

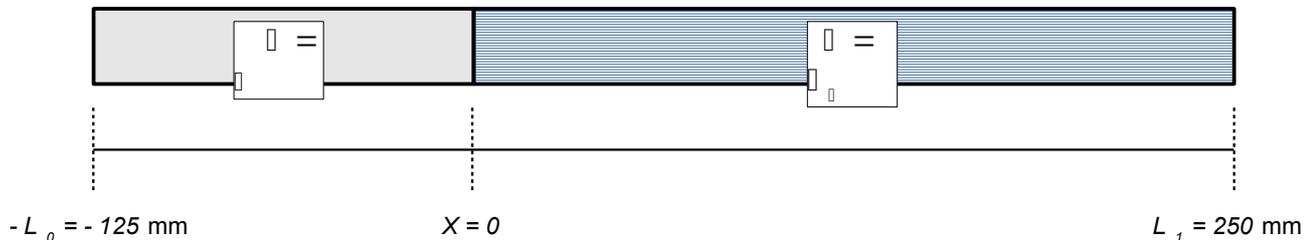
SSNL125 - Traction of a fragile bar: damage with gradient

Summary:

This test allows the checking of the laws of fragile damage gradient `ENDO_SCALAIRE` modeling A) and `ENDO_FISS_EXP` (modeling B) in a nonhomogeneous unidimensional situation. From its character 1D, this problem admits an analytical solution which displays two modes of boundary layers: one finite length (existence of a free border between the damaged zone and the healthy zone) and the other infinite length (it extends to the border from the part).

1 Problem of reference

1.1 Geometry



The studied structure is a bar of 375 mm of length. The problem being purely 1D, its section is without influence.

1.2 Properties of material

In modeling A, the material obeys a law of fragile elastic behavior (ENDO_SCALAIRE) with gradient of damage (modeling *_GRAD_VARI).

ELAS	ENDO_SCALAIRE	NON_LOCAL
$E = 30\,000 \text{ MPa}$ NAKED = 0	$SY = 3 \text{ MPa}$ GAMMA = 4	$C_GRAD_VARI = 1.875 \text{ N}$ PENA_LAGR = 1.5

In modeling B, the material obeys a law ENDO_FISS_EXP for which one informs the following parameters: $E = 30,000 \text{ Mpa}$, $NAKED = 0$, $ft = 3 \text{ MPa}$, $fc=30 \text{ MPa}$ (without incidence on the 1D result), $Gf=0.1 \text{ N/mm}$, $p=1.5$, $D=50 \text{ mm}$.

1.3 Conditions of loading

One imposes on the left part of the bar (125 mm of length) to remain rigid (blocking of the degrees of freedom of displacement). As for the right part of the bar, it is subjected to a uniform axial deformation ϵ_0 , i.e. with an imposed displacement whose spatial is linear. Only one parameter thus controls the intensity of the loading: the level of imposed deformation ϵ_0 .

In the directions perpendicular to the axis of the bar, displacements are blocked: the problem is purely 1D. Moreover, as the Poisson's ratio is null, no constraint of fastening does not develop in these directions.

2 Reference solution

We consider one of the laws of behavior to gradient of damage [R5.04.01] of which the density of free energy arises in the following form:

$\Phi(\varepsilon, a) = A(a)w(\varepsilon) + \omega(a) + c/2(\nabla a)^2$, where $0 \leq a \leq 1$ indicate the variable of damage and $w(\varepsilon)$ the elastic deformation energy. For the law ENDO_SCALAIRE, $\omega(a) = ka$ and the prefactor of damage is written like $A(a) = \left(\frac{1-a}{1+\gamma a}\right)^2$. The solution depends then on three parameters material γ , c , and k (see [R5.03.18]).

In the case general, two partial derivative equations must be solved: the equilibrium equation ($\delta \Phi(\varepsilon, a)/\delta \varepsilon = 0$) and the equation of behavior $\delta \Phi(\varepsilon, a)/\delta a = 0$. To obtain an analytical solution proves generally delicate, even for unidimensional structures. To validate nevertheless this model, one sticks to a simpler problem for which the equilibrium equation does not require to be solved, i.e. the field of displacement is fixed everywhere. The equation of behavior is then controlled by the elastic deformation energy w known in any point of space. In this simplified case, the semi-analytical solution is obtained by a digital integration of a differential equation with the initial conditions of Cauchy. As for the law ENDO_FISS_EXP, it is completely equivalent to the law ENDO_SCALAIRE in a situation of uniaxial traction. The analytical solution is thus identical to that for ENDO_SCALAIRE.

