SSLV111 - Estimator of error on a plate perforated in linear elasticity

Summary:

This test validates and compares the 2 versions of the estimator of error of Zhu-Zienkiewicz (version 1 of 1987, noted $ZZ1$, and version 2 of 1992, noted $ZZ2$) applied to the system of linear elasticity, in statics.

It comprises 5 modelings in plane constraints, corresponding each one to a kind of finite element (TRIA3, QUAD4, TRIA6, QUAD8, QUAD9).
The analytical solution is known and makes it possible to compare the errors estimated with the exact error.

This test also validates version 1 of the estimator of error of Zhu-Zienkiewicz applied to 2 modelings in 3D. The first modeling comprises finite elements HEXA8, PYRAM5 and TETRA4 and the second, of finite elements PENTA6. The solution does not depend on the value along axis Z. The results are thus compared with the analytical values used in 2D.

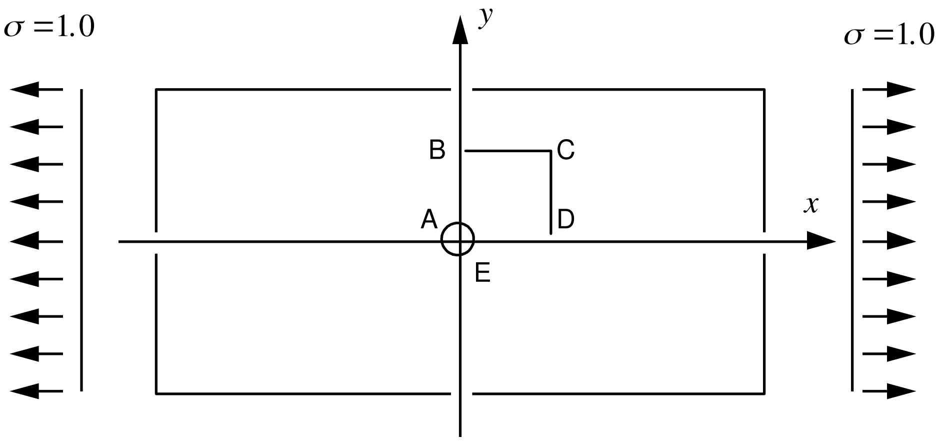
The interest of the test resides:

- the comparison enters the constraints smoothed with $ZZ1$ (continuous total smoothing) and $ZZ2$ (local smoothing with patchs of elements),
- in the comparison of the estimators between them,
- in the qualitative and quantitative analysis of the results (relative errors total and local)
- in the validation of the estimator in smoothing of Zhu-Zienkiewicz version 1 to the 3D.

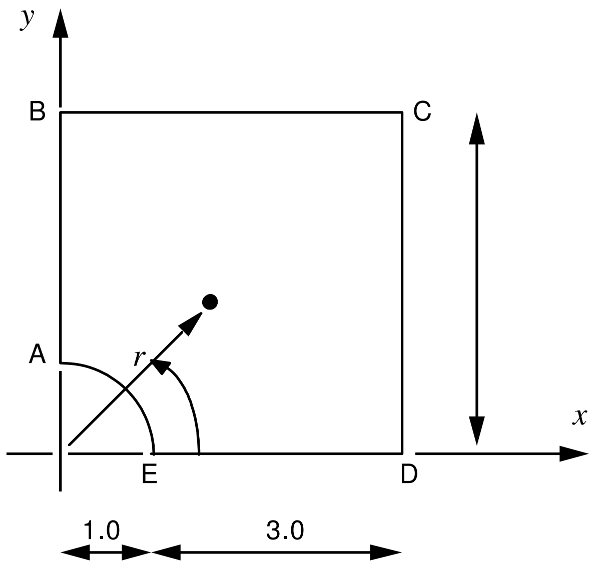
The test highlights the good behavior of $ZZ2$ on all the types of elements and the bad results of $ZZ1$ on quadratic elements when the solution does not present a strong singularity, which is the case.

1 Problem of reference

1.1 Geometry



(a)



(h)

1.2 Material properties

$E = 1\ 000\ MPa$

$\nu = 0.3$

1.3 Boundary conditions and loadings

On AB , $u_X = 0$

On ED , $u_y = 0$

On CD tractions $\begin{cases} F_x = \sigma_{xx}(x=4.) \\ F_y = \sigma_{xy}(x=4.) \end{cases}$

On BC tractions $\begin{cases} F_x = \sigma_{xy}(y=4.) \\ F_y = \sigma_{yy}(y=4.) \end{cases}$

According to the analytical solution [2].

2 Reference solution

2.1 Method of calculating used for the reference solution

One considers a portion of an infinite plate with a circular central hole, subjected to an one-way unit loading in the direction $Ox\, \sigma=1\, e_x \otimes e_x$.

