Modelings second gradient

Summary

One presents here modelings second gradient exit of work of Chambon et al. ([bib1], [bib2], [bib3]) and second gradient of dilation resulting from work of Fernandes et al. ([bib4], [bib5]).

These modelings lie within the scope of the mediums with microstructure and have like digital objective to give convergent results compared to the smoothness of the space discretization to avoid obtaining localised solutions. They must be used since the mechanical rheological laws considered present a softening of the behavior translating the damage or the degradation of a material before cracking.

Modeling second gradient of dilation is, in fact, a simplified approach of the second gradient restricted with materials fixed with the phenomenon of dilatancy. The digital objective is to reduce to a significant degree the computing times. It lends itself particularly well to the géomatériaux one.

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1.1 Brief presentation of the mediums with microstructure

At the origin of this theory, one finds work of Mindlin ([bib6], [bib7]) within the framework of linear elasticity. This work was then resumed by Germain ([bib8], [bib9]) who gave of it an expression by application of the principle of virtual work, bases digital methods for an application by finite elements.

This theory implies the definition of an enriched kinematics. Besides the classical field of displacements u_i , one considers the tensor of the second order, noted f_{ij} and called microscopic kinematic gradient, which models at the same time the deformations and rotations at the level of the grains of the structure. One draws here attention to the fact that, within the framework of the mediums with microstructure, the gradient of microscopic deformation does not have any raison d'être related to the gradient of any field depend on macroscopic displacement. It is not necessarily symmetrical. The microscopic gradient of deformation f_{ij} is a variable as well as macroscopic displacement u_i , contrary to the classical field of deformation (macroscopic), which is obtained to him by derivation

$$\varepsilon_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)$$

A fact essential with the base of the writing of this theory relates to the statement of the principle of objectivity or also of material indifference:

The virtual power of the interior efforts to a system is worthless in any virtual movement rigidifying the system at the moment considered.

By neglecting the expression of the efforts external of volume for reasons of simplification of writing, the consequence of the axiom of the virtual powers of the interior efforts conduit with the expression of the variational formulation, for any field kinematically acceptable $\left(u_i^*, f_{ij}^*\right)$

$$\int_{\Omega}\left(\sigma_{ij}\frac{\partial u_i^*}{\partial x_j}+\tau_{ij}\left(f_{ij}^*-\frac{\partial u_i^*}{\partial x_j}\right)+\Sigma_{ijk}\frac{\partial f_{ij}^*}{\partial x_k}\right)dv=\int_{\partial\Omega}\left(t_i u_i^*+T_{ij} f_{ij}^*\right)ds \qquad (1)$$

where t_i and T_{ij} are respectively the forces of traction and the double forces corresponding to the boundary conditions, on the border $\partial\Omega$, combined variable kinematics.

The variational formulation (1) is another means of expressing the relations of balance which are expressed

$$\frac{\partial\left(\sigma_{ij}-\tau_{ij}\right)}{\partial x_j}=0 \qquad (2)$$

$$\frac{\partial\Sigma_{ijk}}{\partial x_k}-\tau_{ij}=0 \qquad (3)$$

and one finds for the expression of the boundary conditions

$$t_i=\left(\sigma_{ij}-\tau_{ij}\right)n_j \quad (4)$$

$$T_{ij}=\Sigma_{ijk}n_k \quad (5)$$

where n_j indicate the outgoing normal at the border $\partial\Omega$.

To supplement the problem, it is necessary to define the laws of behavior which will bind the static variables σ_{ij} , τ_{ij} , Σ_{ijk} respectively with the history of variable kinematics of $\frac{\partial u_i}{\partial x_j}$, $\left(f_{ij}-\frac{\partial u_i}{\partial x_j}\right)$ and $\frac{\partial f_{ij}}{\partial x_k}$.

These models already proved that they were effective from the point of view of the regularization. However, they are complex in their use because of various laws of behavior to specify. Moreover, the discretization by the finite element method in 3D induces the addition of 9 additional degrees of freedom per node corresponding to the components f_{ij} . The computing times are then relatively important and consequently not-compatible with the type of studies which we wish to carry out.

1.2 The model second gradient

On the basis of the preceding model, expressed by the relation (1), one can restrict kinematics by forcing the microscopic gradient to be equal to the macroscopic gradient (see Chambon et al. [bib2] for a detailed analysis)

$$f_{ij}=\frac{\partial u_i}{\partial x_j} \quad (6)$$

The advantage of this assumption is to reduce the number of independent variables and to introduce simpler laws of behavior. The expression of the virtual powers altered after some algebraic handling is written then for any field kinematically acceptable u_i^*

$$\int_\Omega\left(\sigma_{ij}\frac{\partial u_i^*}{\partial x_j}+\Sigma_{ijk}\frac{\partial^2 u_i^*}{\partial x_j\partial x_k}\right)dv=\int_{\partial\Omega}\left(p_i u_i^*+P_i Du_i^*\right)ds \quad (7)$$

where p_i and P_i are the boundary conditions defined by

$$p_i=\sigma_{ij}n_j-n_k n_j D\Sigma_{ijk}-\frac{D\Sigma_{ijk}}{Dx_k}n_j-\frac{D\Sigma_{ijk}}{Dx_j}n_k+\frac{Dn_l}{Dx_l}\Sigma_{ijk}n_j n_k-\frac{Dn_j}{Dx_k}\Sigma_{ijk} \quad (8)$$

$$P_i=\Sigma_{ijk}n_j n_k \quad (9)$$

with

Dq who indicates the normal derivative of the variable q : $Dq=\frac{\partial q}{\partial x_j}n_j$

$\frac{Dq}{Dx_j}$ who indicates the tangential derivative of the variable q : $\frac{Dq}{Dx_j}=\frac{\partial q}{\partial x_j}-n_j Dq$

The assumption on the equality between field of deformations microscopic and macroscopic (6) a direct impact has on the expression of the boundary conditions because the variables u_i^* and f_{ij}^* are not independent any more.

