Modeling élasto (visco) plastic with isotropic work hardening in great deformations

Summary

One describes here a thermoelastoplastic relation between behaviour and isotropic work hardening written in great deformations and proposed by Simo and Miehe. This model is available in the order STAT_NON_LINE via the keyword RELATION: 'VMIS_ISOT_TRAC', 'VMIS_ISOT_PUIS' or 'VMIS_ISOT_LINE' under the keyword factor BEHAVIOR and with the keyword DEFORMATION: 'SIMO_MIEHE'. A viscous version of this model is also proposed: 'VISC_ISOT_TRAC' and 'VISC_ISOT_LINE'.

This model is established for three-dimensional modelings (3D), axisymmetric (AXIS) and in plane deformations (D_PLAN).

One presents the writing and the digital processing of this law, as well as the associated variational formulation. It is about a variational formulation eulérienne, with reactualization of the geometry and which takes account of the rigidity of behavior and geometrical rigidity.

Contents

1 Introduction

We present here a thermoelastoplastic law of behavior written in great deformations proposed by SIMO J.C and MIEHE C. [bib1] which tends in small deformations towards the elastoplastic model of behavior to isotropic work hardening and criterion of Von Mises, described in [R5.03.02]. The kinematics choices allow, as with the simple reactualization available via the keyword `PETIT_REAC`, to treat great displacements and great deformations but also of great rotations in an exact way.
Specificities of this model are the following ones:

- just like in small deformations, one supposes the existence of a slackened configuration, i.e. locally free of constraint, which makes it possible to break up the total deflection into a thermoelastic part and a plastic part,
- the decomposition of this thermoelastic deformation into cubes parts and plastic is not additive any more as in small deformations (or for the models great deformations written in rate of deformation with for example a derivative of Jaumann) but multiplicative,
- the elastic strain are measured in the current configuration (deformed) while the plastic deformations are measured in the initial configuration,
- as in small deformations, the constraints depend only on the thermoelastic deformations,
- the plastic deformations are done with constant volume. The variation of volume is then only due to the thermoelastic deformations,
- this model led during its digital integration to a model incrémentalement objective (cf [§3.3]) what makes it possible to obtain the exact solution in the presence of great rotations.

A viscous version of this model is also available (law in hyperbolic sine like in the case of the model of Rousselier `ROUSS_VISC`, cf [R5.03.07]).

Thereafter, one briefly points out some concepts of mechanics in great deformations, then one presents the relations of behavior of the model and his digital integration to treat the equilibrium equations.

One proposes a variational formulation eulérienne, with reactualization of the geometry. For this reason, one expresses the work of the interior efforts and his variation (with an aim of a resolution by the method of Newton) for the continuous problem, which respectively provide after discretization by finite elements the vector of the interior forces and the tangent matrix.

Nota bene :

One will find in [bib2] or [bib3] a presentation deepened on the great deformations.
This document is extracted from [bib4] where one makes a more detailed presentation of the elastoplastic model, of his digital integration and where some examples of validation are given.

2 Notations

One will note by:

$\mathbf{Id}$	matrix identity
$\mathrm{tr}\, A$	trace of the tensor $\mathbf{A}$
$\mathbf{A}^{\mathbf{T}}$	transposed of the tensor $\mathbf{A}$
$\det A$	determinant of $\mathbf{A}$
$\langle X \rangle$	positive part of X
$\tilde{A}$	deviatoric part of the tensor $\mathbf{A}$ defined by $\tilde{A}=A-(\frac{1}{3} tr\, A)\mathbf{Id}$
$:$	doubly contracted product: $\mathbf{A}:\mathbf{B}=\sum_{i,j} A_{ij} B_{ij}=tr(\mathbf{A}\mathbf{B}^{T})$
$\otimes$	tensorial product: $(\mathbf{A}\otimes\mathbf{B})_{ijkl}=A_{ij} B_{kl}$
A_{eq}	equivalent value of von Mises defined by $A_{eq}=\sqrt{\frac{3}{2}\tilde{A}:\tilde{A}}$
$\nabla_{\mathbf{X}}\mathbf{A}$	gradient: $\nabla_X \mathbf{A}=\frac{\partial \mathbf{A}}{\partial \mathbf{X}}$
$\mathrm{div}_x \mathbf{A}$	divergence: $(\mathrm{div}_x \mathbf{A})_i=\sum_j \frac{\partial A_{ij}}{\partial x_j}$
λ, μ, E, ν, K	moduli of the isotropic elasticity
σ_y	elastic limit
α	coefficent of thermal dilation
T	temperature
T_{ref}	temperature of reference

In addition, within the framework of a discretization in time, all the quantities evaluated at the previous moment are subscripted by $^-$, quantities evaluated at the moment $t+\Delta t$ are not subscripted and the increments are indicated by Δ. One has as follows:

$$\Delta Q=Q-Q^-$$

3 Recalls on the great deformations

3.1 Kinematics

Let us consider a solid subjected to great deformations. That is to say Ω_0 the field occupied by the solid before deformation and $\Omega(t)$ the field occupied at the moment t by the deformed solid.

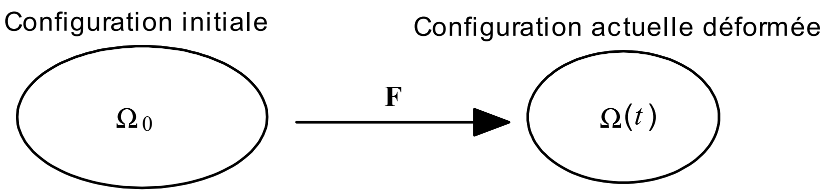


Figure 3.1-a: Representation of the initial and deformed configuration

In the initial configuration Ω_0 , the position of any particle of the solid is indicated by $\mathbf{X}$ (Lagrangian description). After deformation, the position at the moment t particle which occupied the position $\mathbf{X}$ before deformation is given by the variable $\mathbf{x}$ (description eulérienne).

The total movement of the solid is defined, with $\mathbf{u}$ displacement, by:

$$\mathbf{x}=\hat{\mathbf{x}}(\mathbf{X},t)=\mathbf{X}+\mathbf{u}$$

To define the change of metric in the vicinity of a point, the tensor gradient of the transformation is introduced $\mathbf{F}$:

$$\mathbf{F}=\frac{\partial\hat{\mathbf{x}}}{\partial\mathbf{X}}=\mathbf{Id}+\nabla_{\mathbf{X}}\mathbf{u}$$

The transformations of the element of volume and the density are worth:

$$d\Omega=Jd\Omega_o \quad \text{with} \quad J=\det F=\frac{\rho_o}{\rho}$$

where ρ_o and ρ are respectively the density in the configurations initial and current.

