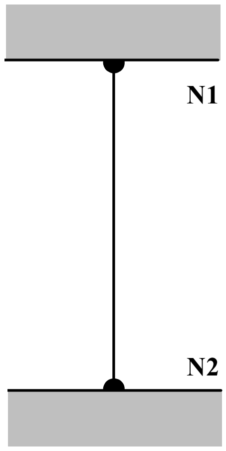
SSNL112 - Bar subjected has a cyclic thermal loading

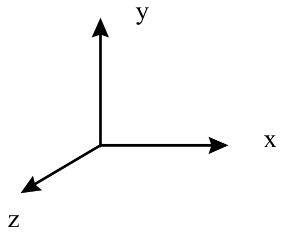
Summary:
This case test enters within the framework of the validation of the relations of behavior in elastoplasticity of the elements bars for the quasi-static mechanics of the structures.
An embedded bar has these two ends undergoes a cyclic thermal loading inducing efforts of traction and compression.

Each modeling makes it possible to validate one of the relations of non-linear behavior introduced: Linear isotropic work hardening with criterion of Von-Settings (modeling A), linear kinematic work hardening with criterion of Von-Settings (modeling B), as well as a model known as of Pinto-Menegotto, representing the cyclic behavior of the steel reinforcements in the reinforced concrete (modelings C and D).

1 Problem of reference

1.1 Geometry

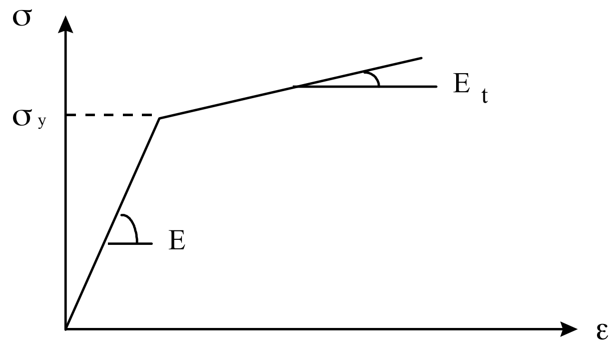




Length of the bar : $1\,m$
Section of the bar : $5\,cm^2$

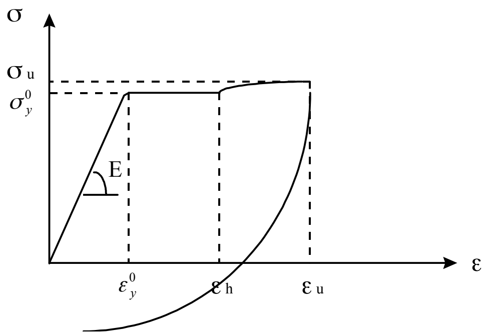
1.2 Properties of materials

1.2.1 Linear work hardenings isotropic and kinematics



Young modulus:	$E=2.10^{11}\,Pa$
Slope of work hardening:	$E_t=2.10^9\,Pa$
Elastic limit:	$\sigma=2.10^8\,Pa$
Poisson's ratio:	$\nu=0,3$
Thermal dilation coefficient:	$\alpha=1.10^{-5}\,K^{-1}$

1.2.2 Model of Pinto-Menegotto



Young modulus:	E	=	2. $10^{11}\,Pa$
Elastic limit:	σ_y^0	=	$2.10^8\,Pa$
Poisson's ratio:	ν	=	0.3
Thermal dilation coefficient:	α	=	$1.10^{-5}\,K^{-1}$
Deformation of work hardening:	ε_h	=	$2.3\,10^{-3}$
Ultimate constraint:	σ_u	=	$2.58\,10^8\,Pa$
Ultimate deformation:	ε_u	=	3.10^{-2}
Coefficient defining the curve ζ :	R_0	=	20
Coefficient defining the curve ζ	A_1	=	18.5
Coefficient defining the curve ζ	A_2	=	0.15
Coefficient of buckling:	C	=	0.5
Coefficient of buckling:	A	=	0,008

