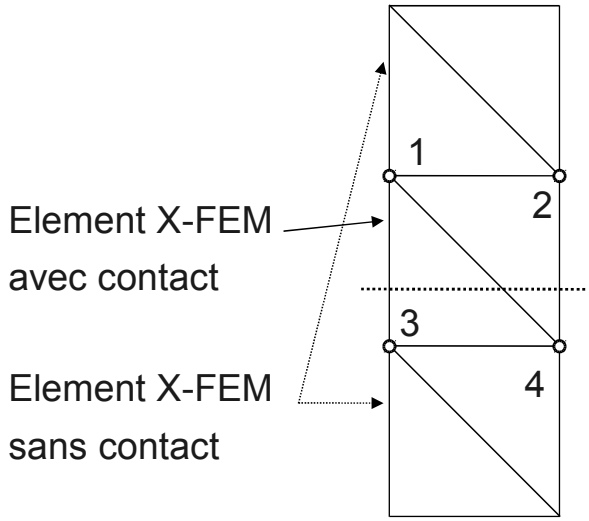


Figure 4.3.1-1 : Nodes carrying the d.o.f. of contact.

FACET CONTACT
contact surface is used only with ends as integration but it requires the installation of algorithm of search of the points of intersection and the points mediums if the element is quadratic.

The algorithm of search of the points of intersection is presented in the following way:
On an element:

•buckle on the edges of the element

are E^1 and the E^2 two ends of the edge

however $lsn\left(E^1\right)lsn\left(E^2\right)\leq 0$ then

•buckles on the two ends

if $lsn\left(E^j\right)=0$ and $lst\left(E^j\right)\leq 0$ then
one adds the point E^j to the list of P_i (with checking of duplicates)
fine if
quadratic element 2D: $lsn\left(E^3\right)=0$ and $lst\left(E^3\right)\leq 0$ then
one adds the point E^3 to the list of P_i (with checking of duplicates)

fine so

•fine buckles

if $lsn(E^k) \neq 0 \ \forall k \in$ many nodes of the edge, then
interpolation of the coordinates of C
so $lst(C) \leq 0$ then
one adds the point C to the list of P_i (with checking of duplicates)
fine so

fine so

fine so

•fine C

buckle Details of the interpolation of the coordinates of the point:

If the element is linear:

$$s = \frac{lsn(E^1)}{lsn(E^1) - lsn(E^2)}$$
$$C = E^1 - s(E^2 - E^1)$$
$$lst(C) = lst(E^1) - s(lst(E^2) - lst(E^1))$$

If the element is quadratic 2D:
The position of the point on the edge informs about the value of one of its coordinates of reference ξ or η, it is then enough to solve the polynomial equation $\sum_{i=1}^{nno} \Phi_i(\xi, \eta)\, lsn_i = 0$ to find the value of the second coordinates of reference. By transition in the real element, one determines his real coordinates (x, y).
end if

If the element is quadratic, it is necessary, besides the points of intersection to interpolate the coordinates of the nodes mediums of the facet of contact. In 2D, the facet of contact is connected with an arc of circle where one knows only the coordinates of his ends following the preceding algorithm. To determine his point medium, one uses the method of Newton which evaluates the point of intersection between the mediator of the segment connecting the two ends and the izo- zero of the level set norm.

4.3.2 Case of the elements in crack tip

For an element containing the crack tip, one needs a particular processing to determine the points of contact. Indeed, such elements are not entirely cut by crack. The points of contact are then of two types:

- either of the intersections between the surface of crack and the edges of the element (general case evoked in the preceding paragraph),
- or of the intersections between the crack tip and the sides of the element (case specific to the elements containing the crack tip).

The points of the 1st type are determined by the preceding algorithm, and the points of the standard 2nd by the algorithm of search of the points of the crack tip (see the paragraph [§2.4] in [R7.02.12]).

The points of contact of the standard 2nd are associated with no node nor edge, and are thus carried natively by no d.o.f. They thus do not intervene in the writing of the approximation. This situation

corresponds to an approximation P1 of the unknowns of contact on the facet of contact out of bottom, having a value zero in crack tip (see diagram from the top of Figure 4.3.2-1). Another solution is to consider a constant connection. In this case, the pressure in crack tip is considered equal to the pressure of the point of contact on an edge nearest. It is this solution which was adopted. In the example of Figure 4.3.2-1, the diagram of bottom illustrates the approximation of the contact pressure on the facet ABC :

$$\begin{aligned}\lambda(x) &= \lambda_A\psi_A+\lambda_B\psi_B+\lambda_C\psi_C \\ &= \lambda_D\psi_A+\lambda_C\psi_B+\lambda_C\psi_C\end{aligned}$$

Since $BC<BD$ and $AD<AC$, the unknown of pressure B is carried from there by the point C and the unknown of pressure A is carried from there by the point D , it thus does not have there an additional d.o.f. on the sides. This approach could be qualified “P0-P1”, because the approximation is a mixture of P0 and P1. The contact pressure is P1 along the crack tip, and P0 along the others with dimensions of the facet.

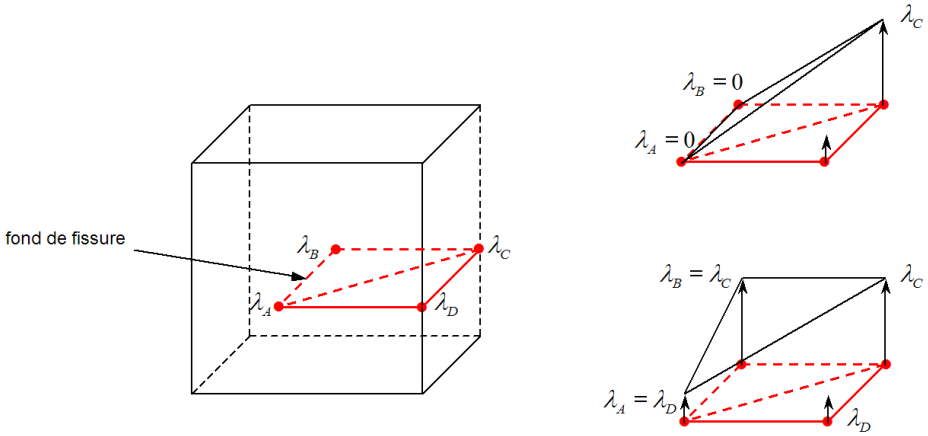


Figure 4.3.2-1 : Various approximations of the contact on the facet in crack tip

Another alternative would be to consider a true approximation P1 on the facets of contact in crack tip. For that, it would be necessary that the degrees of freedom of contact on the points of the bottom are true independent degrees of freedom. They could be for example carried by the nodes tops of the opposite edges. On the example of Figure 4.3.2-2, the contact pressure B would be carried from there by the node F and the contact pressure in A by the node E . This case is generalizable with any type of configuration. The interest is to have degrees of freedom of independent contact. Such an approximation P1 would improve the accuracy compared to a constant approximation.

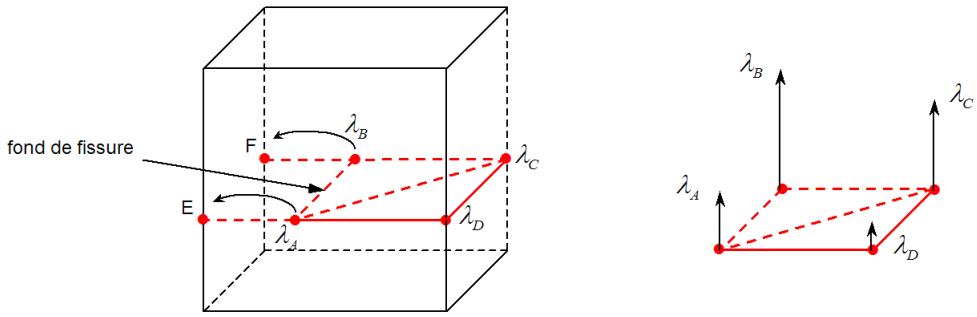


Figure 4.3.2-2 : Approximation P1 on the facet of contact in crack tip

4.3.3 Under-cutting in triangular facets of contact

For each element, for the list of the points of contact P_i, it is necessary to create a under-cutting in triangular facets. For that, one sorts the points of contact per element with the same process as that being used to direct the points of the bottom fissures (see the paragraph [§2.4] in [R7.02.12]). One determines n the average of norms at nodes (based on the gradients of the level sets). One determines G the barycenter of the points of contact on this element. One projects the points of contact in the plane of norm n and passer by par. G For each one of projected, one determines the angle θ compared to the 1st point of the list, then one sorts the points of the list according to θ crescent.

To illustrate under-cutting in triangular facets, let us take a hexahedron occupying the region $[0,1]^3$. That is to say the plane of Cartesian equation $4x-11y-9z+d=0$. Let us examine the intersections between this plane and the hexahedron, for various values of the parameter d.

For $d=-1$, it 3 points of intersection enters there the plane and the hexahedron. The trace of the plane in the hexahedron is a triangle, which corresponds to the facet of contact. For $d=4$, it 4 points of intersection enters there the plane and the hexahedron. The trace of the plane in the hexahedron is a quadrangle, is cut out in two triangular facets of contact. For $d=6$, it 5 points of intersection enters there the plane and the hexahedron. The trace of the plane in the hexahedron is a pentagon, is cut out in three triangular facets of contact. For $d=8$, it 6 points of intersection enters there the plane and the hexahedron. The trace of the plane in the hexahedron is a hexagon, is cut out in four triangular facets of contact. Figure 4.3.3-1 presents the various diagrams for the values of d previously evoked. Moreover, when the crack tip is contained in an element, it may be that adds a point of intersection moreover. For example for $d=8$, if the crack tip cuts segments P1P2 and P2P3 then that adds a P2a point located on P1P2 and another P2b point located on P2 P3 and removes the point P2 (2 added points and 1 removed point, i.e. a total of 1 added point). The maximum number of points of intersection east then 7.

This example illustrates the various cases which can occur. In a general way, one can gather the various cases according to the number of points of intersection met. Tableau 4.3.3-1 gathers the under-cuttings carried out according to number of points of intersection found between the element (that the element is a tetrahedron, a pentahedron or a hexahedron) and surfaces it crack.