Various tensors of deformations can be obtained by eliminating rotation in the local transformation. For example, by directly calculating the variations length and angle (variation of the scalar product), one obtains:

$$\mathbf{E}=\frac{1}{2}(\mathbf{C}-\mathbf{Id}) \quad \text{with} \quad \mathbf{C}=\mathbf{F}^{\mathrm{T}}\mathbf{F}$$

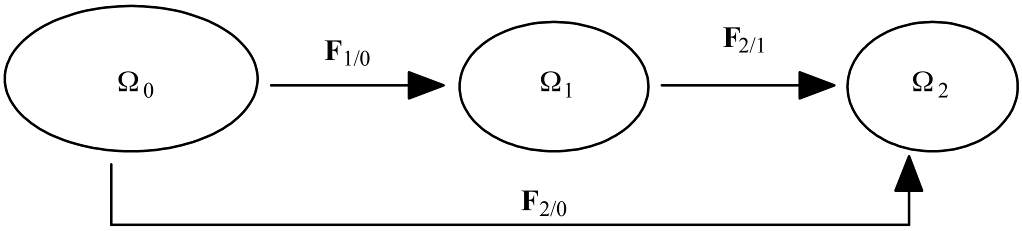
$$\mathbf{A}=\frac{1}{2}(\mathbf{Id}-\mathbf{b}^{-1}) \quad \text{with} \quad \mathbf{b}=\mathbf{F}\mathbf{F}^{T}$$

$\mathbf{E}$ and $\mathbf{A}$ are respectively the tensors of deformation of Green-Lagrange and Euler-Almansi and $\mathbf{C}$ and $\mathbf{b}$, tensors of right and left Cauchy-Green respectively.

In Lagrangian description, one will describe the deformation by the tensors $\mathbf{C}$ or $\mathbf{E}$ because they are quantities defined on Ω_0 , and of description eulérienne by the tensors $\mathbf{b}$ or $\mathbf{A}$ (definite on Ω).

Notice :

That is to say a solid undergoing two successive transformations, for example the first transformation makes pass the solid of the initial configuration Ω_0 with a configuration Ω_1 (tensor gradient $\mathbf{F_{1/0}}$ and vector displacement $\mathbf{u_{1/0}}$), then the second transformation of the configuration Ω_1 with Ω_2 (tensor gradient $\mathbf{F_{2/1}}$ and vector displacement $\mathbf{u_{2/1}}$).



The passage of the configuration Ω_0 with Ω_2 is given by the tensor gradient $\mathbf{F_{2/0}}$ (displacement $\mathbf{u_{2/0}}=\mathbf{u_{2/1}}+\mathbf{u_{1/0}}$) such as:

$$\mathbf{F_{2/0}}=\mathbf{F_{2/1}}\mathbf{F_{1/0}}$$

One obtains then, for example, for the tensor of deformation of Green-Lagrange $\mathbf{E}$

$$\mathbf{E_{2/0}}=\mathbf{F_{1/0}^T}\mathbf{E_{2/1}}\mathbf{F_{1/0}}+\mathbf{E_{1/0}}$$

where $\mathbf{E_{2/0}}$, $\mathbf{E_{1/0}}$ and $\mathbf{E_{2/1}}$ are the deformations of Green-lagrange of the configurations Ω_2 compared to Ω_0 associated with $F_{2/0}$, Ω_1 compared to Ω_0 associated with $F_{1/0}$ and Ω_2 compared to Ω_1 associated with $\mathbf{F_{2/1}}$, respectively.

This constitutes one of the difficulties encountered at the time of the writing of a law of behavior in great deformations because one cannot write any more one formula similar to that written in small deformations, namely $\boldsymbol{\varepsilon}_{2/0}=\boldsymbol{\varepsilon}_{2/1}+\boldsymbol{\varepsilon}_{1/0}$ where $\boldsymbol{\varepsilon}$ is the linearized tensor of total deflection.

To find $\boldsymbol{\varepsilon}_{2/0}=\boldsymbol{\varepsilon}_{2/1}+\boldsymbol{\varepsilon}_{1/0}$ in small deformations starting from the expression of $\mathbf{E_{2/0}}$, it is necessary to neglect all the terms of order 2 of $\nabla\mathbf{u_{2/0}}$, $\nabla\mathbf{u_{1/0}}$ and $\nabla\mathbf{u_{2/1}}$. In this case, one has $\mathbf{E_{2/0}}\simeq\boldsymbol{\varepsilon}_{2/0}$, $\mathbf{E}_{1/0}\simeq\boldsymbol{\varepsilon}_{1/0}$ and $\mathbf{F_{1/0}^T}\mathbf{E_{2/1}}\mathbf{F_{1/0}}\simeq\boldsymbol{\varepsilon}_{2/1}$.

3.2 Constraints

For the model describes here, the tensor of the constraints used is the tensor eulérien of Kirchhoff $\boldsymbol{\tau}$ defined by:

$$J\boldsymbol{\sigma}=\boldsymbol{\tau}$$

where $\boldsymbol{\sigma}$ is the tensor eulérien of Cauchy. The tensor $\boldsymbol{\tau}$ thus result from a “scaling” by the variation of volume of the tensor of Cauchy $\boldsymbol{\sigma}$; this is not the case of other tensors of constraints used (first and second tensor of Piola-Kirchhoff).

In description eulérienne, the equilibrium equations are given by:

$$\begin{aligned} &\mathrm{div}_x\boldsymbol{\sigma}+\rho\,\mathbf{f}=0 \ \text{ sur } \Omega \\ &\boldsymbol{\sigma}\,.\,\mathbf{n}=\mathbf{t^d} \qquad \text{ sur } \partial\Omega^f \end{aligned}$$

where f is the voluminal force applied to the field Ω , $\mathbf{n}$ the normal external with the border $\partial\Omega^f$ and $\partial\Omega^f$ the part of the border of the field Ω where are applied the surface forces $\mathbf{t^d}$.