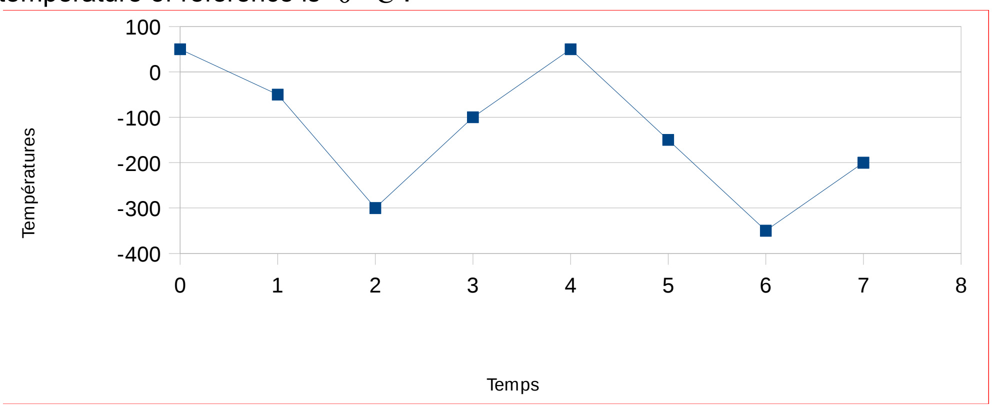
1.3 Boundary conditions and loading

Boundary conditions:
The bar is embedded. Displacements are thus blocked in the three directions.
In $N1$ and $N2$: $DX=DY=DZ=0$

Loading:
The way of loading is described by the change of the temperature, uniform in the bar:

t	0	1	2	3	4	5	6	7
$T(°C)$	50	– 50	– 300	– 100	50	– 150	– 350	– 200

The temperature of reference is $0°C$.



2 Reference solutions

2.1 Method of calculating used for the reference solutions

2.1.1 Linear work hardenings

Isotropic work hardening

For a uniaxial traction, the criterion of plasticity is written:

$$\left|\sigma_L\right| - R(p) \le 0$$

where p is the cumulated plastic deformation

$$R(p) = R' p + \sigma^y \quad \text{and} \quad R' = \frac{E\ E_1}{E - E_1}$$

The criterion is written then:

$$\left|\sigma_L\right| - R' p - \sigma^y \le 0$$

The tensor of the constraints is obtained by:

$$\sigma = \boldsymbol{A}.\left(\varepsilon(\boldsymbol{u}) - \varepsilon^p\right) - 3\mathrm{K}\alpha\left(T - T^{ref}\right)\boldsymbol{Id}$$

One from of thus deduced the expression from σ_L

$$\sigma_L = E(\varepsilon - \alpha T) - E\varepsilon^p \qquad \left(T^{ref} = 0\right)$$

In our case, $\varepsilon = 0$ thus:

$$\sigma_L = E\varepsilon_L - E\varepsilon^p \qquad \text{with } \varepsilon_L = -\alpha T$$

Thus:

- If $\left|\sigma_L\right| - R(p) < 0$:

  p=0 and $\sigma_L = E\varepsilon_L$
- If $\left|\sigma_L\right| - R(p) = 0$:

  $$p = \left(\frac{|\sigma| - \sigma^y}{R'}\right)$$

  $$\sigma_L = E\varepsilon_L - E\varepsilon^p$$

Application to the way of loading

Moment 1:

$\sigma = E\varepsilon = 200\mathrm{MPa}$ and $R(p) = R' p + \sigma^y = 100\mathrm{MPa}$ because *p=0*.

One has well $\sigma_L - R(p) \le 0$.

The criterion is not crossed, the evolution is elastic: $\sigma_L = 100\mathrm{MPa}$ and $N = 100\mathrm{kN}$

Moment 2:

The criterion is reached:

$$\sigma_L = \frac{E}{E + R'}(R'\varepsilon_L + \sigma^y) = \frac{2\cdot 10^{11}}{2\cdot 10^{11} + 2.02\cdot 10^9}(2.02\cdot 10^9 \times 3.5\cdot 10^{-3} + 2\cdot 10^8) = 205\mathrm{MPa}$$

$N = 102.5\mathrm{kN}$

and $p = 2.475\cdot 10^{-3}$

Moment 3:
One discharges elastically:
$$\sigma_L = E\,\varepsilon_L - E\,\varepsilon^p = 2\cdot 10^{11}(1.5\cdot 10^{-3} - 2.475\cdot 10^{-3}) = -195\,MPa$$
$$N = -97.5\,kN$$

Moment 4:
One plasticizes again:
The criterion is written: $|\sigma| - R'\,p - \sigma^y = 0$ with $p = p_1 + p_2$ where $p_1 = 2.475\cdot 10^{-3}$
One thus obtains:
$$p_2 = \frac{|\sigma| - \sigma^y}{R'} - p_1$$
$$\sigma = -E\,\varepsilon^p = -E(p_1 - p_2)$$
$$\sigma = \left(\frac{R'}{R'+E}\right)\left(-2E\;p_1 - \frac{E\sigma^y}{R'}\right) = -207.9\text{MPa}$$

