# SDND104 - Calculation of the power of wear of a mass rubbing under harmonic seismic excitation

**Summary:**

One considers a mass in contact rubbing with a rigid plan on which one imposes a vibratory movement of harmonic type. Friction is modelled by the law of Coulomb. The calculation of the answer of the mass is of transitory type nonlinear. One calculates the power of wear resulting from the phases of slip between the mass and the rigid plan. The calculation of the power of wear being developed in *Aster* that for modal calculations, the analysis is carried out on the basis of modal system (commonplace). In order to avoid the digital problems resulting from the nullity of the single mode of rigid body of the mass, a spring far from stiff is introduced, flexible the mass at a point interdependent of the vibrating rigid plan.

The reference solution is a quasi analytical calculation of the transitory answer, whose digital estimates are programmed with Maple.

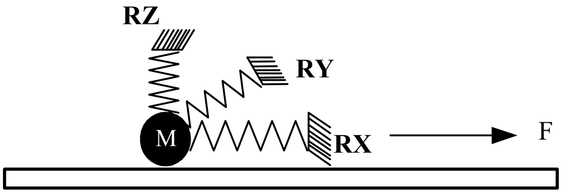
Single modeling *Aster* reserve tests the explicit algorithms of integration with constant step of Euler (order 1), Devogeleare (order 4) and the algorithms with variable step `ADAPT_ORDRE2` (order 2) and `RUNGE_KUTTA` (orders 54 and 32) developed in the operator dedicated to vibratory dynamics, for various amplitudes of the harmonic acceleration of seismic excitation of the rigid plan of support. According to this amplitude, the mode of the answer of the mass is of the adherent type for any time (stick), successively member and slipping (stick-slip), or always slipping with inversion of the direction of slip (slipway-slipway).

An account is given owing to the fact that in the case of a sufficiently low amplitude of excitation (first mode, permanent adherence), the power of wear is strictly worthless.

# 1 Problem of reference

## 1.1 Geometry

The system considered consists of a simple heavy mass posed on a rigid support subjected to an imposed vibration of type seismic, sinusoidal. The contact, as well as solid friction are modelled by penalization. The system thus has two degrees of freedom of translation (horizontal and vertical).




A very weak spring of stiffness connects the mass to the support in the three directions. This spring is an artifice of calculation, intended to avoid the nullity of the frequency associated with the rigid mode with horizontal adjustment with the mass. Results *Aster* taking into account the presence of this spring are not very different from the results which one would get without spring.

## 1.2 Properties of the model

| | |
|---|---|
| Stiffness of the spring (according to the three directions): | $k=3.10^{-5} N/m$ |
| mass: | $m=1\,kg$ |
| gravity: | $g=10\,m/s^2$ |
| coefficient of Coulomb: | $\mu=0{,}1$ |

## 1.3 Boundary conditions, conditions initial and loadings

The mass rests on the rigid level with the dimension $z=0$ .

The harmonic acceleration imposed on the base has as an equation $a=a_0 \sin(\omega t)$ . In particular, it is worthless at the initial moment. The displacement of the support satisfies the equation $X(t) = -(a_0 / \omega^2) \sin(\omega t)$ , and thus its movement starts towards the left, with nonworthless initial speed $\dot{X}(0) = -a_0 / \omega$ .

Initial displacement (with $t=0$ ) mass is taken null. The mass is regarded as in a state of adherence at the initial moment. It thus has same nonworthless speed as the support with $t=0$ .

Calculations are carried out for various values of maximum acceleration:

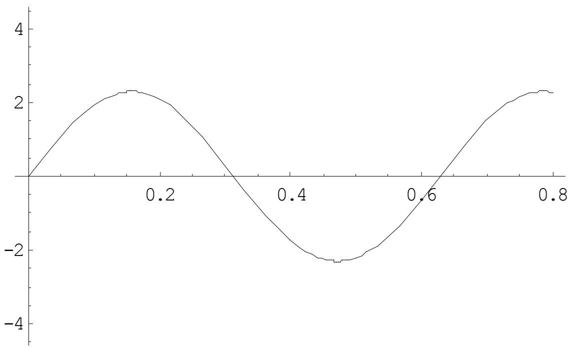
$a_0 = 15$ m/s $^2$ , $a_0 = 1{,}5$ m/s $^2$ , $a_0 = 1{,}01$ m/s $^2$ and $a_0 = 0{,}99$ m/s $^2$

and a value of pulsation: $\omega=2\pi$ .

