

## SSNL112 - Bar subjected has a cyclic thermal loading

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### Summary:

This case test enters within the framework of the validation of the relations of behavior in elastoplasticity of the elements bars for the quasi-static mechanics of the structures.

An embedded bar has these two ends undergoes a cyclic thermal loading inducing efforts of traction and compression.

Each modeling makes it possible to validate one of the relations of non-linear behavior introduced: Linear isotropic work hardening with criterion of Von-Settings (modeling A), linear kinematic work hardening with criterion of Von-Settings (modeling B), as well as a model known as of Pinto-Menegotto, representing the cyclic behavior of the steel reinforcements in the reinforced concrete (modelings C and D).

## 1 Problem of reference

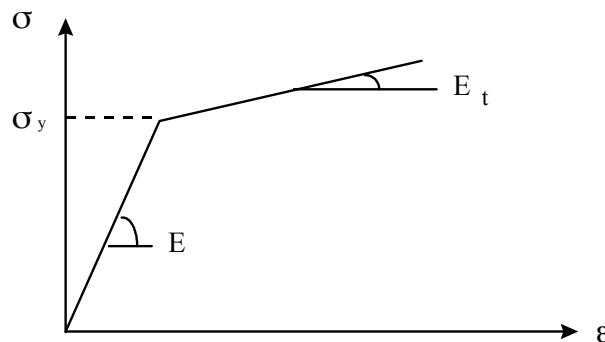
### 1.1 Geometry



Length of the bar : 1 m  
Section of the bar : 5 cm<sup>2</sup>

### 1.2 Properties of materials

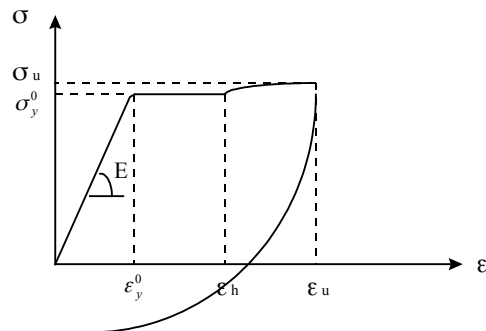
#### 1.2.1 Linear work hardenings isotropic and kinematics



Young modulus:  $E = 2 \cdot 10^{11} \text{ Pa}$   
Slope of work hardening:  $E_t = 2 \cdot 10^9 \text{ Pa}$   
Elastic limit:  $\sigma = 2 \cdot 10^8 \text{ Pa}$   
Poisson's ratio:  $\nu = 0,3$   
Thermal dilation coefficient:  $\alpha = 1 \cdot 10^{-5} \text{ K}^{-1}$

This test makes it possible to validate the operation of MFront (behavior plasticity) for the bars (methods of Deborst).

## 1.2.2 Model of Pinto-Menegotto



Young modulus:	$E$	=	$2. 10^{11} Pa$
Elastic limit:	$\sigma_y^0$	=	$2.10^8 Pa$
Poisson's ratio:	$\nu$	=	0.3
Thermal dilation coefficient:	$\alpha$	=	$1.10^{-5} K^{-1}$
Deformation of work hardening:	$\epsilon_h$	=	$2.3 10^{-3}$
Ultimate constraint:	$\sigma_u$	=	$2.58 10^8 Pa$
Ultimate deformation:	$\epsilon_u$	=	$3.10^{-2}$
Coefficient defining the curve $\zeta$ :	$R_0$	=	20
Coefficient defining the curve $\zeta$	$A_1$	=	18.5
Coefficient defining the curve $\zeta$	$A_2$	=	0.15
Coefficient of buckling:	$C$	=	0.5
Coefficient of buckling:	$A$	=	0,008

## 1.3 Boundary conditions and loading

### Boundary conditions:

The bar is embedded. Displacements are thus blocked in the three directions.

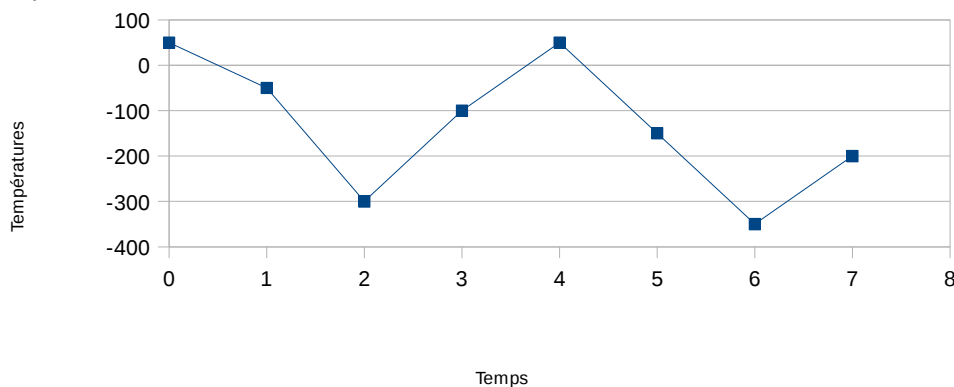
In  $N1$  and  $N2$  :  $DX = DY = DZ = 0$