And thus $N = -103.95\text{kN}$

Moment 5:
One discharges elastically:
$$\sigma_L = E\,\varepsilon_L - E\,\varepsilon^p = 2\cdot 10^{11}(2\cdot 10^{-3} - 1.0395\cdot 10^{-3}) = 192.1\,MPa$$
$$N = 96.05\,kN$$

Moments 6 and 7:
The reasoning is identical
One finds:
$$N_{(inst.6)} = 105.87\text{kN}$$
$$N_{(inst.7)} = -44.13\text{kN}$$

Kinematic work hardening
The method of calculating is identical, but in this case, the criterion of plasticity is written:
$$\sigma - X_{eq} - \sigma^y \le 0 \text{ with } X_{eq} = C(\varepsilon^p)_{eq} = \frac{3}{2}C\,\varepsilon^p = \frac{E\,E_t}{E - E_t}\varepsilon^p$$
With the preceding notations, the criterion is written:
$\left|\sigma_L - R'\,\varepsilon^p\right| - \sigma^y \le 0$ from where $\sigma_L = R'\,\varepsilon^p \pm \sigma^y$ (according to the direction of the flow).

Application to the way of loading
Moment 1:
The criterion is not crossed, the evolution is elastic: $\sigma_L = 100\text{MPa}$ and $N = 100\text{kN}$

Moment 2:
The criterion is reached: $|\sigma_L - R'\,\varepsilon^p| - \sigma^y = 0$
$$\sigma_L = R'\,\varepsilon^p + \sigma^y = 2.02\cdot 10^9 \times 2.475\cdot 10^{-3} + 2\cdot 10^8 = 205\text{MPa}$$

Moment 3:
One discharges elastically:
$$\sigma_L = E\,\varepsilon_L - E\,\varepsilon^p = 2\cdot 10^{11}(1.5\cdot 10^{-3} - 2.475\cdot 10^{-3}) = -195\text{MPa}$$
$$N = -97.5\text{kN}$$

Moment 4:

$$|\sigma - R'\varepsilon^p| - \sigma^y = 0 \quad \text{with } p_1 = 2.475\cdot 10^{-3}$$

$$\varepsilon^p = p_1 - p_2$$

$$p_2 = p_1 - \frac{|\sigma + \sigma^y|}{R'}$$

$$\sigma = -E\varepsilon^p = -E(p_1 - p_2)$$

$$\sigma = -E\left(\frac{|\sigma + \sigma^y|}{R'}\right) = -198\text{MPa}$$

$$N = -99\text{kN}$$

Moment 5:
One discharges elastically:

$$\sigma_L = E\varepsilon_L - E\varepsilon^p = 2\cdot 10^{11}(2\cdot 10^{-3} - 9.9\cdot 10^{-4}) = 202\,MPa$$

$$N = 101\text{kN}$$

Moments 6 and 7:
The reasoning is identical. One finds:

$$N_{(inst.6)} = 103\text{kN}$$

$$N_{(inst.7)} = -47\text{kN}$$

2.1.2 Model of Pinto-Menegotto

This model is described in the Manuel de Référence of *Code_Aster* [R5.03.09] [bib1]. The law constitutive of steels is made up of two distinct parts: the monotonous loading composed of three successive zones (linear elasticity, plastic stage and work hardening) and the cyclic loading where the way between two points of inversion (semi-cycle) is described by an analytical curve of expression of the type $\sigma = f(\varepsilon)$.

As previously the imposed deformations are thermal deformations: $\varepsilon = -\alpha T$

2.1.2.1 Case without buckling

First loading

- Linear elasticity: $\sigma = E\varepsilon$

Moment 1:

$$N = E\varepsilon S = 2\cdot 10^{11}\times 1\cdot 10^{-3}\times 5\cdot 10^{-4} = 100\text{kN}$$

- Plastic stage: $\sigma = \sigma^y$
- Polynomial of degree 4: $\sigma = \sigma_u - (\sigma_{su} - \sigma_y^0)\left(\frac{\varepsilon_u - \varepsilon}{\varepsilon_u - \varepsilon_h}\right)^4$

Moment 2:

$\varepsilon = 3.5\cdot 10^{-3} > \varepsilon_h = 2.3\cdot 10^{-3}$, one uses the polynomial of degree 4:

$\sigma = 209.416\text{MPa}$ and $N = 104.708\text{kN}$

Cycles

Semi-cycle 1:

One determines ζ_p^0 :

$\zeta_p^0 = \varepsilon_r^0 - \varepsilon_y^0 = 3.5\cdot 10^{-3} - 1.10^{-3} = 2.5\cdot 10^{-3}$ because $\varepsilon_r^0 = \varepsilon_{(inst.2)}$

Then $\Delta\sigma^0$:

$$\Delta\sigma^0 = E_h \zeta_p^0 = 2\cdot10^9 \times 2.5\cdot10^{-3} = 5\mathrm{MPa}$$

From where $\sigma_y^1 = \sigma_y^0 \cdot sign(-\zeta_p^0) + \Delta\sigma^0 = -200+5 = -195\mathrm{MPa}$

One calculates then ε_y^1 :

$$\varepsilon_y^1 = \varepsilon_r^0 + \frac{\sigma_y^1 - \sigma_r^0}{E} = 3.5\cdot10^{-3} + \frac{(-195-209.416)\,10^6}{2.0\cdot10^{11}} = 1.477\cdot10^{-3}$$

One determines thus $\sigma^* = f(\varepsilon^*)$, defined by:

$$\sigma^* = b\,\varepsilon^* + \left(\frac{1-b}{(1+(\varepsilon^*)^R)^{(1/R)}}\right)\varepsilon^* \quad \text{, with } b = \frac{E_h}{E}$$

$$\varepsilon^* = \frac{\varepsilon - \varepsilon_r^0}{\varepsilon_y^1 - \varepsilon_r^0}$$

$$\sigma^* = \frac{\sigma - \sigma_r^0}{\sigma_y^1 - \sigma_r^0}$$

$$\xi_p^0 = \frac{\zeta_p^0}{|\varepsilon_y^1 - \varepsilon_r^0|} \quad \text{and} \quad R^1 = R_0 - \frac{A_1 \cdot \xi_p^0}{A_2 + \xi_p^0}$$

One obtains $\xi_p^0 = -1.23$ and $R^1 = 3.51$

One can then calculate the value of σ at moments 3 and 4:

Moment 3:

$$\varepsilon^* = \frac{\varepsilon_{(inst.3)} - \varepsilon_r^0}{\varepsilon_y^1 - \varepsilon_r^0} = \frac{1.5\cdot10^{-3} - 3.5\cdot10^{-3}}{1.477\cdot10^{-3} - 3.5\cdot10^{-3}} = 0.988$$

$$\sigma^* = b\,\varepsilon^* + \left(\frac{1-b}{(1+(\varepsilon^*)^R)^{(1/R)}}\right)\varepsilon^* = 0.01\times0.988 + \left(\frac{1-0.01}{(1+(0.988)^{3.51})^{(1/3.51)}}\right) = 0.82$$

and $\sigma = \sigma^*(\sigma_y^1 - \sigma_r^0) + \sigma_r^0 = 0.82\times(-195-209.416) + 209.416 = -122\,MPa$

from where $N = -61\mathrm{kN}$

Moment 4:

One uses the same method, with $\varepsilon = 0$.

$\varepsilon^* = 1.73$

$\sigma^* = 0.56$

$\sigma = -20\,MPa$

$N = -10\mathrm{kN}$

Semi-cycle 2: Moment 5 and 6:

The method of calculating is identical, one determines:

ζ_p^1 , σ_y^2 , ε_y^2 , ξ_p^1 , R^2 , then $\sigma^*_{(inst.5)} = f(\varepsilon^*_{(inst.5)})$ and $\sigma^*_{(inst.6)} = f(\varepsilon^*_{(inst.6)})$

and finally $\sigma_{(inst.5)}$ and $\sigma_{(inst.6)}$.

Semi-cycle 3: Moment 7 : Idem

2.1.2.2 Case with buckling

First loading

Identical to the preceding case.

