SSNP152 – Inclusion of two crowns

Summary:

This test is used to evaluate the performances of Aster with regard to the treatment of the contact between two structures with geometrical compatibility enters surfaces Master and slave at the initial moment and various positions of geometrical incompatibility in the course of time. It makes it possible to validate the treatment of the contact with the method continues by taking into account great rotations.

One considers a structure made up of two concentric crowns. The external crown is subjected to a uniform pressure on its free edge whereas one imposes a rotation finished on the interior crown. Their rigidity, represented by their Young moduli plays an important role in the evaluation of the value of the deformations and the fluctuations of the contact pressure. One also seeks to know which are the effects of the use of a grid of a higher nature.

An analytical solution was developed for this problem in order to validate the calculated digital results. The validation of this test relates to the values of the contact pressure.

1 Problem of reference

1.1 Geometry

The structure is made up of two concentric circular rings. The internal ray R_2 external crown is equal to the external ray of the interior crown (Figure 1.1-1).
Dimensions structural feature are:

$R_1=1,0\,m\,;\,R_2=0,6\,m\,;\,R_3=0,2\,m$

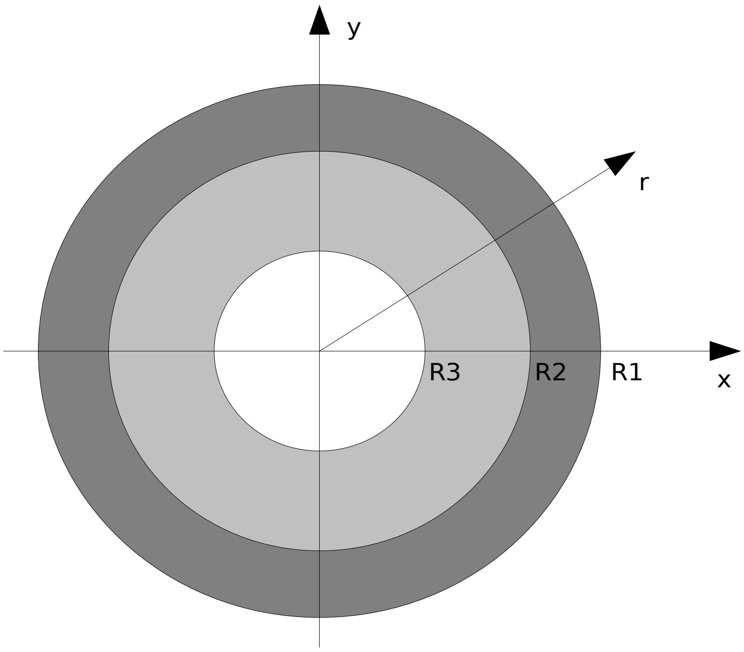

Figure 1.1-1: Geometry of the structure

1.2 Properties of materials

The Young modulus and the Poisson's ratio of material of the external crown are given by E_1 and ν_1 (respectively E_2 and ν_2 for the interior crown).

1.3 Boundary conditions and loadings

1.3.1. Boundary conditions in the case without rotation

The external crown is subjected to a displacement are equivalent to the application of a pressure p on its edge ($r=R_1$) whereas the edge of the interior crown ($r=R_3$) is left free of constraint.

$$\xi_X(r)=f(r)\cos(\arctan(\frac{Y}{X}))$$
$$\xi_Y(r)=f(r)\sin(\arctan(\frac{Y}{X}))$$

The function $f(r)$ radial displacement is given according to the properties of materials and the pressure p .
In the case of plane deformations (`MODELING = 'D_PLAN'`) one a:

$$f(R_1)=\frac{1+\nu_1}{E_1}(A_1(1-2\nu_1)R_1+\frac{B_1}{R_1})$$
$$f(R_3)=\frac{1+\nu_2}{E_2}(A_2(1-2\nu_2)R_3+\frac{B_2}{R_3}) \quad \text{éq 1.1}$$

with:

$$A_1=\frac{-p R_1^2+\lambda R_2^2}{R_1^2-R_2^2};\ B_1=(-p+\lambda)\frac{R_1^2R_2^2}{R_1^2-R_2^2}$$
$$A_2=-\lambda\frac{R_2^2}{R_2^2-R_3^2};\ B_2=-\lambda\frac{R_2^2R_3^2}{R_2^2-R_3^2} \quad \text{éq 1.2}$$