It was already shown that this model corrects the dependence thickness of the bands of localization compared to the discretization of the grid (see Chambon et al. [bib1] or Matsushima et al. [bib3]). For that, the model can be used by taking of account two laws of behavior different, one to describe the part classical first gradient and the other for the second gradient. With regard to the latter, any relation can-being considered, but until today, it is in general of the linear elasticity which was selected.

1.3 Space discretization by finite elements

Written in its form of the equation 7 , the discretization by the finite element method of the expression of the second gradient supposes that the fields u_i and u_i^* are twice derivable. The numeric work implementation of such a condition implies the integration of C1-continuous finite elements (as that was proposed by Chambon et al. [bib1] within the framework of this unidimensional formulation second gradient or Zervos et al. [bib10] in a similar approach in gradient of deformation).

One second approach consists in introducing a mixed formulation by the means of multipliers of Lagrange. That consists in weakening the mathematical constraint (6) in the writing of the variational formulation (7) . One obtains then for any field kinematically acceptable $\left(u_i^*, f_{ij}^*, \lambda_{ij}^*\right)$

$$\int_\Omega \left(\sigma_{ij} \frac{\partial u_i^*}{\partial x_j} + \Sigma_{ijk} \frac{\partial f_{ij}^*}{\partial x_k} + \lambda_{ij} \left(f_{ij}^* - \frac{\partial u_i^*}{\partial x_j} \right) - \lambda_{ij}^* \left(f_{ij} - \frac{\partial u_i}{\partial x_j} \right) \right) dv = \int_{\partial\Omega} \left(t_i u_i^* + T_{ij} f_{ij}^* \right) ds \qquad (10)$$

where λ_{ij} are the multipliers of Lagrange. The advantage of this new expression comes owing to the fact that the interpolations from the nodal unknown factors, that are $\left(u_i, f_{ij}, \lambda_{ij}\right)$, only conditions of C0-continuity require. The disadvantage is due, on the other hand, with the addition of new degrees of freedom which does not make it possible to make decrease the number of unknown factors of the problem.

The numeric work implementation suggested in Code_Aster is detailed in the chapter 3 . One specifies here only spaces of approximation of the variables defining the kinematic field. The polynomial interpolations are the following ones:

1) functions of forms of the second order for the variables of displacements u_i ,
2) functions of first order forms for the microscopic tensor of the deformations f_{ij}
3) constant functions by element for the multipliers of Lagrange λ_{ij} .

One speaks about a formulation second gradient P2-P1-P0. Modeling is currently available in Code_Aster only under the assumption of the plane deformations. One gives in figure 1.3-a a representation of the finite element associated with this combination of interpolations.

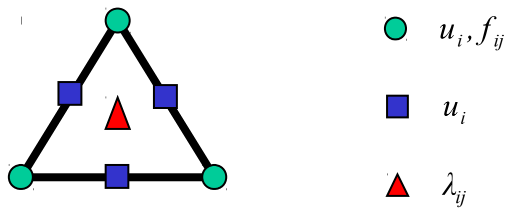

Figure **1.3-a** : Discretization finite element of the model second gradient for the plane deformations

2 The model second gradient of dilation

The principle of the mediums with microstructure is based on the taking into account of the microscopic deformations to introduce into the expression of the model an internal length. If the effectiveness of the regularization brought by the model second gradient is not any doubt, the computing times of simulations can become prohibitory. To decrease them within the particular framework of dilating materials (for which the voluminal deformation evolves according to the loading) one restricts the general information of the effect regularizing in the dilating mediums with microstructure.

2.1 Dilating mediums with microstructure

The kinematics of these mediums is defined by the field of usual displacement u_i , noted microscopic voluminal variation χ and its gradients. By duality with this enriched kinematics, the static variables of the classical macroscopic constraints are introduced σ_{ij} , the microscopic constraint of dilation κ and double vectorial constraints of dilation S_j . The variable κ is the combined component of the relative voluminal deformation (of the macroscopic field compared to microscopic) $\varepsilon_V-\chi$, while components S_j define a vector which is combined gradient of microscopic dilation $\frac{\partial\chi}{\partial x_j}$.

In a way similar to the expression of the virtual power of the mediums with microstructure of the chapter 1.1 , and by again neglecting the expression of the efforts external of volume for analytical simplification, one finds for the dilating mediums with microstructure, for any field kinematically acceptable $\left(u_i^*,\chi^*\right)$

$$\int_\Omega \left(\sigma_{ij}\frac{\partial u_i^*}{\partial x_j}+\kappa\left(\varepsilon_V^*-\chi^*\right)+S_j\frac{\partial\chi^*}{\partial x_j}\right)dv=\int_{\partial\Omega}\left(t_i u_i^*+m\chi^*\right)ds \quad (11)$$

For which

$$t_i=\left(\sigma_{ij}+\kappa\delta_{ij}\right)n_j \quad (12)$$

$$m=S_j n_j \quad (13)$$

are the boundary conditions, expressed on the border $\partial\Omega$, combined by duality with variable kinematics u_i and χ respectively.