The analytical solution of this problem is [bib1]:

$$
\begin{aligned}
\sigma_{xx} &= 1-\frac{a^2}{r^2}\left[\frac{3}{2}\cos\theta+\cos(4\theta)\right]+\frac{3}{2}\frac{a^4}{r^4}\cos(4\theta)\\
\sigma_{yy} &= -\frac{a^2}{r^2}\left[\frac{1}{2}\cos(2\theta)-\cos(4\theta)\right]-\frac{3}{2}\frac{a^4}{r^4}\cos(4\theta)\\
\sigma_{xy} &= -\frac{a^2}{r^2}\left[\frac{1}{2}\sin(2\theta)+\sin(4\theta)\right]+\frac{3}{2}\frac{a^4}{r^4}\sin(4\theta)
\end{aligned}
$$

where:

- a is the ray of the hole,
- (r, θ) polar coordinates.

2.2 Uncertainty on the solution

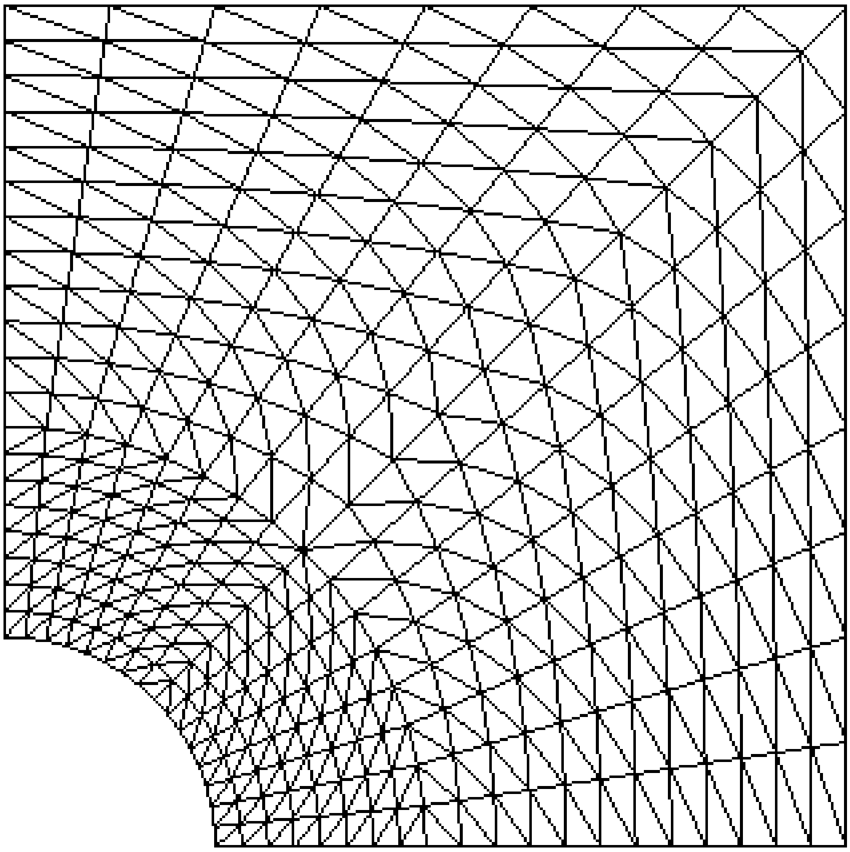
Analytical solution.

2.3 Bibliographical references

1) Zhu-Zienkiewicz: The superconvergent patch recovery and a posteriori error estimates - Share 1: the technical recovery (Int. J. for Num. Methods in Engineering vol. 33, p. 1355 (May 1992)).

3 Modeling A

3.1 Characteristics of modeling



3.2 Characteristics of the grid

Many nodes: 357.

Many meshs and types: 640 TRIA3.

3.3 Values tested

Identification	Reference	*Aster*	% difference	tolerance
With σ_{xx} ZZ1	3.	2,823	– 5.91	0.1
σ_{xx} ZZ2	3.	2,884	– 3.85	0.1
σ_{yy} ZZ1	0.	0,261	-	0.3
σ_{yy} ZZ2	0.	0,207	-	0.3
σ_{xy} ZZ1	0.	$-7.4\ 10^{-3}$	-	0.1
σ_{xy} ZZ2	0.	$-6.1\ 10^{-2}$	-	0.1
P σ_{xx} ZZ1	1.15625	1,152	– 0.37	0.1
σ_{xx} ZZ2	1.15625	1,145	– 0.98	0.1