	3 points of intersection	4 points of intersection	5 points of intersection	6 points of intersection	7 points of intersection
triangular Facets	P1 P2 P3	P1 P2 P3 P1 P3 P4	P1 P2 P3 P1 P3 P4 P1 P4 P5	P1 P2 P3 P1 P3 P5 P1 P5 P6 P3 P4 P5	P1 P2 P3 P1 P3 P5 P3 P4 P5 P1 P5 P7 P5 P6 P7

Tableau 4.3.3-1 : Cutting in triangular facets

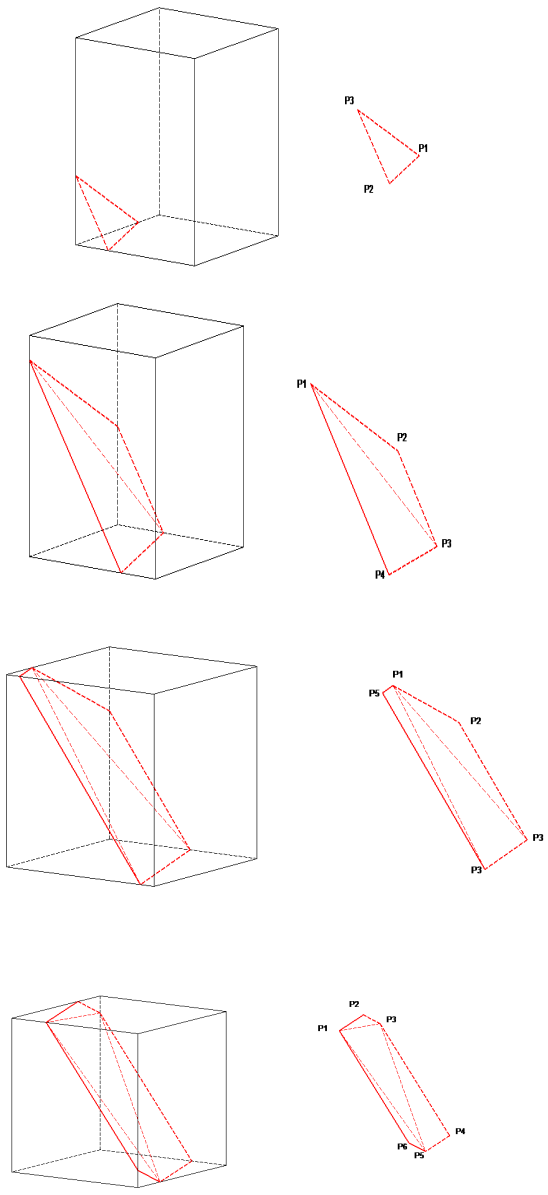


Figure 4.3.3-1 : Intersections and cuttings for D = -1,4,6 and 8 (from top to bottom)

4.4 Zero setting of the inactive degrees of freedom

This procedure is used to put at zero the value of the degrees of freedom of which do not intervene in the equations. Several solutions can be considered.

Note:

> *The discussion which follows is strongly influenced by the possibilities (and the restrictions) of Code_Aster; the solutions considered could not then be exhaustive.*

4.4.1 Not to introduce

First of all, couldn't one simply avoid introducing these degrees of freedom which for nothing and which are not used will have to be cancelled thereafter? One could for example initially imagine an element without degree of freedom of contact, and once the intersections between the edges of the determined element and crack, to add the degrees of freedom of contact necessary. That would like to say some

nodes to transform the finite element into finite element having of the degrees of freedom of contact in precise (for example with the nodes $N1\ \ B\ \ C$, $N4$ of the fig. 4.3.3-1). One easily foresees the number of possibilities of that represents, and the number of different finite elements that would imply (21 for a tetrahedron with 3 or 4 150 and points of intersection, more than for a hexahedron with 3,4,5 or 6 points of intersection). This solution is thus isolated. One is thus well obliged to have an element of contact with degrees of freedom of contact on all the top nodes and medium.

4.4.2 Elimination

to eliminate the degrees of freedom from contact which are not used for nothing, the simplest idea is that which consists in withdrawing total system of equations and the lines corresponding columns (on paper, that amounts “striping” the lines and the useless columns). The system obtained is thus size lower than the total system, and of the same size than that which one would have obtained with the method of the preceding paragraph. However, the fact of withdrawing total system of the lines and columns is not possible at the present time in *Code_Aster*.

4.4.3 Substitution

If one cannot eliminate from the degrees of freedom, value 0 should be affected to them. For that one can decide to modify the tangent matrix and the second member, by putting an unspecified actual value k_0 (for example 1) on the diagonal from the matrix and 0 in the second member, with the position corresponding to the degree of freedom to be cancelled. There is thus of an the same system cuts, but numerically badly-conditioned because of the unspecified value selected on the diagonal. Indeed, on the level of the computation of the elementary matrix, one does not know the stiffness matrix overall, therefore the optimal value of the parameter k_0 (in term of conditioning) is not known.

4.4.4 Dualisation

to mitigate this kind of problem, key word DDL_IMPO of operator AFFE_CHAR_MECA makes it possible to impose on a degree of freedom of a node a preset value. To solve the linear system under stresses thus obtained, the technique of double the Lagrange multipliers is used biberon542, which allows a better conditioning than with the simplistic technique of the preceding paragraph, because the choice of the parameters k_0 is carried out knowing the assembled matrix. The principal disadvantage is that two additional equations are added with the imposed system for each d.o.f.

4.4.5 Generalized substitution

the method by substitution is generalized with the imposition of d.o.f. to an unspecified value (other that 0) in *Code_Aster* (operator AFFE_CHAR_CINE). However, this operator does not function at the present time with the nonlinear operator of static (STAT_NON_LINE) used to solve the system (the non-linearity of with the dealt problem is due to contact-friction).

4.4.6 Selected solution

elimination is useful for meshes bilinear and the crack tips. It by means of is done the method by substitution: however this choice must be the object of a follow-up, even of a study thorough of the robustness, because there are possible impacts on the stability of convergence. For the rest, namely the relations of equality, the solution which was adopted is that of the double Lagrange multipliers. Let us note that with the use of Mumps as solver, only one multiplying is taken into account. However, one is return-account that the argument on the bad conditioning which led us not to choose the method by substitution does not hold. Admittedly, the matrix can be badly-conditioned, but that does not have a consequence on the quality of the results because the matrix is diagonal per block (for example, the diagonal matrix $diag(1,10.e16)$ is badly conditioned but does not pose a problem). One thus thinks of using in the future the method by substitution (either while putting at the hand of the 1 on the diagonal, or by means of AFFE_CHAR_CINE when it is functional , which is equivalent) instead of the dualisation.

Note:

The cancellation of the Heaviside degrees of freedom and ace-tip nouveau riches in excess is made by means of the method of substitution. Indeed for problems where the mesh is refined in the zone of crack, the number of equations added to cancel them in the case of the choice of the dualisation would generate considerable additional computing times.

4.5 Computation of the norm to the facet at the points of integration

As long as the fields of the level sets are interpolated by linear shape functions, one can admit a single n norm on the facet of contact, from the cross product on the sides of this facet. When one goes up in order, the crack is nonplane and it is necessary to consider a new norm in each point of integration. This one is resulting from the gradient of the level set norm, which results from the approximation to the nodes of the facet of the gradients to the nodes. The gradients with the nodes are themselves resulting from an average to the nodes of the elementary gradients of the elements connected to the node.

4.6 Conditioning for the penalized method

a good conditioning of the balance equation of the penalized formulation imposes a “beach advised” for the definition of the coefficients of penalization, which is left at the discretion of the user. We have:

$K_{\text{méca}} \sim E\,h$ et $A_u \sim \kappa_n h^2$, which imposes κ_n reasonable in front $\frac{E}{h}$.

$K_{\text{méca}} \sim E\,h$ et $B_u \sim \sigma\,\mu\,\kappa_\tau h^2$, which imposes κ_τ reasonable in front $\frac{E}{\mu\,\sigma\,h}$.

In the tests, one takes $\kappa_n \sim 10^5 \frac{E}{h}$ and $\kappa_\tau \sim 10^5 \frac{E}{\mu\,\sigma\,h}$.

5 Strategy of resolution

the strategy of resolution is the same one as that used by the continuous method in the frame finite elements classic [biberon142]. The only difference is that with X-FEM, no pairing is not necessary.

5.1 Algorithm of Model

5.1.1 resolution of contact-friction

With X-FEM, the points in opposite are known *a priori* and this intrinsic “pairing” does not evolve during computation (assumption of small displacements). Thus, it is not necessary to carry out a phase of pairing as in the classical frame. The geometrical loop is also removed since there is no reactualization. For each time step, there remain 3 overlapping loops. The loop on the thresholds of friction makes it possible to solve the problem of friction by a fixed search for point on the thresholds of friction of Tresca. The loop on the statutes of contact (related with the method of the active stresses) makes it possible to determine the space of the effective points of contact. At the major level, the loop on the iterations of Newton makes it possible to solve remaining non-linearity, that due to projection on the ball unit.

For one time step:

Initialization of the thresholds of friction λ_s .

•Buckle on the thresholds of frottementInitialisation

of the statutes of contact χ

•Buckles on the statutes of contact

•Iterations of Newton

Computation of the tangent matrix and the second Fine

•member of the iterations of Newton

Actualization of the statutes of Fine χ

•contact of the loop of the active stresses

Actualization of the thresholds of Fine λ_s

•friction of the loop on the thresholds of friction

5.1.2 cohesive Model.

In the implementation of the cohesive models, as well mixed as regularized, we do not introduce fields of statutes as we had been able to do it as for the contact: the various modes are managed by the routine of behavior itself, directly in the method of Newton. The only operation to be realized besides the iterations of Newton is thus the actualization of the local variable.

For one time step:

•Iterations of Newton

Computation of the tangent matrix and the second Fine

•member of the iterations of Newton

Actualization of the local variable α

One could legitimately wonder why the local variable is not brought up to date during iterations of Newton. In fact, as it is about a parameter measuring the irreversibility, and determined by a maximum in the course of time, he should be updated only with each time step converged. Indeed, in the contrary case, if this parameter exceeds its value of equilibrium during an iteration of Newton, the algorithm of Newton will be then unable to decrease it to find the value of equilibrium.

5.2 Stopping criteria of the loop on the statutes of lack of contact

contact For the supposed points $(\chi=0)$, one check the condition of noninterpenetration $(d_n \leq 0)$. The test is the following: if there is interpenetration $(d_n > 0)$ then the point is supposed contacting $(\chi=1)$ at the time of the iteration of active stresses following. Numerically, the test is written $d_n > 10^{-16}$.

For the supposed points contacting $(\chi=1)$, one checks that the value of the reaction of contact following the norm is negative $(\lambda_n \leq 0)$. The test is the following: if there is separation $(\lambda_n > 0)$ then the point is supposed lack of contact $(\chi=0)$ at the time of the iteration of active stresses following. Numerically, the test is written $\lambda_n > -10^{-3}$.

Note:

> *The statute of contact being defined independently for each point of integration of each facet of contact, the tests of stop are carried out of each one of these points.*

Note:

Clearance in these points of integration is calculated thanks to the interpolation of the field of displacement on the element relative (3D) whereas the reaction of contact is calculated thanks to the interpolation of the ddls of contact on the facet of contact (2D). One could also calculate clearance by interpolation of displacement at the tops of the facet of contact (this last being given with the shape functions of the element relative).

Note:

The numerical values of the 2 tests of stops are delicate to determine. In certain configurations, oscillations of statute of contact appear and prevent the convergence of the algorithm. This phenomenon should be identified, and if the values in the 2 cases (contacting and lack of contact) are close, one could consider that the convergence of the loop of active stresses is reached.

5.3 Stopping criteria of the loop on the thresholds of friction

One consider that the loop on the thresholds of friction converged if the solution does not change too much from one iteration to another. More precisely, that is to say v the mixed vector solution (displacement, multipliers of contact, semi-multipliers of friction), one converged with the iteration i if:

$$\frac{\max|v^i - v^{i-1}|}{\max|v^{i-1}|} < 10^{-3}$$

where $\max(u)$ the max means on all the components of the vector u .

5.4 Writing of the formulation during an iteration of Newton

during Let us rewrite the weak formulation at three fields described in the paragraph [3§3] an iteration of Newton. It is necessary to take account of the loop on the thresholds of friction, of that on the active stresses. Thus, on this level, the thresholds of friction become noted constants λ_s , the statutes of contact $\chi(g_n)$ become constants χ . Moreover, the problem remaining being nonlinear (because of projection on the ball unit), it is linearized by the method of Newton-Raphson.