Where λ is the contact pressure whose analytical expression is:

$$\lambda=2p(1-\nu_1)\frac{\frac{R_1^2}{R_1^2-R_2^2}}{\frac{R_1^2+R_2^2(1-2\nu_1)}{R_1^2-R_2^2}+\frac{E_1}{E_2}\frac{(1+\nu_1)}{(1+\nu_2)}\frac{R_2^2(1-2\nu_2)+R_3^2}{R_2^2-R_3^2}} \quad \text{éq. 1.3}$$

It is noticed that in the case of plane deformations the Poisson's ratios does not intervene in the equations, whereas in the case of forced plane (MODELING = 'C_PLAN') one a:

$$f(R_1)=\frac{1}{E_1}(A_1(1-\nu_1)R_1+\frac{B_1(1+\nu_1)}{R_1})$$
$$f(R_3)=\frac{1}{E_2}(A_2(1-\nu_2)R_3+\frac{B_2(1+\nu_2)}{R_3}) \quad \text{éq 1.4}$$

Expressions of A_1, B_1, A_2 and B_2 are the same ones as those of equations 1.2 while the contact pressure is given by:

$$\lambda=2p\frac{\frac{R_1^2}{R_1^2-R_2^2}}{\frac{R_1^2(1+\nu_1)+R_2^2(1-\nu_1)}{R_1^2-R_2^2}+\frac{E_1}{E_2}\frac{R_2^2(1-\nu_2)+R_3^2(1+\nu_2)}{R_2^2-R_3^2}} \quad \text{éq 1.5}$$

1.3.2. Boundary conditions in the case with rotation

In addition to radial displacement, one imposes on the internal surface of the interior crown ($r=R_3$) a displacement involving the rotation of this one. The expression of displacement must take into account the contraction of the interior crown due to the application of the pressure, its new internal ray being $R_3+f(R_3)$:

$$\xi_X(X,Y,i)=(R_3+f(R_3))\cos(\arctan(\frac{Y}{X})+\omega(i))-X$$
$$\xi_Y(X,Y,i)=(R_3+f(R_3))\sin(\arctan(\frac{Y}{X})+\omega(i))-Y \quad \text{éq 1.6}$$
$$\omega(i)=\frac{2\pi}{N_tN_e}i,\ i\in\{n\in\mathbb{Z},0\leqslant n\leqslant N_t\}$$

where N_t is the number of steps of time and N_e the number of elements along the surface of contact. As a rotation of angle is imposed $\frac{2\pi}{N_e}$, at the last moment there is a configuration similar to the initial configuration but where the interior crown was shifted of an element. In this last configuration the grids Master and slave of surfaces of contact are again in opposite.

2 Reference solution

We develop here an analytical solution to the problem presented above. This solution will be developed on the assumption of small deformations by considering that the materials of the crowns isotropic, are governed by a linear elastic law without temperature variation.
Because of symmetries of the problem, the solution in displacement of the problem has the following generic form:

$$u=u_r(r,z).\underline{e_r}+u_z(r,z).\underline{e_z}$$

2.1.1. Case 1: plane deformations

By using symmetries of the problem and the assumption of invariance according to Z of the plane constraints, the solution of the problem takes the following shape:

$$\begin{aligned} u_r&=u_r(r)\\ u_\theta&=0\\ u_z&=0 \end{aligned} \quad \text{éq 2.1}$$

By using the equation of Lamé-Navier:

$$(\lambda+\mu)\underline{grad}(div(\underline{u}))+\mu\Delta\underline{u}+\underline{fd}=\underline{0} \quad \text{éq 2.2}$$

where $\underline{fd}=\underline{0}$ are here the worthless voluminal efforts, and the F ormule of the Laplacian:

$$\Delta\underline{u}=\underline{grad}(div(\underline{u}))+\underline{rot}\,\underline{rot}(\underline{u}) \quad \text{éq 2.3}$$

One can write éq 2.2 pennies the form:

$$(\lambda+2\mu)\underline{grad}(div(\underline{u}))+\mu\,\underline{rot}\,\underline{rot}(\underline{u})+\underline{fd}=\underline{0} \quad \text{éq 2.4}$$

that is to say still while using $\underline{rot}(\underline{u})=\vec{0}$ et $\underline{fd}=\vec{0}$ et $u=u_r(r).\underline{e_r}$:

$$\begin{aligned} div(\underline{u})&=\frac{d}{dr}u_r(r)+\frac{1}{r}u_r(r)\\ \underline{grad}(div\,\underline{u})&=\frac{d}{d_r}[\frac{1}{r}\frac{d}{d_r}(r\,u_r(r))].\underline{e}_r\\ \text{soit encore }(\lambda+2\mu)&\frac{d}{d_r}[\frac{1}{r}\frac{d}{d_r}(r\,u_r(r))]=0 \end{aligned} \quad \text{éq 2.5}$$

by integrating the equation, one obtains for the crowns (external and interior) the following shape of the field of displacement:

$$u_r=C_i r+\frac{D_i}{r}\quad u_\theta=0\quad u_z=0 \quad \text{éq 2.6}$$

To determine C_i and D_i, it remains us to impose the limiting conditions in pressure and displacement. For that, the deformations should initially be calculated then constraints associated with the field with displacement.