σ_{yy} ZZ1	– 0.15625	– 0,150	– 3.81	0.1
σ_{yy} ZZ2	– 0.15625	– 0,145	– 7.00	0.1
σ_{xy} ZZ1	– 0,125	– 0,117	– 6.11	0.1
σ_{xy} ZZ2	– 0,125	– 0,124	– 0.68	0.1
M1 mesh e_{abs} ZZ1		$1.33\ 10^{-4}$		1.10^{-3}
e_{abs} ZZ2		$8.13\ 10^{-5}$		1.10^{-3}
e_{rel} ZZ1		6.63%		0.1
e_{rel} ZZ2		4.05%		0.1
Ω e_{abs} ZZ1	$0,445\ 10^{-2}$	$0,424\ 10^{-2}$	– 4.76	
e_{abs} ZZ2	$0,445\ 10^{-2}$	$0,451\ 10^{-2}$	+1.31	
e_{rel} ZZ1	3.44%	3.28%		
e_{rel} ZZ2	3.44%	3.49%		
θ ZZ1		0,952		
θ ZZ2		1,013		

3.4 Remarks

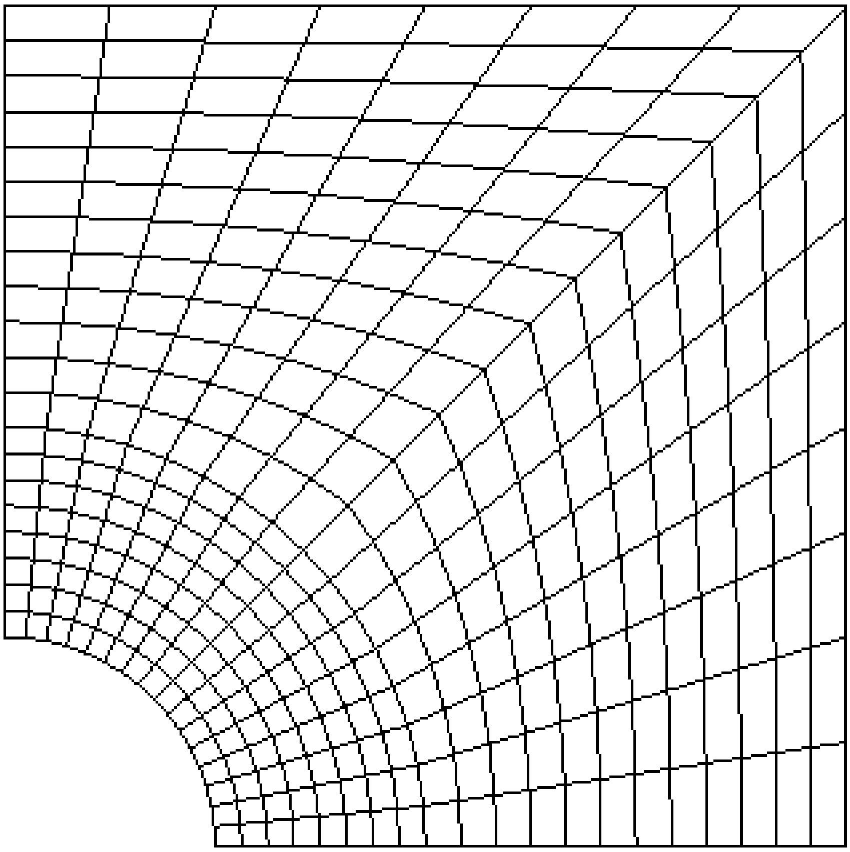
$\theta = \dfrac{\text{erreur estimée}}{\text{erreur exacte}}$ is an indication of effectiveness of the estimator.

Contents of the files results:

- absolute errors and relative total by the 2 methods,
- maximum and minimal values of the constraints and the errors,
- lists of the meshs where the relative error is higher than 10%.

4 Modeling B

4.1 Characteristics of modeling



4.2 Characteristics of the grid

Many nodes: 357.

Many meshs and types: 320 QUAD4.

4.3 Values tested

Identification	Reference	*Aster*	% difference	tolerance
With σ_{xx} ZZ1	3.	3,017	0.57	0.1
σ_{xx} ZZ2	3.	2,971	- 0.95	0.1
σ_{yy} ZZ1	0.	0.17	-	0.3
σ_{yy} ZZ2	0.	0,136	-	0.3
σ_{xy} ZZ1	0.	$2.8\ 10^{-3}$	-	0.1
σ_{xy} ZZ2	0.	$-1.04\ 10^{-2}$	-	0.1
P σ_{xx} ZZ1	1.15625	1,168	0.98	0.1
σ_{xx} ZZ2	1.15625	1,153	– 0.26	0.1