In the unidimensional case, the method of Newton is an iterative process making it possible to approach the zeros of a continuous and differentiable function. One is reduced to the resolution of $F(x)=0$. One builds a succession of points x^k by doing one develop of Taylor of F in the vicinity of x^k , which gives to the first order:

$$F(x^{k+1}) \approx F(x^k) + F'(x^k)(x^{k+1} - x^k)$$

By noting $\delta x^k = x^{k+1} - x^k$ the increment between two successive iterations, the equation linearized with the iteration $k+1$ is then the following one:

$$F'(x^k)\,\delta x^k = -F(x^k)$$

In the case of the finite element method, $F'(x^k)$ are connected with the tangent matrix, which can be calculated with each iteration so necessary, δx^k is the vector of the increments of the nodal unknowns, and $F(x^k)$ is the second member. It is noted that $F'(x^k)$ and $F(x^k)$ only quantities of the iteration utilize k , which are thus known quantities.

Projection on the ball unit is written:

$$P_{B(0,1)}(x) = \begin{cases} x & \text{si } x \in B(0,1) \\ \dfrac{x}{\|x\|} & \text{sinon} \end{cases}$$

The differential of this application, in any point not located on edge of $B(0,1)$, is defined by:

$$\partial_x P_{B(0,1)}(x)\delta x = K(x)\delta x$$

with

$$K(x)=\begin{cases} I_d & \text{si } x\in B(0,1) \quad (\text{adhérence}) \\ \dfrac{1}{\|x\|}\left(I_d-\dfrac{x\cdot x^T}{\|x\|^2}\right) & \text{sinon} \quad (\text{glissement}) \end{cases}$$

Thus, the differential of $P_{B(0,1)}(g_\tau)$ will be:

$$K(g_\tau)\delta g_\tau = K\left(\Lambda+\frac{\rho_\tau}{\Delta t}\Delta[[u]]_\tau\right)\left(\delta\Lambda+\frac{\rho_\tau}{\Delta t}[[\delta u]]_\tau\right)$$

where g_τ is the semi-multiplier of friction resulting from the preceding iteration of Newton and δg_τ the increment of unknowns. The knowledge of the semi-multiplier of friction resulting from the preceding iteration of Newton makes it possible to know easily if the point is in the state adherent or slipping.

In the same way, in the case of regularized cohesive model CZM_EXP_REG, for example, the differential of $\boldsymbol{t}_c(\llbracket \boldsymbol{u} \rrbracket)$ will be $\dfrac{\partial \boldsymbol{t}_c}{\partial \llbracket \boldsymbol{u} \rrbracket}\cdot\llbracket \delta \boldsymbol{u} \rrbracket$ with:

However $\dfrac{\partial \boldsymbol{t}_c}{\partial \llbracket \boldsymbol{u} \rrbracket}=H(\llbracket u \rrbracket_{\text{èq}}-\alpha)\dfrac{\partial \boldsymbol{\sigma}_{lin}}{\partial \llbracket \boldsymbol{u} \rrbracket}+(1-H(\llbracket u \rrbracket_{\text{èq}}-\alpha))\dfrac{\partial \boldsymbol{\sigma}_{dis}}{\partial \llbracket \boldsymbol{u} \rrbracket}+\dfrac{\partial \boldsymbol{\sigma}_{pen}}{\partial \llbracket \boldsymbol{u} \rrbracket}$

We re-use the statements of these three partial derivatives which are given in [R7.02.11], with $\boldsymbol{\delta}=-\llbracket \boldsymbol{u} \rrbracket$. In the statement of $\boldsymbol{\sigma}_{dis}$, it is necessary to write $\alpha=\llbracket u \rrbracket_{èq}$, which becomes thus a variable to be taken into account in derivative. In practice, in the code, one distinguishes four cases for clearness from reading:

- $\llbracket u \rrbracket_{èq}\geq\alpha$ et $\llbracket u_n \rrbracket<0$ (dissipative lack of contact). We have then:

$$\frac{\partial \boldsymbol{t}_c}{\partial \llbracket \boldsymbol{u} \rrbracket}=-\sigma_c\exp\left(-\frac{\sigma_c}{G_c}\llbracket u \rrbracket_{èq}\right)\left(\frac{\mathbf{Id}}{\llbracket u \rrbracket_{èq}}-\frac{\llbracket \boldsymbol{u} \rrbracket}{\llbracket u \rrbracket_{èq}}\otimes\frac{\llbracket \boldsymbol{u} \rrbracket}{\llbracket u \rrbracket_{èq}}\left(\frac{\sigma_c}{G_c}+\frac{1}{\llbracket u \rrbracket_{èq}}\right)\right)$$

- $\llbracket u \rrbracket_{èq}<\alpha$ et $\llbracket u_n \rrbracket\geq 0$ (elastic contacting). With $(\boldsymbol{\tau}_1,\boldsymbol{\tau}_2)$ a base of the tangent plane, we have:

$$\frac{\partial \boldsymbol{t}_c}{\partial \llbracket \boldsymbol{u} \rrbracket}=\frac{\partial \boldsymbol{\sigma}_{lin}(\llbracket \boldsymbol{u}_\tau \rrbracket)}{\partial \llbracket \boldsymbol{u} \rrbracket}+\frac{\partial \boldsymbol{\sigma}_{pen}(\llbracket \boldsymbol{u}_n \rrbracket)}{\partial \llbracket \boldsymbol{u} \rrbracket}=-C\,\boldsymbol{n}\otimes\boldsymbol{n}-\frac{\sigma_c}{\alpha}\exp\left(\frac{-\sigma_c}{G_c}\alpha\right)(\boldsymbol{\tau}_1\otimes\boldsymbol{\tau}_1+\boldsymbol{\tau}_2\otimes\boldsymbol{\tau}_2)$$

- $\llbracket u \rrbracket_{èq}\geq\alpha$ et $\llbracket u_n \rrbracket\geq 0$ (dissipative contacting). By a similar reasoning by replacing $\boldsymbol{\sigma}_{lin}$ by $\boldsymbol{\sigma}_{dis}$, we obtain:

$$\frac{\partial \boldsymbol{t}_c}{\partial \llbracket \boldsymbol{u} \rrbracket}=-C\,\boldsymbol{n}\otimes\boldsymbol{n}-\exp\left(\frac{-\sigma_c}{G_c}\llbracket u \rrbracket_{èq}\right)\left[\frac{\sigma_c}{\llbracket u \rrbracket_{èq}}(\boldsymbol{\tau_1}\otimes\boldsymbol{\tau_1}+\boldsymbol{\tau_2}\otimes\boldsymbol{\tau_2})-\left(\frac{\sigma_c^2}{G_c}+\frac{\sigma_c}{\llbracket u \rrbracket_{èq}}\right)\frac{\llbracket \boldsymbol{u} \rrbracket_\tau\otimes\llbracket \boldsymbol{u} \rrbracket_\tau}{\llbracket u \rrbracket_{èq}^{\,2}}\right]$$

- $\llbracket u \rrbracket_{èq}<\alpha$ et $\llbracket u_n \rrbracket<0$ (elastic lack of contact) .formule

$$\frac{\partial \boldsymbol{t}_c}{\partial \llbracket \boldsymbol{u} \rrbracket}=-\frac{\sigma_c}{\alpha}\exp\left(\frac{-\sigma_c}{G_c}\alpha\right)\mathbf{Id}$$

5.5 Linearization of the problem

5.5.1 integral Writing with the method of Augmented Lagrangian

the linear system of the three equations to the iteration of Newton $k+1$ is written in the following way (by weighing down the writing, the references to the iteration of Newton are omitted, because in an obvious way, the unknowns are noted with δ a front, and the fields tests from now on are noted with a star):

To find $(\delta u\ ,\ \delta\lambda\ ,\ \delta\Lambda)\in V_0\times H\times H$ such as:
$\forall(u^*,\lambda^*,\Lambda^*)\in V_0\times H\times H$

Balance equation

$$\int_\Omega \sigma(\delta u):\varepsilon(u^*)\,d\Omega$$
$$-\int_{\Gamma_c}\chi\delta\lambda[[u^*]]\cdot n d\Gamma_c+\int_{\Gamma_c}\chi\rho_n[[\delta u]]\cdot n\,[[u^*]]\cdot n d\Gamma_c$$
$$-\int_{\Gamma_c}\chi\mu\lambda_s K(g_\tau)\delta g_\tau[[u^*]]_\tau d\Gamma_c$$
$$=-\int_\Omega\sigma(u):\varepsilon(u^*)d\Omega+\int_\Omega f\cdot u^*\,d\Omega+\int_{\Gamma_t}t\cdot u^*\,d\Gamma_t$$
$$+\int_{\Gamma_c}\chi(\lambda-\rho_n[[u]]\cdot n)[[u^*]]\cdot n d\Gamma_c$$
$$+\int_{\Gamma_c}\chi\mu\lambda_s P_{B(0,1)}(g_\tau)\cdot[[u^*]]_\tau d\Gamma_c$$

Model of contact

$$-\int_{\Gamma c}\frac{(1-\chi)}{\rho_n}\delta\lambda\ \lambda^*d\Gamma_c-\int_{\Gamma c}\chi[[\delta u]]\cdot n\lambda^*d\Gamma_c$$
$$=\int_{\Gamma c}\frac{(1-\chi)}{\rho_n}\lambda\lambda^*d\Gamma_c+\int_{\Gamma c}\chi[[u]]\cdot n\lambda^*d\Gamma_c$$

Friction law

$$\int_{\Gamma c}\frac{\chi\mu\lambda_s\Delta t}{\rho_\tau}\left[\delta\Lambda-K(g_\tau)\delta g_\tau\right]\Lambda^*d\Gamma_c+\int_{\Gamma c}(1-\chi)\delta\Lambda\Lambda^*d\Gamma_c$$
$$=-\int_{\Gamma c}\frac{\chi\mu\lambda_s\Delta t}{\rho_\tau}\left[\Lambda-P_{B(0,1)}(g_\tau)\right]\Lambda^*d\Gamma_c-\int_{\Gamma c}(1-\chi)\Lambda\Lambda^*d\Gamma_c$$

5.5.2 integral Writing with the penalized method

the linear system of the three equations to the iteration of Newton $k+1$ is written in the following way (by weighing down the writing, the references to the iteration of Newton are omitted, because in an obvious way, the unknowns are noted with δ a front, and the fields tests from now on are noted with a star):

To find $(\delta u\ ,\ \delta\lambda\ ,\ \delta\Lambda)\in V_0\times H\times H$ such as:
$\forall(u^*,\lambda^*,\Lambda^*)\in V_0\times H\times H$

Balance equation

$$\int_\Omega \sigma(\delta u):\varepsilon(u^*)\,d\Omega$$
$$-\int_{\Gamma_c}\chi\,\delta\lambda\,[[u^*]]\cdot n d\Gamma_c$$
$$-\int_{\Gamma_c}\chi\mu\lambda_s\kappa_\tau K(\kappa_\tau\nu_\tau)\delta\nu_\tau[[u^*]]_\tau d\Gamma_c$$
$$=-\int_\Omega\sigma(u):\varepsilon(u^*)d\Omega+\int_\Omega f\cdot u^*\,d\Omega+\int_{\Gamma_t}t\cdot u^*\,d\Gamma_t$$
$$+\int_{\Gamma_c}\chi\lambda[[u^*]]\cdot n d\Gamma_c$$
$$+\int_{\Gamma_c}\chi\mu\lambda_s P_{B(0,1)}(\kappa_\tau\nu_\tau)\cdot[[u^*]]_\tau d\Gamma_c$$

Model of contact

$$-\int_{\Gamma c}\frac{(1-\chi)}{\kappa_n}\delta\lambda\ \lambda^*\,d\Gamma_c-\int_{\Gamma c}\chi\left(\frac{\delta\lambda}{\kappa_n}+[[\delta u]]\cdot n\right)\lambda^*\,d\Gamma_c$$
$$=\int_{\Gamma c}\frac{(1-\chi)}{\kappa_n}\lambda\lambda^*\,d\Gamma_c+\int_{\Gamma c}\chi\left(\frac{\lambda}{\kappa_n}+[[u]]\cdot n\right)\lambda^*\,d\Gamma_c$$

Friction law

$$\int_{\Gamma c}\frac{\chi\mu\lambda_s}{\kappa_\tau}\left[\delta\Lambda-K\left(\kappa_\tau\nu_\tau\right)\kappa_\tau\delta\nu_\tau\right]\Lambda^*\,d\Gamma_c+\int_{\Gamma c}(1-\chi)\delta\Lambda\Lambda^*\,d\Gamma_c$$
$$=-\int_{\Gamma c}\frac{\chi\mu\lambda_s}{\kappa_\tau}\left[\Lambda-P_{B(0,1)}\left(\kappa_\tau\nu_\tau\right)\right]\Lambda^*\,d\Gamma_c-\int_{\Gamma c}(1-\chi)\Lambda\Lambda^*\,d\Gamma_c$$

One chose here to solve the problems of contact and implicit friction of way (the semi-multipliers of contact and friction are expressed according to the jump of displacement of the current iteration of Newton). This choice makes the stiffness matrix asymmetric, the block B_r (see [§5.6.15.6.1]) being non-zero because of term in red in the writing of the friction law whereas the block B_r^T is null.