3.3 Objectivity

When a law of behavior in great deformations is written, one must check that this law is objective, i.e. invariant by any change of space reference frame of the form:

$$\mathbf{x}^*=\mathbf{c}(t)+\mathbf{Q}(t)\mathbf{x}$$

where $\mathbf{Q}$ is an orthogonal tensor which represents the rotation of the reference frame and $\mathbf{c}$ a vector which represents the translation.
More concretely, if one carries out a tensile test in the direction $\mathbf{e_1}$, for example, followed by a rotation of 90° around $\mathbf{e_3}$, which amounts carrying out a tensile test according to $\mathbf{e_2}$, then the danger with a nonobjective law of behavior is not to find a uniaxial tensor of the constraints in the direction $\mathbf{e_2}$ (what is in particular the case with kinematics `PETIT_REAC`).

4 Presentation of the model of behavior

4.1 Kinematic aspect

This model supposes, just like in small deformations, the existence of a slackened configuration Ω^r , i.e. locally free of constraint, which then makes it possible to break up the total deflection into cubes rubber band parts and plastic, this decomposition being multiplicative.
Thereafter, one will note by $\mathbf{F}$ the tensor gradient which makes pass from the initial configuration Ω_0 with the current configuration $\Omega(t)$, by $\mathbf{F^p}$ the tensor gradient which makes pass from the configuration Ω_0 with the slackened configuration Ω^r , and $\mathbf{F^e}$ configuration Ω^r with $\Omega(t)$. The index p refers to the plastic part, the index e with the elastic part.

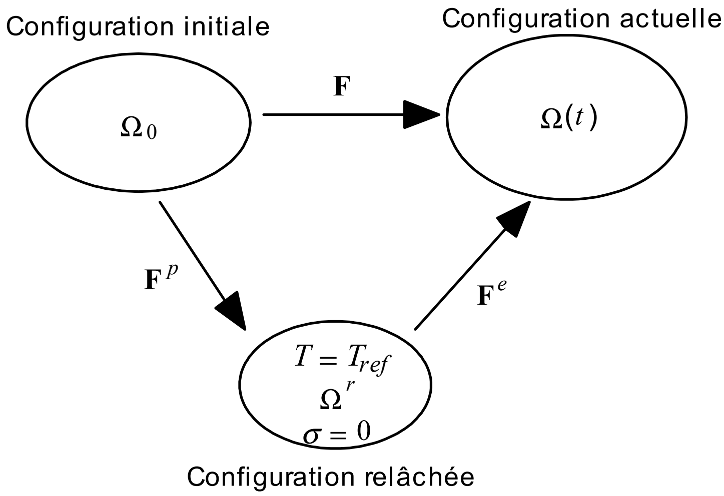


Figure 4.1-a: Decomposition of the tensor gradient $\mathbf{F}$ in an elastic part $\mathbf{F^e}$ and plastic $\mathbf{F^p}$

By composition of the movements, one obtains the following multiplicative decomposition:

$$\mathbf{F}=\mathbf{F^e}\,\mathbf{F^p}$$

The elastic strain are measured in the current configuration with the left tensor eulérien of Cauchy-Green $\mathbf{b}^e$ and plastic deformations in the initial configuration by the tensor $\mathbf{G}^p$ (Lagrangian description). These two tensors are defined by:

$$\mathbf{b}^e=\mathbf{F}^e\,\mathbf{F}^{eT}\ ,\ \mathbf{G}^p=(\mathbf{F}^{pT}\,\mathbf{F}^p)^{-1}\quad \text{from where}\quad \mathbf{b}^e=\mathbf{F}\mathbf{G}^p\,\mathbf{F}^T$$

The model presented is written in order to distinguish the isochoric terms from the terms of change of volume. One introduces for that the two following tensors:

$$\bar{\mathbf{F}}=J^{-1/3}\,\mathbf{F}\quad \text{and}\quad \bar{\mathbf{b}}^e=J^{-2/3}\,\mathbf{b}^e\quad \text{with}\quad J=\det\mathbf{F}$$

By definition, one a: $\det\bar{\mathbf{F}}=1$ and $\det\ \bar{\mathbf{b}}^e=1$.

4.2 Relations of behavior

The law presented is a model thermoélasto (visco) plastic with isotropic work hardening which tends under the assumption of the small deformations towards the model [R5.03.02] with criterion of Von Mises (it is the plastic model). The plastic deformations are done with constant volume so that:

$$J^p=\det F^p=1\quad \text{from where}\quad J=J^e=\det\mathbf{F}^e$$

The relations of behavior are given by:

- the definition of the hyperelastic free energy:
  $\psi=\psi_{ther}(T\,,J)+\psi_{elas}(J\,,\bar{\mathbf{b}}^e)+K(\alpha)$,
  with in particular $\psi_{elas}(J\,,\bar{\mathbf{b}}^e)=\dfrac{1}{2}\dfrac{E}{3(1-2\nu)}\left[\dfrac{1}{2}(J^2-1)-\ln J\right]+\dfrac{\mu}{2}(tr\,\bar{\mathbf{b}}^e-3)$
  and $\psi_{ther}(T\,,J)=-3\,\alpha\Delta\,T\left(J-\dfrac{1}{J}\right)$
- thermoelastic relation stress-strain:

$$\tilde{\boldsymbol{\tau}}=\mu\,\tilde{\bar{\mathbf{b}}}^e$$
$$\mathrm{tr}\,\boldsymbol{\tau}=\frac{3\mathrm{K}}{2}(J^2-1)-\frac{9\mathrm{K}}{2}\alpha(T-T_{ref})(J+\frac{1}{J})$$

- threshold of plasticity (it is admitted that it is expressed with the constraints of Kirchhoff):

$$f=\tau_{eq}-R(p)-\sigma_y$$

where R is the variable of work hardening isotropic, function of the cumulated plastic deformation p.

- laws of flow:

$$\bar{\mathbf{F}}\dot{\mathbf{G}}^p\,\bar{\mathbf{F}}^T=-\dot{\lambda}\frac{3}{\tau_{eq}}\tilde{\boldsymbol{\tau}}\,\bar{\mathbf{b}}^e=-3\,\dot{\lambda}\left(\frac{1}{3}\mathrm{tr}\ \bar{\mathbf{b}}^e+\frac{\tilde{\boldsymbol{\tau}}}{\mu}\right)\frac{\tilde{\boldsymbol{\tau}}}{\tau_{eq}}$$
$$\dot{p}=\dot{\lambda}$$

For the model of plasticity, the plastic multiplier is obtained by writing the condition of coherence $\dot{f}=0$ and one a:

$$\dot{p}\geq 0, f\leq 0\ \text{et}\ \dot{p}\,f=0$$

In the viscous case, one takes $\dot{p}$ equalize with:

$$\dot{p}=\dot{\varepsilon}_0\left[\mathrm{sh}\left(\frac{\langle f\rangle}{\sigma_0}\right)\right]^m$$

where $\dot{\varepsilon}_0$, σ_0 and m are the viscosity coefficients. Let us announce that this law is reduced to a law of the type Norton when the 2 parameters materials $\dot{\varepsilon}_0$ and σ_0 are very large.