Déformations are the symmetrical part of the gradient of displacements. One obtains:

$$\epsilon_{rr}=C_i-\frac{D_i}{r^2}$$
$$\epsilon_{\theta\theta}=C_i+\frac{D_i}{r^2} \quad \text{éq 2.7}$$
$$\epsilon_{zz}=\epsilon_{r\theta}=\epsilon_{rz}=\epsilon_{\theta z}=0$$

By applying the law of Hooke:

$$\underline{\underline{\sigma}}=\lambda tr(\underline{\underline{\epsilon}})\underline{\underline{1}}+2\mu\underline{\underline{\epsilon}} \quad \text{éq 2.8}$$

one obtains the following general form for the constraints:

$$\sigma_{rr}=\frac{E}{1+\nu}\left(\frac{C_i}{1-2\nu}-\frac{D_i}{r^2}\right)$$
$$\sigma_{\theta\theta}=\frac{E}{1+\nu}\left(\frac{C_i}{1-2\nu}+\frac{D_i}{r^2}\right) \quad \text{éq 2.9}$$
$$\sigma_{zz}=\frac{2\nu EC_i}{(1+\nu)(1-2\nu)}$$
$$\sigma_{r\theta}=\sigma_{rz}=\sigma_{\theta z}=\underline{0}$$

While posing:

$$A_i=\frac{E_i}{(1+\nu_i)(1-2\nu_i)}C_i \qquad B_i=\frac{E_i}{1+\nu_i}D_i \quad \text{éq 2.10}$$

the nonworthless constraints become:

$$\sigma_{rr}=A_i-\frac{B_i}{r^2}$$
$$\sigma_{\theta\theta}=A_i+\frac{B_i}{r^2} \quad \text{éq 2.11}$$
$$\sigma_{zz}=2\nu A_i$$

It any more but does not remain us to calculate the values of A_i and B_i for each crown. One will note λ_n the contact pressure between the two crowns such as:

$$\underline{\underline{\sigma}}_{1rr}(R_2).(-\underline{e}_r)=\lambda_n\underline{e}_r$$
$$\underline{\underline{\sigma}}_{2rr}(R_2).\underline{e}_r=-\lambda_n\underline{e}_r \quad \text{éq 2.12}$$

with the boundary conditions:

$$\underline{\underline{\sigma}}_{1rr}(R_1).\underline{e}_r=-p.\underline{e}_r$$
$$\underline{\underline{\sigma}}_{2rr}(R_3).(-\underline{e}_r)=\underline{0} \quad \text{éq 2.13}$$

The condition of continuity on displacement with the interface between the two groups of contact gives moreover:

$$u_{r;1}(R2)=u_{r;2}(R2) \qquad \text{éq 2.14}$$

We thus have 5 equations for 5 unknown factors $A_1, B_1, A_2, B_{2,}\lambda_n$.

The system of the first 4 equations enables us to obtain:

$$A_1=\frac{-p\,R_1^2+\lambda_n R_2^2}{R_1^2-R_2^2};\,B_1=(-p+\lambda_n)\frac{R_1^2R_2^2}{R_1^2-R_2^2}$$
$$A_2=-\lambda_n\frac{R_2^2}{R_2^2-R_3^2};\,B_2=-\lambda_n\frac{R_2^2R_3^2}{R_2^2-R_3^2} \qquad \text{éq 2.15}$$

and the equation of continuity on displacement finally makes it possible to have the contact pressure:

$$\lambda_n=\frac{2\,p\,R_1^2(1-\nu_1)}{R_1^2+R_2^2(1-2\nu_1)+\frac{E_1}{E_2}\frac{1+\nu_2}{1+\nu_1}\frac{R_1^2-R_2^2}{R_2^2-R_3^2}(R_2^2(1-2\nu_2)+R_3^2)} \qquad \text{éq 2.16}$$