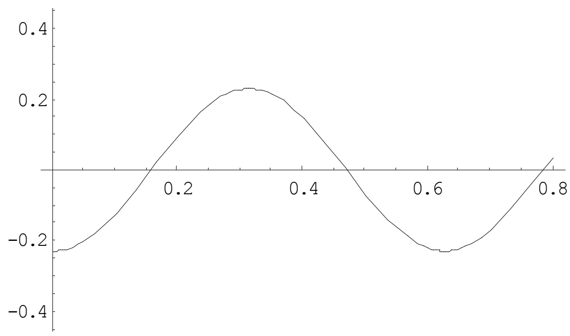
# 2 Reference solution

The reference solution, which is analytical, is calculated in the following way.
That is to say $x(t)$ the X-coordinate of the mass in the fixed reference mark and $X(t)$ the X-coordinate of the vibrating support in this same reference mark.

Initially, it is supposed that the mass is adherent on its support. It remains to it a certain time then after the initial moment $t=0$. It undergoes of this fact the acceleration imposed by the rigid support, that is to say $\ddot{x}(t)=\ddot{X}(t)=a_0 \sin\omega t$. The tangential force exerted by the mass on the support is then $F_T=-m\ddot{x}(t)=-ma_0 \sin\omega t$ (worthless at the moment initial, which justifies the starting assumption that initially, the mass is adherent on its support). The mass remains adherent as long as $|F_T|=ma_0|\sin\omega t|\leq\mu F_N=\mu mg$. If $a_0\leq\mu g$, the mass thus remains indefinitely adherent on its support, and its movement is exactly the same one as this one. By introducing the adimensional coefficient $\eta=\frac{\mu g}{a_0}$, the condition of permanent adherence is written $\eta\geq 1$. The curve of acceleration of the mass, like support, then takes the following form according to time:

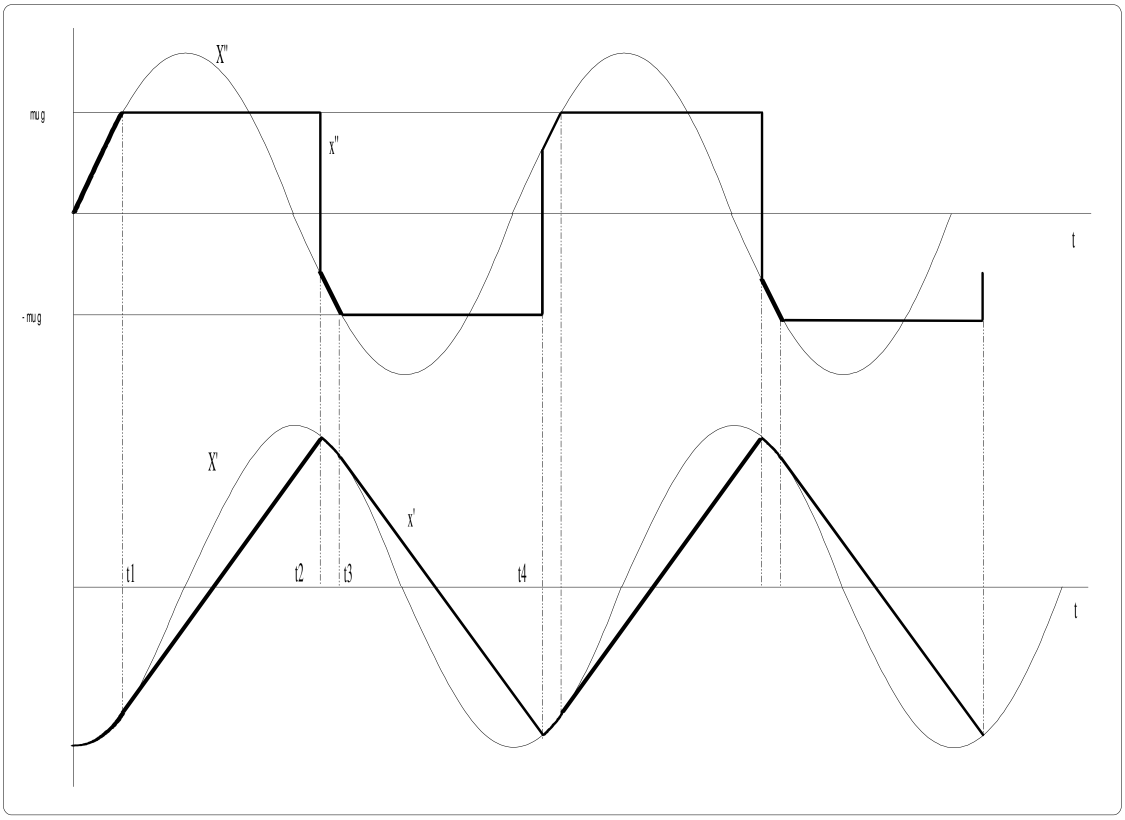



As for speed, it takes the following form (single primitive of worthless average):



If $a_0 > \mu g$ , there exists a smaller time $t = t_1$ such as $\left|F_T\right| = ma_0\left|\sin\omega t_1\right| = \mu mg$ . This smaller time is necessarily such as $\sin\omega t_0 > 0$, which makes it possible to remove the absolute value in the preceding expression, and to obtain explicit expression $t_1 = \frac{1}{\omega}\arcsin\frac{\mu g}{a_0} = \frac{1}{\omega}\arcsin\eta$ . In particular, $t_1 \leq \frac{T}{4} = \frac{2\pi}{4\omega} = \frac{\pi}{2\omega}$ .

After this moment, the mass slips towards the left compared to the support, therefore it checks the dynamic equation $\ddot{x}(t) = \mu g$ , that is to say $\dot{x}(t) = \mu g(t - t_1) + \dot{x}(t_1)$ . Its speed thus increases linearly with time, while leaving to $t_1$ negative value $\dot{x}(t_1) = -\frac{a_0}{\omega}\cos\omega t_1 = -\frac{a_0}{\omega}\sqrt{1-\eta^2}$ (indeed, $\sin\omega t_1 = \eta$ ).




**Movement for $\eta > \eta^*$ , mode of "stick-slip", succession of adherence and slip**

Necessarily, for a certain value of time $t_2$ satisfying $\pi / 2\omega \leq t_2 \leq 2\pi / \omega$ , the speed of the mass becomes again equal at the speed of the support. At this moment, the movement becomes again adherent *if and only if* the acceleration which the mass at the beginning of adherence undergoes is lower in absolute value than $\mu g$ . One examines the translation of this condition in the continuation. One expresses to begin the value of $t_2$ .

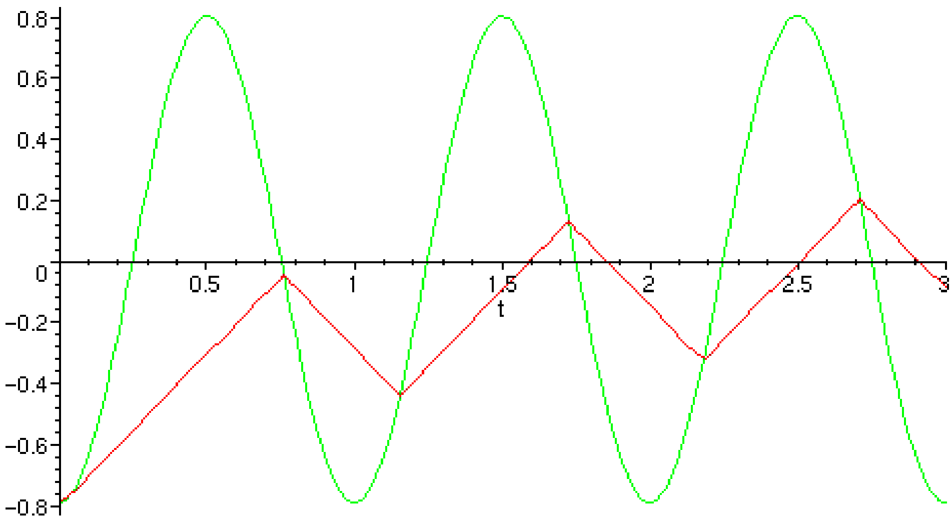
Time $t_2$ satisfied the equation $\dot{x}(t_2) = \dot{X}(t_2)$, that is to say $\mu g(t_2 - t_1) - \frac{a_0}{\omega}\cos\omega t_1 = -\frac{a_0}{\omega}\cos\omega t_2$, or $\eta\omega(t_2 - t_1) - \cos\omega t_1 + \cos\omega t_2 = 0$.