It is pointed out that:

$$\bar{\mathbf{b}}^e=J^{-2/3}\,\mathbf{b}^\mathbf{e}$$

$$\bar{\mathbf{F}}=J^{-1/3}\,\mathbf{F}$$

and that the partition of the deformations is written:

$$\bar{\mathbf{b}}^e=\bar{\mathbf{F}}\,\mathbf{G}^\mathbf{p}\,\bar{\mathbf{F}}^T$$

For metallic materials where the report τ_{eq}/μ is small in front of 1, the expression of the law of flow can be approximate by:

$$\bar{\mathbf{F}}\dot{\mathbf{G}}^p\bar{\mathbf{F}}^T=-\dot{\lambda}\,\mathrm{tr}\;\bar{\mathbf{b}}^\mathbf{e}\frac{\tilde{\boldsymbol{\tau}}}{\tau_{eq}}+O\left(\frac{\tau_{eq}}{\mu}\right) \quad \text{éq.4.2-1}$$

where $O\left(\frac{\tau_{eq}}{\mu}\right)$ is negligible in front of the first term.

It is this last expression which is established in *Code_Aster*.

Notice :

> *If the deformations are small, one a:*
>
> $$J\simeq 1+\mathrm{tr}\,\boldsymbol{\varepsilon}$$
> $$\mathbf{b}^\mathbf{e}\simeq\mathbf{Id}+2\,\boldsymbol{\varepsilon}^\mathbf{e}$$
> $$\mathbf{G}^\mathbf{p}\simeq\mathbf{Id}-2\,\boldsymbol{\varepsilon}^\mathbf{p}$$
>
> *where* $\boldsymbol{\varepsilon}$ *is the total deflection,* $\boldsymbol{\varepsilon}^\mathbf{e}$ *elastic strain and* $\boldsymbol{\varepsilon}^\mathbf{p}$ *plastic deformation in small deformations.*

By replacing these three expressions in the equations of the law of behavior presented here, one finds well the thermoelastoplastic classical model with isotropic work hardening and criterion of Von Mises.

4.3 Correction of elastic energy in the presence of thermics

The expression of hyperelastic energy ψ_{elas} raise some difficulties. Indeed, it depends on J ; in the presence of thermal deformation, J included a thermal component which disturbs the expression and which it is advisable to correct.

The approach of the correction of the energy of Simo-Miehe in the presence of thermics is the following one:

1. cancellation of pure energy in thermics when the constraint is worthless;
2. resolution of the equation in J obtained, which one calls J_0 the solution.
3. final calculation of the elastic energy of Simo-Miehe in the presence of thermics:

$$\psi_{elas}^{corrigee}=\psi_{elas}(J,\bar{\mathbf{b}}^{\mathbf{e}})+\psi_{ther}(T,J)+\psi_{elas}(J,\bar{\mathbf{b}}^{\mathbf{e}})-\psi_{elas}(J_{0,}\bar{\mathbf{b}}^{\mathbf{e}}=\boldsymbol{I_d})-\psi_{ther}(T,J_0)$$

The cancellation of the pure constraint in thermics leads to the equation:

$$\mathrm{tr}\,\boldsymbol{\tau}=0\Leftrightarrow(J^2-1)-3\alpha(T-T_{ref})(J+\frac{1}{J})=0\Leftarrow\left\{\begin{array}{c}J^3-J-3\alpha\Delta T(J^2-1)\\ J\neq 0\end{array}\right.$$

This equation, under the assumption $3\alpha\Delta T\ll 1$, has solutions close to -1.0 and 1. Only largest is physically acceptable. One thus poses:

$$J_0=MAX\{J\ tq\ J^3-J-3\alpha\Delta T(J^2-1)=0\}$$

The energy corrected in the presence of thermics is written finally:

$$\psi_{elas}^{corrigee}=\psi_{elas}(J,\bar{\mathbf{b}}^{\mathbf{e}})+\psi_{ther}(T,J)+\psi_{elas}(J,\bar{\mathbf{b}}^{\mathbf{e}})-\psi_{elas}(J_{0,}\bar{\mathbf{b}}^{\mathbf{e}}=\boldsymbol{I_d})-\psi_{ther}(T,J_0)$$

4.4 Choice of the function of work hardening

This relation of behavior is available in the operator STAT_NON_LINE, under the keyword factor BEHAVIOR and the argument 'SIMO_MIEHE' keyword factor DEFORMATION. One can choose for the function of work hardening, a linear work hardening or provide a traction diagram. Five relations can be used.

```
RELATION =    /   'VMIS_ISOT_TRAC'
              /   'VMIS_ISOT_PUIS'
              /   'VMIS_ISOT_LINE'
              /   'VISC_ISOT_TRAC'
              /   'VISC_ISOT_LINE'
```

For a purely thermoelastic behavior, the user chooses the argument 'VMIS_ISOT_LINE' for example, with SY very large (the behavior is then hyperelastic); for an isotropic work hardening given by a traction diagram, the user chooses the argument 'VMIS_ISOT_TRAC' in the plastic case or 'VISC_ISOT_TRAC' in the viscous case and for a linear isotropic work hardening, the argument 'VMIS_ISOT_LINE' in the plastic case or 'VISC_ISOT_LINE' in the viscous case. For an elastoplastic behavior whose curve of work hardening (diagram traction rational) is given by a law in power, form

$R(p)=\sigma_y+\sigma_y\left(\frac{E}{a\sigma_y}p\right)^{\frac{1}{n}}$, the user chooses the argument 'VMIS_ISOT_PUIS'.

The various characteristics of material are indicated in the operator DEFI_MATERIAU ([U4.23.01]) under the keywords:

- ELAS some is the law (one gives the Young modulus, the Poisson's ratio and possibly the thermal dilation coefficient),
- TRACTION for 'VMIS_ISOT_TRAC' and 'VISC_ISOT_TRAC' (the rational traction diagram is given),
- ECRO_PUIS for 'VMIS_ISOT_PUIS' (one gives the parameters of the law power),
- ECRO_LINE for 'VMIS_ISOT_LINE' and 'VISC_ISOT_LINE' (the elastic limit and the slope of work hardening are given),
- VISC_SINH for 'VISC_ISOT_TRAC' and 'VISC_ISOT_LINE' (one gives the three viscosity coefficients).