2.1.2. Case 2: plane constraints

We suppose initially that there are no constraints according to the direction perpendicular to the plan of the crowns ($\underline{\sigma}_i\,.\,\underline{e}_z=\underline{0}$). Symmetries of the problem bring us to a stress field which can be written in the form:

$$\underline{\underline{\sigma_i}}(r,z)=\sigma_{rr;i}\underline{e}_r\otimes\underline{e}_r+\sigma_{\theta\theta;i}\underline{e}_\theta\otimes\underline{e}_\theta+\sigma_{r\theta;i}\underline{e}_r\otimes\underline{e}_\theta$$

where the index i 1 for the external crown and 2 for the interior crown is worth. In the absence of voluminal forces and by considering the quasi-static problem (one neglects the effects of acceleration), one a:

$$div\,\underline{\underline{\sigma}}=\underline{0}$$

By using the solution in displacement of the generic problem:

$$u=u_r(r,z).\underline{e_r}+u_z(r,z).\underline{e_z}$$

The field of deformation is written then:

$$\begin{gathered}\epsilon_{rr}=\frac{\partial u_r(r,z)}{\partial r}\\ \epsilon_{\theta\theta}=\frac{u_r}{r}\\ \epsilon_{zz}=\frac{\partial u_z(r,z)}{\partial z}\\ \epsilon_{\theta z}=\epsilon_{r\theta}=0\\ \epsilon_{rz}=\frac{1}{2}\left(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right)\end{gathered}\quad \text{éq 2.17}$$

Like:

$$\begin{gathered}div\,\underline{\underline{\sigma}}=\underline{0}\ \text{ devient }\ \frac{\partial\sigma_{rr}}{\partial r}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}=0\\ \text{avec}\\ \sigma_{rr}=\frac{E}{1+\nu}\left(\epsilon_{rr}+\frac{\nu}{1-2\nu}(\epsilon_{rr}+\epsilon_{\theta\theta}+\epsilon_{zz})\right)\\ \sigma_{\theta\theta}=\frac{E}{1+\nu}\left(\epsilon_{\theta\theta}+\frac{\nu}{1-2\nu}(\epsilon_{rr}+\epsilon_{\theta\theta}+\epsilon_{zz})\right)\\ \sigma_{zz}=\frac{E}{1+\nu}\left(\epsilon_{zz}+\frac{\nu}{1-2\nu}(\epsilon_{rr}+\epsilon_{\theta\theta}+\epsilon_{zz})\right)\\ \sigma_{zz}=\sigma_{z\theta}=\sigma_{zr}=0\\ \text{impliquant}\\ \epsilon_{zr}=0=\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\end{gathered}\quad \text{éq 2.18}$$

The fact that $\sigma_{zz}=0$ we gives:

$$\begin{gathered}\frac{E}{1+\nu}\left(\epsilon_{zz}+\frac{\nu}{1-2\nu}(\epsilon_{rr}+\epsilon_{\theta\theta}+\epsilon_{zz})\right)=0\\ \epsilon_{zz}(1-\nu)=-\nu(\epsilon_{rr}+\epsilon_{\theta\theta})\\ \epsilon_{zz}=\frac{-\nu}{(1-\nu)}(\epsilon_{rr}+\epsilon_{\theta\theta})\end{gathered}\quad \text{éq 2.19}$$

what implies:

$$\begin{gathered}\sigma_{rr}-\sigma_{\theta\theta}=\frac{E}{1+\nu}(\epsilon_{rr}-\epsilon_{\theta\theta})\\ tr\,\underline{\underline{\epsilon}}=\frac{\nu-1}{\nu}\epsilon_{zz}+\epsilon_{zz}\\ tr\,\underline{\underline{\epsilon}}=\frac{2\nu-1}{\nu}\epsilon_{zz}\end{gathered}\quad \text{éq 2.20}$$

and allows to write:

$$tr\,\underline{\underline{\epsilon}}=\frac{1-2\nu}{1-\nu}(\epsilon_{rr}+\epsilon_{\theta\theta})\quad \text{éq 2.21}$$

One replaces then $tr\,\underline{\underline{\epsilon}}$ by its value in σ_{rr} and $\sigma_{\theta\theta}$ to obtain:

$$\sigma_{rr}=\frac{E}{1+\nu}\left(\epsilon_{rr}+\frac{\nu}{1-2\nu}\frac{1-2\nu}{1-\nu}(\epsilon_{rr}+\epsilon_{\theta\theta})\right)$$
$$\sigma_{\theta\theta}=\frac{E}{1+\nu}\left(\epsilon_{\theta\theta}+\frac{\nu}{1-2\nu}\frac{1-2\nu}{1-\nu}(\epsilon_{rr}+\epsilon_{\theta\theta})\right)$$

