Harmonic response

Abstract

This document presents the theoretical bases of the computation of the permanent mode of the response of a mechanical system complexes, with linear behavior, subjected to a harmonic dynamic stress. The computation relates indifferently directly to the system modelled in finite elements, or represented by a modal base; in this last case if modal base is the product of the technique of substructuring see document [R4.06.03].

Contents

1 Introduction

Into the harmonic problems, the studied system are subjected to an excitation varying like the product of an unspecified function of space by a sinusoidal function of time.
To search the response consists in calculating the field of the quantities represented by the ddls of the modelization in finite elements of the system. When the system has a linear behavior the response of the field of the quantities observed tends quickly (because of extinction of its transitory component by dissipation interns) towards a permanent mode: the resulting field varies finally harmonically like the excitation. It is this permanent mode of the response that one proposes to calculate.

General notations:

t	:	time
P	:	Not running of the model
ω	:	Pulsation ($rad.s^{-1}$)
j	:	Imaginary pure unit $\left(j^2=-1\right)$
$\mathbf{M}$	:	Mass matrix resulting from the modelization finite elements
$\mathbf{K}$	:	Stiffness matrix resulting from the modelization finite elements
$\mathbf{C}$	:	Damping matrix resulting from the modelization finite elements
$\mathbf{q}$	:	Vector of the degrees of freedom resulting from the modelization finite elements
$\mathbf{f}_{\mathbf{ext}}$	:	Vector of the external forces to the system
$\boldsymbol{\Phi}$	:	Stamp vectors of the base of substructures
$\boldsymbol{\eta}$	:	Vector of the generalized degrees of freedom

2 harmonic Equation

We establish the dynamic equation in the case of a harmonic request for three kinds of mechanical systems:

pure structures (without fluid),
pure fluids (without structure) with linear “acoustic” behavior,
analog and digital systems structures and fluids in interaction fluid-structure.

2.1 Harmonic equation of structures

the vibratory behavior of a pure structure results from the external forces which are applied to him.
The quantity with calculating is displacement in any point P of the model.

2.1.1 Direct computation

In the case of direct computation on the model in finite elements we can write:

$$\mathbf{M}\,\ddot{\mathbf{u}}+\mathbf{C}\,\dot{\mathbf{u}}+\mathbf{K}\;\mathbf{u}=\mathbf{f}_{ext}(P,t) \qquad \text{éq 2.1.1-1}$$

where:

$\mathbf{M}$	is the matrix (real) of mass resulting from the modelization finite elements from S ,
$\mathbf{C}$	is the matrix (real) of damping exit of the modelization finite elements of S ,
$\mathbf{K}$	is the matrix (real) of stiffness resulting from the modelization finite elements from S ,
$\mathbf{f}_{ext}(P,t)$	is the vector (complex) of field of the external forces applied to S ,
$\mathbf{u}$, $\dot{\mathbf{u}}$ and $\ddot{\mathbf{u}}$	are the vectors (complex) displacement, velocity and acceleration, functions of P and t , resulting from the modelization finite elements.

In a harmonic problem, one imposes a loading dynamic, spatially unspecified, but sinusoidal in time. One is interested then has the stabilized answer of the system, without taking account of the transitory part.

The field of the external forces is written :

$$\mathbf{f}_{ext}(P,t)=\{\mathbf{f}_{ext}(P)\}e^{j\omega t}$$

The field of displacements is written:

$$\mathbf{u}(P,t)=\{\mathbf{u}(P)\}e^{j\omega t}$$

The velocity fields and of acceleration are written:

$$\dot{\mathbf{u}}(P,t)=j\omega\{\mathbf{u}(P)\}e^{j\omega t}$$
$$\ddot{\mathbf{u}}(P,t)=-\omega^2\{\mathbf{u}(P)\}e^{j\omega t}$$

Finally the structure S checks the following equation:

$$(\mathbf{K}+j\omega\mathbf{C}-\omega^2\mathbf{M})\{u\}=\{\mathbf{f}_{ext}(P)\}$$ éq 2.1.1-2

Typical case : if damping is of hysteretic **type** “total” the equation [éq 2.1.1-1] becomes:

$$\mathbf{M}\ddot{\mathbf{u}}+\mathbf{K}(1+j\eta)\mathbf{u}=\mathbf{f}_{ext}(P,t)$$ éq 2.1.1-3

where η is a total loss ratio (cf [R5.05.04]).
Then the equation [éq 2.1.1-2] is replaced by:

$$(\mathbf{K_c}-\omega^2\mathbf{M})\{\mathbf{u}\}=\{\mathbf{f}_{ext}(P)\}$$ éq 2.1.1-4

where:

$\mathbf{M}$	is the matrix (real) of mass resulting from the modelization finite elements from S ,
$\mathbf{K_c}=\mathbf{K}+j\eta\mathbf{K}$	is a complex *stiffness matrix* .