The relations of balance of this problem are written

$$\frac{\partial\sigma_{ij}}{\partial x_j}+\frac{\partial\kappa}{\partial x_i}=0 \quad (14)$$

$$\kappa+\frac{\partial S_j}{\partial x_j}=0 \quad (15)$$

The system of equations made up of [(12), (13), (14) and (15)] is obtained classically by application of the theorem of the divergence and an integration by part of (11).

2.2 The model second gradient of dilation

According to a principle similar to that implemented for the model second gradient of the chapter 1.2 , one introduces a mathematical constraint to force the equality between the voluminal deformations macroscopic ε_V and microscopic χ

$$\chi=\varepsilon_V \tag{16}$$

The expression of the virtual powers is then the following one, for any field kinematically acceptable u_i^*

$$\int_\Omega \left(\sigma_{ij}\frac{\partial u_i^*}{\partial x_j}+S_j\frac{\partial^2 u_i^*}{\partial x_i \partial x_j}\right)dv=\int_{\partial\Omega}\left(p_i u_i^*+Pn_i Du_i^*\right)ds \tag{17}$$

where p_i and P are the boundary conditions defined by

$$p_i=\sigma_{ij}n_j-n_i n_j DS_j-\frac{DS_j}{Dx_j}n_i-\frac{DS_j n_j}{Dx_i}+\frac{Dn_p}{Dx_p}S_j n_j n_i \tag{18}$$

$$P=S_j n_j \tag{19}$$

As for the model second gradient, the assumption on the equality between field of voluminal deformations microscopic and macroscopic (16) a direct impact has on the expression of the boundary conditions because the variables u_i^* and χ^* are not independent any more.

For reasons of simplicity, one supposes that $P=0$. The consequence of this assumption is that $S_j n_j=0$ on the border, which reduces the expression (18) with

$$p_i=\sigma_{ij}n_j-\frac{\partial S_j}{\partial x_j}n_i \tag{20}$$

A remarkable property of this simplification comes owing to the fact that it expression of the boundary conditions breaks up then into the classical part $\sigma_{ij}n_j$ and a second term which does not induce components of shearing.

The equilibrium equation is written

$$\frac{\partial\sigma_{ij}}{\partial x_j}-\frac{\partial^2 S_j}{\partial x_i \partial x_j}=0 \tag{21}$$

2.3 Space discretization by finite elements

The goal of the approach of the second gradient of dilation is to define a regularizing model, making it possible to ensure the independence of the results compared to the space discretizations, by introducing a minimum of nodal unknown factors into its approach by finite elements. For that one limits oneself to the applications considering of dilating materials. However, to discretize by the method of the elements stop the expression (17) has as consequence to impose that the field of the unknown factors of displacement as its divergence are continuous and derivable. That returns has to take into account C1-continuous finite elements.

One then proposes to introduce the mathematical constraint (16) in the expression of the mediums with microstructure dilating by means of a coupling of multipliers of Lagrange and penalization. One obtains then for any field kinematically acceptable $\left(u_i^*, \chi^*, \lambda^*\right)$

$$\int_\Omega \left(\sigma_{ij} \frac{\partial u_i^*}{\partial x_j} + S_j \frac{\partial \chi^*}{\partial x_j} - \lambda\left(\varepsilon_V^* - \chi^*\right) + \lambda^*\left(\varepsilon_V - \chi\right) + r\left(\varepsilon_V - \chi\right)\left(\varepsilon_V^* - \chi^*\right)\right) dv = \int_{\partial\Omega} \left(p_i u_i^* + P n_i D u_i^* \right) ds \quad (22)$$

The numeric work implementation suggested in Code_Aster is detailed in the chapter 3 . One specifies here only spaces of approximation of the variables defining the kinematic field. The polynomial interpolations are the following ones:

A modeling P2-P1-P0 in plane deformations

1) functions of forms of the second order for the variables of displacements u_i
2) functions of first order forms for the tensor of the voluminal deformations microscopic χ
3) constant functions by element for the multipliers of Lagrange λ .

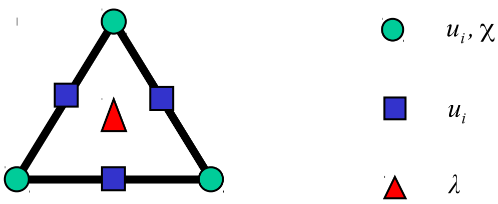

Figure **2.3-a** : Discretization finite element of the model second gradient of dilation for the plane deformations

A modeling P2-P1-P1 in 3D

1) functions of forms of the second order for the variables of displacements u_i
2) functions of first order forms for the tensor of the voluminal deformations microscopic χ and for the multipliers of Lagrange λ .

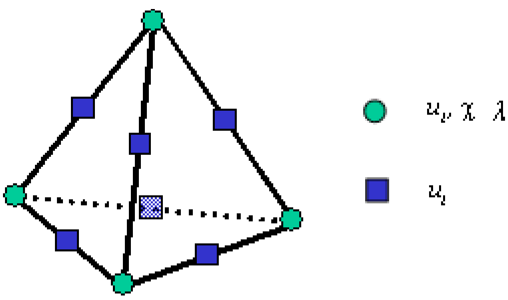

Figure **2.3-b** : Discretization finite element of the model second gradient of dilation for the 3D.