### Loading:

The way of loading is described by the change of the temperature, uniform in the bar:

$t$	0	1	2	3	4	5	6	7
$T(^{\circ}C)$	50	- 50	- 300	- 100	50	- 150	- 350	- 200

The temperature of reference is  $0^{\circ}C$  .



## 2 Reference solutions

### 2.1 Method of calculating used for the reference solutions

#### 2.1.1 Linear work hardenings

##### Isotropic work hardening

For a uniaxial traction, the criterion of plasticity is written:

$$|\sigma_L| - R(p) \leq 0$$

where  $p$  is the cumulated plastic deformation

$$R(p) = R' p + \sigma^y \quad \text{and} \quad R' = \frac{E E_1}{E - E_1}$$

The criterion is written then:

$$|\sigma_L| - R' p - \sigma^y \leq 0$$

The tensor of the constraints is obtained by:

$$\sigma = A \cdot (\varepsilon(\mathbf{u}) - \varepsilon^p) - 3K \alpha (T - T^{ref}) \mathbf{Id}$$

One from of thus deduced the expression from  $\sigma_L$

$$\sigma_L = E(\varepsilon - \alpha T) - E \varepsilon^p \quad (T^{ref} = 0)$$

In our case,  $\varepsilon = 0$  thus:

$$\sigma_L = E \varepsilon_L - E \varepsilon^p \quad \text{with} \quad \varepsilon_L = -\alpha T$$

Thus:

- If  $|\sigma_L| - R(p) < 0$  :  
 $p=0$  and  $\sigma_L = E \varepsilon_L$

- If  $|\sigma_L| - R(p) = 0$  :

$$p = \left( \frac{|\sigma_L| - \sigma^y}{R'} \right)$$

$$\sigma_L = E \varepsilon_L - E \varepsilon^p$$

##### Application to the way of loading

###### Moment 1:

$$\sigma = E \varepsilon = 200 \text{MPa} \quad \text{and} \quad R(p) = R' p + \sigma^y = 100 \text{MPa} \quad \text{because} \quad p=0.$$

One has well  $\sigma_L - R(p) \leq 0$ .

The criterion is not crossed, the evolution is elastic:  $\sigma_L = 100 \text{MPa}$  and  $N = 100 \text{kN}$

###### Moment 2:

The criterion is reached:

$$\sigma_L = \frac{E}{E + R'} (R' \varepsilon_L + \sigma^y) = \frac{2 \cdot 10^{11}}{2 \cdot 10^{11} + 2.02 \cdot 10^9} (2.02 \cdot 10^9 \times 3.5 \cdot 10^{-3} + 2 \cdot 10^8) = 205 \text{MPa}$$

$$N = 102.5 \text{kN}$$

$$\text{and} \quad p = 2.475 \cdot 10^{-3}$$

**Moment 3:**

One discharges elastically:

$$\sigma_L = E \varepsilon_L - E \varepsilon^p = 2 \cdot 10^{11} (1.5 \cdot 10^{-3} - 2.475 \cdot 10^{-3}) = -195 \text{ MPa}$$
$$N = -97.5 \text{ kN}$$

**Moment 4:**

One plasticizes again:

The criterion is written:  $|\sigma| - R' p - \sigma^y = 0$  with  $p = p_1 + p_2$  where  $p_1 = 2.475 \cdot 10^{-3}$

One thus obtains:

$$p_2 = \frac{|\sigma| - \sigma^y}{R'} - p_1$$
$$\sigma = -E \varepsilon^p = -E (p_1 - p_2)$$
$$\sigma = \left( \frac{R'}{R' + E} \right) \left( -2 E p_1 - \frac{E \sigma^y}{R'} \right) = -207.9 \text{ MPa}$$

And thus  $N = -103.95 \text{ kN}$

**Moment 5:**

One discharges elastically:

$$\sigma_L = E \varepsilon_L - E \varepsilon^p = 2 \cdot 10^{11} (2 \cdot 10^{-3} - 1.0395 \cdot 10^{-3}) = 192.1 \text{ MPa}$$
$$N = 96.05 \text{ kN}$$