σ_{yy} ZZ1	– 0.15625	– 0,158	1.28	0.1
σ_{yy} ZZ2	– 0.15625	– 0,152	– 2.83	0.1
σ_{xy} ZZ1	– 0,125	– 0,121	– 2.99	0.1
σ_{xy} ZZ2	– 0,125	– 0,124	– 0.94	0.1
M1 mesh e_{abs} ZZ1		$1.57\ 10^{-4}$		1.10^{-3}
e_{abs} ZZ2		$2.40\ 10^{-4}$		1.10^{-3}
e_{rel} ZZ1		5.79%		0.1
e_{rel} ZZ2		8.83%		0.1
Ω e_{abs} ZZ1	$0{,}320\ 10^{-2}$	$0{,}294\ 10^{-2}$	– 8.1	
e_{abs} ZZ2	$0{,}320\ 10^{-2}$	$0{,}307\ 10^{-2}$	– 4.2	
e_{rel} ZZ1	2.48%	2.28%		
e_{rel} ZZ2	2.48%	2.37%		
θ ZZ1		0,919		
θ ZZ2		0,958		

4.4 Remarks

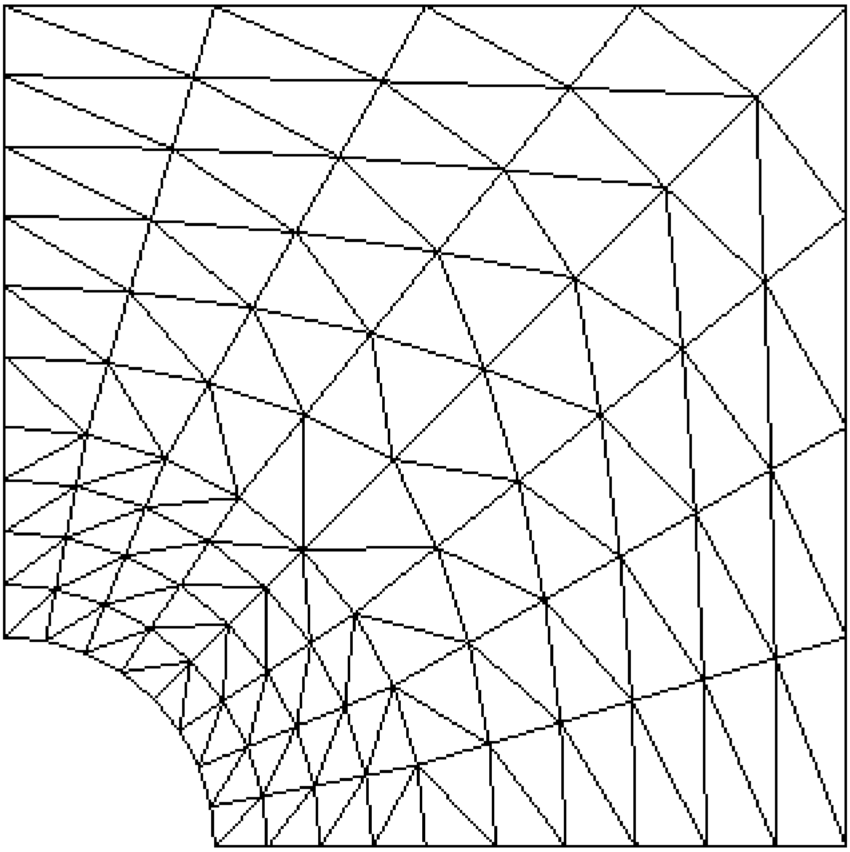
$\theta = \frac{\text{erreur estimée}}{\text{erreur exacte}}$ is an indication of effectiveness of the estimator.

Contents of the files results:

- absolute errors and relative total by the 2 methods,
- maximum and minimal values of the constraints and the errors,
- lists of the meshs where the relative error is higher than 10%.

5 Modeling C

5.1 Characteristics of modeling



5.2 Characteristics of the grid

Many nodes: 357.

Many meshs and types: 160 TRIA6.

5.3 Values tested

Identification	Reference	*Aster*	% difference	tolerance
With σ_{xx} ZZ1	3.	2,975	– 0.83	0.1
σ_{xx} ZZ2	3.	2,957	– 1.43	0.1
σ_{yy} ZZ1	0.	$6.86\ 10^{-2}$	-	0.3
σ_{yy} ZZ2	0.	$7.52\ 10^{-2}$	-	0.3
σ_{xy} ZZ1	0.	$-3.12\ 10^{-2}$	-	0.3
σ_{xy} ZZ2	0.	– 0,155	-	0.3
P σ_{xx} ZZ1	1.15625	1,166	0.85	0.1
σ_{xx} ZZ2	1.15625	1,153	– 0.25	0.1