3 Digital integration of the models second gradient

3.1 Modelings second gradients in Code_Aster: “regularizing patchs”

Modelings finite elements put in work within the framework as of chapters 1 (second gradient) and 2 (second gradient of dilation) follow an atypical protocol compared to the existing procedures in Code_Aster. goal is to define the models second gradients like “patchs regularizing” to simplify at the same time the data-processing development and to generalize the validity of method with the unit of the existing laws of behaviors in Code_Aster. One interprets these two points in the continuation of this chapter.

The principle - identical for two modelings - thus consists with partitionner numerically the respective variational formulations (second gradient and second gradient of dilation) of classical a part known as “local” and a “regularizing” part in the following way (in the case of the second gradient of dilation)

$$\int_\Omega \left(\sigma_{ij} \frac{\partial u_i^*}{\partial x_j} \right) dv + \int_\Omega \left(S_j \frac{\partial \chi^*}{\partial x_j} - \lambda \left(\varepsilon_V^* - \chi^* \right) + \lambda^* \left(\varepsilon_V - \chi \right) + r \left(\varepsilon_V - \chi \right) \left(\varepsilon_V^* - \chi^* \right) \right) dv = \int_{\partial\Omega} \left(p_i u_i^* + P n_i D u_i^* \right) ds \qquad (23)$$

The calculation of the regularizing term is independent of the local part under condition that the laws of behaviors introduced to define the first order static variables σ_{ij} and of the second order S_j are independent. This restrictive assumption is thus currently obligatory and is the principal disadvantage of this strategy of design.

On the other hand, the interest comes owing to the fact that the “patch regularizing” to introduce is independent of the local part. It is thus enough to apply it to any type of modeling (mechanical, coupled hydro-mechanical, thermo-hydro-mechanics…). Moreover, numerically there is no data-processing impact on the part regularizing following integration of new laws of behaviors first gradient.

3.2 Modeling second gradient

While thus following the stated principle to the chapter 3.1 , the “patch regularizing” representative of the finite element second gradient describes in the chapter 1.3 results in the digital integration of the following formulation

$$\int_\Omega \left(\Sigma_{ijk} \frac{\partial f_{ij}^*}{\partial x_k} - \lambda_{ij} \left(\frac{\partial u_i^*}{\partial x_j} - f_{ij}^* \right) + \lambda_{ij}^* \left(\frac{\partial u_i}{\partial x_j} - f_{ij} \right) \right) dv \qquad (24)$$

One details below the fields of strains, stresses associated and the tangent matrix. There is no internal variable in the description of this finite element. For recall the nodal variables defining the degrees of freedom are the following ones: $\left(u_i, f_{ij}, \lambda_{ij} \right)$.

3.2.1 The field of deformations

$$E = \begin{pmatrix} \left(\dfrac{\partial u_i}{\partial x_j} - f_{ij} \right) \\ \dfrac{\partial f_{ij}}{\partial x_k} \\ \lambda_{ij} \end{pmatrix}$$

3.2.2 The field of the constraints

$$\Sigma=\begin{pmatrix}\lambda_{ij}\\ \Sigma_{ijk}\\ -\left(\frac{\partial u_i}{\partial x_j}-f_{ij}\right)\end{pmatrix}$$

3.2.3 The tangent matrix

The elementary tangent matrix of modeling second gradient is composed, inter alia, of the tangent matrix of elementary rigidity T^{2g} associated with the law of behavior second gradient which binds the double constraints Σ_{ijk} with the gradients of the microscopic deformations $\frac{\partial f_{ij}}{\partial x_k}$.

$$K^{el}=\begin{pmatrix}0 & 0 & (\mathbf{Id})_{\dim(ij)}\\ 0 & (T^{2g})_{\dim(ijk)} & 0\\ (-\mathbf{Id})_{\dim(ij)} & 0 & 0\end{pmatrix}$$

3.3 Modeling second gradient of dilation

While again following the stated principle to the chapter 3.1 , the "patch regularizing" representative of the finite element second gradient of dilation describes in the chapter 2.3 results in the digital integration of the formulation

$$\int_{\Omega}\left(S_j\frac{\partial\chi^*}{\partial x_j}-\lambda\left(\varepsilon_V^*-\chi^*\right)+\lambda^*\left(\varepsilon_V-\chi\right)+r\left(\varepsilon_V-\chi\right)\left(\varepsilon_V^*-\chi^*\right)\right)dv \quad (25)$$

One details below the fields of strains, stresses associated and the tangent matrix. There is no internal variable in the description of this finite element. For recall the nodal variables defining the degrees of freedom are the following ones: (u_i,χ,λ) .