**Moments 6 and 7:**

The reasoning is identical

One finds:

$$N_{(inst.6)} = 105.87 \text{ kN}$$
$$N_{(inst.7)} = -44.13 \text{ kN}$$

**Kinematic work hardening**

The method of calculating is identical, but in this case, the criterion of plasticity is written:

$$\sigma - X_{eq} - \sigma^y \leq 0 \quad \text{with} \quad X_{eq} = C (\varepsilon^p)_{eq} = \frac{3}{2} C \varepsilon^p = \frac{E E_t}{E - E_t} \varepsilon^p$$

With the preceding notations, the criterion is written:

$$\left| \sigma_L - R' \varepsilon^p \right| - \sigma^y \leq 0 \quad \text{from where} \quad \sigma_L = R' \varepsilon^p \pm \sigma^y \quad (\text{according to the direction of the flow}).$$

**Application to the way of loading****Moment 1:**

The criterion is not crossed, the evolution is elastic:  $\sigma_L = 100 \text{ MPa}$  and  $N = 100 \text{ kN}$

**Moment 2:**

The criterion is reached:  $|\sigma_L - R' \varepsilon^p| - \sigma^y = 0$

$$\sigma_L = R' \varepsilon^p + \sigma^y = 2.02 \cdot 10^9 \times 2.475 \cdot 10^{-3} + 2 \cdot 10^8 = 205 \text{ MPa}$$

**Moment 3:**

One discharges elastically:

$$\sigma_L = E \varepsilon_L - E \varepsilon^p = 2 \cdot 10^{11} (1.5 \cdot 10^{-3} - 2.475 \cdot 10^{-3}) = -195 \text{ MPa}$$
$$N = -97.5 \text{ kN}$$

## Moment 4:

$$|\sigma - R' \varepsilon^p| - \sigma^y = 0 \quad \text{with } p_1 = 2.475 \cdot 10^{-3}$$

$$\varepsilon^p = p_1 - p_2$$

$$p_2 = p_1 - \frac{|\sigma + \sigma^y|}{R'}$$

$$\sigma = -E \varepsilon^p = -E(p_1 - p_2)$$

$$\sigma = -E \left( \frac{|\sigma + \sigma^y|}{R'} \right) = -198 \text{MPa}$$

$$N = -99 \text{kN}$$

## Moment 5:

One discharges elastically:

$$\sigma_L = E \varepsilon_L - E \varepsilon^p = 2 \cdot 10^{11} (2 \cdot 10^{-3} - 9.9 \cdot 10^{-4}) = 202 \text{MPa}$$

$$N = 101 \text{kN}$$

## Moments 6 and 7:

The reasoning is identical. One finds:

$$N_{(inst.6)} = 103 \text{kN}$$

$$N_{(inst.7)} = -47 \text{kN}$$

## 2.1.2 Model of Pinto-Menegotto

This model is described in the Manuel de Référence of Code\_Aster [R5.03.09] [bib1]. The law constitutive of steels is made up of two distinct parts: the monotonous loading composed of three successive zones (linear elasticity, plastic stage and work hardening) and the cyclic loading where the way between two points of inversion (semi-cycle) is described by an analytical curve of expression of the type  $\sigma = f(\varepsilon)$ .

As previously the imposed deformations are thermal deformations:  $\varepsilon = -\alpha T$

### 2.1.2.1 Case without buckling

#### First loading

- Linear elasticity:  $\sigma = E \varepsilon$

#### Moment 1:

$$N = E \varepsilon S = 2 \cdot 10^{11} \times 1 \cdot 10^{-3} \times 5 \cdot 10^{-4} = 100 \text{kN}$$

- Plastic stage:  $\sigma = \sigma^y$
- Polynomial of degree 4:  $\sigma = \sigma_u - (\sigma_{su} - \sigma_y^0) \left( \frac{\varepsilon_u - \varepsilon}{\varepsilon_u - \varepsilon_h} \right)^4$

#### Moment 2:

$$\varepsilon = 3.5 \cdot 10^{-3} > \varepsilon_h = 2.3 \cdot 10^{-3}, \text{ one uses the polynomial of degree 4:}$$

$$\sigma = 209.416 \text{MPa and } N = 104.708 \text{kN}$$

## Cycles

### Semi-cycle 1:

One determines  $\zeta_p^0$ :

$$\zeta_p^0 = \varepsilon_r^0 - \varepsilon_y^0 = 3.5 \cdot 10^{-3} - 1.1 \cdot 10^{-3} = 2.5 \cdot 10^{-3} \text{ because } \varepsilon_r^0 = \varepsilon_{(inst.2)}$$