Notice :

The user must make sure well that the "experimental" traction diagram used, either directly, or to deduce the slope from it from work hardening is well given in the plan forced rational $\sigma=F/S$ *- deformation logarithmic curve* $\ln(1+\Delta l/l_0)$ where l_0 *is the initial length of the useful part of the test-tube,* Δl *variation length after deformation,* F *the force applied and* S *current surface. It will be noticed that* $\sigma=F/S=\frac{F}{S_0}\frac{l_0}{l}\frac{1}{J}$ *from where* $\tau=J\sigma=\frac{F}{S_0}\frac{l_0}{l}$ *. In general, it is well the quantity* $\frac{F}{S_0}\frac{l_0}{l}$ *who is measured by the experimenters and this gives the constraint of Kirchhoff directly used in the model of Simo and Miehe.*

4.5 Internal constraints and variables

The constraints are the constraints of Cauchy $\boldsymbol{\sigma}$, thus calculated on the current configuration (six components in 3D, four in 2D).
Internal variables produced in *Code_Aster* are:

- $V1$, cumulated plastic deformation p ,
- $V2$ the indicator of plasticity (0 if the last calculated increment is elastic, 1 if not).
- $V3$ with $V8$: opposite of the elastic strain $\bar{\mathbf{b}}^e$

Notice :

If the user wants to possibly recover deformations in postprocessing of his calculation, it is necessary to trace the deformations of Green-Lagrange $\mathbf{E}$ *, which represents a measurement of the deformations in great deformations (options* EPSG_ELGA *or* EPSG_ELNO *). Linearized deformations* $\boldsymbol{\varepsilon}$ *classics measure deformations under the assumption of the small deformations and do not have a direction in great deformations.*

4.6 Field of application

The choice of a kinematics DEFORMATION : 'PETIT_REAC' also allows to treat a thermoelastoplastic law of behavior with isotropic work hardening and criterion of Von Mises in great deformations. The law is written in small deformations and the taking into account of the great deformations is done by reactualizing the geometry.

Between the law presented here (SIMO_MIEHE) and PETIT_REAC,

- there is no difference if the deformations are small
- there is no difference if the deformations are large but small rotations
- there are differences if rotations are important.

In particular, the solution obtained with kinematics PETIT_REAC can deviate notably from the exact solution in the presence of great rotations and this whatever the size of the steps of time chosen by the user, contrary to kinematics SIMO_MIEHE.

4.7 Integration of the law of behavior

In the case of an incremental behavior, keyword factor BEHAVIOR, knowing the constraint $\boldsymbol{\sigma}^-$, cumulated plastic deformation p^-, the trace divided by three of the tensor of deformations rubber bands $\frac{1}{3}\mathrm{tr}\,\bar{\mathbf{b}}^{e-}$, displacements $\mathbf{u}^-$ and $\Delta\mathbf{u}$ and temperatures T^- and T, one seeks to determine $(\boldsymbol{\sigma}, p, \frac{1}{3}\mathrm{tr}\,\bar{\mathbf{b}}^e)$.

Displacements being known, gradients of the transformation of Ω_0 with Ω^-, noted $\mathbf{F}^-$, and of Ω^- with Ω, noted $\Delta\mathbf{F}$, are known.

Discretization *implicit* law gives:

$$\mathbf{F}=\Delta\mathbf{F}\,\mathbf{F}^-$$
$$J=\det\mathbf{F}$$
$$\bar{\mathbf{F}}=J^{-1/3}\mathbf{F}$$
$$\bar{\mathbf{b}}^e=\bar{\mathbf{F}}\,\mathbf{G}^{\mathbf{p}}\,\bar{\mathbf{F}}^T$$
$$J\boldsymbol{\sigma}=\boldsymbol{\tau}$$
$$\tilde{\boldsymbol{\tau}}=\mu\,\tilde{\bar{\mathbf{b}}}^e$$
$$\frac{1}{3}\mathrm{tr}\,\boldsymbol{\tau}=\frac{1}{2}K(J^2-1)-\frac{3}{2}K\alpha(T-T_{ref})(J+\frac{1}{J})$$
$$f=\boldsymbol{\tau}_{eq}-R(p^-+\Delta p)-\sigma_y$$
$$\bar{\mathbf{F}}(\mathbf{G}^{\mathbf{p}}-\mathbf{G}^{\mathbf{p}-})\bar{\mathbf{F}}^T=-\mathrm{tr}\,\bar{\mathbf{b}}^e\frac{\tilde{\boldsymbol{\tau}}}{\tau_{eq}}\Delta p \text{ from where } \bar{\mathbf{b}}^e=\bar{\mathbf{F}}\,\mathbf{G}^{\mathbf{p}-}\,\bar{\mathbf{F}}^T-\mathrm{tr}\,\bar{\mathbf{b}}^e\frac{\tilde{\boldsymbol{\tau}}}{\tau_{eq}}\Delta p$$

In the plastic case: $\Delta p\geq 0, f\leq 0$ et $f\Delta p=0$

In the viscous case: $\langle\tau_{eq}-R(p^-+\Delta p)-\sigma_y\rangle-\sigma_0\,\mathrm{sh}^{-1}\left[\left(\frac{\Delta p}{\dot{\varepsilon}_0\Delta t}\right)^{\frac{1}{m}}\right]=0$

Notice :

> *This formulation is incrémentalement objective because the only incremental tensorial quantity which intervenes in the discretization is* $\dot{\mathbf{G}}^p$ *. Like* $\mathbf{G}^{\mathbf{p}}$ *and* $\mathbf{G}^{\mathbf{p}-}$ *are measured on the same configuration, i.e. the initial configuration, the discretization of* $\dot{\mathbf{G}}^p$ *, that is to say* $\Delta\mathbf{G}^{\mathbf{p}}=\mathbf{G}^{\mathbf{p}}-\mathbf{G}^{\mathbf{p}-}$ *, is incrémentalement objective.*