3.3.1 The field of deformations

$$E=\begin{pmatrix}(\varepsilon_V-\chi)\\ \frac{\partial\chi}{\partial x_j}\\ \lambda\end{pmatrix}$$

3.3.2 The field of the constraints

$$\Sigma=\begin{pmatrix}r(\varepsilon_V-\chi)+\lambda\\ S_j\\ -(\varepsilon_V-\chi)\end{pmatrix}$$

3.3.3 The tangent matrix

The elementary tangent matrix of modeling second gradient is composed, inter alia, of the tangent matrix of elementary rigidity S^{2d} associated with the law of behavior second gradient of dilation which binds the double constraints S_j with the gradients of the microscopic voluminal deformations $\frac{\partial \chi}{\partial x_j}$.

$$K^{el}=\begin{pmatrix} r & 0 & 1 \\ 0 & \left(S^{2d}\right)_{\dim(j)} & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

4 The Councils/Procedure for the use of the models second gradient

Some simple handling is to be defined in the command file Code_Aster to use modelings second gradients. The approach is on the other hand same whatever the local modeling. Examples are available in the base of the CAS-tests of validation of Code_Aster. One can quote, inter alia, the biaxial tests in linear elasticity in compression referred ssll117, the construction of a column of ground into non-linear (law of Hujeux behavior in hydraulic coupled porous environment) referred wtnv132 or the triaxial compression test in linear elasticity referred sslv117.

4.1 Linear elasticity second gradient

To use modelings second gradients it is necessary to define two laws of behaviors respectively to describe the macroscopic relations between constraints σ_{ij} and deformations ε_{ij} and microscopic relations between double constraints (Σ_{ijk} for the second gradient or S_j for the second gradient of dilation) and gradient of deformations $\frac{\partial f_{ij}}{\partial x_k}$ for the second gradient or voluminal gradient of deformation $\frac{\partial \chi}{\partial x_j}$ for the second gradient of dilation.

For the choice of the behavior of the type first gradient – that which intervenes in the local modeling described in section 3.1 – there is no restriction: all the laws of behaviors are possible. On the other hand, one currently lays out of only one relation of behavior of the type second gradient. It is the linear elasticity suggested by Mindlin ([bib6], [bib7]) of which here the writing in 2D

$$\begin{bmatrix} \Sigma_{111} \\ \Sigma_{112} \\ \Sigma_{121} \\ \Sigma_{122} \\ \Sigma_{211} \\ \Sigma_{212} \\ \Sigma_{221} \\ \Sigma_{222} \end{bmatrix} = \begin{bmatrix} a^{12345} & 0 & 0 & a^{23} & 0 & a^{12} & a^{12} & 0 \\ 0 & a^{145} & a^{145} & 0 & a^{25} & 0 & 0 & a^{12} \\ 0 & a^{145} & a^{145} & 0 & a^{25} & 0 & 0 & a^{12} \\ a^{23} & 0 & 0 & a^{34} & 0 & a^{25} & a^{25} & 0 \\ 0 & a^{25} & a^{24} & 0 & a^{34} & 0 & 0 & a^{23} \\ a^{12} & 0 & 0 & a^{25} & 0 & a^{145} & a^{145} & 0 \\ a^{12} & 0 & 0 & a^{25} & 0 & a^{145} & a^{145} & 0 \\ 0 & a^{12} & a^{12} & 0 & a^{23} & 0 & 0 & a^{12345} \end{bmatrix} \cdot \begin{bmatrix} \chi_{111} \\ \chi_{112} \\ \chi_{121} \\ \chi_{122} \\ \chi_{211} \\ \chi_{212} \\ \chi_{221} \\ \chi_{222} \end{bmatrix} \quad (26)$$

where $\chi_{ijk}=\frac{\partial f_{ij}}{\partial x_k}$ and all the terms of the matrix depend on five constants according to the following relation

$$\begin{cases} a^{12345}=2\left(a^1+a^2+a^3+a^4+a^5\right) \\ a^{23}=a^2+2\mathrm{a}^3 \\ a^{12}=a^1+\frac{a^2}{2} \\ a^{145}=\frac{a^1}{2}+a^4+\frac{a^5}{2} \\ a^{25}=\frac{a^2}{2}+a^5 \\ a^{34}=2\left(a^3+a^4\right) \end{cases}$$

Fernandes et al. [bib5] applied this model for the second gradient of dilation by simplifying the expression (26)) in 2D by

$$\begin{bmatrix} S_1 \\ S_2 \end{bmatrix}=\begin{bmatrix} 3\mathrm{a}^1 & 0 \\ 0 & 3\mathrm{a}^1 \end{bmatrix}\cdot\begin{bmatrix} \frac{\partial \chi}{\partial x_1} \\ \frac{\partial \chi}{\partial x_2} \end{bmatrix} \quad (27)$$

4.2 Names of modelings in Code_Aster

The assumption of the plane deformations is currently possible for modelings second gradients in Code_Aster. For the second gradient the name of modeling is D_PLAN_2DG while for the second gradient of dilation the name of modeling is D_PLAN_DIL.

The accepted meshs are QUAD9 and TRIA7 for two modelings P2-P1-P0 second gradients. For the second gradient of dilation it is also possible to use meshs QUAD8 and TRIA6 in which case modeling is then defined without multipliers of Lagrange and the mathematical constraint (16)) is only ensured by the penalization.

For the applications 3D, the name of modeling is 3D_DIL for the second gradient of dilation. The meshs accepted are TETRA10, PENTA15 or HEXA20 for interpolations P2-P1-P1.

4.3 An example with accompanying notes

Here an example of command file to the Code_Aster format. The comments introduced by the character *$* are specific to the “regularizing patchs” introduced by modelings second gradients.