Cycles

Semi-cycle 1 (compression):

The method of calculating is identical, but the value of the slope of the asymptote is modified:
A new coefficient is calculated b_c :

$$b_c = a(5.0 - L/D) e\left(b\xi' \frac{E}{\sigma^y - \sigma^\infty}\right) = 0.006\times(5.0-5.9)e^{\left(0.01\times 1.477\cdot 10^{-3}\frac{2\cdot 10^{11}}{2\cdot 10^{8}-1.36\cdot 10^{8}}\right)} = -0.0057$$

It is necessary then, as in the model without buckling, to determine σ_y^n. The reasoning is identical, but a complementary constraint is added σ_s^* in order to correctly position the curve compared to the asymptote.

$$\sigma_s^* = \gamma_s b E \frac{b-b_c}{1-b_c} = 0.028\times 0.01\times 2\cdot 10^{11}\times \frac{0.01+0.0057}{1+0.0057} = 0.87\,MPa$$

where γ_s is given by: $\gamma_s = \dfrac{11.0 - L/D}{10\left(e^{c(L/D)} - 1.0\right)} = 0.028$

Semi-cycle 2 (traction):

- In traction, a reduced Young modulus is adopted:
  $E_r = E\left(a_5 + (1.0 - a_5)e^{-a_6\xi''^2}\right) = 2\cdot 10^{11}\times\left(0{,}88 + (1-0.88)e^{(-620\times 1.473\cdot 10^{-6})}\right) = 1.99\cdot 10^{11}\,MPa$
  with $a_5 = 1.0 + (5.0 - L/D)/7.5 = 0.88$

The rest of the method is identical.

2.2 Results of reference

Normal effort N constant on the bar

2.3 Uncertainty on the solution

No, the solution is analytical

2.4 Bibliographical references

[1] Handbook of reference of *Code_Aster* [R5.03.09].
[2] S. ANDRIEUX: Thermoelastoplastic TD 1 Three bars perfect Von Mises. In “Initiation with thermoplasticity in *Code_Aster* ”, HI-74/96/November 13th, 1996 (manual of reference of the course).

3 Modeling A

3.1 Characteristics of modeling

The model is composed of an element of bar (BAR).
Law of behavior: elastoplasticity with linear isotropic work hardening - Criterion of Von Mises

3.2 Characteristics of the grid

2 nodes.
1 mesh SEG2

3.3 Sizes tested and results

Identification	Moments	Reference	Variation (%)
normal effort NR	1	1.0000E+05	0
normal effort NR	2	1.0250E+05	0
normal effort NR	3	– 9.7500E+04	0
normal effort NR	4	– 1.0395E+05	0
normal effort NR	5	9.6050E+04	0
normal effort NR	6	1.0587E+05	0
normal effort NR	7	– 4.4129E+04	0

4 Modeling B

4.1 Characteristics of modeling

The model is composed of an element of bar (BAR).
Law of behavior: elastoplasticity with linear kinematic work hardening - Criterion of Von Mises

4.2 Characteristics of the grid

2 nodes.
1 mesh SEG2

4.3 Sizes tested and results

Identification	Moments	Reference	Variation (%)
normal effort NR	1	1.0000E+05	0
normal effort NR	2	1.0250E+05	0
normal effort NR	3	– 9.7500E+04	0
normal effort NR	4	– 9.9000E+04	0
normal effort NR	5	1.0100E+05	0
normal effort NR	6	1.0300E+05	0
normal effort NR	7	– 4.7000E+04	0

5 Modeling C

5.1 Characteristics of modeling

The model is composed of an element of bar (BAR).
Law of behavior: model of Pinto-Menegotto without buckling (value of DASH lower than 5).

5.2 Characteristics of the grid

2 nodes.
1 mesh SEG2

5.3 Sizes tested and results

Identification	Moments	Reference	Variation (%)
normal effort NR	1	1.0000E+05	0
normal effort NR	2	1.0470E+05	0
normal effort NR	3	– 6.0777E+04	0
normal effort NR	4	– 9.1430E+04	0
normal effort NR	5	7.6082E+04	0
normal effort NR	6	1.0125E+05	0
normal effort NR	7	– 3.7965E+04	0

6 Modeling D

6.1 Characteristics of modeling

The model is composed of 1 element of bar (BAR).
Law of behavior: model of Pinto-Menegotto with buckling (value of DASH higher than 5).

6.2 Characteristics of the grid

2 nodes.
1 mesh SEG2

6.3 Sizes tested and results

Identification	Moments	Reference	Variation (%)
normal effort NR	1	1.0000E+05	0
normal effort NR	2	1.0470E+05	0
normal effort NR	3	– 6.0556E+04	0
normal effort NR	4	– 8.9078E+05	0
normal effort NR	5	7.6905E+05	0
normal effort NR	6	1.0125E+05	0
normal effort NR	7	– 3.8119E+04	0

7 Summary of the results

Results calculated by *Code_Aster* are in excellent agreement with the analytical solutions.