One introduces $\boldsymbol{\tau}^{Tr}$, the tensor of Kirchhoff which results from an elastic prediction (Tr: trial, in English test):

$$\tilde{\boldsymbol{\tau}}^{Tr}=\mu\,\tilde{\bar{\mathbf{b}}}^{eTr}$$

where

$$\bar{\mathbf{b}}^{eTr}=\bar{\mathbf{F}}\,\mathbf{G}^{\mathbf{p}-}\,\bar{\mathbf{F}}^T=\Delta\,\bar{\mathbf{F}}\,\bar{\mathbf{b}}^{e-}\Delta\,\bar{\mathbf{F}}^T\;,\;\Delta\,\bar{\mathbf{F}}=(\Delta\,J)^{-1/3}\Delta\,\mathbf{F}\;\text{ and }\;\Delta\,J=\det(\Delta\,\mathbf{F})$$

One obtains $\bar{\mathbf{b}}^{e-}$ starting from the constraints $\boldsymbol{\tau}^-$ by the thermoelastic relation stress-strain and trace of the tensor of the elastic strain.

$$\bar{\mathbf{b}}^{e-}=\frac{\tilde{\boldsymbol{\tau}}^-}{\mu^-}+\frac{1}{3}\,\mathrm{tr}\;\bar{\mathbf{b}}^{e-}$$

Notice :

The interest of this formulation is that it is not necessary to calculate the plastic deformation $\mathbf{G}^{\mathbf{p}-}$ who would oblige us to reverse the gradient of the transformation $\bar{\mathbf{F}}$. One needs to only know $\bar{\mathbf{F}}\mathbf{G}^{\mathbf{p}-}\bar{\mathbf{F}}^T$.

If $\boldsymbol{\tau}_{eq}^{Tr}<R(p^-)+\sigma_y$, one remains elastic. In this case, one a:

$$p=p^-\;,\;\boldsymbol{\tau}=\tilde{\boldsymbol{\tau}}^{Tr}+\frac{1}{3}\,\mathrm{tr}\,\boldsymbol{\tau}^{Tr}\,\mathbf{Id}\;\text{ and }\;\frac{1}{3}\,\mathrm{tr}\;\bar{\mathbf{b}}^e=\frac{1}{3}\,\mathrm{tr}\;\bar{\mathbf{b}}^{eTr}$$

if not, one obtains:

$\mathrm{tr}\;\bar{\mathbf{b}}^e=\mathrm{tr}\,\bar{\mathbf{b}}^{eTr}$, thanks to simplification on the law of flow: $\bar{\mathbf{b}}^e=\bar{\mathbf{b}}^{e^{Tr}}-\mathrm{tr}\,\bar{\mathbf{b}}^e\,\dfrac{\tilde{\boldsymbol{\tau}}}{\tau_{eq}}\Delta\,p$

By taking the deviatoric parts of this equation, and by multiplying them by μ one leads to:

$$\tilde{\boldsymbol{\tau}}^{Tr}=\tilde{\boldsymbol{\tau}}\,(1+\frac{\mu\,\mathrm{tr}\;\bar{\mathbf{b}}^{eTr}\Delta\,p}{\tau_{eq}})$$

While calculating the equivalent constraint, one brings back oneself to a nonlinear scalar equation in $\Delta\,p$:

$$\tau_{eq}^{Tr}-\tau_{eq}-\mu\,\mathrm{tr}\,\bar{\mathbf{b}}^{eTr}\Delta\,p=0$$

In the plastic case : $\tau_{eq}=\sigma_y+R(p^-+\Delta\,p)$, which leads to $\Delta\,p$ solution of the equation:

$$\tau_{eq}^{Tr}-\sigma_y-R(p^-+\Delta\,p)-\mu\,\mathrm{tr}\,\bar{\mathbf{b}}^{eTr}\Delta\,p=0$$

In the viscoplastic case: $\tau_{eq}=\sigma_y+R(p^-+\Delta\,p)+\sigma_0\,\mathrm{sh}^{-1}\left[\left(\dfrac{\Delta\,p}{\dot{\varepsilon}_0\Delta\,t}\right)^{\frac{1}{m}}\right]$, which leads to $\Delta\,p$ solution of the equation:

$$\tau_{eq}^{Tr}-\sigma_y-R(p^-+\Delta\,p)-\sigma_0\,\mathrm{sh}^{-1}\left[\left(\frac{\Delta\,p}{\dot{\varepsilon}_0\Delta\,t}\right)^{\frac{1}{m}}\right]-\mu\,\mathrm{tr}\,\bar{\mathbf{b}}^{eTr}\Delta\,p=0$$

If work hardening linear, or is given by a point by point defined traction diagram, therefore closely connected per pieces, the equation to be solved is linear. The solution directly is obtained. $\Delta\,p$ In the other cases, the resolution is carried out in *Code_Aster* by a method of the secants with interval of research (cf [R5.03.05]). Integration can be controlled by the parameters `RESI_INTE_RELA` and `ITER_INTE_MAXI`.

Once $\Delta\,p$ known, one can then deduce the tensor from it from Kirchhoff, that is to say:

$$\boldsymbol{\tau}=\frac{\tau_{eq}}{\tau_{eq}^{Tr}}\tilde{\boldsymbol{\tau}}^{Tr}+\left[\frac{K}{2}(J^2-1)-\frac{3K}{2}(T-T_{ref})(J+\frac{1}{J})\right]\mathbf{Id}$$

Once calculated the cumulated plastic deformation, the tensor of the constraints and the tangent matrix, one carries out a correction on the trail of tensor of the elastic strain $\bar{\mathbf{b}}^e$ to take account of the plastic incompressibility, which is not preserved with the simplification made on the law of flow [éq 4.2.1]. This correction is carried out by using a relation between the invariants of $\bar{\mathbf{b}}^e$ and $\tilde{\bar{\mathbf{b}}}^e$ and by exploiting the plastic condition of incompressibility $J^p=1$ (or in an equivalent way $\det\bar{\mathbf{b}}^e=1$). This relation is written:

$$x^3-\bar{J}_2^e x-(1-\bar{J}_3^e)=0$$

with $\bar{J}_2^e=\frac{1}{2}\bar{b}_{eq}^{e^2}=\frac{\tau_{eq}^2}{2\mu^2}$, $\bar{J}_3^e=\det\tilde{\bar{\mathbf{b}}}^e=\det\frac{\tilde{\boldsymbol{\tau}}}{\mu}$ and $x=\frac{1}{3}\operatorname{tr}\bar{\mathbf{b}}^e$

The solution of this cubic equation makes it possible to obtain $\operatorname{tr}\bar{\mathbf{b}}^e$ and consequently thermoelastic deformation $\bar{\mathbf{b}}^{e-}$ with the step of next time. If this equation admits several solutions, one takes the solution nearest to the solution of the step of previous time. It is besides why one stores in an internal variable $\frac{1}{3}\operatorname{tr}\bar{\mathbf{b}}^e$.