2.1 Characterization of the solution

More precisely, one considers a bar of which a part is obligation to remain without deformation while the other is subjected to a homogeneous deformation. One then studies the boundary layer of damage which develops with the interface of these two zones. The differential equation of behavior is the following one in the zones where the criterion is reached, i.e. where the damage evolves:

$$c \frac{\partial^2 a}{\partial x^2} - \frac{\partial A}{\partial a} w - k = 0, \text{ where } \frac{\partial A}{\partial a}(a) = -2(I + \gamma) \frac{(1-a)}{(1 + \gamma a)^3}, \quad \text{éq 2.1-1}$$

where x indicate the variable of space.

The elastic deformation energy w is known in each of the two parts of the bar: $w = 0$ on the left and $w(\varepsilon_0) = \text{const}$ on the right. The solution is given by a derivable continuous function $a \in C^1(-L_0, L_1)$.

One has three boundary conditions following:

$$\frac{da}{dx}(-L_0) = \frac{da}{dx}(L_1) = 0 \text{ and } a(-L_0) = 0 \quad \text{éq 2.1-2}$$

like two conditions of interface:

$$[|a|]_{x=0} = 0 \quad \left[\left[\frac{da}{dx} \right] \right]_{x=0} = 0 \quad \text{éq 2.1-3}$$

On the whole, we thus have five boundary conditions for five unknown factors. In spite of this fact, the problem "is badly posed" to be able to be solved numerically, because the boundary conditions are defined in the various points of space. We proceed in following the identification of the unknown factors using the boundary conditions finally to obtain an ordinary differential equation of Cauchy-Lipschitz type.

2.2 Resolution of the problem in the discharged zone

It is supposed that the left end of the bar $x = -L_0$ is located sufficiently far from the interface $x = 0$ so that the assumption of a boundary layer finite length in the discharged zone remains valid. Let us note then $b_0 > 0$ the length (*a priori* dependent on the level of deformation ε_0 in the zone charged) on which this boundary layer develops.

In the part $x \in [-L_0, -b_0]$ the damage does not evolve and remains null:

$$a(x) = 0 \text{ on } [-L_0, -b_0] \quad \text{éq 2.2-1}$$

In the part $x \in [-b_0, 0]$, the criterion is reached and the field of damage evolves according to the equation [éq 2.1-1], which is simplified considerably in this part of the bar since the elastic deformation energy is worthless there:

$$c \frac{d^2 a}{dx^2} = k \text{ on } [-b_0, 0] \quad \text{éq 2.2-2}$$

By definition of b_0 , one has the following conditions:

$$a(-b_0) = 0 \text{ and } \frac{da}{dx}(-b_0) = 0 \quad \text{éq 2.2-4}$$

One can then analytically express the variable of damage while integrating [éq 2.2-2]:

$$a(x) = \frac{k}{2c} (x + b_0)^2 \text{ on } [-b_0, 0] \quad \text{éq 2.2-5}$$

One then knows the values of the damage and his derivative to the interface $x = 0$, according to the unknown factor b_0 :

$$a(0^-) = \frac{kb_0^2}{2c} \text{ and } a'(0^-) = \frac{kb_0}{c} \quad \text{éq 2.2-6}$$

and the damage with the interface a_0 being strictly understood enters 0 and 1 , one from of deduced the following framing the length b_0 :

$$0 < a_0 < 1 \Rightarrow 0 < b_0 < \sqrt{\frac{2c}{k}} \quad \text{éq 2.2-7}$$

It is thus enough to take the left part of the bar longer than $\sqrt{2c/k}$ to be certain to have the profile of damage confined on the left.