5.5.3 Integral writing for a formulation with regularized cohesive model

the linear system of the three equations to the iteration of Newton $k+1$ is written in the following way (by weighing down the writing, the references to the iteration of Newton are omitted, because in an obvious way, the unknowns are noted with δ in front of, and the fields tests from now on are noted with a star):

To find $(\delta u\ ,\ \delta\lambda\ ,\ \delta\Lambda)\in V_0\times H\times H$ such as:

$\forall(u^*,\lambda^*,\Lambda^*)\in V_0\times H\times H$

Balance equation

$$\int_\Omega\sigma(\delta u):\varepsilon(u^*)\,d\Omega$$
$$-\int_{\Gamma_c}\left[\frac{\partial t_{c,n}}{\partial[[u]]_n}[[\delta u]]_n+\frac{\partial t_{c,n}}{\partial[[u]]_\tau}[[\delta u]]_\tau\right][[u^*]]_n\,d\Gamma_c$$
$$-\int_{\Gamma_c}\left[\frac{\partial t_{c,\tau}}{\partial[[u]]_n}[[\delta u]]_n+\frac{\partial t_{c,\tau}}{\partial[[u]]_\tau}[[\delta u]]_\tau\right][[u^*]]_\tau\,d\Gamma_c$$
$$=-\int_\Omega\sigma(u):\varepsilon(u^*)\,d\Omega+\int_\Omega f\cdot u^*\,d\Omega+\int_{\Gamma_t}t\cdot u^*\,d\Gamma_t$$
$$+\int_{\Gamma_c}\left(t_{c,n}[[u^*]]_n+t_{c,\tau}[[u^*]]_\tau\right)d\Gamma_c$$

Interfaces: normal part

$$\int_{\Gamma c}\lambda^*\left(\lambda+\delta\lambda-t_{c,n}\cdot n\right)d\Gamma_c=0$$

Interfaces: tangential part

$$\int_{\Gamma c}\Lambda^*\left(\Lambda+\delta\Lambda-t_{c,\tau}\right)d\Gamma_c=0$$

5.5.4 integral Writing for a formulation with mixed cohesive model

the linear system of the three equations to the iteration of Newton $k+1$ is written in the following way (by weighing down the writing, the references to the iteration of Newton are omitted, because in an obvious way, the unknowns are noted with δ in front of, and the fields tests from now on are noted with a star):

To find $(\delta u\ ,\ \delta\lambda\ )\in V_0\times H$ such as:

$\forall(u^*,\lambda^*)\in V_0\times H$

Balance equation

$$\int_\Omega \sigma(\delta u):\epsilon(u^*)\,d\,\Omega-\int_\Gamma\left(Id-r\frac{\partial\delta}{\partial p}\right)\cdot\delta\lambda\cdot[\![u^*]\!]\,d\,\Gamma$$
$$+\int_\Gamma r\left(Id-r\frac{\partial\delta}{\partial p}\right)\cdot[\![\delta u]\!]\cdot[\![u^*]\!]d\,\Gamma$$
$$=-\int_\Omega \sigma(u):\epsilon(u^*)\,d\,\Omega+\int_\Omega f\cdot u^*\,d\,\Omega+\int_{\Gamma_t} t\cdot u^*\,d\,\Gamma$$
$$+\int_{\Gamma_c}\left[\lambda-r\left(\ [\![u]\!]+\delta(p)\ \right)\right]\cdot[\![u^*]\!]d\,\Gamma$$

Model of interface

$$-\int_{\Gamma_c}\left(1-r\frac{\partial\delta}{\partial p}\right)\cdot[\![u]\!]\cdot\lambda^*\,d\,\Gamma-\int_\Gamma\frac{\partial\delta}{\partial p}\cdot\delta\lambda\cdot\lambda*\,d\,\Gamma$$
$$=\int_\Gamma([\![u]\!]+\delta)\cdot\lambda^*\,d\,\Gamma$$

5.6 elementary Terms of contact rubbing

5.6.1 matric Writing of the problem linearized

By taking again the notations of biberon142, and by considering the unified writing adopted for the models of contact friction and cohesive regularized, the system linearized such as it is solved with the iteration $k+1$ of Newton can be put in matric form:

Balance equation

$$\{u^*\}[K_{méca}](\delta u)$$
$$+\{u^*\}[A]^T(\delta\lambda)+\{u^*\}[A_u](\delta u)$$
$$+\{u^*\}[B_r]^T(\delta\Lambda)+\{u^*\}[B_u](\delta u)$$
$$+\{u^*\}[D_u](\delta u)$$
$$=\{u^*\}(L^1_{méca})$$
$$+\{u^*\}(L^1_{cont})$$
$$+\{u^*\}(L^1_{frott})$$
$$+\{u^*\}(L^1_{coh})$$

Model of contact

$$\{\lambda^*\}[C](\delta\lambda)+\{\lambda^*\}[A](\delta u)$$
$$=\{\lambda^*\}(L^2_{cont})+\{\lambda^*\}(L^2_{coh})$$

Friction law

$$\{\Lambda^*\}[F_r](\delta\Lambda)+\{\Lambda^*\}[B_r](\delta u)$$
$$=\{\Lambda^*\}(L^3_{frott})+\{\Lambda^*\}(L^3_{coh})$$

where the vectors column are noted (x) and the vectors line $\{x\}=(x)^T$. This system can be put in the following matric form:

$$\begin{bmatrix} K_{méca}+A_u+B_u+D_u & A^T & B_r^T \\ A & C & 0 \\ B_r & 0 & F_r \end{bmatrix}\begin{pmatrix}\delta u\\ \delta\lambda\\ \delta\Lambda\end{pmatrix}=\begin{pmatrix} L^1_{méca}+L^1_{cont}+L^1_{frott}+L^1_{coh}\\ L^2_{cont}+L^2_{coh}\\ L^3_{frott}+L^3_{coh}\end{pmatrix}$$

The unknown is the increment compared to the preceding iteration of Newton. One voluntarily omitted the reference to the number of the iteration of Newton.

Of course, the terms of cohesion D_u , L^1_{coh} , L^2_{coh} and L^3_{coh} appears only in the formulation for regularized cohesive model. If such is the case, all the other terms are null, except for the terms $K_{méca}$ and $L^1_{méca}$ of course, but also except for C , F_r , L^2_{cont} and L^3_{frott} which are used for postprocessing, and which are deduced from the statements detailed to follow while considering $\chi=0$ and $\rho_n=1$.

$K_{méca}$ is the mechanical stiffness matrix defined in the paragraph [§3.2] of [R7.02.12].
A_u is the increased stiffness matrix due to the contact.
B_u is the increased stiffness matrix due to friction.
D_u is the stiffness matrix due to the forces of cohesion.
A is the matrix binding the terms of displacement to those of contact (matrix of the model of contact).
B_r is the matrix binding the terms of displacement to those of friction (matrix of the friction laws). This matrix is noted B in biberon142, but not to confuse it with the matrix of derivatives of the shape functions, we will note it B_r .
C is the matrix allowing to determine contact pressures in the lack of contact case.
F_r is the matrix making it possible to determine the multipliers of friction in the case of NON-rubbing contact.
$L^1_{méca}$ is the second member representing the internal forces and the increments of loadings.
L^1_{cont} and L^2_{cont} are the second members due to the contact.
L^1_{frott} and L^3_{frott} are the second members due to friction.
L^1_{coh} , L^2_{coh} and L^3_{coh} are the second members due to the forces of cohesion.

Note:

> *It is pointed out that the system solved by Code_Aster is not of the type* $[K][U]=[F]$ *but of the type* $[K][U]+[F]=0$ *. There thus exists a minus sign between the second members given in this document and those coded in the files fortrans.*

5.6.2 Form of the elementary matrixes of contact

5.6.2.1 Method of Augmented Lagrangian

Taking into account as of discretizations the fields evoked in the paragraphs [§3.2] of [R7.02.12] and [§4.14.1] of this document, the continuous matric system above is replaced by a discrete system. More precisely, the matrix A has the following form:

$$[A]_{ij} = \begin{array}{c} \begin{matrix} \delta a_j & \delta b_j & \delta c^1_j & \delta c^2_j & \delta c^3_j & \delta c^4_j \end{matrix} \\ \begin{bmatrix} 0 & x & x & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{matrix} \\ \lambda^*_i \end{matrix}$$

Indeed, this is due to the fact that the terms of contact are cancelled for the ddls whose shape function associated is continuous. Indeed,

$$\begin{aligned}
&\left(\lambda^*\right)_i [A]_{ij} (\delta u)_j = -\int_{\Gamma^1} \chi \psi_i \lambda^*_i \phi_j \left(\delta a_j + H\,\delta b_j + F^1 \delta c^1_j + F^2 \delta c^2_j + F^3 \delta c^3_j + F^4 \delta c^4_j\right) \cdot n \, d\Gamma \\
&+ \int_{\Gamma^2} \chi \psi_i \lambda^*_i \phi_j \left(\delta a_j + H\,\delta b_j + F^1 \delta c^1_j + F^2 \delta c^2_j + F^3 \delta c^3_j + F^4 \delta c^4_j\right) \cdot n \, d\Gamma \\
&= -\int_{\Gamma^1} \chi \psi_i \lambda^*_i \phi_j \left(\delta a_j - \delta b_j - \sqrt{r}\,\delta c^1_j\right) \cdot n \, d\Gamma + \int_{\Gamma^2} \chi \psi_i \lambda^*_i \phi_j \left(\delta a_j + \delta b_j + \sqrt{r}\,\delta c^1_j\right) . n \, d\Gamma \\
&= \int_{\Gamma} \chi \psi_i \lambda^*_i \phi_j \left(2\,\delta b_j + 2\sqrt{r}\,\delta c^1_j\right) \cdot n \, d\Gamma
\end{aligned}$$

Here, one clearly sees appearing to the product of the shape functions ψ_i of the triangle with the shape functions ϕ_j of the voluminal element relative.

