Solution of a differential equation of the second order by the method of NIGAM

Summarized:

We present in this document, a method of resolution of the linear differential equation of the second order obtained during computation of an oscillator spectrum.

1 Introduction

During the computation of an oscillator spectrum, one is brought to solve a differential equation of the second order whose solution is an integral of DUHAMEL.

If this integral can be calculated exactly using the transform of LAPLACE for certain simple analytical functions (Dirac, Sine, Cosine, Heavyside,…) [bib1] it must be integrated numerically in the general case.

This document presents an effective method to solve this problem.

This method is implemented in *Code_Aster*, in the operator `CALC_FONCTION`, key word factor `SPEC_OSCI`.

2 Analytical solution of the equation

During the computation of the oscillator spectrum of an accelerogram [R4.05.03], one is brought to solve the linear differential equation of the second order:

$$\ddot{q}+2\xi\omega\dot{q}+\omega^2 q=-\alpha(t)$$

where	$q(t)$	is relative displacement
	$\alpha(t)$	is the acceleration of the motion imposed on the base
	ω	is the pulsation of the oscillator
	ξ	is the reduced damping of the oscillator

With initial conditions on q and $\dot{q}$.

The solution of this equation is written in the form:

$$q(t)=+\int_0^t h(t-\tau).\alpha(\tau)\,d\tau+q(0)\,g(t)+\dot{q}(0)\,h(t) \quad \text{éq 2-1}$$

where $q(0)$ and $\dot{q}(0)$ is displacement and the velocity at initial time.

- Statement of $h(t)$ and $g(t)$ according to the value of reduced damping ξ .
  - If $\xi<1$ (damping sub-critical):

$$\begin{aligned} h(t)&=\frac{e^{-\xi\omega t}}{\omega\sqrt{1-x^2}}\sin\left(\omega t\sqrt{1-\xi^2}\right)\\ g(t)&=e^{-\xi\omega t}\left[\cos\left(\omega t\sqrt{1-\xi^2}\right)+\frac{\xi}{\sqrt{1-\xi^2}}\sin\left(\omega t\sqrt{1-\xi^2}\right)\right]\end{aligned} \quad \text{éq 2-2}$$

- If $\xi=1$ (critical damping):

$$h(t)=te^{-\omega t}$$
$$g(t)=(1-\omega)e^{\omega t}$$

- If $\xi>1$ (supercritical damping):

$$h(t)=\frac{e^{-\xi\omega t}}{\omega\sqrt{\xi^2-1}}.sh\left(\omega t\sqrt{\xi^2-1}\right)$$
$$g(t)=e^{-\xi\omega t}\left[ch\left(\omega t\sqrt{\xi^2-1}\right)+\frac{\xi}{\sqrt{\xi^2-1}}sh\left(\omega t\sqrt{\xi^2-1}\right)\right]$$

3 Numerical method

the numerical method established in *Code_Aster* was proposed by NIGAM and JENNINGS [bib2] in the case of the damping sub-critical which corresponds to our initial seismic problem [R4.05.03].

By introducing the formulation [éq 2-2] in [éq 2-1] one is thus led to solve the differential equation:

$$\ddot{q}(t)+2\xi\omega\dot{q}(t)+\omega^2 q(t)=-\alpha(t)$$

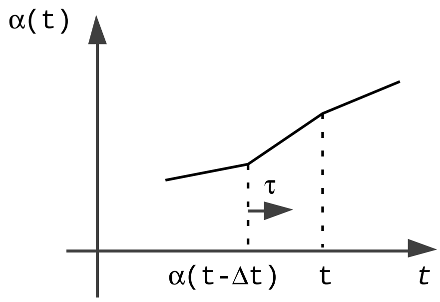
with null initial conditions, whose solution is written:

$$q(t)=\frac{1}{\omega_d}\int_0^t e^{-\xi\omega(t-\tau)}\sin\left[\omega_d(t-\tau)\right]\alpha(\tau)\,d\tau$$

with $\omega_d=\omega\sqrt{1-\xi^2}$

By supposing that $\alpha(t)$ varies linearly inside each interval $\Delta(t)$, one can then write:

$$\alpha(\tau)=\alpha(t-\Delta t)+\frac{\tau}{\Delta t}\left[\alpha(t)-\alpha(t-\Delta t)\right] \text{ pour } \tau\in\left[0,\Delta t\right]$$



from where the equation to be solved (expressed in the new variable τ):

$$\ddot{q}(t)+2\xi\omega\dot{q}(\tau)+\omega^2 q(\tau)=a+b\tau \text{ pour } \tau\in[0,\Delta t]$$

where $a=\alpha(t-\Delta t)$
$b=[\alpha(t)-\alpha(t-\Delta t)]/\Delta t$

with the initial conditions: $q(0)=q(t-\Delta t)$
$\dot{q}(0)=\dot{q}(t-\Delta t)$

The solution of this equation is the superposition of a particular solution and solutions of the homogeneous problem.