<u>reading of the grid (quadratic)</u>

```
MY = LIRE_MAILLAGE ()
```

$ Duplication of the grid (quadratic). Only the meshs are duplicated, the nodes remain common. The goal is to associate with each one of these grids a different modeling only on the structure part. As regards the application of the boundary conditions one does not duplicate the meshs, it is directly the expression (20) who will be applied.

```
MAIL=CRÉA_MAILLAGE (MAILLAGE=MA,
                              CREA_GROUP_MA= (_F (NOM=' ROCHE_REG',
```

```
GROUP_MA=' ROCHE')))
```

$ Introduction of central nodes to the finite elements of the new grid for an interpolation P2-P1-P0. Procedure necessary to take into account the interpolations of the multipliers of Lagrange.

```
MAILLAGE=CRÉA_MAILLAGE (MAILLAGE=MAIL,
                       MODI_MAILLE= (_F (GROUP_MA = 'ROCHE_REG',
                                        OPTION = 'QUAD8_9'),
                                     _F (GROUP_MA = 'ROCHE_REG',
                                        OPTION = 'TRIA6_7')))
```

$ Assignment of a modeling to each grid.

```
MODELE=AFFE_MODELE (GRID = GRID,
                   AFFE = (_F (GROUP_MA = 'ROCK',
                              PHENOMENON = 'MECHANICAL',
                              MODELING = 'D_PLAN'),
                           _F (GROUP_MA = 'ROCHE_REG',
                              PHENOMENON = 'MECHANICAL',
                              MODELING = 'D_PLAN_DIL')))
```

Taking into account of the boundary conditions:
$ Attention the expression of the boundary conditions with taking into account of modelings second gradients differs from that of local modelings (see chapter 1).

```
CHCI=AFFE_CHAR_MECA (MODELE=MODELE,
                    DDL_IMPO= (...),…)
```

Definition of the parameters materials for the law of behavior first gradient

```
SOL1 = DEFI_MATERIAU (…)
```

$ Definition of the parameters materials for the law of behavior second gradient. Currently was introduced only the linear elasticity second gradient suggested by Mindlin.

```
SOL2 = DEFI_MATERIAU (ELAS_2NDG =_F (A1=10, A2=0, A3=0, A4=0, A5=0),…)
```

$ Assignment of the parameters materials according to the same procedure as for the definition of modelings.

```
= AFFE_MATERIAU (GRID SUBDUE = GRID,
                     AFFE = (_F (ALL = 'ROCK',
                     MATER =  SOL1,),
                     _F (GROUP_MA = 'ROCHE_REG',
                        MATER = SOL2)))
```

$ Definition of nonlinear static calculation with a law of behavior associated with each one of modeling: behavior of the Drucker-Prager type for the first gradient, and linear elasticity for the second gradient

```
U1 = STAT_NON_LINE (MODEL = MODEL,
                   CHAM_MATER = SUBDUE,
                   EXCIT = (_F (LOAD = CHCI),…),
                   SOLVEUR = (_F (METHODE=' MUMPS',)),
                   BEHAVIOR = (_F (GROUP_MA=' ROCHE',
                                  RELATION=' DRUCK_PRAGER',),
                               _F (GROUP_MA=' ROCHE_REG',
                                  RELATION=' ELAS',),),
                   NEWTON = _F (MATRIX   = 'TANGENT', REAC_ITER = 1),
                   INCREMENT = _F (LIST_INST = TIME))
```

4.4 Estimate of the parameter of regularization `A1` (2Nd `gradient of dilation`)

With an aim of characterizing the value as well as possible to be assigned to the parameter of regularization, `A1`, of the 2Nd gradient of dilation, it is possible to determine a limit higher than this parameter according to the tangent matrix of speed of very model of lenitive behavior.
In this way, the user will be able to determine according to the size of the meshs of with the problem which he deals the value of the parameter `A1` best adapted its problem has.

This calculation is based on an analytical problem $2\mathrm{D}$ of a band of shearing [bib2 and bib11].

The rupture in the géomatériaux ones is often characterized by the formation of zones to located deformations, reporting a passage of zones of homogeneous deformations towards modes of nonhomogeneous deformations. The appearance of this phenomenon can be seen on the theoretical level like the spontaneous change of the mode of deformation, compared to a junction of a branch of balance, i.e. the intersection of two branches of solutions functions of the parameters of control. Within the framework of the continuous mediums, it is possible to release under restricted conditions of the criteria of junction making it possible to identify the parameter of control causing the appearance of the band of shearing, as well as the potential orientations of this one. On the other hand, for the classical continuous mediums, the mode of “post-localization”, in particular the bandwidth of shearing, cannot be characterized. The experimental results show nevertheless that this width is an intrinsic element with the properties of material, related to its microstructure (form and size of the grains, Desrues and Viggiani [bib12]) and its initial state (index of the vacuums and state of stresses). The use of the theory of the second local gradient of dilation makes it possible to enrich kinematics by the medium thus translating the effects of the microstructure on a total scale. In this section, we will extract from a two-dimensional analytical problem the elements allowing to characterize the bandwidth of shearing lasting the mode of “post-localization”.

The formulation of the problem of speed is written then in the following form, by considering the constant forces of volume:

$$\int_{\Omega} \dot{\sigma}_{ij}\dot{\epsilon}_{ij}(u^{*})+\dot{S}_{j}\eta_{j}(u^{*})dv=\int_{\partial\Omega} \dot{F}_{i}u_{i}^{*}ds \quad \forall u^{*}\in\mathcal{V}_{0} \tag{28}$$

$\dot{\sigma}_{ij}$, $\dot{S}_{j}$ and $\dot{F}_{i}$ are them derived temporal of the terms introduced into the equation 17 .