Note:

One notes following this computation the statement of the jump of displacement according to the degrees of freedom nouveau riches X-FEM:

$$\begin{aligned}[[u]]_j &= \left(a_j + H\,b_j + F^1 c_j^1 + F^2 c_j^2 + F^3 c_j^3 + F^4 c_j^4\right)_1 \\ &\quad - \left(a_j + H\,b_j + F^1 c_j^1 + F^2 c_j^2 + F^3 c_j^3 + F^4 c_j^4\right)_2 \\ &= \left(a_j - b_j - \sqrt{r}\,c_j^1\right) - \left(a_j + b_j + \sqrt{r}\,c_j^1\right) \\ &= -\left(2\,b_j + 2\sqrt{r}\,c_j^1\right)\end{aligned}$$

In the same way, the increased stiffness matrix due to the contact has the following form:

$$[A_u]_{ij} = \begin{array}{c} \begin{array}{cccccc} \delta a_j & \delta b_j & \delta c_j^1 & \delta c_j^2 & \delta c_j^3 & \delta c_j^4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \times & \times & 0 & 0 & 0 \\ 0 & \times & \times & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} a_i^* \\ b_i^* \\ c_i^{1*} \\ c_i^{2*} \\ c_i^{3*} \\ c_i^{4*} \end{array}$$

$$\left\{u^*\right\}_i \left[A_u\right]_{ij} (\delta u)_j = \int_\Gamma \chi \rho_n \phi_i \left\{\left(2\,b_{i^*} + 2\sqrt{r}\,c_i^{1^*}\right)\cdot n\right\} \phi_j \left(2\,\delta b_j + 2\sqrt{r}\,\delta c_j^1\right)\cdot n\, d\Gamma$$

The form of the matrix C does not change compared to the classical case without X-FEM:

$$\left\{\lambda^*\right\}_i [C]_{ij} (\delta\lambda)_j = -\int_\Gamma \frac{1}{\rho_n}(1-\chi)\,\psi_i \lambda_i^* \psi_j\, \delta\lambda_j\, d\Gamma$$

5.6.2.2 Penalized method

the matrix C has as a statement:

$$\left\{\lambda^*\right\}_i [C]_{ij} (\delta\lambda)_j = -\int_\Gamma \frac{1}{\kappa_n} \psi_i \lambda_i^* \psi_j\, \delta\lambda_j\, d\Gamma$$

The matrix A_u is null.

The matrix A has as a statement:

$$\left\{\lambda^*\right\}_i [A]_{ij} (\delta u)_j = \int_\Gamma \chi\, \psi_i \lambda_i^* \phi_j \left(2\,\delta b_j + 2\sqrt{r}\,\delta c_j^1\right)\cdot n\, d\Gamma$$

5.6.2.3 Formulation for regularized cohesive model

a matrix C necessary to the postprocessing is used, whose statement that of Lagrangian is increased where one wrote $\chi = 0$ and $\rho_n = 1$. We obtain as follows:

$$\left\{\lambda^*\right\}_i [C]_{ij} (\delta\lambda)_j = \int_\Gamma \psi_i \lambda_i^* \psi_j\, \delta\lambda_j\, d\Gamma$$

5.6.3 Statement of the second members of contact

5.6.3.1 Method of Augmented Lagrangian

the These statements utilize quantities with the preceding iteration of Newton (iteration $k-1$). Therefore one made appear the reference to the index explicitly $k-1$:

$$\left[u^*\right]_i\left(L^1_{cont}\right)_i=-\int_\Gamma \chi\,\phi_i\left\{\left(2\mathrm{b}_i^*+2\sqrt{r}\,c_i^{1^*}\right)\cdot n\right\}\left(\lambda^{k-1}-\rho_n\,d_n^{k-1}\right)d\Gamma$$

The statement of the vector L^2_{cont} does not change compared to the classical case without X-FEM:

$$\left[\lambda^*\right]_i\left(L^2_{cont}\right)_i=\int_\Gamma \psi_i\,\lambda_i^*\left(\frac{1-\chi}{\rho_n}\,\lambda^{k-1}+\chi d_n^{k-1}\right)d\Gamma$$

5.6.3.2 Penalized method

These statements utilize quantities with the preceding iteration of Newton (iteration $k-1$). Therefore one made appear the reference to the index explicitly $k-1$:

$$\left[u^*\right]_i\left(L^1_{cont}\right)_i=-\int_\Gamma \chi\,\phi_i\left\{\left(2\mathrm{b}_i^*+2\sqrt{r}\,c_i^{1^*}\right)\cdot n\right\}\lambda^{k-1}\,d\Gamma$$

$$\left[\lambda^*\right]_i\left(L^2_{cont}\right)_i=\int_\Gamma \frac{1}{\kappa_n}\psi_i\,\lambda_i^*\,\lambda^{k-1}\,d\Gamma+\int_\Gamma \chi\,\psi_i\,\lambda_i^*\,d_n^{k-1}\,d\Gamma$$

5.6.3.3 Formulation for regularized cohesive model

a vector L^2_{cont} necessary to the postprocessing is used, whose statement that of Lagrangian is increased where one wrote $\chi=0$ and $\rho_n=1$. We obtain as follows:

$$\left[\lambda^*\right]_i\left(L^2_{cont}\right)_i=\int_\Gamma \psi_i\,\lambda_i^*\,\lambda^{k-1}\,d\Gamma$$

5.6.4 Form of the matrixes of friction

5.6.4.1 Method of Augmented Lagrangian

In order to expressing the quantities in the tangent plane, one uses the statement of the paragraph [§2.32.3], which one writes in matric form:

$$u_\tau=\left(\mathrm{Id}-n\otimes n\right)u=\left[P\right]u$$

In this statement, the matrix P appoints the operator of projection as regards norm n . The matrix of this symmetric operator has as a statement:

$$\left[P\right]=\begin{bmatrix}1-n_x^2 & -n_x n_y & -n_x n_z\\ -n_x n_y & 1-n_y^2 & -n_y n_z\\ -n_x n_z & -n_y n_z & 1-n_z^2\end{bmatrix}$$

where n_x n_y , n_z are the coordinates of the norm n as defined in [§4.54.5]. With the choice of a constant norm per facet, this matrix, depending only on the norm, has the same value in each Gauss point and can be calculated only once for each facet.

The increased stiffness matrix due to friction is written in the following way:

$$\left[u^*\right]_i\left[B_u\right]_{ij}\left(\delta u\right)_j=-\int_\Gamma \chi\mu\lambda_s\frac{\rho_\tau}{\Delta t}\,\phi_i\left\{2\mathrm{b}_i^*+2\sqrt{r}\,c_i^{1^*}\right\}\left[P\right]^T\left[K_n\right]\phi_j\left(2\,\delta b_j+2\sqrt{r}\,\delta c_j^1\right)\left[P\right]d\Gamma$$

where the matrix K_n represents the tangent matrix of projection on the ball unit of the semi-multiplier of friction increased with the preceding iteration of Newton: $K_n = K\left(g_\tau\right)$. It is a known matrix.

The matrix B_r has as a statement:

$$\left\{\Lambda^*\right\}_i\left[B_r\right]_{ij}\left(\delta u\right)_j = \int_\Gamma \chi\mu\lambda_s\psi_i\left\{\Lambda_i^{1*}\tau_i^1 + \Lambda_i^{2*}\tau_i^2\right\}\left[K_n\right]\left[P\right]\phi_j\left(2\mathrm{b}_j + 2\sqrt{r}\,c_j^1\right)d\Gamma$$

The matrix F_r has as a statement:

$$\left\{\Lambda^*\right\}_i\left[F_r\right]_{ij}\left(\delta\Lambda\right)_j = \int_\Gamma \frac{\chi\mu\lambda_s\,\Delta t}{\rho_\tau}\psi_i\left\{\Lambda_i^{1*}\tau_i^1 + \Lambda_i^{2*}\tau_i^2\right\}\left[I_d - K_n\right]\psi_j\left(\Lambda_j^1\tau_j^1 + \Lambda_j^2\tau_j^2\right)d\Gamma$$

$$+\int_\Gamma (1-\chi)\psi_i\left\{\Lambda_i^{1*} \quad \Lambda_i^{2*}\right\}\cdot\begin{bmatrix}\tau_i^1\tau_j^1 & \tau_i^1\tau_j^2\\ \tau_i^2\tau_j^1 & \tau_i^2\tau_j^2\end{bmatrix}\cdot\psi_j\begin{pmatrix}\Lambda_j^1\\ \Lambda_j^2\end{pmatrix}d\Gamma$$

5.6.4.2 Penalized method

the matrix F_r has as a statement:

$$\left\{\Lambda^*\right\}_i\left[F_r\right]_{ij}\left(\delta\Lambda\right)_j = \int_\Gamma\left((1-\chi)+\chi\frac{\mu\lambda_s}{\kappa_\tau}\right)\psi_i\left\{\Lambda_i^{1*} \quad \Lambda_i^{2*}\right\}\cdot\begin{bmatrix}\tau_i^1\tau_j^1 & \tau_i^1\tau_j^2\\ \tau_i^2\tau_j^1 & \tau_i^2\tau_j^2\end{bmatrix}\cdot\psi_j\begin{pmatrix}\Lambda_j^1\\ \Lambda_j^2\end{pmatrix}d\Gamma$$

The matrix B_u has as a statement:

$$\left\{u^*\right\}_i\left[B_u\right]_{ij}\left(\delta u\right)_j = -\int_\Gamma \chi\,\mu\lambda_s\frac{\kappa_\tau}{\Delta t}\phi_i\left\{2\mathrm{b}_i^* + 2\sqrt{r}\,c_i^{1^*}\right\}\left[P\right]^T\left[K_n\right]\phi_j\left(2\,\delta b_j + 2\sqrt{r}\,\delta c_j^1\right)\left[P\right]d\Gamma$$

where the matrix K_n represents the tangent matrix of projection on the ball unit of the semi-multiplier of friction increased with the preceding iteration of Newton: $K_n = K\left(\kappa_\tau v_\tau\right)$. It is a known matrix.

The matrix B_r has as a statement:

$$\left\{\Lambda^*\right\}_i\left[B_r\right]_{ij}\left(\delta u\right)_j = \int_\Gamma \chi\mu\lambda_s\psi_i\left\{\Lambda_i^{1*}\tau_i^1 + \Lambda_i^{2*}\tau_i^2\right\}\left[K_n\right]\left[P\right]\phi_j\left(2\mathrm{b}_j + 2\sqrt{r}\,c_j^1\right)d\Gamma$$

In penalized method, the stiffness matrix is not symmetric. One does not have $B_r^T = B_r$ but a matrix null instead of B_r^T .

5.6.4.3 Formulation for regularized cohesive model

an E stamps F_r necessary to postprocessing is used E , whose statement that of Lagrangian is increased where formula $\chi = 0$. We obtain as follows:

$$\left\{\Lambda^*\right\}_i\left[F_r\right]_{ij}\left(\delta\Lambda\right)_j = \int_\Gamma \psi_i\left\{\Lambda_i^{1*} \quad \Lambda_i^{2*}\right\}\cdot\begin{bmatrix}\tau_i^1\tau_j^1 & \tau_i^1\tau_j^2\\ \tau_i^2\tau_j^1 & \tau_i^2\tau_j^2\end{bmatrix}\cdot\psi_j\begin{pmatrix}\Lambda_j^1\\ \Lambda_j^2\end{pmatrix}d\Gamma$$

5.6.5 Statement of the second members of friction

5.6.5.1 Method of Augmented Lagrangian

the second members of friction have the following statements:

$$\left\{u^*\right\}_i\left(L^1_{frott}\right)_i=-\int_\Gamma \chi\mu\lambda_s\phi_i\left\{2\mathrm{b}_i^*+2\sqrt{r}\,c_i^{1*}\right\}[P]^T P_{B(0,1)}\left(g_\tau^{k-1}\right)d\Gamma$$

$$\left\{\Lambda^*\right\}_i\left(L^3_{frott}\right)_i=-\int_\Gamma \frac{\chi\mu\lambda_s\Delta t}{\rho_\tau}\psi_i\left\{\Lambda_i^{1*}\tau_i^1+\Lambda_i^{2*}\tau_i^2\right\}\left(\Lambda^{k-1}-P_{B(0,1)}\left(g_\tau^{k-1}\right)\right)d\Gamma$$

$$-\int_\Gamma(1-\chi)\psi_i\left\{\Lambda_i^{1*}\tau_i^1+\Lambda_i^{2*}\tau_i^2\right\}\Lambda^{k-1}d\Gamma$$

where $k-1$ the index of the preceding iteration of Newton represents.