éq 2.22

$$\sigma_{rr}=\frac{E}{1+\nu}\left(\epsilon_{rr}+\frac{\nu}{1-\nu}(\epsilon_{rr}+\epsilon_{\theta\theta})\right)$$
$$\sigma_{\theta\theta}=\frac{E}{1+\nu}\left(\epsilon_{\theta\theta}+\frac{\nu}{1-\nu}(\epsilon_{rr}+\epsilon_{\theta\theta})\right)$$

The equilibrium equation becomes as follows:

$$\frac{\partial\sigma_{rr}}{\partial r}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}=0 \quad\rightarrow\quad \frac{\partial\epsilon_{rr}}{\partial r}+\frac{\nu}{1-\nu}\left(\frac{\partial}{\partial r}(\epsilon_{rr}+\epsilon_{\theta\theta})\right)+\frac{\epsilon_{rr}-\epsilon_{\theta\theta}}{r}=0$$ éq 2.23

As we have:

$$\epsilon_{rr}=\frac{\partial u_r(r,z)}{\partial r}$$
$$\epsilon_{\theta\theta}=\frac{u_r}{r}$$

éq 2.24

the substitution of the constraints by the deformations in the equilibrium equation makes it possible to write finally:

$$\frac{\partial^2 u_r(r,z)}{\partial r^2}+\frac{\nu}{1-\nu}\left(\frac{\partial^2 u_r(r,z)}{\partial^2 r}+\frac{\partial u_r(r,z)}{\partial r}\right)+\frac{1}{r}\left(\frac{\partial u_r(r,z)}{\partial r}-\frac{u_r(r,z)}{r}\right)=0$$
$$\frac{\partial^2 u_r(r,z)}{\partial r^2}+\frac{\nu}{1-\nu}\left(\frac{\partial^2 u_r(r,z)}{\partial r^2}+\frac{1}{r}\frac{\partial u_r(r,z)}{\partial r}-\frac{1}{r^2}u_r(r,z)\right)+\frac{1}{r}\left(\frac{\partial u_r(r,z)}{\partial r}-\frac{u_r(r,z)}{r}\right)$$
$$\frac{\partial^2 u_r(r,z)}{\partial r^2}+\frac{1}{r}\frac{\partial u_r(r,z)}{\partial r}-\frac{1}{r^2}u_r(r,z)=0$$
$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r.u_r(r,z))\right]=0$$

éq 2.25

By successive integrations of éq 2.25 we obtain the following shape of the field $u_r(r,z)$:

$$\frac{1}{r}\frac{\partial}{\partial r}(r.u_r)=f(z)$$
$$\frac{\partial}{\partial r}(r.u_r)=r.f(z)$$
$$r.u_r=\frac{r^2}{2}f(z)+g(z)$$
$$u_r=\frac{r}{2}f(z)+\frac{g(z)}{r}$$

éq 2.26

That is to say:

$$u_r(r,z)=C(z)r+\frac{D(z)}{r} \quad \text{éq 2.27}$$

As we have:

$$\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}=0$$
$$\epsilon_{rr}=\frac{u_r(r,z)}{\partial r}=C(z)-\frac{D(z)}{r^2}$$
$$\epsilon_{\theta\theta}=\frac{u_r}{r}=C(z)+\frac{D(z)}{r^2} \quad \text{éq 2.28}$$
$$\epsilon_{zz}=\frac{\partial u_z}{\partial z}=\frac{\nu}{1-\nu}(\epsilon_{rr}+\epsilon_{\theta\theta})$$
$$\epsilon_{zz}=\frac{-2\nu}{1-\nu}C(z)$$

One leads thus to:

$$\frac{\partial u_z}{\partial z}=\frac{-2\nu}{1-\nu}C_1 z+C_2 \quad \text{éq 2.29}$$

One can thus write while integrating ϵ_{zz} that $u_z(r,z)=f(z)+g(r)$. By using the first relation of éq 2.28 one obtains then:

$$\frac{\partial u_z}{\partial r}=g'(r)=\frac{-\partial u_r}{\partial z}$$
$$g'(r)=-C'(z)r-\frac{D'(z)}{r}$$
$$C'(z)=cte\rightarrow C(z)=C_1 z+C_2 \quad \text{éq 2.30}$$
$$D'(z)=cte\rightarrow D(z)=D_1 z+D_2$$
$$g(r)=-C_1\frac{r^2}{2}-D_1\ln(r)+C_0$$