4.4.1 Analytical resolution

The two-dimensional analytical problem approached below was solved by [bib2] for a model of second complete gradient with the elastoplastic local model of Mohr-Coulomb described by [bib13]. The example consists in applying the model of second gradient of dilation in a sheared layer and to determine the solutions of the problem of speed.

One considers a layer defined in the plan (x, y) with z the axis perpendicular to the plan. Chambon and al. [bib2] consider a layer infinite understood between two plans defined by $x=0$ and $x=l$ (Figure 4.4-a). Champs of displacement is noted u in the direction x and v in the direction y . $\dot{u}$ and $\dot{v}$ are supposed to depend only on x and their derivative compared to x will be noted $'$.

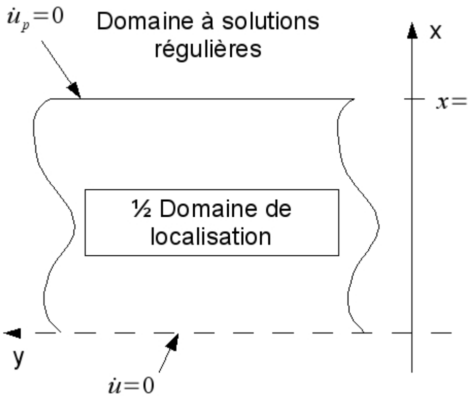

Figure 4.4-a : Studied field

The initial state of material is defined homogeneous. The evolutions of the boundary conditions applied to the field are the following ones:

- for $x=0$, $\dot{u}=0$, $\dot{v}=0$

- for $x=l$, $\dot{F}_i$ are known.

The forces of volume are supposed to be constant.
Conditions of symmetry applied to the border $x=0$ indicate that it field studied with a total width of $2l$.
This problem can be seen as analysis of the behavior of a band of localization, where the orientation of the band is supposed and the state of stresses is left free of any restriction. border of the field in $x=l$ defines the zone of transition between the zone from localization (field studied) and the field of an unspecified solid where the solutions are supposed to be regular and stable.

The gradient of the field of displacement in the field takes form following:

$$\dot{u}_{i,j}=\begin{bmatrix}\dot{u}' & 0\\ \dot{v}' & 0\end{bmatrix} \tag{29}$$

The macroscopic field of deformation is written explicitly:

$$\dot{\epsilon}_{ij}=\frac{1}{2}\left(\dot{u}_{i,j}+\dot{u}_{j,i}\right)=\begin{bmatrix}\dot{u}' & \frac{1}{2}\cdot\dot{v}'\\ \frac{1}{2}\cdot\dot{v}' & 0\end{bmatrix} \tag{30}$$

The only term not no one of the gradient of the deformations voluminalS is :

$$\frac{\partial\dot{\epsilon}_v}{\partial x_1}=\dot{u}'' \tag{31}$$

The application of the principle of virtual work to the field of displacement imposed after 2 integrations by parts the following equilibrium equations for the problem of speed gives:

$$\frac{\partial\dot{\sigma}_{ij}}{\partial x_j}-\frac{\partial^2\dot{S}_j}{\partial x_i\partial x_j}=0 \;\Rightarrow\; \begin{cases}\dot{\sigma}'_{11}-\dot{S}''_1 &= 0\\ \dot{\sigma}'_{12} &= 0\end{cases} \tag{32}$$

The boundary conditions provide in $x=l$:

$$\begin{cases}\dot{\sigma}_{11}-\dot{S}'_1 &= \dot{F}_1\\ \dot{\sigma}_{12} &= \dot{F}_2\\ 3a^1\dot{u}''' &= 0\end{cases} \tag{33}$$

After space integration of the equilibrium equations by taking account of conditions in extreme cases for a homogeneous problem, the solutions of the problem respect the following equations:

$$\begin{cases}\dot{\sigma}_{11}-\dot{S}'_1 &= \dot{F}_1\\ \dot{\sigma}_{12} &= \dot{F}_2\end{cases} \tag{34}$$

By using the laws constitutive of the first and second gradients, the equations become :

$$\begin{cases}H_{1111}\dot{u}'+H_{1112}\dot{v}'-3a^1\dot{u}''' &= \dot{F}_1\\ H_{1211}\dot{u}'+H_{1212}\dot{v}' &= \dot{F}_2\end{cases} \tag{35}$$

The system of coupled differential equations can be reduced to an equation, second equation expressing a linear relation between the first gradients of vertical and horizontal displacements in the band.

$$\begin{cases} H_{1111}\dot{u}' + H_{1112}\dfrac{\dot{F}_2 - H_{1211}\dot{u}'}{H_{1212}} - 3a^1\dot{u}''' &= \dot{F}_1 \\ \dfrac{\dot{F}_2 - H_{1211}\dot{u}'}{H_{1212}} &= \dot{v}' \end{cases} \quad (36)$$

Solutions of this system of linear differential equations of order 2 express themselves like the sum of a particular solution answering the boundary conditions imposed and of a partial solution established starting from the roots of the polynomial characteristic of this system: $\dot{u}' = \dot{u}'_0 + \dot{u}'_p$