5.6.5.2 Penalized method

the second members of friction have the following statements:

$$\left\{\Lambda^*\right\}_i\left(L^3_{frott}\right)_i=-\int_\Gamma \psi_i\left\{\Lambda_i^{1*}\tau_i^1+\Lambda_i^{2*}\tau_i^2\right\}\left((1-\chi)\Lambda^{k-1}+\chi\frac{\mu.\lambda_k}{\kappa_\tau}\left(\Lambda^{k-1}-P_{B(0,1)}(\kappa_\tau\nu_\tau^{k-1})\right)\right)d\Gamma$$

$$\left\{u^*\right\}_i\left(L^1_{frott}\right)_i=-\int_\Gamma \chi\,\mu\lambda_s\phi_i\left\{2\mathrm{b}_i^*+2\sqrt{r}\,c_i^{1^*}\right\}\left[P_T\right]P_{B(0,1)}\left(\kappa_\tau\nu_\tau^{k-1}\right)d\Gamma$$

where $k-1$ the index of the preceding iteration of Newton represents.

5.6.5.3 Formulation for cohesive model

a vector L^3_{frott} necessary to the postprocessing is used, whose statement that of Lagrangian is increased where formula $\chi=0$. We obtain as follows:

$$\left\{\Lambda^*\right\}_i\left(L^3_{frott}\right)_i=\int_\Gamma \psi_i\left\{\Lambda_i^{1*}\tau_i^1+\Lambda_i^{2*}\tau_i^2\right\}\Lambda^{k-1}d\Gamma$$

5.6.6 Form of the matrixes of cohesion

Are two directions of the fixed base I and J, of unit vectors $\boldsymbol{e}_I$ and $\boldsymbol{e}_J$. Let us introduce the tangent matrix of the cohesive model into the fixed base $[K_{gl}]$ coefficients $[K_{gl}]_{IJ}=\boldsymbol{e}_I\cdot\frac{\partial\boldsymbol{\delta}}{\partial[\![\boldsymbol{u}]\!]}\cdot\boldsymbol{e}_J$.

With the statement of $\frac{\partial\boldsymbol{t}_c}{\partial[\![\boldsymbol{u}]\!]}$ given to [§5.424], we have the tangent matrix of the cohesive model $[K_{loc}]$ in the local base (see Doc. [R7.02.11]). We obtain formula $[K_{gl}]$ formula $[K_{gl}]=[Q]^T[K_{loc}][Q]$, where $[Q]$ is an orthonormal transition matrix defined by:

$$[Q]=\begin{bmatrix}n_x & n_y & n_z\\ \tau_x^1 & \tau_y^1 & \tau_z^1\\ \tau_x^2 & \tau_y^2 & \tau_z^2\end{bmatrix}$$

Are i and j two nodes nouveau riches. That is to say Γ the intersection of the supports of i and j. The matrix $[D_u]$ then given by:

$$\{u^*\}_i[D_u]_{ij}\{\delta u\}_j=-\int_\Gamma 2b_i^*\phi_i[K_{gl}]2b_j\phi_j d\Gamma$$

5.6.7 Statement of the second members of cohesion

the second members of cohesion have the following statements:

$$\left\{u_i\right\}^*\left(L^1_{coh}\right)_i=-\int_\Gamma 2b_i^*\phi_i\boldsymbol{t}_c^{k-1}d\Gamma$$

$$\left\{\lambda^*\right\}_i\left(L^2_{coh}\right)_i=-\int_\Gamma \psi_i \lambda_i^*\left(\boldsymbol{t}_{c,n}^{k-1}\cdot\boldsymbol{n}\right)d\Gamma$$

$$\left\{\Lambda^*\right\}_i\left(L^3_{coh}\right)_i=-\int_\Gamma \psi_i\left(\Lambda_i^{1*}\tau_i^1+\Lambda_i^{2*}\tau_i^2\right)\boldsymbol{t}_{c,\tau}^{k-1}\,d\Gamma$$

where $k-1$ represents the index of the preceding iteration of Newton.
To express $\boldsymbol{t}_c$ in the global database, one can use $[Q]$.

5.6.8 Matric writing of the problem with mixed cohesive model

the matric system such as it is solved with the iteration $k+1$ Newton can be put in following matric form:

$$\begin{bmatrix} K_{méca}+A_u & A^T \\ A & C \end{bmatrix}\begin{pmatrix}\delta u\\ \delta\lambda\end{pmatrix}=\begin{pmatrix}L^1_{méca}+L^1_{coh}\\ L^2_{coh}\end{pmatrix}$$

5.6.8.1 Form of the elementary matrixes of cohesion:

Are two directions of the fixed base I and J, of unit vectors $\boldsymbol{e}_I$ and $\boldsymbol{e}_J$. We introduce as previously the tangent matrix of the cohesive model in the fixed base $[K_{gl}]$ coefficients $[K_{gl}]_{IJ}=\boldsymbol{e}_I\cdot\dfrac{\partial\boldsymbol{\delta}}{\partial\boldsymbol{p}}\cdot\boldsymbol{e}_J$. We lay out, by the cohesive constitutive law, of the tangent matrix $[K_{loc}]$ the local base (see Doc. [R7.02.11]). We obtain formula $[K_{gl}]$ formula $[K_{gl}]=[Q]^T[K_{loc}][Q]$, where $[Q]$ is an orthonormal transition matrix defined by:

$$[Q]=\begin{bmatrix} n_x & n_y & n_z\\ \tau_x^1 & \tau_y^1 & \tau_z^1\\ \tau_x^2 & \tau_y^2 & \tau_z^2\end{bmatrix}$$

Having introduced these notations, we have:

$$\left\{u^*\right\}_i[A_u]_{ij}\{\delta u\}_j=\int_\Gamma 2\phi_i b_i^* r\left([Id]-r[K_{gl}]\right)2\phi_j b_j\,d\Gamma$$

In addition, we choose to discretize $\boldsymbol{\lambda}$ in local base. The coefficient $(1,J)$ matrix $\{\lambda_n^*\}[A_u]_{ij}\{\delta u\}_j$ is then $\int_\Gamma 2\phi_i\lambda_i^*\boldsymbol{n}\cdot\left(Id-r\dfrac{\partial\delta}{\partial p}\right)\cdot\boldsymbol{e}_j 2\phi_j b_j\,d\Gamma$. For the coefficients $(2,J)$ and $(3,J)$, the formula is the same one while replacing $\boldsymbol{n}$ by $\boldsymbol{\tau_1}$ and $\boldsymbol{\tau_2}$, respectively. By exploiting the notations introduced into the preceding paragraph, we deduce some:

$$\{\lambda^*\}_i[A_u]_{ij}\{\delta u\}_j=\int_\Gamma \varphi_i\,\lambda_i^*\left([Id]-r[K_{loc}]\right)\cdot[Q]2\,\varphi_j b_j\,d\Gamma$$

As for the matrix $[C]$, it is written simply:

$$\{\lambda^*\}_i[C]_{ij}\{\delta\lambda\}_j=-\int_\Gamma \phi_i\lambda_i^*[K_{loc}]\phi_j\lambda_j\,d\Gamma$$

5.6.8.2 Statement of the elementary vectors of cohesion:

The coefficient I vector $\{u\}_i^*\left(L_{coh}^1\right)_i$ has as a statement $\int_\Gamma 2\, b_i^* \phi_i\left(-\boldsymbol{\lambda}\cdot\boldsymbol{e_I}+r\left(\llbracket u\rrbracket+\delta\right)\cdot\boldsymbol{e_I}\right)d\,\Gamma$. With the notations indroduites in the preceding ones part, we deduce:

$$\{u_i\}^*\left(L_{coh}^1\right)_i=\int_\Gamma 2\, b_i^* \phi_I\left(-[Q]^T\cdot\{\lambda\}+r\ \left(\{\llbracket u\rrbracket\}+[Q]^T\cdot\{\delta\}\right)\ \right)d\,\Gamma$$

where $\{\delta\}$ and $\{\lambda\}$ are given in local base, and $\{\llbracket u\rrbracket\}$ in fixed base.

The coefficient 1 vector $\{\lambda\}_i^*\left(L_{coh}^2\right)_i$ is written $\int_\Gamma \lambda_i^*\left(\llbracket u\rrbracket\cdot n+\delta\cdot n\right)d\,\Gamma$. For the coefficients 2 and 3, the formula is the same one by replacing $\boldsymbol{n}$ by $\boldsymbol{\tau_1}$ and $\boldsymbol{\tau_2}$, respectively. We deduce some:

$$\{\lambda\}_i^*\left(L_{coh}^2\right)_i=\int_\Gamma \lambda_i^*\left([Q]\{\llbracket u\rrbracket\}+\{\delta\}\right)d\,\Gamma$$

where $\{\delta\}$ is given in local base, and $\{\llbracket u\rrbracket\}$ in fixed base.

6 LBB condition

the selected approximations on the one hand for displacement and on the other hand for contact pressures do not seem by satisfying the condition *inf-sup* in all the cases. The non-observance of the LBB condition generates oscillations of contact pressures, phenomenon comparable to that met of incompressibility biberon1342. Physically, in the case of the Lagrangian contact, that amount wanting to impose the contact in too many points of the interface (overstrained), making the system hyperstatic. To slacken it, it is necessary to restrict the space of the Lagrange multipliers, as that is done in biberon1442 for the conditions of Dirichlet with X-FEM. The algorithm proposed by Moës biberon1442 to reduce the oscillations is extended to 3D case. Its goal is to impose relations of equality between Lagrange multipliers. This algorithm was the object of an improvement to make it more physical and more effective.
In the case of the penalized contact one finds the same oscillations. For a mesh triangle for example, one can show that it does not exist combination of the degrees of freedom of the Heaviside which allow a rotation of the mean surface of crack without generating oscillations of the jump of displacement. For a stiffness of raised interface, as it is the case in penalization, this generates oscillations of pressure. To cure it, it is necessary to recover the explicit pressure in the degree of freedom λ, to apply the condition of LBB to him and to make it go up in the equilibrium, which explains the formulation given to the § 12. The actualization of the statutes of contact is done as in the Lagrangian case, where λ and not d_n is tested to pass from a state contacting to a state not contacting, this in order to avoid the oscillations on the statutes of contact.