	σ_{yy} ZZ1	– 0.15625	– 0,167	6.92	0.1
	σ_{yy} ZZ2	– 0.15625	– 0,153	– 1.87	0.1
	σ_{xy} ZZ1	– 0,125	– 0,127	1.52	0.1
	σ_{xy} ZZ2	– 0,125	– 0,124	– 0.58	0.1
M1 mesh	e_{abs} ZZ1		$1.81\ 10^{-4}$		1.10^{-3}
	e_{abs} ZZ2		$2.92\ 10^{-4}$		1.10^{-3}
	e_{rel} ZZ1		4.69%		0.1
	e_{rel} ZZ2		7.56%		0.1
Ω	e_{abs} ZZ1	$0,152\ 10^{-2}$	$0,123\ 10^{-2}$	– 19	
	e_{abs} ZZ2	$0,152\ 10^{-2}$	$0,167\ 10^{-2}$	+9.9	
	e_{rel} ZZ1	1.17%	0.95%		
	e_{rel} ZZ2	1.17%	1.29%		
	θ ZZ1		0,810		
	θ ZZ2		1,099		

5.4 Remarks

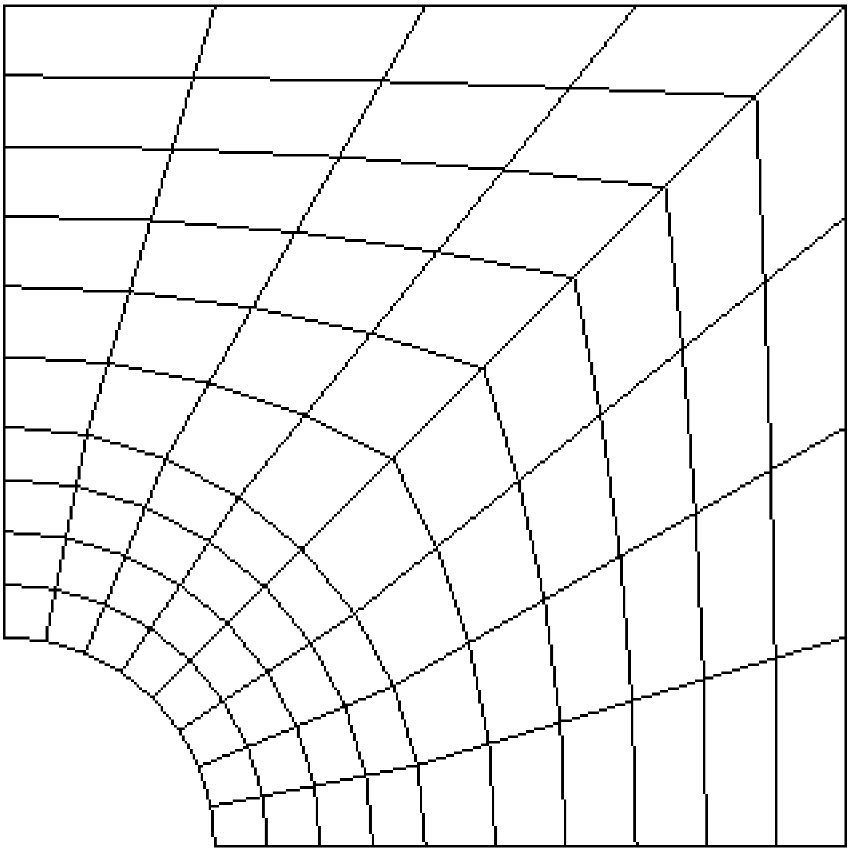
$\theta = \frac{\text{erreur estimée}}{\text{erreur exacte}}$ is an indication of effectiveness of the estimator.

Contents of the files results:

- absolute errors and relative total by the 2 methods,
- maximum and minimal values of the constraints and the errors,
- lists of the meshs where the relative error is higher than 10%.

6 Modeling D

6.1 Characteristics of modeling



6.2 Characteristics of the grid

Many nodes: 277.

Many meshs and types: 80 QUAD8.