One leads thus to:

$$\frac{\partial u_z}{\partial r}=g'(r) \quad \text{éq 2.31}$$

By integrating the two partial derivative equations éq 2.29 and éq 2.31 one obtains for each of the two crowns i=1 external and i=2 interior:

$$u_z=\frac{-2\nu}{1-\nu}(C_1\frac{z^2}{2}+C_2 z)-C_1\frac{r^2}{2}-D_1\ln(r)+C_0$$
$$u_z(r,0)=g(r)+f(0)=0\rightarrow C_0=C_1=D_1=0 \quad \text{éq 2.32}$$
$$u_z(r,z)=\frac{-2\nu}{1-\nu}C_2 z$$

from where one obtains:

$$\sigma_{rr}=\frac{E}{1+\nu}(\epsilon_{rr}+\frac{\nu}{1-\nu}(\epsilon_{rr}+\epsilon_{\theta\theta}))$$
$$\sigma_{rr}=\frac{E}{1+\nu}(\frac{\partial u_r(r,z)}{\partial r}+\frac{\nu}{1-\nu}(\frac{\partial u_r(r,z)}{\partial r}+\frac{u_r}{r})) \quad \text{éq 2.33}$$
$$\sigma_{rr}=\frac{E}{1+\nu}(\frac{1+\nu}{1-\nu}C_i+\frac{D_i}{r^2})$$

While posing:

$$A_i=\frac{E}{1+\nu}\frac{1+\nu}{1-\nu}C_i \qquad B_i=\frac{E}{1+\nu}D_i \quad \text{éq 2.34}$$

the field of the constraints is written:

$$\sigma_{rr;i}(r)=A_i-\frac{B_i}{r^2}$$
$$\sigma_{\theta\theta;i}(r)=A_i+\frac{B_i}{r^2} \quad \text{éq 2.35}$$
$$\underline{\underline{\sigma}}_i(r)=A_i\underline{\underline{1}}-\frac{B_i}{r^2}(\underline{e}_r\otimes\underline{e}_r-\underline{e}_\theta\otimes\underline{e}_\theta)$$

It any more but does not remain us to calculate the values of A_i and B_i for each crown. One will note λ_n the contact pressure between the two crowns such as:

$$\underline{\underline{\sigma}}_1(R_2).(-\underline{e}_r)=\lambda_n\underline{e}_r$$
$$\underline{\underline{\sigma}}_2(R_2).\underline{e}_r=-\lambda_n\underline{e}_r \quad \text{éq 2.36}$$

with the boundary conditions:

$$\underline{\underline{\sigma}}_1(R_1).\underline{e}_r=-p.\underline{e}_r$$
$$\underline{\underline{\sigma}}_2(R_3).(-\underline{e}_r)=\underline{0} \quad \text{éq 2.37}$$

one a:

$$A_1=\frac{-p\,R_1^2+\lambda_n R_2^2}{R_1^2-R_2^2};\ B_1=(-p+\lambda_n)\frac{R_1^2R_2^2}{R_1^2-R_2^2}$$
$$A_2=-\lambda_n\frac{R_2^2}{R_2^2-R_3^2};\ B_2=-\lambda_n\frac{R_2^2R_3^2}{R_2^2-R_3^2} \quad \text{éq 2.38}$$

in addition the law of behavior of materials makes it possible to write:

$$\underline{\underline{\epsilon}}_i=\frac{1+\nu_i}{E_i}\underline{\underline{\sigma}}_i-\frac{\nu_i}{E_i}Tr(\underline{\underline{\sigma}}_i)\underline{\underline{1}}=\frac{1}{E_i}[A_i(1-\nu_i)\underline{\underline{1}}-B_i(1+\nu_i)\frac{1}{r^2}(\underline{e}_r\otimes\underline{e}_r-\underline{e}_\theta\otimes\underline{e}_\theta)]$$
$$=\frac{\partial u_{r;i}}{\partial r}\underline{e}_r\otimes\underline{e}_r+\frac{1}{2}(\frac{1}{r}\frac{\partial u_{r;i}}{\partial\theta}+\frac{\partial u_{\theta;i}}{\partial r}-\frac{u_{\theta;i}}{r})(\underline{e}_r\otimes\underline{e}_\theta+\underline{e}_\theta\otimes\underline{e}_r)+(\frac{1}{r}\frac{\partial u_{\theta;i}}{\partial\theta}+\frac{u_{r;i}}{r})\underline{e}_\theta\otimes\underline{e}_\theta \quad \text{éq 2.39}$$

what makes it possible to obtain:

$$\underline{u}_i=\frac{1}{E_i}[A_i(1-\nu_i)r+B_i(1+\nu_i)\frac{1}{r}]\underline{e}_r=f_{CP}(r)\underline{e}_r \quad \text{éq 2.40}$$