6.1 Description of the algorithm of Moës (algorithme1)

the algorithm introduced by Moës biberon1442 is presented there with an aim of imposing conditions of Dirichlet on an interface in the frame of X-FEM. It shows that the technique of the Lagrange multipliers to impose conditions of Dirichlet must be used carefully, because the condition *inf-sup* is not always observed. Paper is restricted with the case 2D, but the algorithm presented is easily generalizable with 3D case. The first phase is a phase of selection of the nodes, in which the selected nodes are those which are “important” for the approximation of the Lagrange multipliers. The other nodes are superabundant and bring to oscillations of the Lagrange multipliers. Once the selected “important” nodes, of the relations of equalities are imposed between the Lagrange multipliers, in order to restrict the space of the multipliers. Thus, the Lagrange multipliers of the edges emanating of the same selected node are equal.
In a more formal way, are E and the N sets containing all the edges and all the nodes of the mesh. The two ends of an edge $e\in E$ are noted $\left(v_1(e),v_2(e)\right)\in N^2$. First of all, one starts with a phase of initialization (iteration $k=0$ of the algorithm). One determines first of all S_e^0, all the edges which are cut by the interface. The interface being represented by the level set norm lsn, an edge $e\in E$ is cut by the interface if and only if $lsn\left(v_1(e)\right)\cdot lsn\left(v_2(e)\right)\leq 0$. Let us note that if the interface coincides with the node $v_1(e)$ or the node $v_2(e)$, the edge e east belongs to S_e^0 :

$$S_e^0=\left\{ e\in E\ ,\ \ lsn\left(v_1(e)\right)\cdot lsn\left(v_2(e)\right)\le 0\right\}$$

That is to say N_e all the nodes connected by the elements of S_e^0 :

$$N_e=\left\{ n\in N\ ,\ \ \exists e\in S_e^0\ \ n=v_1(e)\ \ \text{ou}\ \ n=v_2(e)\right\}$$

That is to say S_n^0 all the nodes selected with the iteration $k=0$ (initialization). These nodes are those which coincide with the interface (this group can be empty):

$$S_n^0=\left\{ n\in N_e\ ,\ \ lsn(n)=0\right\}$$

After this phase of initialization, the algorithm reiterates $k=1,\text{nmax\_iter}$.
With each iteration, the following stages are carried out:

- Update of the group of all the edges: one removes those which are connected to a node selected with the preceding iteration

$$S_e^k=S_e^{k-1}\backslash\left\{ e\in S_e^{k-1}\ ,\ \ v_1(e)\in S_n^{k-1}\ \text{ou}\ \ v_2(e)\in S_n^{k-1}\right\}$$

- Computation of the score of the nodes: for each node in N_e , one calculates a score made up of 2 digits: the first figure corresponds to the numbers of edges in S_e^k connected, and the second corresponds to the absolute value of the level set norm in this node. This score sc_no is thus a matrix with two columns whose lines represent the node.

$$\forall\, n\in N_e\ \ \begin{cases} sc\_no^k(n,1)=\text{nombre d'arêtes connectées au noeud } n \\ sc\_no^k(n,2)=|lsn(n)| \end{cases}$$

- Computation of the score of the edges: for each edge in S_e^k , one calculates a score made up of 2 digits: the first figure corresponds to the absolute value of the difference of the 1st figure of the score of the 2 nodes ends, and the second corresponds to a relationship between the values of the 2nd figure of the 2 nodes ends (i.e. a value ratio of lsn . This score sc_ar is thus a matrix with two columns whose lines represent the edge.

$$\forall\, e\in S_e^k,\ \ \forall\, j\in(1,2),\ \ s_j=sc\_no^k\left(v_j(e),1\right),\ \ l_j=sc\_no^k\left(v_j(e),2\right)$$

$$\begin{cases} sc\_ar^k(e,1)=|s_1-s_2| \\ sc\_ar^k(e,2)=\begin{cases} \dfrac{l_1}{l_1+l_2} & \text{si}\ \ s_1<s_2 \\ \dfrac{l_1}{l_1+l_2} & \text{si}\ \ s_1>s_2 \\ \dfrac{\min\left(l_1,l_2\right)}{l_1+l_2} & \text{si}\ \ s_1=s_2 \end{cases} \end{cases}$$

- Search “better edge” b_e : it is the edge whose 1st figure of its score is largest. In the event of equality between 2 edges, it is that whose 2nd figure of its score is largest.

•Searching for "better node" b_n : it is ending node of b_e which the 1st figure of its score is largest. In the event of equality, it is the node whose 2nd figure of the score is smallest (the node nearest to the interface). The node b_n is the only node selected with this iteration:

$$S_n^k=\left\{b_n\right\}$$

The algorithm so during stops an iteration, the group S_e^k becomes the empty whole. The final whole of the selected nodes will be then:

$$W=\underset{k}{\cup} S_n^k$$

After this phase of selection of the nodes, the algorithm builds the space of the Lagrange multipliers, whose size is equal to that of W . Thus, the space of the multipliers is:

$$S_\lambda=\left\{\lambda^i, i\in\left\{1, card\left(W\right)\right\}\right\}$$

With this algortihme any node of N_e / W is connected by an edge of S_e^0 to a node of W . One then imposes a relation of equality between these two nodes. In the event of conflict (connections with several nodes of W by as many edges), one discriminates the nodes of W by criterion 2 of the nodes (weaker level-set norm).

6.2 Description of the modified algorithm (algo2)

Based on similar ideas, a new algorithm was proposed. Thus, in the new version, one leaves all the edges on which the level set norm is cancelled at least in a point. These edges connect points on both sides of the interface (or possibly of the points on the interface). The algorithm seeks the minimal subset of edges making it possible to connect all the points ends of the edges. Then, groups of connected edges are extracted from it. The imposed relations are then the following ones:

the multipliers on the nodes tops of each group are imposed equal,

In a more formal way, are E and the N sets containing all the edges and all the nodes of the mesh. The two ends of an edge $e\in E$ are noted $\left(v_1(e), v_2(e)\right)\in N^2$. One determines first of all S_e , all the edges which are strictly cut by the interface. The interface being represented by the level set norm lsn , an edge $e\in E$ is strictly cut by the interface if and only if $lsn\left(v_1(e)\right)\cdot lsn\left(v_2(e)\right)<0$. Let us note that if the interface coincides with the node $v_1(e)$ or the node $v_2(e)$, the edge e is not in S_e :

$$S_e=\left\{e\in E,\ \ lsn\left(v_1(e)\right)\cdot lsn\left(v_2(e)\right)<0\right\}$$

That is to say N_e all the nodes connected by the elements of S_e . One separates N_e in two parts: nodes "below" and "above" the crack, according to the sign of lsn :

$$N_e=\left\{n\in N,\ \ \exists e\in S_e\ \ n=v_1(e)\ \text{ ou }\ n=v_2(e)\right\}$$
$$N_e^+=\left\{n\in N_e,\ \ lsn(n)>0\right\}\ \text{ et }\ N_e^-=\left\{n\in N_e,\ \ lsn(n)<0\right\}$$

The algorithm searches S_{ve} , the minimal subset of S_e which makes it possible to connect the nodes in N_e^+ to the nodes in N_e^- . Each node in N_e^+ must be connected to at least a node in N_e^- , and each node in N_e^- must be connected to at least a node in N_e^+ . The edges in S_{ve} are called "vital edges", because if one of these edges disappears, at least a node in N_e will be orphan. This set of

vital edges is not necessarily single. In the presence of choice, the vital edge shortest is privileged. As it thereafter will be seen, that amounts minimizing the region of P0 approximation. For the search of the group S_{ve} , we chose an algorithm based on the notions of scores of nodes and edge, notion that one finds in the algorithme1. The algorithm will remove with all NON-vital edges, until there remain nothing any more but vital edges. More precisely, one associates a score with each node, which corresponds to the number of edges connected to this node. With each edge, one associates a score, which corresponds at least of the scores of the two nodes ends. That is to say e the edge having the score more raised (with identical score, the longest edge is privileged). If the score of e is equal to 1, then all the edges which remain are vital edges. S_{ve} is determined and the algorithm stops. If the score of e is strictly higher than 1, the edge e is a symbolically removed NON-vital edge, and it list of the edges S_e . The algorithm starts again, with a new computation of the score of the nodes, and so on until there remain nothing any more but vital edges.

It is important to note that S_{ve} is composed of certain disconnected edges, and certain edges connected between them. These groups of connected vital edges are extracted from S_{ve} . Let us note that in such a group, all the edges are connected by a single node (see Figure 6.2-1). That is to say G^i_{cve} the group of vital edges connected by the node i . Then G^i_{cve} is defined by:

$$G^i_{cve}=\left\{e\in S_{ve},\ \ i=v_1(e)\ \ \text{ou}\ \ i=v_2(e)\right\}$$

Now, one imposes relations between the Lagrange multipliers. All the multipliers carried by the nodes tops of the same group are imposed equal.

We 2D illustrate these algorithms on the case of Figure 6.2-1. The nodes groups bound by relations between equality and version 1 are marks by the blue circles. The nodes groups bound by relations between equality and version 2 are marked by the full edges connected between them.

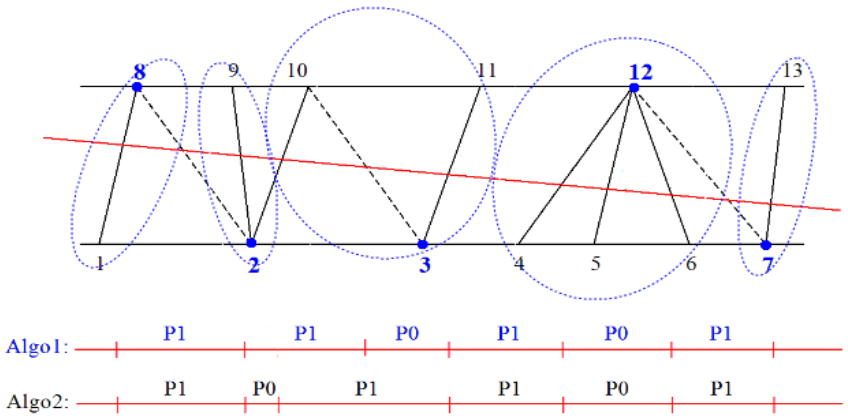


Figure 6.2-1 : Example of edges cut by an interface and approximation resulting

6.3 Relations imposed between the semi-multipliers from friction

When friction is taken into account, one observes the same phenomenon of oscillations on the semi-multipliers of friction as in the preceding case on the multipliers of contact. So that the reaction of contact does not oscillate any more, it is necessary to remove the oscillations of the normal reaction (contact pressure) and of tangential reaction (thus of the semi-multipliers of friction). For that, it is thus necessary also to activate the algorithm of restriction of spaces of the multipliers for the semi-multipliers of friction.

The relations are to be imposed on the tangential reactions, and utilize the unknowns of friction Λ_1 and Λ_2 as well as the vectors of the base covariante τ_1 and τ_2 .

In the case of imposition of a relation of equality between the nodes A and B, the relation is written:

$$\Lambda_1^A \tau_1^A + \Lambda_2^A \tau_2^A = \Lambda_1^B \tau_1^B + \Lambda_2^B \tau_2^B$$

The two unknowns to be determined (Λ_1^A and Λ_2^A for example) being scalar, it is necessary to transform the preceding vectorial relation into two scalar relations. This is done by projection on the basis $\left(\tau_1^A, \tau_2^A\right)$. The two relations to be imposed are thus finally:

$$\begin{cases} \Lambda_1^A \left(\tau_1^A \cdot \tau_1^A\right) + \Lambda_2^A \left(\tau_2^A \cdot \tau_1^A\right) = \Lambda_1^B \left(\tau_1^B \cdot \tau_1^A\right) + \Lambda_2^B \left(\tau_2^B \cdot \tau_1^A\right) \\ \Lambda_1^A \left(\tau_1^A \cdot \tau_2^A\right) + \Lambda_2^A \left(\tau_2^A \cdot \tau_2^A\right) = \Lambda_1^B \left(\tau_1^B \cdot \tau_2^A\right) + \Lambda_2^B \left(\tau_2^B \cdot \tau_2^A\right) \end{cases} \qquad \textbf{éq 6.3-1}$$

This choices is called into question by the introduction of the great slidings [R5.03.53]: indeed the base of contact changes with each geometrical iteration. The preceding relations introduced into tough by means of home base thus do not have any more a meaning. To solve the conflict, one rather introduces the two relations of equalities on the components:

$$\begin{cases} \Lambda_1^A = \Lambda_1^B \\ \Lambda_2^A = \Lambda_2^B \end{cases} \qquad \textbf{éq 6.3-2}$$

By considering that the bases $\left(\tau_1^A, \tau_2^A\right)$ and $\left(\tau_1^B, \tau_2^B\right)$ are almost identical because of proximity of the points A and B, equations 6.3-1 and 6.3-2 are almost equivalent.