- a particular solution: $q_p(t)=-\frac{a}{\omega^2}+\frac{2\xi b}{\omega^3}-\frac{b}{\omega^2}\tau$
- solutions of the homogeneous problem: $q_h(t)=e^{-\xi\omega\tau}[C_1.\cos(\omega_d\tau)+C_2.\sin(\omega_d\tau)]$

Consequently: $q(\tau)=e^{-\xi\omega\tau}[C_1.\cos(\omega_d\tau)+C_2.\sin(\omega_d\tau)]-\frac{a}{\omega^2}+2\frac{\xi b}{\omega^3}-\frac{b.\tau}{w^2}$

and by deriving q (compared to t) one a:

$$\dot{q}(\tau)=(-\xi\omega)e^{-\xi\omega\tau}(C_1\cos\omega_d\tau+C_2\sin\omega_d\tau)+e^{-\xi\omega\tau}(-C_1\omega_d\sin\omega_d\tau+C_2\omega_d\cos\omega_d\tau)-\frac{b}{\omega^2}$$

the coefficients C_1 and C_2 are then determined by the initial conditions at the beginning of the interval (it is - with-to say for $\tau=0$).

$$C_1=q(t-\Delta t)+\frac{a}{\omega^2}-\frac{2\xi b}{\omega^3}$$
$$C_2=\frac{1}{\omega_d}\left[\dot{q}(t-\Delta t)+\xi\omega q(t-\Delta t)+\frac{\xi a}{\omega}-\frac{2\xi^2-1}{\omega^2}b\right]$$

and while deferring C_1 and C_2 in the statement of q and $\dot{q}$ one obtains the matric equality for $\tau=\Delta t$:

$$\begin{Bmatrix} q(t) \\ \dot{q}(t) \end{Bmatrix}=A(\xi,\omega,\Delta t)\begin{Bmatrix} q(t-\Delta t) \\ \dot{q}(t-\Delta t) \end{Bmatrix}+B(\xi,\omega,\Delta t)\begin{Bmatrix} \alpha(t-\Delta t) \\ \alpha(t) \end{Bmatrix}$$

4 Coefficients of the matrixes A and B of the system to solve

Matrix A **:**

$$a_{11}=e^{-\xi\omega\Delta t}\left[\frac{\xi}{\sqrt{1-\xi^2}}\sin\left(\omega_d\Delta t\right)+\cos\left(\omega_d\Delta t\right)\right]$$

$$a_{12}=\frac{e^{\xi\omega\Delta t}}{\omega_d}\sin\left(\omega_d\Delta t\right)$$

$$a_{21}=-\frac{\omega}{\sqrt{1-\xi^2}}e^{-\xi\omega\Delta t}\sin\left(\omega_d\Delta t\right)$$

$$a_{22}=e^{-\xi\omega\Delta t}\left[\cos\left(\omega_d\Delta t\right)-\frac{\xi}{\sqrt{1-\xi^2}}\sin\left(\omega_d\Delta t\right)\right]$$

Stamp B **:**

$$b_{11}=e^{-\xi\omega\Delta t}\left[\left(\frac{2\xi^2-1}{\omega^2\Delta t}+\frac{\xi}{\omega}\right).\frac{\sin\left(\omega^d\Delta t\right)}{\omega^d}+\left(\frac{2\xi}{\omega^3\Delta t}+\frac{1}{\omega^2}\right)\cos\left(\omega^d\Delta t\right)\right]-\frac{2\xi}{\omega^3\Delta t}$$

$$b_{12}=e^{-\xi\omega\Delta t}\left[\frac{2\xi^2-1}{\omega^2\Delta t}.\frac{\sin\left(\omega^d\Delta t\right)}{\omega^d}+\frac{2\xi}{\omega^3\Delta t}.\cos\left(\omega^d\Delta t\right)\right]-\frac{1}{\omega^2}+\frac{2\xi}{\omega^3\Delta t}$$