6.3 Values tested

Identification	Reference	*Aster*	% difference	tolerance
With σ_{xx} ZZ1	3.	3,063	2.11	0.1
σ_{xx} ZZ2	3.	3,037	1.24	0.1
σ_{yy} ZZ1	0.	0,101	-	0.3
σ_{yy} ZZ2	0.	$2.47\ 10^{-2}$	-	0.3
σ_{xy} ZZ1	0.	$-5.8\ 10^{-3}$	-	0.3
σ_{xy} ZZ2	0.	$-2.41\ 10^{-2}$	-	0.3
P σ_{xx} ZZ1	1.15625	1,170	1.19	0.1
σ_{xx} ZZ2	1.15625	1,153	– 0.29	0.1

σ_{yy} ZZ1	– 0.15625	– 0,162	3.54	0.1
σ_{yy} ZZ2	– 0.15625	– 0,153	– 1.87	0.1
σ_{xy} ZZ1	– 0,125	– 0,124	– 1.09	0.1
σ_{xy} ZZ2	– 0,125	– 0,124	– 0.84	0.1
M1 mesh e_{abs} ZZ1		$6.1\ 10^{-5}$		1.10^{-3}
e_{abs} ZZ2		$2.1\ 10^{-4}$		1.10^{-3}
e_{rel} ZZ1		1.45%		0.1
e_{rel} ZZ2		5.01%		0.1
Ω e_{abs} ZZ1	$9.01\ 10^{-4}$	$2.90\ 10^{-4}$	+67.9 (!)	
e_{abs} ZZ2	$9.01\ 10^{-4}$	$8.88\ 10^{-4}$	– 1.5	
e_{rel} ZZ1	0,697%	0.22%		
e_{rel} ZZ2	0,697%	0,687%		
θ ZZ1		0,321		
θ ZZ2		0,985		

6.4 Remarks

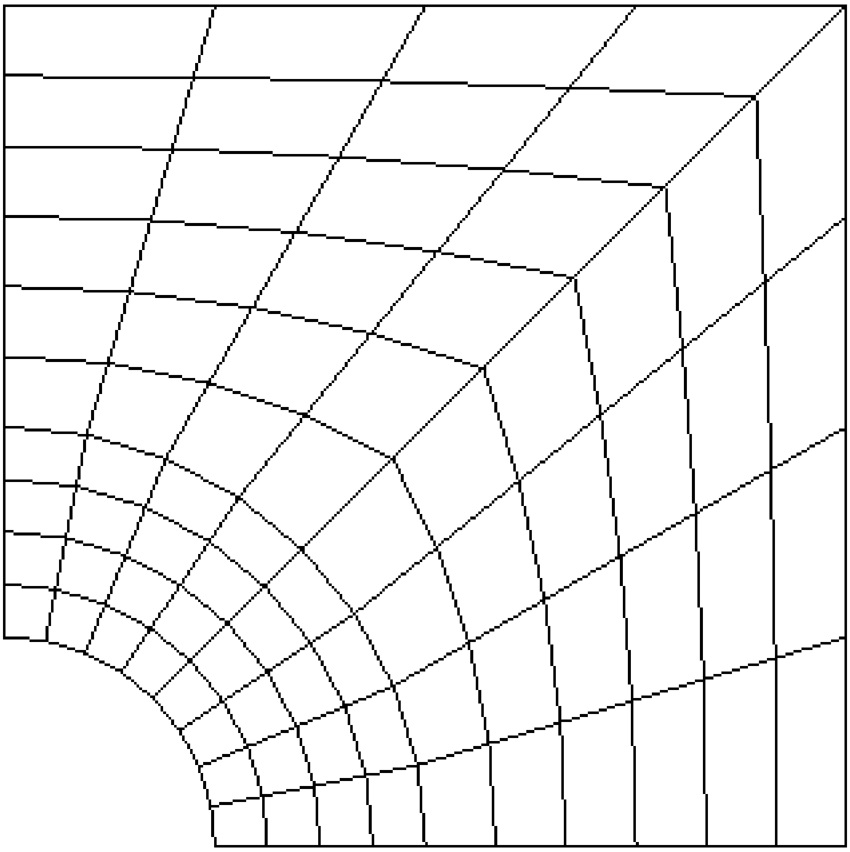
$\theta = \dfrac{\text{erreur estimée}}{\text{erreur exacte}}$ is an indication of effectiveness of the estimator

Contents of the files results:

- absolute errors and relative total by the 2 methods,
- maximum and minimal values of the constraints and the errors,
- lists of the meshs where the relative error is higher than 10%.

7 Modeling E

7.1 Characteristics of modeling



7.2 Characteristics of the grid

Many nodes: 357.

Many meshs and types: 80 QUAD9.