5 Variational formulation

Insofar as the constraints provided by the law of behavior are eulériennes, one chooses a variational formulation written on the current configuration (eulérienne) and not on the initial configuration, that is to say:

$$\underbrace{\int_\Omega \sigma\,\nabla_x\,\delta\mathbf{v}\,d\Omega}_{\mathbf{F}_{\text{int}}\cdot\delta\mathbf{v}}=\underbrace{\int_\Omega \rho\,\mathbf{f}\,\delta\mathbf{v}\,d\Omega+\int_{\partial\Omega^f}\mathbf{t}^{\mathbf{d}}\,\delta\mathbf{v}\,\mathrm{dS}}_{\mathbf{F}_{\text{ext}}\cdot\delta\mathbf{v}}\qquad \delta\mathbf{v}\ \text{Kinematically acceptable}$$

We are interested only here in work of the interior forces and its variation in optics of a resolution by the method of Newton. One will find in [feeding-bottle 4] the demonstration of the expressions presented.

5.1 Case of the continuous medium

One rewrites here the work of the interior efforts in indicielle form, that is to say:

$$\mathbf{F}_{\text{int}}\cdot\delta\mathbf{v}=\int_\Omega \sigma_{ij}\frac{\partial\delta v_i}{\partial x_j}d\Omega$$

We need also to express the variation of the interior efforts in the current configuration Ω that is to say:

$$\delta\mathbf{F}_{\text{int}}\cdot\delta\mathbf{u}\cdot\delta\mathbf{v}=\int_\Omega\left[\sigma_{ij}\frac{\partial\delta u_p}{\partial x_p}-\sigma_{ik}\frac{\partial\delta u_j}{\partial x_k}\right]\frac{\partial\delta v_i}{\partial x_j}d\Omega\quad\text{rigidité géométrique}$$

$$+\int_\Omega\left[\frac{\partial\sigma_{ij}}{\partial\Delta F_{pq}}\frac{\partial\delta u_p}{\partial x_q^-}\right]\frac{\partial\delta v_i}{\partial x_j}d\Omega\qquad\text{rigidité de comportement}$$

¿

where $\mathbf{x}^-$ are the punctual coordinates on the configuration Ω^-.

5.2 Discretization by finite elements

Displacements are discretized $\mathbf{u}$ and virtual displacements $\mathbf{v}$ by finite elements. The notations are the following ones, by adopting the convention of summation of the repeated indices:

$$u_i(x)=N^n(x)\,U_i^n \qquad \frac{\partial u_i}{\partial x_j}=D_j^n(x)\,U_i^n \qquad \frac{\partial u_i}{\partial x_j^-}=D_j^{-n}(x)\,U_i^n$$

where:

$N^n(x)$ is the function of form associated with the node n
U_i^n , the component *I* nodal displacement of the node n
$D_j^n(x)$, components of the gradient of the functions of form on the configuration Ω
$D_j^{-n}(x)$, components of the gradient of the functions of form on the configuration Ω^-

One obtains for the vector of the interior forces:

$$(F_{\mathrm{int}})_i^n=\int_\Omega \sigma_{ij}\,D_j^n\,d\Omega$$

and for the tangent matrix, which is not a priori symmetrical:

$$K_{i\ p}^{n\,m}=\int_\Omega \left[D_p^m\sigma_{ij}D_j^n-D_k^m\sigma_{ik}D_p^n\right]d\Omega + \int_\Omega \left[D_q^{-n}\frac{\partial\sigma_{ij}}{\partial\Delta F_{pq}}D_j^n\right]d\Omega$$

In the case of a two-dimensional modeling (deformation planes), the expressions of the vector of the interior forces and the tangent matrix are identical to this ready that the indices corresponding to the components only vary from 1 to 2.

In the case of an axisymmetric modeling, by numbering the axes in the order (r,z,θ), the vector of the interior forces is written:

$$(F_{\mathrm{int}}^{axi})_\alpha^n=\int_\Omega\left[\sigma_{\alpha\beta}D_\beta^n+\sigma_{33}\frac{N^n}{r}\delta_{\alpha 1}\right]d\Omega \quad , \ \alpha\in\{1,2\} \ , \ \beta\in\{1,2\}$$

and the tangent matrix:

$$[\mathbf{K}^{axi}]=[\mathbf{K}]+[\mathbf{K}^{corr}]$$

with:

$$\left[\mathbf{K}_{(1)}^{\mathbf{corr}}\right]_{1\beta}^{nm}=\int_\Omega \frac{N^n}{r}\sigma_{\beta\gamma}D_\gamma^m\,d\Omega+\int_\Omega\frac{N^n}{r^-}\frac{\partial\sigma_{\beta\gamma}}{\partial\Delta F_{33}}D_\gamma^m\,d\Omega$$

$$\left[\mathbf{K}_{(2)}^{\mathbf{corr}}\right]_{\alpha 1}^{nm}=\int_\Omega D_\alpha^n\sigma_{33}\frac{N^m}{r}d\Omega+\int_\Omega D_\gamma^{-n}\frac{\partial\sigma_{33}}{\partial\Delta F_{\alpha\gamma}}\frac{N^m}{r}d\Omega$$

$$\left[\mathbf{K}_{(3)}^{\mathbf{corr}}\right]_{11}^{nm}=\int_\Omega\frac{N^n}{r^-}\frac{\partial\sigma_{33}}{\partial\Delta F_{33}}\frac{N^m}{r}$$

From an algorithmic point of view, the tangent elementary matrix $\mathbf{K}$ is not symmetrical a priori. The total resolution will thus be made by default with a nonsymmetrical solvor. It is however possible to symmetrize the total tangent matrix before resolution (keyword SOLVEUR), which makes it possible to save time calculation but can degrade convergence.

5.3 Form of the tangent matrix of the behavior

One gives the form of the tangent matrix here (option FULL_MECA during iterations of Newton, option RIGI_MECA_TANG for the first iteration). This one is obtained by linearizing the system of equations which governs the law of behavior. We give here the final result of this linearization. One will find in [bib4] the detail of this calculation.