Then  $\Delta \sigma^0$  :

$$\Delta \sigma^0 = E_h \zeta_p^0 = 2 \cdot 10^9 \times 2.5 \cdot 10^{-3} = 5 \text{ MPa}$$

From where  $\sigma_y^1 = \sigma_y^0 \cdot \text{sign}(-\zeta_p^0) + \Delta \sigma^0 = -200 + 5 = -195 \text{ MPa}$

One calculates then  $\varepsilon_y^1$  :

$$\varepsilon_y^1 = \varepsilon_r^0 + \frac{\sigma_y^1 - \sigma_r^0}{E} = 3.5 \cdot 10^{-3} + \frac{(-195 - 209.416) 10^6}{2.0 \cdot 10^{11}} = 1.477 \cdot 10^{-3}$$

One determines thus  $\sigma^* = f(\varepsilon^*)$ , defined by:

$$\sigma^* = b \varepsilon^* + \left( \frac{1-b}{(1+(\varepsilon^*)^R)^{(1/R)}} \right) \varepsilon^* \quad , \text{ with } b = \frac{E_h}{E}$$

$$\varepsilon^* = \frac{\varepsilon - \varepsilon_r^0}{\varepsilon_y^1 - \varepsilon_r^0}$$

$$\sigma^* = \frac{\sigma - \sigma_r^0}{\sigma_y^1 - \sigma_r^0}$$

$$\xi_p^0 = \frac{\zeta_p^0}{|\varepsilon_y^1 - \varepsilon_r^0|} \quad \text{and} \quad R' = R_0 - \frac{A_1 \cdot \xi_p^0}{A_2 + \xi_p^0}$$

One obtains  $\xi_p^0 = -1.23$  and  $R' = 3.51$

One can then calculate the value of  $\sigma$  at moments 3 and 4:

### Moment 3:

$$\varepsilon^* = \frac{\varepsilon_{(inst.3)} - \varepsilon_r^0}{\varepsilon_y^1 - \varepsilon_r^0} = \frac{1.5 \cdot 10^{-3} - 3.5 \cdot 10^{-3}}{1.477 \cdot 10^{-3} - 3.5 \cdot 10^{-3}} = 0.988$$

$$\sigma^* = b \varepsilon^* + \left( \frac{1-b}{(1+(\varepsilon^*)^R)^{(1/R)}} \right) \varepsilon^* = 0.01 \times 0.988 + \left( \frac{1-0.01}{(1+(0.988)^{3.51})^{(1/3.51)}} \right) = 0.82$$

and  $\sigma = \sigma^* (\sigma_y^1 - \sigma_r^0) + \sigma_r^0 = 0.82 \times (-195 - 209.416) + 209.416 = -122 \text{ MPa}$

from where  $N = -61 \text{ kN}$

### Moment 4:

One uses the same method, with  $\varepsilon = 0$ .

$$\varepsilon^* = 1.73$$

$$\sigma^* = 0.56$$

$$\sigma = -20 \text{ MPa}$$

$$N = -10 \text{ kN}$$

### Semi-cycle 2: Moment 5 and 6:

The method of calculating is identical, one determines:

$$\zeta_p^1, \sigma_y^2, \varepsilon_y^2, \xi_p^1, R^2, \text{ then } \sigma_{(inst.5)}^* = f(\varepsilon_{(inst.5)}^*) \text{ and } \sigma_{(inst.6)}^* = f(\varepsilon_{(inst.6)}^*)$$

and finally  $\sigma_{(inst.5)}$  and  $\sigma_{(inst.6)}$ .

**Semi-cycle 3: Moment 7** : Idem

## 2.1.2.2 Case with buckling

### First loading

Identical to the preceding case.

### Cycles

#### Semi-cycle 1 (compression):

The method of calculating is identical, but the value of the slope of the asymptote is modified:

A new coefficient is calculated  $b_c$  :

$$b_c = a(5.0 - L/D) e \left( b \xi' \frac{E}{\sigma^y - \sigma^\infty} \right) = 0.006 \times (5.0 - 5.9) e^{(0.01 \times 1.477 \cdot 10^{-3} \frac{2 \cdot 10^{11}}{2 \cdot 10^8 - 1.36 \cdot 10^8})} = -0.0057$$

It is necessary then, as in the model without buckling, to determine  $\sigma_y^n$ . The reasoning is identical, but a complementary constraint is added  $\sigma_s^*$  in order to correctly position the curve compared to the asymptote.