This equation, transcendent, allows the determination of $t_2$ according to $t_1$ and $\eta$, that is to say finally, taking into account the expression of $t_1$, determination of $t_2$ according to the physical parameters of the system $\eta$ and $\omega$. If the acceleration of the support in $t_2$ is lower in absolute value than $\mu g$, the movement remains adherent then up to one moment $t_3$ for which the acceleration of the support and mass reach the value $-\mu g$, moment which for reasons of clear symmetries on the graphs above, exactly satisfied $t_3 = t_1 + \pi/\omega$. The mass then starts a phase of slip up to one moment $t_4$, after which the movement reproduces periodically.

One understands that for sufficiently small values of $\eta$, the movement will not be able to become adherent as from time $t_2$, because the acceleration of the mass would exceed the threshold $\mu g$. There thus exists a breaking value $\eta^*$ such as for $\eta > \eta^*$, the movement of the mass passes without phase of adherence of a slip to a shift in opposite meaning. A reflection on the continuity of the function answer of speed of the mass compared to the parameter $\eta$ watch that for $\eta \leq \eta^*$, the later movement is always slipping (mode of "slipway-slipway", of alternate directions). For $\eta < \eta^*$, the movement periodically alternates phases of adherence and slip.

The breaking value $\eta^*$ admits a simple analytical expression. Indeed, for $\eta = \eta^*$, moments $t_2$ and $t_3$ are confused. Thus $t_2 - t_1 = t_3 - t_1 = \pi/\omega$ and the equation $\eta\omega(t_2 - t_1) - \cos\omega t_1 + \cos\omega t_2 = 0$ becomes $\pi\eta^* = 2\cos\omega t_1 = 2\sqrt{1-\eta^{*2}}$. While passing squared, one obtains $\pi^2\eta^{*2} = 4 - 4\eta^{*2}$, that is to say $\eta^* = \frac{2}{\sqrt{\pi^2+4}} \approx 0{,}537$.

For $\eta \leq \eta^*$, the movement is not **that asymptotically** periodical. The continuation $(t_n)$ moments of change of direction of slip checks $t_{n+1} - t_n \rightarrow \pi/\omega$ when $n$ tends towards the infinite one. Figure Ci - below watch typical pace (broken line) the speed of the mass in the situation of slipway-slipway.



**Movement for $\eta \leq \eta^{*}$ : mode of "slipway-slipway", no adherence**

Let us summarize the conclusions:

There is the adimensional coefficient $\eta = \frac{\mu g}{a_0}$ and its value criticizes  * such as $\eta^* = \frac{2}{\sqrt{\pi^2+4}} \approx 0{,}537$.

| | |
|---|---|
| If $\eta^* < \eta < 1$ | the established mode is of standard "stick-slip": alternation of phases of adherence and slip; |
| If $\eta < \eta^*$, | the established mode is of standard "slipway-slipway": alternate permanent slip; |
| If $\eta > 1$, | the established mode is of standard "stick": permanent adherence with the base. |

In the results of analytical comparison calculation*Aster* who follow, the choices of the amplitude $a_0$ are such as these three situations are visited. One takes indeed $m=1\,kg$, $g=10\,m/s^2$, $\mu=0.1$,

$a_0=15\,m/s^2$, $a_0=1.5\,m/s^2$, $a_0=1.01\,m/s^2$ and $a_0=0.99\,m/s^2$.