7.3 Values tested

Identification	Reference	*Aster*	% difference	tolerance
With σ_{xx} ZZ1	3.	3,070	2.33	0.1
σ_{xx} ZZ2	3.	3,004	0.14	0.1
σ_{yy} ZZ1	0.	0,113	-	0.3
σ_{yy} ZZ2	0.	0.04	-	0.3
σ_{xy} ZZ1	0.	$-1.4\ 10^{-3}$	-	0.1
σ_{xy} ZZ2	0.	$1.89\ 10^{-2}$	-	0.1
P σ_{xx} ZZ1	1.15625	1,170	1.19	0.1
σ_{xx} ZZ2	1.15625	1,152	– 0.33	0.1

σ_{yy} ZZ1	$-$ 0.15625	$-$ 0,162	3.44	0.1
σ_{yy} ZZ2	$-$ 0.15625	$-$ 0,153	$-$ 2.11	0.1
σ_{xy} ZZ1	$-$ 0,125	$-$ 0,124	$-$ 1.06	0.1
σ_{xy} ZZ2	$-$ 0,125	$-$ 0,124	$-$ 0.94	0.1
M1 mesh e_{abs} ZZ1		$6.1\ 10^{-5}$		1.10^{-3}
e_{abs} ZZ2		$2.1\ 10^{-4}$		1.10^{-3}
e_{rel} ZZ1		1.45%		0.1
e_{rel} ZZ2		5.01%		0.1
Ω e_{abs} ZZ1	$8.99\ 10^{-4}$	$2.75\ 10^{-4}$	+69.4 (!)	
e_{abs} ZZ2	$8.99\ 10^{-4}$	$8.55\ 10^{-4}$	$-$ 4.9	
e_{rel} ZZ1	0,695%	0.21%		
e_{rel} ZZ2	0,695%	0.66%		
θ ZZ1		0,306		
θ ZZ2		0,951		

7.4 Remarks

$\theta = \frac{\text{erreur estimée}}{\text{erreur exacte}}$ is an indication of effectiveness of the estimator

Contents of the files results:

- absolute errors and relative total by the 2 methods,
- maximum and minimal values of the constraints and the errors,
- lists of the meshs where the relative error is higher than 10%.

8 Summary of the results in 2D

		TRIA3	QUAD4	TRIA6	QUAD8	QUAD9
e_{rel}	exact	3.44%	2.48%	1.17%	0,697%	0,695%
	$ZZ1$	3.28%	2.28%	0.95%	0.22%	0.21%
	$ZZ2$	3.49%	2.37%	1.29%	0,687%	0.66%
θ	$ZZ1$	0,952	0,919	0,810	0,321	0,306
	$ZZ2$	1,013	0,958	1,099	0,985	0,951

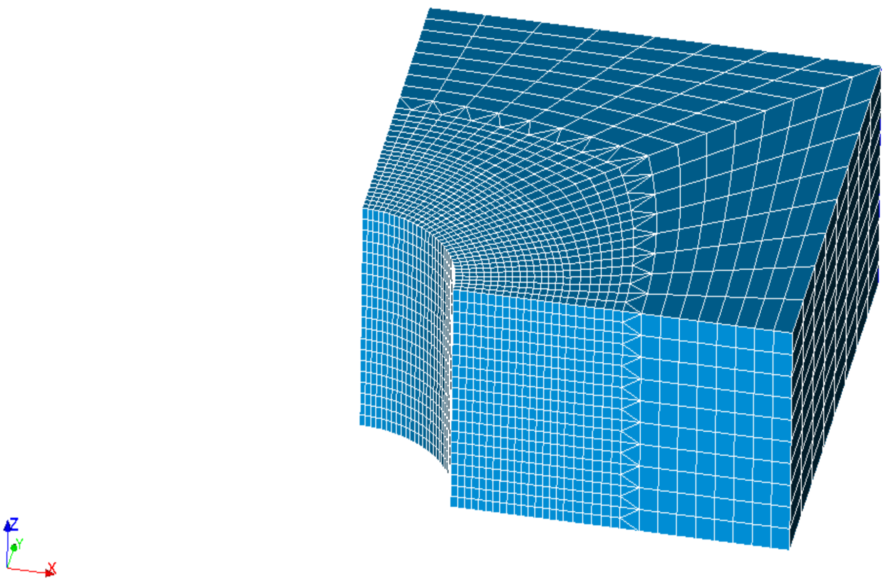
The constraints with the nodes, as a whole, are approximated better with $ZZ2$, especially for the elements of order 2. If one makes tend h towards 0, rates of convergence with h of $\|\sigma^{ex} - \sigma^{*}\|$ are higher by the method $ZZ2$ for all the types of elements to those of the method $ZZ1$ (σ^{*} is the smoothed constraint).

The estimator $ZZ1$ is not reliable for the elements of order 2, the nodal constraints remain correct. One can check in this typical case that $\theta \rightarrow 0$ when $h \rightarrow 0$, which shows that continuous total smoothing proves to be insufficient to estimate the error in the case of a solution without singularity (case of this test).

$ZZ2$ is on the other hand reliable and asymptotically exact ($\theta \rightarrow 1$ when $h \rightarrow 0$).

