Relation of nonlinear elastic behavior in great displacements

Summary:

One proposes to describe here a relation of nonlinear elastic behavior which coincides with the elastoplastic law of Hencky-Von Put (isotropic work hardening) in the case of a loading which induces a radial and monotonous evolution constraints in any point of the structure. This model is selected in the order STAT_NON_LINE via the keyword RELATION=' ELAS_VMIS_LINE' or 'ELAS_VMIS_TRAC' under the keyword factor BEHAVIOR.

One extends then this relation of behavior to great displacements and great rotations, insofar as it derives from a potential (hyperelastic law); this functionality is selected via the keyword DEFORMATION=' GROT_GDEP'. It is available for all the isoparametric elements 2D and 3D.

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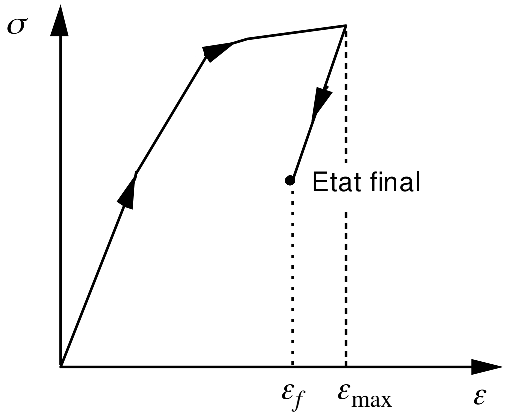
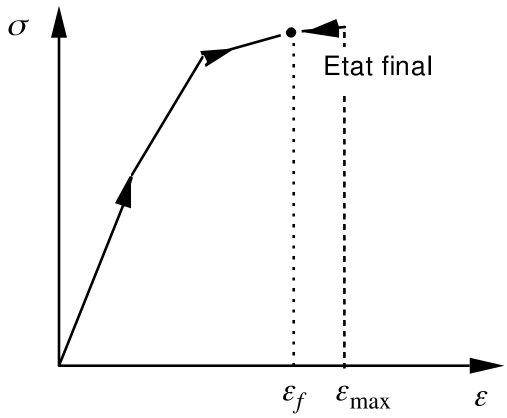
1.1 Objective

Within the framework of the comprehensive approach in breaking process, one can give a direction to the rate of refund of energy only for hyperelastic relations of behavior, i.e. which derive from a potential, free energy. In order to be able nevertheless to deal with plastic problems élasto -, one proposes a relation of nonlinear elastic behavior which in the case of leads to results identical to those obtained by the relation of plastic behavior of Hencky-Von Put (isotropic work hardening) an evolution of loading radial and monotonous in any point. The definition of the characteristics of the material (keyword `DEFI_MATERIAU`) is identical to that of the isotropic plastic behavior. For further information on the model, one will be able to refer to [bib1]. To illustrate the common points and the differences between the models plastic and rubber band, one presents Ci - below a traction diagram then compression obtained for a unidimensional bar.



Elasto-plasticité



Elasticité non linéaire

1.2 Relation of behavior

After integration in time of the relation of behavior of Hencky-Von Put, formulated in speeds of strains and stresses in [R5.03.02] which one adopts the notations, the expression of the constraints according to the deformations is:

$$\begin{cases} \boldsymbol{\sigma} = K(\mathrm{tr}\,\varepsilon)\,\mathbf{Id} + G(\varepsilon_{eq})\,\tilde{\varepsilon} & \text{éq } 1.2-1 \\ -\mathrm{si} \quad \varepsilon_{eq} \le \dfrac{\sigma^y}{2\mu} & \\ \quad G = 2\mu \quad \text{et} \quad p = 0 & \\ -\mathrm{si} \quad \varepsilon_{eq} > \dfrac{\sigma^y}{2\mu} & \\ \quad G = \dfrac{R(p)}{\varepsilon_{eq}} \quad \text{et} \quad p \text{ tel que : } \quad p + \dfrac{R(p)}{3\mu} = \dfrac{2\,\varepsilon_{eq}}{3} & \text{éq } 1.2-2 \end{cases}$$

In a way similar to plasticity, the function of work hardening $R(p)$ is deduced from the abundant data by a simple tensile test (linear work hardening with the keyword ELAS_VMIS_LINE or defined well by points with the keyword ELAS_VMIS_TRAC, cf [R5.03.02]).