2.3 Resolution of the problem in the zone charged

This part of the bar sees a homogeneous deformation $\varepsilon_0 > 0$ associated with an elastic deformation energy w_0 . In this zone, the boundary layer is not any more limited and asymptotically extends towards the homogeneous answer. It will thus be supposed that the right end of the bar $x = L_1$ is

located sufficiently far from the interface $x=0$ to be able to make the approximation of a worthless derivative for the field of damage "ad infinitum". In the vicinity of the right end of the bar, the equation of behavior [éq 2.1-1] is reduced then to:

$$\frac{-\partial A}{\partial a}(a_\infty)w_0=k \quad \text{éq 2.3-1}$$

where a_∞ indicate the asymptotic value of the field of damage. Taking into account the expression of $\partial A/\partial a$ (cf. [éq 2.1-1]), one can then parameterize the level of loading by the only value a_∞ :

$$\text{for } a_\infty \in]0,1[, \quad w_0 = \frac{k}{2(1+\gamma)} \frac{(1+\gamma a_\infty)^3}{(1-a_\infty)} \quad \text{éq 2.3-2}$$

In addition, the nonlinear differential equation [éq 2.1-1] can be written in the following form:

$$\frac{c}{2} 2a'a'' = a'(A'(a)w - k) \quad \text{éq 2.3-3}$$

she thus admits an integral first:

$$\forall x \in [0, L_1], \quad \left[\frac{c}{2} \left(\frac{da}{dx}(s) \right)^2 \right]_0^x = [A(a(s)) + k a(s)]_0^x \quad \text{éq 2.3-4}$$

It is pointed out that the condition of interface [éq 2.1-3] imposes a connection C^1 in $x=0$, by keeping flexible account as of expressions $a(0^+)$ and $a'(0^+)$ with the length b_0 boundary layer one can then write:

$$a_0 = a(0^+) = \frac{kb_0^2}{2c} \quad \text{and} \quad a'_0 = a'(0^+) = \sqrt{\frac{2k}{c} a_0} \quad \text{éq 2.3-5}$$

Moreover, the boundary conditions [éq 2.1-2] ensure the nullity of the derivative of the damage in $x=L_1$, by making the approximation $a(L_1)=a_\infty$, the integral first [éq 2.3-4] evaluated in $x=L_1$ cost with:

$$A(a_0)w_0 + k a_0 - \frac{c}{2} a'^2_0 = A(a_\infty)w_0 + k a_\infty \quad \text{éq 2.3-6}$$

The equation [éq 2.3-6] is simplified according to [éq 2.3-5], it is reduced then to trinomial of the second following degree of unknown factor a_0 :

$$\left(f(a_\infty)\gamma^2 - 1 \right) a_0^2 + 2 \left(f(a_\infty)\gamma + 1 \right) a_0 + \left(f(a_\infty) - 1 \right) = 0 \quad \text{where } f(a_\infty) = A(a_\infty) + \frac{k a_\infty}{w_0} \quad \text{éq 2.3-7}$$

One thus has a simple relation between the parameter of loading a_∞ and the value a_0 damage as well as its derivative a'_0 with the interface.

The resolution of the nonlinear EDO [éq 2.1-1] on $[0, L_1]$ was realized while bringing back it to a differential connection of a nature 1, integrated numerically with a diagram of Runge-Kutta into order 4

(in space). The setting in œuvre of this diagram requires only the knowledge of the initial conditions (a_0, a'_0) associated with each parameter of loading a_∞ , which is ensured by the relation [éq 2.3-7]

2.4 Resolution of the problem in the whole bar

In order to calculate the reference solution on all the bar, the following diagram is adopted:

- choice of a parameter of loading $a_\infty \in]0, 1[$
- Resolution in the part charged:
 - determination with [éq 2.3-2] of the elastic deformation energy w_0 associated
 - determination with [éq 2.3-7] of the set of boundary conditions (a_0, a'_0)
 - digital integration with a diagram of Runge-Kutta to order 4
- Resolution in the discharged part:
 - determination with [éq 2.3-7] and [éq 2.3-5] of the size b_0 boundary layer in the discharged zone
 - obtaining with [éq 2.2-5] the analytical field $a(x)$ in this zone

2.5 Digital application

For elasticity, work hardening and the nonlocal parameter, one adopts the following characteristics:

$$\begin{aligned} E &= 3.10^4 \text{ MPa} & \sigma^y &= 3. \text{ MPa} & c &= 1.875 \text{ N} & \text{éq 2.5-1} \\ \nu &= 0. & \gamma &= 4. & & & \end{aligned}$$

These choices lead to an elastic deformation energy threshold:

$$w_y = 1.5 \cdot 10^{-4} \text{ MPa} \quad \text{éq 2.5-2}$$

2.6 Results of reference

The reference solution is obtained by taking a bar length $-L_0 = -125 \text{ mm}$ and $L_1 = 250 \text{ mm}$. One examines the value of the field of damage a for three levels of loading and in two places, one in the discharged zone, the other in the zone charged.