The power of wear is physically worthless at the time of the phases of adherence.
In *Code_Aster*, with the operator `DYNA_VIBRA` used here, adherence is not detected because the integration of the movement is made by regularization of the law of friction. The respect of the null result of the power of wear during phases of adherence required the introduction of a criterion on the speed of slip, so that below a certain value, it must regarded as worthless, and the adherent movement. One can consult the reference material Operator of calculation of wear/Model of Archard [R7.04.10].

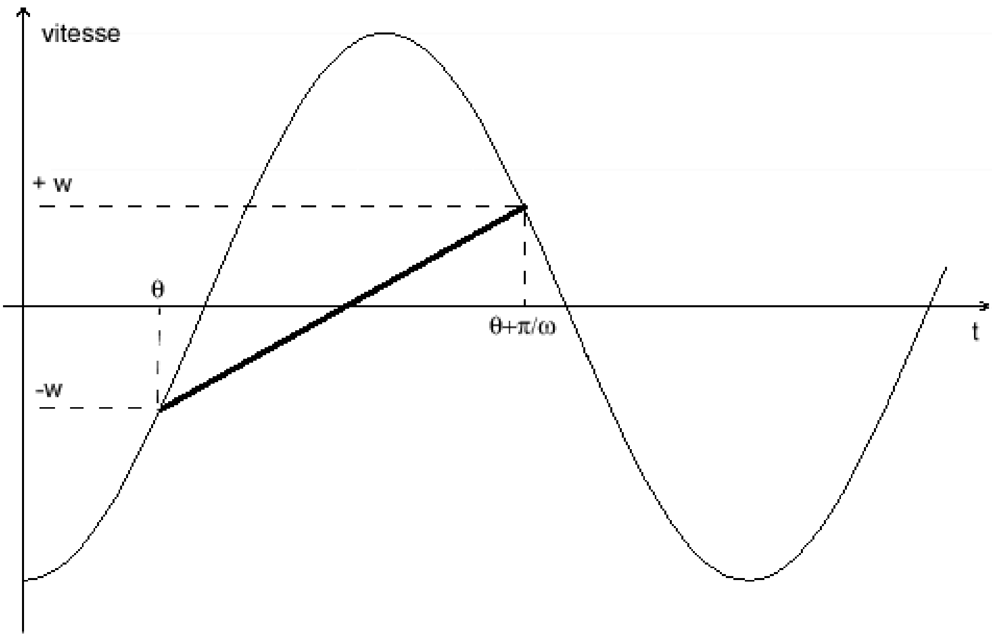
During the phases of slip, the power of wear follows the law $P_u(t) = \mu mg|V_R(t)|$, where $V_R(t) = \dot{x}(t) - \dot{X}(t)$ is the relative speed of slip of the mass on the support. In the situation of the mode of stick-slip, for which the movement becomes strictly periodic at the end of a finished time, the energy of wear during a half-period is exactly

$$E_u = \int_{t_1}^{t_2} mg|V_R(t)|dt = mg\int_{t_1}^{t_2} |\dot{X}(t) - \dot{x}(t)|dt = mg\int_{t_1}^{t_2} (-\frac{a_0}{\omega}\cos\omega t - (\mu g(t-t_1) - \frac{a_0}{\omega}\cos\omega t_1))dt$$

$$= mg\left[\frac{a_0}{\omega}((t_2-t_1)\cos\omega t_1 - \frac{1}{\omega}(\sin\omega t_2 - \eta)) - \frac{\mu g}{2}(t_2-t_1)^2\right].$$

The transcendent formulation of $t_2$ apparently does not allow to simplify the expression of this energy of wear. Power of average wear $\overline{P}_u$ is simply the energy of wear $E_u$ divided above by the half-period of the answer $T/2 = \pi/\omega$.