As for the variable p, it deserves a few moments of attention. In the plastic model, its significance is clear. It is the cumulated plastic deformation, always increasing; it is an internal variable of the model. On the other hand, in the elastic case, it does not have any more the internal statute of variable, since there is no dissipation. Moreover, it decrease during discharges. In fact, its value coincides with that obtained in plasticity as long as the evolution of the loading is radial and monotonous.

In addition to the relation of behavior itself, it is necessary to know the value of the free energy for a state given for calculations of rate of refund of energy. Without demonstration, this potential of which derives the relation of behavior is worth:

- If $\varepsilon_{eq} \leq \frac{\sigma^y}{2\mu}$ $\quad \psi(\varepsilon)=\frac{1}{2}K(\mathrm{tr}\,\varepsilon)^2+\frac{2\mu}{3}\varepsilon_{eq^2}$
- If $\varepsilon_{eq} > \frac{\sigma^y}{2\mu}$ $\quad \psi(\varepsilon)=\frac{1}{2}K(\mathrm{tr}\,\varepsilon)^2+\frac{R(p(\varepsilon_{eq}))^2}{6\mu}+\int_0^{p(e_{eq})} R(s)\,ds$

éq 1.2-3

1.3 Resolution of the equation in p

One could note in the preceding paragraph that the expression of the constraints requires the solution of an equation relating to the variable p. Insofar as the function of work hardening R is increasing, this equation can be written by gathering the terms where appear p in the first member (who is then increasing with p):

More precisely, the first member is linear per pieces in p. To solve the equation, it is then enough sequentially to traverse each interval until finding that in which the solution is. An equation closely connected provides the value then of p.

1.4 Calculation of the relation of behavior and tangent rigidity

The calculation of the constraints and tangent rigidity, i.e. the variation of the constraints compared to the deformations, is carried out according to the algorithm presented below. By adopting the convention of *Code_Aster*, the constraints and the deformations are arranged in a vector with six components, while tangent rigidity is a matrix 6×6 .

$$\{\boldsymbol{\varepsilon}\}=\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \sqrt{2}\,\varepsilon_{xy}\\ \sqrt{2}\,\varepsilon_{xz}\\ \sqrt{2}\,\varepsilon_{yz}\end{Bmatrix}\{\boldsymbol{\sigma}\}=\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sqrt{2}\,\sigma_{xy}\\ \sqrt{2}\,\sigma_{xz}\\ \sqrt{2}\,\sigma_{yz}\end{Bmatrix}\{\mathbf{1}\}=\begin{Bmatrix}1\\1\\1\\0\\0\\0\end{Bmatrix}$$

Relation of behavior:

$$\{\boldsymbol{\sigma}\}=K(\operatorname{tr}\boldsymbol{\varepsilon})\{\mathbf{1}\}+G\{\tilde{\boldsymbol{\varepsilon}}\}$$

Tangent rigidity:

$$\frac{d\{\boldsymbol{\sigma}\}}{d\{\boldsymbol{\varepsilon}\}}=\mathbf{K}=[\mathbf{K_1}]+[\mathbf{K_2}]$$

- $$[\mathbf{K_1}]=\frac{3\mathrm{K}-G}{3}\{\mathbf{1}\}\otimes\{\mathbf{1}\}+G[\mathbf{Id}]$$
- $$[\mathbf{K_2}]=\begin{cases}[\mathbf{0}] & \text{si } \boldsymbol{\varepsilon}_{eq}\le\dfrac{\sigma^y}{2\mu}\\ \dfrac{3}{2\varepsilon_{eq}^2}\left[\dfrac{2\mu R'(p)}{R'(p)+3\mu}-G\right]\{\tilde{\boldsymbol{\varepsilon}}\}\otimes\{\tilde{\boldsymbol{\varepsilon}}\} & \text{si } \varepsilon_{eq}>\dfrac{\sigma^y}{2\mu}\end{cases}$$