6.4 Remarks on the relations imposed by algorithm 1 or 2

6.4.1 On the multipliers of contact

Is the linear relation between the Lagrange multipliers of contact λ_1, λ_2 and λ_3 :

$$\lambda_2 = \alpha \lambda_1 + (1-\alpha)\lambda_3$$

The relation relates to the value of the pressure and not the vector contact pressure. In the event of structure a relation curves on the vectors pressure of the type:

$$\lambda_2 n_2 = \alpha \lambda_1 n_1 + (1-\alpha)\lambda_3 n_3$$

is not inevitably possible because the vector n_2 is not an unknown.

6.4.2 On the semi-multipliers of friction

the unknown for friction is vectorial. One could imagine to bind the vectors tangent reactions between them:

$$r_{\tau_2} = \alpha r_{\tau 1} + (1-\alpha) r_{\tau 3}$$

however

$$r_{\tau_i} = \mu \lambda_i \Lambda_i$$

what gives

$$\Lambda_2 = \frac{\alpha \lambda_1 \Lambda_1 + (1-\alpha)\lambda_3 \Lambda_3}{\alpha \lambda_1 + (1-\alpha)\lambda_3} = \beta \Lambda_1 + (1-\beta)\Lambda_3$$

with $\beta=\frac{\alpha\lambda_1}{\alpha\lambda_1+(1-\alpha)\lambda_3}\approx\alpha$ if the mesh is rather fine

This choice of relation is impossible because in 3D, if items 1 and 3 are in slipping contact, then item 2 will not be it! Indeed, the norm of Λ_2 will be strictly lower than 1 if the directions of sliding are not colinéaires (see Figure 6.4.2-1).

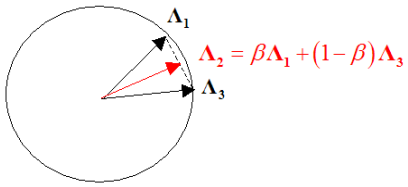


Figure 6.4.2-1 : Case of a point adhering between two slipping points

the suggested solution is to impose a linear relation between the norms of the vectors tangent reactions, which is equivalent to impose a relation between the norms of the semi-multipliers of friction:

$$\|r_{\tau 2}\|=\alpha\|r_{\tau 1}\|+(1-\alpha)\|r_{\tau 3}\| \Leftrightarrow \|\Lambda_2\|=\beta\|\Lambda_1\|+(1-\beta)\|\Lambda_3\| \qquad \textbf{éq. 6.4.2-1}$$

This relation is nonlinear because of norm. The method of Newton makes it possible to be reduced to the successive imposition of the linear relations. With the iteration of Newton I , the relation is:

$$\delta\Lambda_2^i.\frac{\Lambda_2^{i-1}}{\|\Lambda_2^{i-1}\|}-\beta\delta\Lambda_1^i.\frac{\Lambda_1^{i-1}}{\|\Lambda_1^{i-1}\|}-(1-\beta)\delta\Lambda_3^i.\frac{\Lambda_3^{i-1}}{\|\Lambda_3^{i-1}\|}=\beta\|\Lambda_1^{i-1}\|+(1-\beta)\|\Lambda_3^{i-1}\|-\|\Lambda_2^{i-1}\|$$

This kind of linear relation is currently not available in *Code_Aster* , where only the linear relations whose coefficients are constant throughout all computation are authorized.

Currently, the relation between the semi-multipliers of friction established is the following one:

$$\Lambda_2=\alpha\Lambda_1+(1-\alpha)\Lambda_3$$

When the sliding or the dependancy is one-way, one finds the equation [éq well. 4.6 – 1] while having substituted α for β .

6.4.3 Nonsimpliciaux elements (quadrangles, hexahedrons…)

In the case of meshes *quad* in 2D, *penta* or *hexa* in 3D, one notes certain configurations where nodes of the mesh do not belong to a cut edge. On the two examples of figure 6.4.3-1 , the red nodes are connected to no other node. In order to satisfy the LBB, the idea is to connect these nodes to the related nodes but while trying not to introduce into this case of the linear relations.

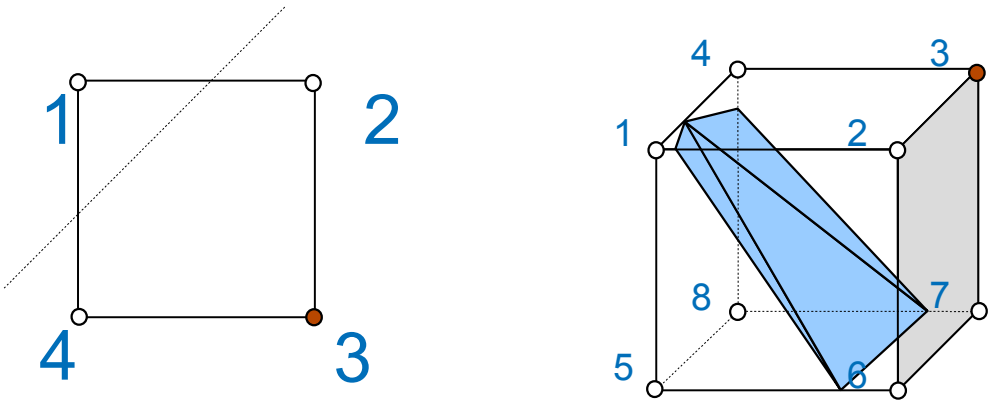


Figure 6.4.3-1 : The red nodes do not belong to a cut edge.

To avoid adding linear relations, the idea is to eliminate from the interpolation the nodes which do not belong to a cut edge. To satisfy the partition with the unit of the contributions of contact, one by means of distributes the shape functions of the nodes eliminated on the other nodes the following uniform distribution:

$$\tilde{\phi}_{\,i\in N_{actif}} = \phi_i + \sum_{j\in N_{elim}} \phi_j / N_{actif} \qquad \text{éq. 6.4.3-1}$$

Where N_{actif} is all the nodes of the element directly cut or pertaining to a cut edge. N_{elim} is all the nodes to be eliminated. For the quadrangle of figure 6.4.3-1 , the modified shape functions are written then $\tilde{\phi}_{\,i=1,2,4} = \phi_i + \phi_3/3$. For the hexahedron they are written $\tilde{\phi}_{\,i=1,2,4,5,6,7,8} = \phi_i + \phi_3/7$.
It is noticed that other choices of distribution are possible: one for example could have chosen to distribute a node eliminated on his related active nodes. For the quadrangle of figure 6.4.3-1 , the shape functions modified would be written then $\tilde{\phi}_1 = \phi_1$ and $\tilde{\phi}_{\,i=2,4} = \phi_i + \phi_3/2$.
The elimination of the degrees of freedom of contact in excess is discussed [§4.44.4]), substitution is chosen (by putting 1 on the diagonal and 0 in the second member).

One can generalize this approach for the crack tips. Indeed on the two examples of figure 6.4.3-2, the edge {1-4} is cut whereas {2-3} is not it: nodes 2 and 3 are thus to eliminate. By means of the approach described previously, the modified shape functions are written $\tilde{\phi}_{\,i=1,4} = \phi_i + (\phi_2 + \phi_3)/2$. That amounts making P0 integration described in the paragraph [§4.3.24.3.2] on the elements crossed containing the point.

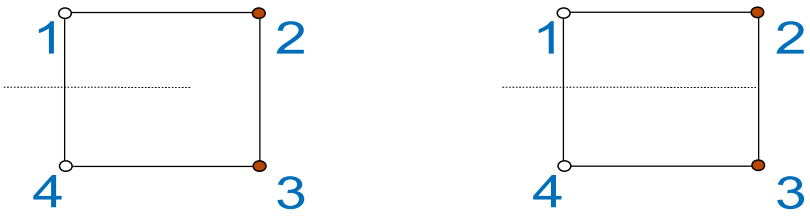


Figure 6.4.3-2 : The red nodes do not belong to a cut edge.

Bibliography

1 "Elements of contacts derived from a continuous hybrid formulation", Documentation of reference of *Code_Aster* n° [R5.03.52]

2 BEN DHIA H., VAUTIER I., "a formulation to treat the contact friction in 3D in *the Code_Aster* ", research report HI-75/99/007/A, June 1999, EDF

3 ALART P., CURNIER A. , " A mixed formulation for frictional contact problems preaches to Newton like solution methods", Methods Computer in Applied Mechanics and Engineering, vol. 92, pp. 353-375, 1991

4 JI H., DOLBOW J.E., "One strategies for enforcing interfacial constraints and evaluating jumps conditions with the extended finite element method", International Newspaper for Numerical Methods in Engineering, vol. 61, pp. 2508-2535, 2004

5 PELLET J., "Dualisation of the boundary conditions", Documentation of reference of *Code_Aster* n° [R3.03.01]

6 ERN A., GUERMOND J.L., Theory and practice of finite elements, Springer, 2004

7 BRENNER S.C., SCOTT L . R., The mathematical theory of finite element methods, 2 Nd ED., Springer, 2002

8 LAURSEN T.A., SIMO J.C. "A continuum element-based formulation for the implicit solution of multi-bodystocking, broad strain frictional contact problem", International Newspaper for Numerical Methods in Engineering , vol. 36, pp. 3451-3485, 1993

9 WRIGGERS P., " Finite element algorithms for contact problem", arch. Of comp. Meth. In Eng., vol. 2, pp. 1-49, 1995

10 CURNIER A, HE, Q.C., KLARBRING A., "Continuum mechanics modelling of broad strain contact with friction", Contact mechanics, ED. Plenum Close, 1995

11 PIETRZAK G., "continuum mechanics modelling and augmented Lagrangian of formulation broad strain frictional contact problems", Thesis of doctorate, Federal Polytechnic School of Lausanne, 1997

12 ALART P., BARBOTEU MR., "Elements contact, method of Newton generalized and decomposition of field" nonlinear Problems Applied, School French atomic energy agency – EDF – INRIA 1999

13 VAULT D., BATHE K.J. , "The Inf-sup test", Computers & Structures, vol. 47, pp. 537-545, 1993

14 MOËS N., BÉCHET E., PEATY MR. , "Imposing essential boundary conditions in the X-FEM", International Newspaper for Numerical Methods in Engineering , 2006

15 DAUX C., MOES N., DOLBOW J., SUKUMAR N., BELYTSCHKO T., "Arbitrary branched and intersecting aces with the extended finite element method", International Newspaper for Numerical Methods in Engineering, 48 (2000), 1741-1760

16 SIAVELIS Mr., "numerical Modelization X-FEM of great slidings with friction the Ion G of a network of discontinuities. " Thesis of doctorate, Central School of Nantes, 2011

17 MESCHKE G., DUMSTORFF cohesive P. Energy-based modelling of and cohesionless aces via X-FEM. Computational Methods in Applied Mechanics Engineering, 196,2338 – 2557,2007.

7 Description of the versions

Index document	Version Aster	Author (S) Organization (S)	Description of the modifications
A	11	S.GENIAUT, P.MASSIN EDF/R & D AMA	initial Text