In the case of a movement always slipping ($\eta \le \eta^*$), the interval of integration to be taken is form $[t_n, t_{n+1}]$ with $n$ sufficient large, so that $t_{n+1} - t_n$ that is to say sufficiently near to the limiting value $\pi/\omega$. One can avoid digital calculation by recurrence of this continuation, knowing that the average asymptotic speed is worthless. Indeed, the continuation $t_n - n\pi/\omega$ has a finished limit $\vartheta$. Satisfied properties by $\vartheta$ are illustrated on the following figure:




The segment of right-hand side has as an equation

$$v = \mu g(t - \vartheta) - w = \mu g(t - \vartheta) - \frac{a_0}{\omega}\cos(\omega\vartheta),$$

and for $t = \vartheta + \pi/\omega$, speed $v$ that is to say to take the opposite value $w = \frac{a_0}{\omega}\cos(\omega\vartheta)$, which gives the equation

$$\mu g\pi/\omega - \frac{a_0}{\omega}\cos(\omega\vartheta) = \frac{a_0}{\omega}\cos(\omega\vartheta),$$

that is to say

$$\mu g\pi = 2a_0\cos(\omega\vartheta)\ ;$$

whose solution is

$$\vartheta = \frac{1}{\omega}\arccos\left(\frac{\mu g\pi}{2a_0}\right) = \frac{1}{\omega}\arccos\left(\frac{\eta\pi}{2}\right).$$

Let us note that one finds although for $\eta = \eta^*$, the acceleration of the support calculated at time $t = \vartheta$ give the limiting value $\mu g$. Indeed

$$a_0\sin(\omega\vartheta) = a_0\sin(\arccos(\eta^*\pi/2)) = a_0\sqrt{1 - \eta^{*2}\pi^2/4} = a_0\sqrt{1 - (1 - \eta^{*2})} = a_0\eta^* = \mu g.$$

In the case of the movement always slipping, the energy of wear during one asymptotic period is given exactly by the formula

$$E_u = \int_{\vartheta}^{\vartheta+\pi/\omega} mg\left|V_R(t)\right|dt$$

that one can clarify according to preceding calculation, while taking $t_1 = \vartheta$ and $t_2 = \vartheta + \pi/\omega$, which gives

$$E_u = mg\left[\frac{a_0}{\omega}\left(t\cos\omega\vartheta - \frac{1}{\omega}\sin\omega t\right) - \frac{\mu g}{2}(t - \vartheta)^2\right]_{\vartheta}^{\vartheta+\pi/\omega} = mg\left[\frac{a_0}{\omega}\left(\frac{\pi}{\omega}\frac{\eta\pi}{2} + \frac{2}{\omega}\sqrt{1 - \frac{\eta^2\pi^2}{4}}\right) - \frac{\mu g}{2}\frac{\pi^2}{\omega^2}\right],$$

that is to say

$$E_u = \frac{mga_0}{\omega^2}\sqrt{4 - \pi^2\eta^2} = \frac{mg}{\omega^2}\sqrt{4a_0^2 - \pi^2\mu^2 g^2} .$$

The power of average wear (over one period) asymptotic is then

$$\overline{P}_u = \frac{E_u}{\pi / \omega} = \frac{mga_0}{\pi\omega}\sqrt{4 - \pi^2\eta^2} = \frac{mga_0}{\omega}\sqrt{\frac{4}{\pi^2} - \eta^2} .$$