The function $f(r)$ radial displacement is given according to the properties of materials and the pressure p . In the case of forced plane (`MODELING = 'C_PLAN'`) one a:

$$\begin{aligned} f(R_1)&=\frac{1}{E_1}(A_1(1-\nu_1)R_1+\frac{B_1(1+\nu_1)}{R_1}) \\ f(R_3)&=\frac{1}{E_2}(A_2(1-\nu_2)R_3+\frac{B_2(1+\nu_2)}{R_3}) \end{aligned} \quad \text{éq 2.41}$$

To obtain the value of λ_n , one imposes the continuity of the vector displacement in $r=R_2$:

$$\begin{aligned} &\underline{u}_1(R_2)=\underline{u}_2(R_2) \\ &\frac{1}{E_1}[A_1(1-\nu_1)R_2+B_1(1+\nu_1)\frac{1}{R_2}]\underline{e}_r=\frac{1}{E_2}[A_2(1-\nu_2)R_2+B_2(1+\nu_2)\frac{1}{R_2}]\underline{e}_r \\ &\frac{1}{E_1}[\frac{-p R_1^2+\lambda_n R_2^2}{R_1^2-R_2^2}(1-\nu_1)R_2+(-p+\lambda_n)\frac{R_1^2 R_2^2}{R_1^2-R_2^2}(1+\nu_1)\frac{1}{R_2}]\underline{e}_r \\ &=\frac{1}{E_2}[-\lambda_n\frac{R_2^2}{R_2^2-R_3^2}(1-\nu_2)R_2-\lambda_n\frac{R_2^2 R_3^2}{R_2^2-R_3^2}(1+\nu_2)\frac{1}{R_2}]\underline{e}_r \end{aligned} \quad \text{éq 2.42}$$

that is to say still:

$$\lambda_n=\frac{2pR_1^2}{R_1^2(1+\nu_1)+R_2^2(1-\nu_1)+\frac{E_1}{E_2}\frac{R_1^2-R_2^2}{R_2^2-R_3^2}(R_2^2(1-\nu_2)+R_3^2(1+\nu_2))} \quad \text{éq 2.43}$$

2.1.3. Notice

Once one calculated the value of displacements and the contact pressure in the case of plane constraints, one can easily calculate these values in plane deformations by replacing the values of the Young modulus and the Poisson's ratio:

$$E_{DP}=\frac{E_{CP}}{1-\nu_{CP}^2};\nu_{DP}=\frac{\nu_{CP}}{1-\nu_{CP}} \quad \text{éq 2.44}$$

Where E_{CP} and ν_{CP} take the values E_i and ν_i §2.1.2.

The values of displacements thus are obtained:

$$\underline{u}_i=\frac{1+\nu_i}{E_i}[A_i(1-2\nu_i)r+\frac{B_i}{r}]\underline{e}_r=f_{DP}(r)\underline{e}_r \quad \text{éq 2.45}$$

and of the contact pressure:

$$\lambda_n=\frac{2\,p\,R_1^2(1-\nu_1)}{R_1^2+R_2^2(1-2\nu_1)+\frac{E_1}{E_2}\frac{1+\nu_2}{1+\nu_1}\frac{R_1^2-R_2^2}{R_2^2-R_3^2}(R_2^2(1-2\nu_2)+R_3^2)}$$ éq 2.46

It is noticed that the values of A_i and B_i remain unchanged.

2.1.4. Calculation of the standard L^2 contact pressure

That is to say Γ_c the surface of contact. The standard L^2 contact pressure is defined by:

$$\|\lambda\|_{L^2}^2=\int_{\Gamma_c}\lambda^2\,dS.$$

In this case, Γ_c L is the circle of center O and of ray R_2 and the contact pressure is contact is uniform. One thus has:

$$\|\lambda\|_{L^2}^2=\lambda^2|\Gamma_c|=2\pi\lambda^2 R_2.$$

And finally:

$$\|\lambda\|_{L^2}=|\lambda|\sqrt{2\pi R_2}.$$

2.1.5. Values tested

One tests the contact pressure to the interface between the 2 crowns. For a node of interface, the analytical solution is obtained by equations 1.3 and 1.5 in plane strains and plane stresses respectively.

The value of the pressure applied to the edge with each step of time is given by the formula:

$$p(t)=p_0 10^{\frac{t}{10}-1,1},\ t\in\{n\in\mathbb{Z},\ 1\leqslant n\leqslant 21\},\ p_0=1,0\,MPa$$ éq 2.47

for $t=1$, the pressure is worth $0,1\,MPa$ and for $t=21$ it is worth $10\mathrm{MPa}$. It is noticed that simulations were made by neglecting the effects of acceleration (order STAT_NON_LINE) what implies that the steps of time have arbitrary units.