9 Modeling F

9.1 Characteristics of modeling



9.2 Geometry

Dimensions along axis X and are the same ones as in 2D there. The plate has a dimension according to Z of 2. The points of observation A, P, C and E have same X and there that in 2D and are located in the z=0 plan.

9.3 Characteristics of the grid

Many nodes: 17622.

Many meshs and types: 15360 HEXA8, 800 PYRAM5, 640 TETRA4.

9.4 Values tested

The constraints are tested at the points A, P, C and E.

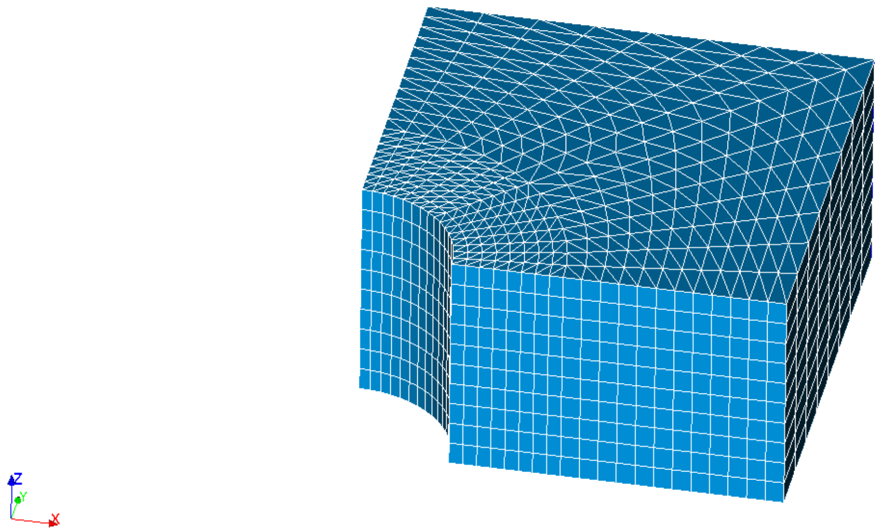
Identification	Reference	tolerance
With σ_{xx} ZZ1	3.	0.06
σ_{yy} ZZ1	0.	0.3
σ_{xy} ZZ1	0.	0.1
P σ_{xx} ZZ1	1.15625	0,015
σ_{yy} ZZ1	-0.15625	0.04
σ_{xy} ZZ1	-0,125	0.07
C σ_{xx} ZZ1	1.02978	0.01
σ_{yy} ZZ1	-0.02978	0,045
σ_{xy} ZZ1	-0.01562	0.09

E	σ_{xx}	ZZ1	-1.0	0.15
	σ_{yy}	ZZ1	0.0	0.3
	σ_{xy}	ZZ1	0.0	0.05

The errors are tested on the M6727 mesh.

Identification	Reference	tolerance
Estimator of error of Zhu-Zienkiewicz:		
e_{abs} with the elements	0.0	1.10^{-3}
e_{rel} with the elements	0.0	10
Estimator of error in mechanics:		
e_{abs} with the elements	0.0	0.02
e_{rel} with the elements	0.0	23.0
e_{abs} with the nodes by element	0.0	0.02
e_{rel} with the nodes by element	0.0	23.0

10 Modeling G



10.1 Geometry

Dimensions along axis X and are the same ones as in 2D there. The plate has a dimension according to Z of 2. The points of observation A, P, C and E have same X and there that in 2D and are located in the z=0 plan.

10.2 Characteristics of the grid

Many nodes: 3927.

Many meshs and types: 6400 PENTA6.

10.3 Values tested

The values tested are located in the z=0 plan.

Identification	Reference	tolerance
With σ_{xx} ZZ1	3.	0.06
σ_{yy} ZZ1	0.	0.33
σ_{xy} ZZ1	0.	0.1
P σ_{xx} ZZ1	1.15625	0.01
σ_{yy} ZZ1	0.15625	0,054
σ_{xy} ZZ1	-0,125	0.07
C σ_{xx} ZZ1	-0.02978	0,021
σ_{yy} ZZ1	1.02978	0.01
σ_{xy} ZZ1	-0.01562	0,095
E σ_{xx} ZZ1	-1.	0.15

σ_{yy} ZZ1	0.0	0.3
σ_{xy} ZZ1	0.0	0.05

The errors are tested on the M2145 mesh.

Identification	Reference	tolerance
Estimator of error of Zhu-Zienkiewicz:		
e_{abs} with the elements	0.0	1.10^{-3}
e_{rel} with the elements	0.0	17.6
Estimator of error in mechanics:		
e_{abs} with the elements	0.0	6.10^{-3}
e_{rel} with the elements	0.0	13.5
e_{abs} with the nodes by element	0.0	6.10^{-3}
e_{rel} with the nodes by element	0.0	13.5