Following the Maple program allows the calculation of the power of exact wear in a specified time interval, as well as the layout of the graph showing the convergence of the function speed of the mass towards a periodic function limits, for any value of the physical parameters and of excitation such as the mode is of standard slipway-slipway ( $\eta \le \eta^*$ ), and the exact value of the average power of wear over one period (the only useful one for what interests us) in the case of the stick-slip.

```
# This program calculates, on the transitory part
# of the beginning of the signal, the power of exact wear,
# until a time specifies at the beginning of program.
Digits: = 20:
pi: = evalf (pi):
T: = 1:                     # period of the movement of the support
Omega: = 2*pi/T:
tmin: = 4:
tmax: = 12:                 # duration of the transient considers
ncycle: = floor (tmax/T) +2:   # iteration count of Ti calculation [I] and tf
[I]
Nmax: = 100*ncycle:      # to replace the function sin by a line brisee
m: = 1:
G: = 10:
driven: = 0.1:
a0: = 1.5:
eta: = mu*g/a0:
Omega: = 2*pi/T:
etaetoile: = 2/sqrt (pi^2+4):
Ti [1]: = 1/omega*arcsin (eta):
dX: = T - > - a0/omega*cos (omega*t):
dxmoins [0]: = dX (T):
lignedx: = [Ti [1], dX (Ti [1])] :
Eusure: = 0:   # wear is worthless on the phase of adherence [0, Ti [1]]
#
# To note that Ti [i+1] is necessarily in the interval [i*T-T/4, i*T+T/2]
# and    that tf [I]   is necessarily in the interval [i*T-3*T/4, i*T].
# These two intervals are recovered, but there is always tf [I] <ti [i+1].
#
yew eta<etaetoile then             # mode of slipway-slipway
  for I from 1 to ncycle C
    dxplus [I]: = mu*g* (T-Ti [I]) + subs (t=ti [I], dxmoins [i-1]):
    tf [I]: = fsolve (dX (T) =dxplus [I], t= (i*T-3*T/4). (i*T)) :
    lignedx: = lignedx, [tf [I], dX (tf [I])] :
    tinf: = max (Ti [I], tmin):
    tsup: = min (tf [I], tmax):
    yew tinf<tsup then
      Eusure: = Eusure + int (m*g* (dX (T) - dxplus [I]), t=tinf. .tsup):
    fi:
```

```
      dxmoins [I]: = - mu*g* (t-tf [I]) + subs (t=tf [I], dxplus [I]):
      Ti [i+1]: = fsolve (dX (T) =dxmoins [I], t= (i*T-T/4). (T/2+i*T)) :
      lignedx: = lignedx, [Ti [i+1], dX (Ti [i+1])] :
      tinf: = max (tf [I], tmin):
      tsup: = min (Ti [i+1], tmax):
      yew tinf<tsup then
        Eusure: = Eusure + int (m*g* (dxmoins [I] - dX (T)), t=tinf. .tsup):
      fi:
    od:
  # courbedX: = stud ([seq ([j*tmax/Nmax, dX (j*tmax/Nmax)], j=0. Nmax)]):
  # courbedx: = stud ([lignedx]):
  # with (studs):
  # display ([courbedX, courbedx]);
    theta: = arccos (pi*eta/2) /omega:
    dxinfini: = T - > mu*g* (T-theta) +dX (theta):
    Vginfini: = dxinfini - dX:
    Eumoyana: = - int (m*g*Vginfini (T), t=theta. (theta+pi/omega)) :
    Eumoyanaana: = m*g*a0/omega^2*sqrt (4-eta^2*pi^2):
    Pumoyana: = 2*Eumoyana/T:
    Pumoyanaana: = 2*Eumoyanaana/T:
    Pusure: = Eusure/(tmax-tmin);
  elif (eta>etaetoile and eta<1) then    # mode of stick-slip
    lignedx: = [Ti [1], dX (Ti [1])] :
    dxplus [1]: = mu*g* (T-Ti [1]) + subs (t=ti [1], dxmoins [0]):
    tf [1]: = fsolve (dX (T) =dxplus [1], t= (T-3*T/4). T):
    dxplus: = unapply (dxplus [1], T):
    Vg: = dxplus - dX:
    Have: = - int (m*g*Vg (T), t=ti [1]. .tf [1]):
    Pusuremoy: = 2*Eu/T;
  else                                   # mode of permanent adherence
    Have: = 0;
  fi:
```

The solution *Aster* considered is the calculation of the power of average wear during a transitional stage going from $4$ with $11{,}99\,secondes$ (of $8\pi/\omega$ with $24\pi/\omega$ ). The energy of wear for this transitory length of time differs somewhat from the energy of average wear (asymptotic) over this duration (as well in situation stick-slip as slipway-slipway). It is thus appropriate, to precisely compare it with the results *Aster*, to do an exact calculation of this energy in the time interval $[4\mathrm{s}, 11{,}99\,s]$.

For $a_0 = 15\mathrm{m}/s^2$, the power of average wear asymptotic is of $15{,}1146144886\,Watt$ whereas power of average wear on the temporal interval $[4\mathrm{s}, 11{,}99\,s]$ is of $15{,}257521794\,Watt$. It is this last value which constitutes the result of reference.

**Note:**

As a calculation of average power, the power of wear calculated on an interval is not obligatorily increasing with the duration of the interval. If one adds to the interval one duration over which there is adherence, the power of average wear will be lower.