One poses:

$$J=\det \mathbf{F} \ , \ J^{-}=\det \mathbf{F}^{-} \ \text{ and } \ \Delta J=\det \Delta \mathbf{F}$$

- For the option FULL_MECA , one a:

$$\mathbf{A}=\frac{\partial \boldsymbol{\sigma}}{\partial \Delta \mathbf{F}}=\frac{(\Delta J)^{-1/3}}{J}\mathbf{H}-\frac{1}{3 J \Delta J}(\mathbf{H}\Delta \bar{\mathbf{F}})\otimes \mathbf{B}-\frac{J^{-}}{J^{2}}\boldsymbol{\tau}\otimes \mathbf{B}$$
$$+\frac{J^{-}}{J}\left[KJ-\frac{3}{2}K\alpha(T-T_{ref})(1-J^{-2})\right]\mathbf{Id}\otimes \mathbf{B}$$

where **B** is worth:

$$\begin{aligned}
B_{11}&=\Delta F_{22}\Delta F_{33}-\Delta F_{23}\Delta F_{32}\\
B_{22}&=\Delta F_{11}\Delta F_{33}-\Delta F_{13}\Delta F_{31}\\
B_{33}&=\Delta F_{11}\Delta F_{22}-\Delta F_{12}\Delta F_{21}\\
B_{12}&=\Delta F_{31}\Delta F_{23}-\Delta F_{33}\Delta F_{21}\\
B_{21}&=\Delta F_{13}\Delta F_{32}-\Delta F_{33}\Delta F_{12}\\
B_{13}&=\Delta F_{21}\Delta F_{32}-\Delta F_{22}\Delta F_{31}\\
B_{31}&=\Delta F_{12}\Delta F_{23}-\Delta F_{22}\Delta F_{13}\\
B_{23}&=\Delta F_{31}\Delta F_{12}-\Delta F_{11}\Delta F_{32}\\
B_{32}&=\Delta F_{13}\Delta F_{21}-\Delta F_{11}\Delta F_{23}
\end{aligned}$$

H and $\mathbf{H}\Delta \bar{\mathbf{F}}$ are given by:

In the elastic case ($f<0$):

$$H_{\mathrm{ijkl}}=\mu\left(\delta_{\mathrm{ik}}\bar{b}^{e-}_{\mathrm{lp}}\Delta \bar{F}_{\mathrm{jp}}+\Delta \bar{F}_{\mathrm{ip}}\bar{b}^{e-}_{\mathrm{pl}}\delta_{\mathrm{jk}}\right)-\frac{2\mu}{3}\delta_{\mathrm{ij}}\Delta \bar{F}_{\mathrm{kp}}\bar{b}^{e-}_{\mathrm{lp}}$$

and

$$\mathbf{H}\Delta \bar{\mathbf{F}}=2\mu\,\tilde{\bar{\mathbf{b}}}^{\mathrm{eTr}}$$

In the plastic case ($f=0$) or viscoplastic:

$$H_{ijkl}=\frac{\mu}{a}(\delta_{ik}\bar{b}^{e-}_{lp}\Delta\bar{F}_{jp}+\Delta\bar{F}_{ip}\bar{b}^{e-}_{pl}\delta_{jk})$$
$$-2\mu\left[\frac{\delta_{ij}}{3a}+\frac{\bar{R}\Delta p\tilde{\tau}_{ij}}{\tau_{eq}(\bar{R}+\mu\,\mathrm{tr}\,\bar{b}^{eTr})}\right]\Delta\bar{F}_{kp}\bar{b}^{e-}_{lp}$$
$$+\frac{3\mu^2\,\mathrm{tr}\,\bar{b}^{eTr}(\bar{R}\Delta p-\tau_{eq})}{a\tau^3_{eq}(\bar{R}+mu\,\mathrm{tr}\,\bar{b}^{eTr})}\tilde{\tau}_{ij}\tilde{\tau}_{kq}\Delta\bar{F}_{qp}\bar{b}^{e-}_{lp}$$

and

$$\mathbf{H}\Delta\bar{F}=\frac{2\mu}{a}\bar{\mathbf{b}}^{eTr}-2\mu\,\mathrm{tr}\,\bar{\mathbf{b}}^{eTr}\left[\frac{\mathbf{Id}}{3a}+\frac{\bar{R}\Delta p\tilde{\boldsymbol{\tau}}}{\tau_{eq}(\bar{R}+\mu\,\mathrm{tr}\,\bar{\mathbf{b}}^{eTr})}\right]+\frac{3\mu^2\,\mathrm{tr}\,\bar{\mathbf{b}}^{eTr}(\bar{R}\Delta p-\tau_{eq})}{a\tau^3_{eq}(\bar{R}+\mu\,\mathrm{tr}\,\bar{\mathbf{b}}^{eTr})}(\tilde{\boldsymbol{\tau}}:\bar{\mathbf{b}}^{eTr})\tilde{\boldsymbol{\tau}}$$

where $a=\frac{\tau^{Tr}_{eq}}{\tau_{eq}}$

and $$\bar{R}=\mathrm{R'}(p)+\underbrace{\sigma_0\times\left(1+\left(\frac{\Delta p}{\dot{\varepsilon}_0\Delta t}\right)^2\right)^{\frac{-1}{2}}\times\frac{1}{m\left(\dot{\varepsilon}_0\Delta t\right)^{\frac{1}{m}}}\times(\Delta p)^{1/m-1}}_{\text{uniquement cas visqueux}},$$

$R'(p)$ being the derivative of isotropic work hardening compared to the cumulated plastic deformation p .

- For the option `RIGI_MECA_TANG`, it is the same expressions as those given for `FULL_MECA` but with $\Delta p=0$ and with all the variables and coefficients of material taken at the moment t^- (in theory, it would be necessary in the viscous case, to take the expressions of `FULL_MECA` in the elastic case, all the variables being taken at the moment t^-). In particular, one will have $\Delta\bar{\mathbf{F}}=\mathbf{Id}$.

6 Bibliography

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7 Description of the versions of the document

Version Aster	Author (S) or contributor (S), organization	Description of the modifications
4.4	V.Cano, E.Lorentz EDF/R & D /AMA	Initial text
6.3	V.Cano, E.Lorentz EDF/R & D /AMA	Card 6396
7.4	S.Michel-Ponnelle EDF/R & D /AMA	Card 8000: addition of viscosity

9.4	J.M.Proix	EDF/R & D /AMA	Addition of VMIS_ISOT_PUIS.
11.4	J.M.Proix	EDF/R & D /AMA	Card 19650 order of the internal variables