One supposes for $\dot{u}'_p$ a solution of the type $\dot{u}'_p = e^{\eta x}$ who allows to establish the following characteristic equation:

$$H_{1111} - H_{1112}\frac{H_{1211}}{H_{1212}} - 3a^1\eta^2 = 0 \quad (37)$$

One can then give the expression of the partial solution according to parameters material of the first and second gradient:

that is to say $\eta^2 = \dfrac{H_{1111}H_{1212} - H_{1112}H_{1211}}{3a^1 H_{1212}}$ (38)

The law constitutive of first gradient being elastoplastic, the solutions will be different depending on the state from the region considered. In a region with elastic behavior, the constitutive law is written $\dot{\sigma} = A\dot{\epsilon}$ and in an area with plastic behavior $\dot{\sigma} = H^{ep}\dot{\epsilon}$.

Finally the solutions of the problem take the following shapes in function sign of

$$\eta^2 = \Delta = \frac{H_{1111}H_{1212} - H_{1112}H_{1211}}{3a^1 H_{1212}} \quad (39)$$

- If $\Delta > 0$, then

$$\dot{u}' = \dot{u}'_0 + \left(C_{11}\exp(\eta_1 x) + C_{12}\exp(\eta_2 x)\right) \quad (40)$$

with $\eta_i = \pm\sqrt{\Delta}$ and $\dot{u}'_0$ obtained starting from the equation of the particular solution:

$$\dot{u}'_0 = \frac{H_{1212}\dot{F}_1 - H_{1112}\dot{F}_2}{H_{1111}H_{1212} - H_{1112}H_{1211}} \quad \text{and} \quad \dot{v}'_0 = \frac{H_{1111}\dot{F}_2 - H_{1211}\dot{F}_1}{H_{1111}H_{1212} - H_{1112}H_{1211}} \quad (41)$$

Moreover, partial solutions $\dot{u}'_p$ check the following equations:

$$\frac{H_{1111}H_{1212} - H_{1112}H_{1211}}{3a^1 H_{1212}}\dot{u}'_p - \dot{u}'''_p = 0 \quad (42)$$

From the conditions of symmetry of the studied field and $\dot{u}'_p(l) = 0$, one from of deduced that coefficients C_{11} and C_{12} are worthless.

- If $\Delta < 0$, then

$$\dot{u}' = \dot{u}'_0 + (C_{21}\cos(\omega x) + C_{22}\sin(\omega x)) \quad (43)$$

with $\omega=\sqrt{\frac{H_{1211}H_{1112}-H_{1212}H_{1111}}{3a^1 H_{1212}}}$. Just as for the solutions previously established for the zones where $\Delta>0$, the partial solution $\dot{u}'_p$ check the equation 42.
The expression of the solution in the zone where $\Delta<0$ fact of appearing goniometrical functions. It is thus clear that a zone of localization can appear in the structure. One can then to admit that the structure privileges a length interns $l_c=2\pi/\omega$, expressing itself according to the first mode of lower energy.
It is also noted that the dependence internal length l_c is in $\sqrt{a^1}$. It is interesting to also notice that the condition of appearance of the solutions of junctions is controlled only by the terms of the model of first gradient. In other words, the taking into account of a kinematics enriched restricted in the mediums by second gradient does not modify the value of the parameter of control causing the appearance of a solution forked in bands of shearing.

4.4.2 Digital resolution

The elementary option of calculation `PDIL_ELGA`, determines, for a given initial state of the constraints and internal variables, the value of `A1_LC2` established according to the following formula:

$$\mathrm{A1\_LC2}=\frac{a^1}{l_c^2}=\frac{\left[H_{1211}H_{1112}-H_{1111}H_{1212}\right]}{3\,H_{1212}(2\pi)^2} \quad (44)$$

for various orientations of the band of shearing. The explicit writing of the components of the tensor of rigidity of the classical local model is obtained via the routines of calculation of the tangent matrices of speed.
A rotation of the band of shearing of one angle θ imply a local rotation applied to the components of the tensor of the constraints. It makes it possible to estimate the new components of the local tensor constitutive of the model of classical behavior, bearing on the first gradient of displacements.
The angular discretization used is first of all $5°$. Around the first raised maximum, one carries out a discretization with $1°$ then $0.2°$. In this way, one ensures oneself to obtain a precise value of the parameter `A1_LC2`.

The user can then define the value of `A1` adapted to the space discretization of the studied structure, knowing that a minimum of 6 elements about `LC` appears necessary to guarantee independence with the grids of the results.

Digital validation of the option of calculation `PDIL_ELGA` is carried in CAS-tests SSNV208A, SSNP125A and WTNV132C.

5 Functionality and checks

The list of the tests of validation for the second gradient:

CAS-test	description
ssll117 (modelings a->e)	Test of `D_PLAN_DIL` (see V3.01.117) for modeling `P2-P1-P0` (see §4.2)
ssll117f	Test of `D_PLAN_2DG` (see V3.01.117) for modeling `P2-P1-P1`
ssll117g	Test of `D_PLAN_DIL` (see V3.01.117) for modeling `P2-P1-P1`
sslv117a	Test of 3D`_DIL` (see V3.04.117) for modeling `P2-P1-P1`

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7 Description of the versions of the document

Index document	Version Aster	Author (S) Organization (S)	Description of the modifications
With	9.3	R.FERNANDES	Initial text