$$b_{21}=e^{-\xi\omega\Delta t}\left[\left(\frac{2\xi^2-1}{\omega^2\Delta t}+\frac{\xi}{\omega}\right).\left(\cos\left(\omega_d\Delta t\right)-\frac{\xi}{\sqrt{1-\xi^2}}\sin\left(\omega^d\Delta t\right)\right)-\right.$$
$$\left.\left(\frac{2\xi}{\omega^3\Delta t}+\frac{1}{\omega^2}\right).\left(\omega_d\sin\left(\omega_d\Delta t\right)+\xi\omega\cos\left(\omega_d\Delta t\right)\right)\right]+\frac{1}{\omega^2\Delta t}$$

$$b_{22}=-e^{-\xi\omega\Delta t}\left[\left(\frac{2\xi^2-1}{\omega^2\Delta t}\right).\left(\cos\left(\omega_d\Delta t\right)-\frac{\xi}{\sqrt{1-\xi^2}}\sin\left(\omega_d\Delta t\right)\right)-\right.$$
$$\left.\left(\frac{2\xi}{\omega^3\Delta t}\right).\left(\omega_d\sin\left(\omega_d\Delta t\right)+\xi\omega\cos\left(\omega_d\Delta t\right)\right)\right]-\frac{1}{\omega^2\Delta t}$$

with $\omega_d=\omega\sqrt{1-\xi^2}$

5 Knowing Computation of $\ddot{q}(\tau)$

acceleration $q(\tau)$ and $\dot{q}(\tau)$, it is consequently possible to give the analytical statement of acceleration $\ddot{q}(\tau)$.

$$\dot{q}(\tau)=-(\xi\omega)e^{-\xi\omega}\left[C_1\cos(\omega_d\tau)+C_2\sin(\omega_d\tau)\right]$$
$$+e^{-\xi\omega}\left(-C_1\omega_d\sin(\omega_d\tau)+C_2\omega_d\cos(\omega_d\tau)\right)-\frac{1}{\omega^2}$$
$$\ddot{q}(\tau)=+(\xi\omega)^2e^{-\xi\omega}\left[C_1\cos(\omega_d\tau)+C_2\sin(\omega_d\tau)\right]$$
$$+(\xi\omega)e^{-\xi\omega}\left(-C_1\omega_d\sin(\omega_d\tau)+C_2\omega_d\cos(\omega_d\tau)\right)$$
$$-(\xi\omega)e^{-\xi\omega}\left(-C_1\omega_d\sin(\omega_d\tau)+C_2\omega_d\cos(\omega_d\tau)\right)$$
$$+e^{-\xi\omega}\left[-C_1\omega_d^2\cos(\omega_d\tau)-C_2\omega_d^2\sin(\omega_d\tau)\right]$$
$$\ddot{q}(\tau)=\left[(\xi\omega)^2-\omega_d^2\right]e^{-\xi\omega}.\left[C_1\cos(\omega_d\tau)+C_2\sin(\omega_d\tau)\right]$$
or
$$\omega_d^2=\omega^2(1-\xi^2)\text{, d'où}$$
$$\ddot{q}(\tau)=\omega^2e^{-\xi\omega}\left[C_1\cos(\omega_d\tau)+C_2\sin(\omega_d\tau)\right]$$

however

$$\omega_d^2=\omega^2(1-\xi^2)$$

from where:

$$\ddot{q}(\tau)=\omega^2e^{-\xi\omega}\left[C_1\cos(\omega_d\tau)+C_2\sin(\omega_d\tau)\right]$$

6 Bibliography

1) R.J. GIBERT: Vibrations of structures, Collection of the Management of the Studies and Searches of Électricité de France, n°69, Eyrolles 1988.

2) N.C. NIGAM & PC JENNINGS: Calculation of Réponse will spectra from motion earthquake Bull. of the Seismological society of America, Vol.59 n°2 pp 909 - 922 April 1969.

3) D. SELIGMANN, L. VIVAN: Seismic response by spectral method [R4.05.03].

7 Description of the versions of the document

Version Aster	Author (S) Organization (S)	Description of the modifications
6	D.Selligmann, EDF/DER/MMN O.Boiteau, EDF-R&D/SINETICS	initial Text