2.1.2 Computation on the basis of harmonic

response modal base The computation by the method of modal synthesis consists has to search the field of unknown displacement, resulting from the modelization finite elements, on an adapted space, of reduced size (transformation of Ritz).
One will refer to the documents [R4.06.02] and [R4.06.03].
If one rather uses this method the equation [éq 2.1.1-2] is projected on the basis of the modal base S and one leads to the following harmonic equation :

$$(\overline{\mathbf{K}}+j\omega\overline{\mathbf{C}}-\omega^2\overline{\mathbf{M}})\{\eta\}=\{\overline{\mathbf{f}}_{ext}\}$$ éq 2.1.2-1

where:

$\bar{\mathbf{M}}=\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{M}\boldsymbol{\Phi}$	is the matrix (real) of generalized mass of S ,
$\bar{\mathbf{C}}=\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{C}\boldsymbol{\Phi}$	is the matrix (real) of damping generalized of S ,
$\bar{\mathbf{k}}=\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{k}\,\boldsymbol{\Phi}$	is the matrix (real) of stiffness generalized of S ,
$\{\bar{\mathbf{f}}_{ext}\}=\boldsymbol{\Phi}^{\mathrm{T}}\{\mathbf{f}_{ext}\}$	is the vector (complex) generalized harmonic external forces applied to S ,
$\boldsymbol{\Phi}$	is the matrix (real) modal vectors of the base of Ritz of S ,
$\{\boldsymbol{\eta}(P)\}$	is the vector (complex) generalized harmonic displacements.

Once $\{\boldsymbol{\eta}(P)\}$ determined by éq 2.1.2-1 one makes a restitution on physical base (cf [R4.06.02]).

2.2 Harmonic equation of the acoustic fluids

the document [R4.02.01] described the modelization by finite elements of a fluid system (without transport) having a linear acoustic behavior.
The fluid system F undergoes a harmonic request acoustic velocity on part of its border. The harmonic response is described by the following equation [éq 2.2-1], where the quantity with calculating is the acoustic pressure in any point P of the model.

$$(\mathbf{K}+j\omega\,\mathbf{C}-\omega^2\mathbf{M})\{\mathbf{p}(P)\}=-j\omega\,\{\mathbf{v_n}(P)\} \qquad \text{éq 2.2-1}$$

where:

$\mathbf{M}$	is the matrix (complex) of “mass” acoustic exit of the modelization finite elements of F ,
$\mathbf{C}$	is the matrix (complex) of “damping” acoustic exit of the modelization finite elements of F , and in the species of the edge $\partial_z F$ where an acoustic impedance is applied,
$\mathbf{K}$	is the matrix (complex) of “stiffness” acoustic exit of the modelization finite elements of F ,
$\mathbf{v_n}(P,t)=\{\mathbf{v_n}(P)\}e^{j\omega t}$	Where $\{\mathbf{V_n}(P)\}$ is the vector (complex) of field normal acoustic velocities applied to the border $\partial_v F$ of F where acoustic velocities are applied,
$\mathbf{p}(P,t)=\{\mathbf{p}(P)\}e^{j\omega t}$	Where $\{\mathbf{p}(P)\}$ is the vector (complex) acoustic pressures resulting from the modelization finite elements from F .

2.3 Harmonic equation of the systems fluid-structures

the document [R4.02.02] described the modelization by finite elements of a system $F+S$ made up of a fluid part (without transport) F in interaction with a part structure S (interaction in $F\cup S$). Fluid and structure have a linear behavior.
The fluid system F undergoes a harmonic request normal acoustic velocity on part of its border. The harmonic response is described by the following equation [éq 2.3-1], where the quantities with calculating are:

acoustic pressure in any point P of the fluid F ,
displacement in any point P of structure S ,
as an auxiliary potential ϕ of displacement in any point P of the fluid F ,

$$(\mathbf{K}-\omega^2\mathbf{M}-j\omega^3\mathbf{I})\begin{Bmatrix}\mathbf{u}(P)\\ \mathbf{p}(P)\\ \phi(P)\end{Bmatrix}=+j\omega\{\mathbf{v_n}(P)\}$$ éq 2.3-1

where:

$\mathbf{M}$	is the matrix (real) of "mass" fluid-structure resulting from the modelization finite elements of the fields F and S
$\mathbf{I}$	is the matrix (real) of "impedance" fluid exit of the modelization finite elements of edge $\partial_z F$ of the field F where one applies an impedance
$\mathbf{K}$	is the matrix (real) of "stiffness" fluid-structure resulting from the modelization finite elements of the fields F and S
$\mathbf{v_n}(P,t)=\{\mathbf{v_n}(P)\}e^{j\omega t}$	Where $\mathbf{v_n}(P)$ is the vector (real) field normal acoustic velocities applied to the border $\partial_v F$ of F
$\mathbf{u}(P,t)=\{\mathbf{u}(P)\}e^{j\omega t}$	is the vector (complex) field of displacement in structure S
$\mathbf{p}(P,t)=\{\mathbf{p}(P)\}e^{j\omega t}$	is the vector (complex) acoustic field of pressure in the fluid F
$\phi(P,t)=\{\phi(P)\}e^{j\omega t}$	is the vector (complex) field of potential of displacement in the fluid F

2.4 general harmonic Equation

With an aim of taking into account all the harmonic cases of equations operator DYNA_LINE_HARM of *Code_Aster* solves the following general harmonic equation (cf [U4.53.11]):

$$(-j\omega^3\mathbf{I}-\omega^2\mathbf{M}+j\omega\mathbf{C}+\mathbf{K})\{\mathbf{q}\}=\left\{\sum_{i=1}^{k}h_i(f)\cdot\omega^{n_i}\cdot e^{j\pi\frac{\phi_i}{180}}\cdot\mathbf{g_i}(P)\right\}$$ éq 2.4-1

where:

$\mathbf{I}$	Stamp fluid "impedance" possible exit of the modelization finite elements,
$\mathbf{M}$	Matrix of "mass" resulting from the modelization finite elements,
$\mathbf{C}$	Matrix of "damping" exit of the modelization finite elements,
$\mathbf{K}$	Matrix of "stiffness" resulting from the modelization finite elements,
$\{\mathbf{q}(P)\}$	Vector of the degrees of freedom resulting from the modelization finite elements,
$\{\mathbf{g_i}(P)\}$	Vector field at nodes corresponding to one or more loads of force or acoustic or potential velocity or imposed motion,
$h_i(f)$	real or complex Function of the frequency f ,
$\omega=2\mathrm{pf}$	Pulsation
n_i	Power of the pulsation when the loading is function of the pulsation,
ϕ_i	Phase in degrees of each component of the excitation compared to a reference of phase.

As example if one takes the case of a fluid system modelled in acoustics, without degrees of freedom imposed, simply solicited on part of his border by a velocity field norm $\mathbf{v_n}(P,t)=\{\mathbf{v_n}(P)\}e^{j\omega t}$, the terms of the equation [**éq 2.4-1**] become:

$\mathbf{I}$	*non-existent* ,
$\mathbf{M}$	Mass matrix resulting from the modelization finite elements acoustics,
$\mathbf{C}$	*Possibly* damping matrix resulting from the modelization finite elements acoustics if impedance on border,
$\mathbf{K}$	Stiffness matrix resulting from the modelization finite elements acoustics
$\{\mathbf{q}(P)\}$	$=\{\mathbf{p}(P)\}$, vector of the pressures to the nodes
$\{\mathbf{g_i}(P)\}$	$=\{\mathbf{v_n}(P)\}$, vector velocity field norm with the sides (finite elements)
$h_i(f)$	$=-1.$ (constant),
$\omega=2\mathrm{pf}$	Pulsation,
n_i	$=1$
ϕ_i	$=0$

Note:

*In addition to the solution of the harmonic equation [**éq 2.4-1**], Code_Aster makes it possible to calculate derivatives of this solution compared to the loading $\{\mathbf{g_i}(P)\}$ or to parameters of mass, stiffness or damping ($\mathbf{M}$ $\mathbf{K}$, $\mathbf{C}$). The equations whose these derivatives are solutions and the relative theoretical developments are in [R4.03.04].*

3 Bibliography

1) R. DAUTRAY, J-L. LIONS, “Analyzes mathematical and numerical computation for sciences and the technology”, Volume 2, Masson, 1985.

2) G. DHATT, G. TOUZOT, “a presentation of the finite element method”, Maloine S.A., Paris, 1984.

4 Description of the versions of the document

Version Aster	Author (S) Organization (S)	Description of the modifications
6	F.STIFKENS EDF-R&D/AMA	initial Text