1.5 Taking into account of deformations of thermal origin

In a way identical to plasticity, one divides the total deflection into a mechanical part which checks the preceding relation of behavior [éq 1.2-1], [éq 1.2-2] and a thermal part, function of the temperature. Let us note moreover that the various characteristics of material can also depend on the temperature.

$$\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}^m+\boldsymbol{\varepsilon}^{th}$$

$$\text{avec}\begin{cases}\boldsymbol{\sigma}=K(\operatorname{tr}\boldsymbol{\varepsilon}^m)\mathbf{Id}+G(\varepsilon_{eq})\tilde{\boldsymbol{\varepsilon}}\\ \boldsymbol{\varepsilon}^{th}=\alpha(T-T_{réf})\mathbf{Id}\end{cases}\quad \textit{éq 1.5.1}$$

α : thermal dilation coefficient

T_{ref} : temperature of reference

It remains to supplement the potential free energy [éq 1.2-3] to include the temperature there. Several choices are possible, depend on the way in which one wishes to define the entropy (derivative of the free energy compared to the temperature). In our case, the adopted potential is:

- If $\varepsilon_{eq} \leq \frac{\sigma^y}{2\mu}$ $\quad \psi(\boldsymbol{\varepsilon}, T) = \frac{1}{2} K \left(\mathrm{Tr}\left(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{th}\right)\right)^2 + \frac{2\mu}{3} \varepsilon_{eq}^2$
- If $\varepsilon_{eq} > \frac{\sigma^y}{2\mu}$ $\quad \psi(\boldsymbol{\varepsilon}, T) = \frac{1}{2} K \left(\mathrm{Tr}\left(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{th}\right)\right)^2 + \frac{R\left(p\left(\varepsilon_{eq}\right)\right)^2}{6\mu} + \int_0^{p(\varepsilon_{eq})} R(s)\, ds$

1.6 Particular treatment of the plane constraints

Usually, one seeks to determine the constraints knowing the deformations and the temperature. However, it is not completely any more the case under the assumption of the plane constraints insofar as three of the components of the tensor of the deformations are henceforth unknown, the dual sizes being fixed:

$$\varepsilon_{xz}, \quad \varepsilon_{yz} \quad \text{and} \quad \varepsilon_{zz} \qquad \text{unknown factors}$$

$$\sigma_{xz} = \sigma_{yz} = \sigma_{zz} \qquad = 0$$

It is thus necessary to start by determining these unknown components. The adopted method is exposed in [bib1] and [R5.03.02]. One can however recall here that the components xz and yz do not pose a problem, being given the form of the relation of behavior [éq 1.2-1]:

$$\varepsilon_{xz} = \varepsilon_{yz} = 0$$

On the other hand determination of the component zz require the solution (digital) of a nonlinear scalar equation.

Lastly, a last warning is essential. Unlike plane deformations, the solutions which one obtains under the assumption of the plane constraints are generally not exact insofar as they do not check the conditions of geometrical compatibility (integrability of the field of deformations). They are only approximate solutions.

2 Elasticity in great transformations

2.1 Objective

Henceforth, one proposes to take into account great displacements and great rotations, functionality accessible by the keyword DEFORMATION=' GROT_GDEP' in the order STAT_NON_LINE. Let us specify as of now that one restricts oneself with isoparametric finite elements (D_PLAN, C_PLAN, AXIS and 3D) for which the discretization of the continuous problem does not raise particular difficulties, cf [R3.01.00].