$$\sigma_s^* = \gamma_s b E \frac{b - b_c}{1 - b_c} = 0.028 \times 0.01 \times 2 \cdot 10^{11} \times \frac{0.01 + 0.0057}{1 + 0.0057} = 0.87 \text{ MPa}$$

$$\text{where } \gamma_s \text{ is given by: } \gamma_s = \frac{11.0 - L/D}{10(e^{c(L/D)} - 1.0)} = 0.028$$

#### Semi-cycle 2 (traction):

- In traction, a reduced Young modulus is adopted:

$$E_r = E \left( a_5 + (1.0 - a_5) e^{-a_6 \xi^{n_2}} \right) = 2 \cdot 10^{11} \times (0.88 + (1 - 0.88) e^{(-620 \times 1.473 \cdot 10^{-6})}) = 1.99 \cdot 10^{11} \text{ MPa}$$

with  $a_5 = 1.0 + (5.0 - L/D)/7.5 = 0.88$

The rest of the method is identical.

## 2.2 Results of reference

Normal effort  $N$  constant on the bar

## 2.3 Uncertainty on the solution

No, the solution is analytical

## 2.4 Bibliographical references

- [1] Handbook of reference of *Code\_Aster* [R5.03.09].
- [2] S. ANDRIEUX: Thermoelastoplastic TD 1 Three bars perfect Von Mises. In "Initiation with thermoplasticity in *Code\_Aster*", HI-74/96/November 13th, 1996 (manual of reference of the course).



## 3 Modeling A

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### 3.1 Characteristics of modeling

The model is composed of an element of bar (BAR).

Law of behavior: elastoplasticity with linear isotropic work hardening - Criterion of Von Mises

### 3.2 Characteristics of the grid

2 nodes.

1 mesh SEG2

### 3.3 Sizes tested and results

Identification	Moments	Reference	Variation ( % )
normal effort NR	1	1.0000E+05	0
normal effort NR	2	1.0250E+05	0
normal effort NR	3	- 9.7500E+04	0
normal effort NR	4	- 1.0395E+05	0
normal effort NR	5	9.6050E+04	0
normal effort NR	6	1.0587E+05	0
normal effort NR	7	- 4.4129E+04	0

## 4 Modeling B

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### 4.1 Characteristics of modeling

The model is composed of an element of bar (BAR).

Law of behavior: elastoplasticity with linear kinematic work hardening - Criterion of Von Mises

### 4.2 Characteristics of the grid

2 nodes.

1 mesh SEG2

### 4.3 Sizes tested and results

Identification	Moments	Reference	Variation ( % )
normal effort NR	1	1.0000E+05	0
normal effort NR	2	1.0250E+05	0
normal effort NR	3	- 9.7500E+04	0
normal effort NR	4	- 9.9000E+04	0
normal effort NR	5	1.0100E+05	0
normal effort NR	6	1.0300E+05	0
normal effort NR	7	- 4.7000E+04	0

## 5 Modeling C

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### 5.1 Characteristics of modeling

The model is composed of an element of bar (BAR).

Law of behavior: model of Pinto-Menegotto without buckling (value of DASH lower than 5).

### 5.2 Characteristics of the grid

2 nodes.

1 mesh SEG2

### 5.3 Sizes tested and results

Identification	Moments	Reference	Variation ( % )
normal effort NR	1	1.0000E+05	0
normal effort NR	2	1.0470E+05	0
normal effort NR	3	- 6.0777E+04	0
normal effort NR	4	- 9.1430E+04	0
normal effort NR	5	7.6082E+04	0
normal effort NR	6	1.0125E+05	0
normal effort NR	7	- 3.7965E+04	0

## 6 Modeling D

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### 6.1 Characteristics of modeling

The model is composed of 1 element of bar (BAR).

Law of behavior: model of Pinto-Menegotto with buckling (value of DASH higher than 5).

### 6.2 Characteristics of the grid

2 nodes.

1 mesh SEG2

### 6.3 Sizes tested and results

Identification	Moments	Reference	Variation ( % )
normal effort NR	1	1.0000E+05	0
normal effort NR	2	1.0470E+05	0
normal effort NR	3	- 6.0556E+04	0
normal effort NR	4	- 8.9078E+05	0
normal effort NR	5	7.6905E+05	0
normal effort NR	6	1.0125E+05	0
normal effort NR	7	- 3.8119E+04	0

## 7 Summary of the results

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Results calculated by Code\_Aster are in excellent agreement with the analytical solutions.