## 2.1 Results of reference

| Value of acceleration $max. a0$ ( $ms^{-2}$ ) | Value of the average power of wear On the interval $[4\mathrm{s}, 11{,}99\,s]$, in Watt |
|---|---|
| 15 (slipway-slipway) | 15.26709959 |
| 1.5 (stick-slip) | 0.40906245 |
| 1.01 (stick-slip) | 2,261641E-4 |
| 0.99 (stick) | 0 |

## 2.2 Uncertainty on the solution

Quasi-analytical solution (presence of transcendent equations solved numerically with an arbitrary precision).

## 2.3 Bibliographical references


1 B. WESTERMO, F. UDWADIA: Periodic Answer of has sliding oscillator system to harmonic excitation. Earthquake Engineeering and structural dynamics Flight 14.135-146 (1983)

2 Documentation of *Code_Aster* [R7.04.10]

# 3 Modeling A

## 3.1 Characteristics of modeling

An element of the type DIS_T on a mesh POI1 is used to model the system.
Calculation is done on modal basis. One blocks displacements in $Y$ and in $Z$, the modal base thus contains only one mode.
One uses the dynamic functionality of calculation on the basis of modal operator DYNA_VIBRA, with the key word SHOCK to model to it not local linearity.
An obstacle of the type PLAN_Z (two parallel plans separated by a game) is used to simulate the slip surface. One chooses to take for generator of this plan $Oy$ that is to say NORM_OBST : $(0.,1.,0.)$.
The origin of the obstacle is ORIG_OBST : $(0.,0.,1.)$, its game which gives the half-spacing between the plans is of $0.5$.
One places oneself in the relative reference mark (loading mono-support) and one applies a loading in acceleration with CALC_CHAR_SEISME.
One uses a step of time of $3.10^{-5} s$ for temporal integration to limit the computing time. This step of time is quite lower than $\min(2/\sqrt{K/M}, 2/\sqrt{K_N/M}) = 7.10^{-4} s$.
The tangential stiffness of friction is taken as large as possible to ensure the stability of the diagram, that is to say $K_T = 900000\, N/m$. The value $K_T = 1000000\, N/m$ conduit with a digital instability.
Normal stiffness $K_N$ must be taken equalizes with $20\, N/m$ to compensate for the weight of the mass exactly. (the value of the game is of $0,50\, m$). Any other value leads to aberrant results.

## 3.2 Characteristics of the grid

Many nodes: 1
Many meshs and types: 1 POI1

# 4 Results of modeling A

## 4.1 Values tested

| Identification | Reference | *Aster* ADAPT ORDRE2 | *Aster* DEVOGE | *Aster* EULER | *Aster* R- K 54 | *Aster* R- K 32 | % difference max |
|---|---|---|---|---|---|---|---|
| $a0=15$ | 15.2671 | 15.2661 | 15.2665 | 15.2668 | 15.2655 | 15.2661 | 0.0065% |
| $a0=1,5$ | 0.409062 | 0.409067 | 0.409067 | 0.409067 | 0.409071 | 0.409068 | 0.0078% |
| $a0=1,01$ | 2,26164E-4 | 2,2715E-4 | 2,26108E-4 | 2,26112E-4 | 2,26105E-04 | 2,31715E-04 | 2.45% |
| $a0=0,99$ | 0 | 0 | 0 | 0 | 0 | 0 | 0% |

# 5 Summary of the results

The cas-test validates the calculation of the power of wear with POST_DYNA_MODA_T after a transitory calculation on modal basis, as well on a diagram with variable steps (ADAPT_ORDRE2, RUNGE_KUTTA54 and RUNGE_KUTTA32) that on diagrams with constant steps (Euler and Devogeleare). In particular the tangential microphone-speeds induced by the model of contact by penalization, at the time of the phases of adherence, are correctly cancelled.

The influence of the added spring remains in on this side precise details obtained.

The tangential stiffness of the contact is the element limiting for a higher precision. The convergence of the results towards the reference solution was checked. The tangential stiffness was taken as large as possible to ensure the stability of the diagram with $dt=10^{-4} s$ .

The tolerances in the tests-resu are taken just above found differences.