ε_0	w_0 (MPa)	$a(x = -7.5 \text{ mm})$	$a(x = +7.5 \text{ mm})$
2,700000000000000E-04	1,093500000000000E-003	1,93274688119012E-02	1,41846675324338E-01
7,34846922834953E-04	8,100539999999999E-003	1,39107889370765E-01	3,80008828951670E-01
1,10464444958548E-02	1,83060308787201E+000	6,14240950943351E-01	9,77312427816067E-01

Table 2.6-1 - Results of reference

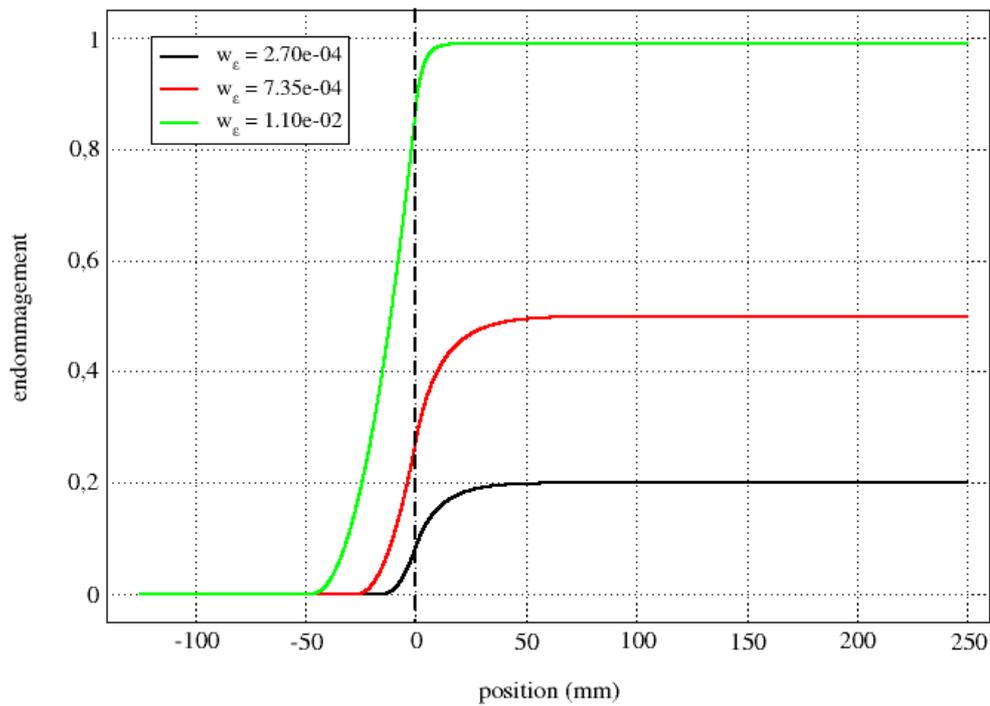


Figure 2.6-1: Profile of damage for the reference solution

3 Modelings A and B

3.1 Characteristics of modeling and the grid

It is about an axisymmetric modeling (AXIS_GRAD_VARI). The corresponding geometry is a rectangle, i.e. the bar is laid out in a vertical way and its section (without influence) is circular.

The grid consists of only one element according to the ray. According to the axis, the elements have a size of 2.5 mm . The grid thus generated finally consists of 150 quadrangular elements with 8 nodes.

3.2 Sizes tested and results

One validates the modeling and the algorithm of integration of nonlocal laws by examining the level of damage (variable internal VI) on the various levels of loading and the various loci listed in [Table 2.6-1]. The results are joined together in the extract of the file of result Ci - below.

Identification	Moment	Reference
$VI(x = -7.5)$	2,700000000000000E-04	1,93274688119012E-02
$VI(x = +7.5)$	2,700000000000000E-04	1,41846675324338E-01
$VI(x = -7.5)$	7,34846922834953E-04	1,39107889370765E-01
$VI(x = +7.5)$	7,34846922834953E-04	3,80008828951670E-01
$VI(x = -7.5)$	1,10464444958548E-02	6,14240950943351E-01
$VI(x = +7.5)$	1,10464444958548E-02	9,77312427816067E-01

4 Summary of the results

One notes very a good agreement between modeling and the analytical solution.