To this end, it is admitted that the second tensor of the constraints of Piola-Kirchhof, S , drift of the potential of Hencky-Von Put expressed using the deformation of Green-Lagrange $\mathbf{E}$:

$$\mathbf{S}=\frac{\partial\Psi}{\partial E}(\mathbf{E})$$

Also let us point out the definitions of $\mathbf{E}$ and $\mathbf{S}$. One can also find additional information in [bib1].

$$\mathbf{F}=\mathbf{Id}+\mathbf{Grad}(\mathbf{u}) \quad \mathbf{E}=\frac{1}{2}\left({}^{T}FF-\mathbf{Id}\right)$$
$$S=\mathrm{Det}(\mathbf{F})\mathbf{F}^{-1}\boldsymbol{\sigma}\,{}^{T}\mathbf{F}^{-1}$$

Such a relation of behavior, known as hyperelastic, makes it possible in any rigour to take into account great deformations and great rotations. However, we limit ourselves to small deformations, and this for two reasons. First of all, the relation of behavior adopted does not present the good properties (polyconvexity) to ensure the existence of solutions and does not control either important compressions. Then, the plastic behavior differs notably from a behavior hyperelastic as soon as the deformations become appreciable. It is for these reasons that we chose to preserve the assumption of small deformations, thus escaping the polemic from the great deformations.

2.2 Virtual work of the external efforts: assumption of the dead loads

To deal with the problem of hyperelastic structural analysis, one seeks to write balance in variational form on the initial configuration. In particular, it is necessary to express the virtual work of the external efforts on this same initial configuration what requires the additional assumption of dead loads: it is supposed that the loading does not depend on the geometrical transformation. Typically, an imposed force is a dead load while the pressure is a following loading since it depends on the orientation of the face of application, therefore of the transformation. Under this assumption, the virtual work of the external efforts is written like a linear form:

$$\delta W_{ext}.\boldsymbol{\delta}\,\mathbf{v}=\int_{\Omega_o}\rho_o F_i\,\delta v_i\,d\Omega_o+\int_{\partial_F\Omega_o}T_i^d\,\delta v_i\,dS_o$$

$\mathbf{F}$: voluminal loading

$\mathbf{T^d}$: surface loading being exerted on the edge $\partial_F\boldsymbol{\Omega}_\mathbf{o}$

2.3 Virtual work of the interior efforts

We will not give here a demonstration of the expressions presented. For that, one will be able to refer to [bib1] and [R7.02.03]. There still, we choose the initial configuration like configuration of reference, to express the work of the interior efforts:

$$SW_{int}.\boldsymbol{\delta}\,\mathbf{v}=-\int_{\Omega_o} F_{ik} S_{kl}\,\delta v_{i,l}\,d\Omega_o$$

with: $\delta v_{i,l}=\frac{\partial dv_i}{\partial X_l}$

In the optics of a resolution by a method of Newton, it is important to also express the variation second of the virtual work of the interior efforts, namely:

$$d^2 W_{\text{int}}.\boldsymbol{\delta}\,\mathbf{u}.\boldsymbol{\delta}\,\mathbf{v}= -\int_{\Omega_o} \delta u_{i,k} S_{kl}\delta v_{i,l}\,d\Omega_o$$ Geometrical rigidity

$$\ldots\ -\int_{\Omega_o} \delta u_{i,q} F_{ip}\frac{\partial^2 y}{\partial E_{pq}\partial E_{kl}} F_{jk}\,\delta v_{j,l}\,d\Omega_o$$ Elastic rigidity

2.4 Variational formulation

We now have at our disposal all the ingredients to write the variational formulation of the problem:

$\delta W_{\text{int}}.\boldsymbol{\delta}\,\mathbf{v}+SW_{ext}.\boldsymbol{\delta}\,\mathbf{v}=0$, $\forall\boldsymbol{\delta}\,\mathbf{v}$ kinematically acceptable

3 Bibliography

1. LORENTZ E.: A nonlinear relation of behavior hyperelastic. Note interns EDF DER, HI-74/95/011/0, 1995.

4 Description of the versions of the document

Version Aster	Author (S) Organization (S)	Description of the modifications
3	E.Lorentz EDF- R&D/MMN	Initial text
10,1	J.M.Proix EDF-R&D/AMA	Change of vocabulary: GREEN becomes GROT_GDEP