The analytical solution is calculated in small deformations. The calculations will be done with the behavior GROT_GDEP, to compare the solutions with and without rotation. The variation compared to the analytical solution will thus increase with the increase in the value of the external pressure applied.

3 Modeling A

3.1 Characteristics of modeling

It is about a modeling in plane constraints (C_PLAN), without rotation of the interior crown and where one imposes a loading increasing exponentially with time, the goal being to measure the difference between the computed values and those analytically obtained in order to better know the field of validity of the solution.

Young moduli $E_1 = E_2$ and Poisson's ratios $\nu_1 = \nu_2$ are respectively $1.0\text{E}9\,Pa$ and 0.2. The pressure applied to the edge of the external crown is worth $1.0\text{E}6\,Pa$ and it varies from 10% to 1000% of its value in the course of time.

The external crown defines main surface.

3.2 Characteristics of the grid

The grid (Figure 3.2-a) comprises:

- 160 meshs of the type SEG2;
- 240 meshs of the type QUAD4.

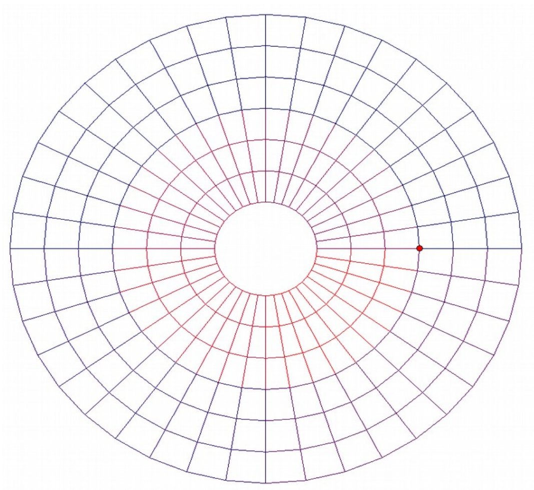

Figure 3.2-1: Grid of modeling A

3.3 Sizes tested and results

One calculates the contact pressure (LAGS_C) for the node A coordinates $(0.6, 0.0)$, that which at the initial moment is more on the right of the interface between the two crowns. For each step of load, one compares the computed value with that given by equation 1.5. The tolerance is fixed at 2% compared to the analytical value.

Identification	Reference	Aster	tolerance
LAGS_C with the node A	$p(t)$	Analytical	2.10^{-2}

The standard is also tested L^2 contact pressure for the moment $t=1$, The analytical value is:

$$\|\lambda\|_{L^2} = p(t=1)\sqrt{2\pi R_2} \approx 179780{,}177088 \text{ Mpa} \times \text{m}^{\frac{1}{2}}.$$

Identification	Type of reference	Value of reference	tolerance
`L normalizes` ^{2}	Analytical	179780.177088	0.1%

3.4 Comments

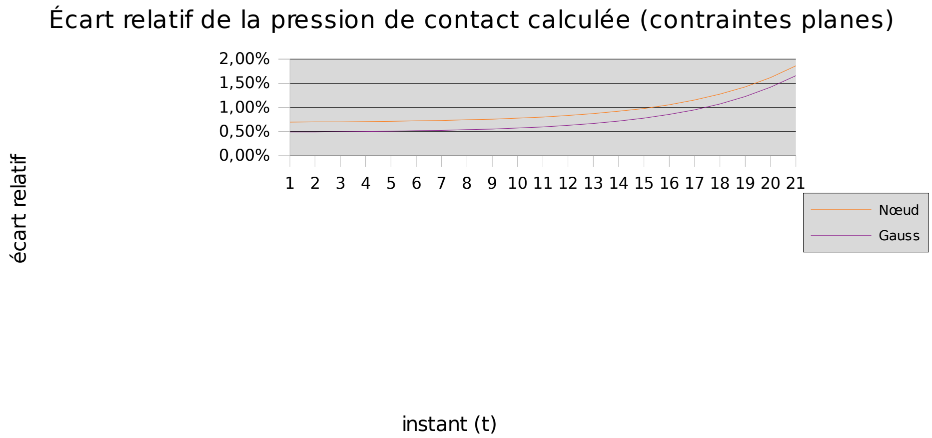

Figure 3.4-1: evolution of the variation enters the analytical solution and the solution given by Code_Aster for an integration to Nœuds and at the points of Gauss.

Since simulation is made on the assumption of an elastic behavior, configuration at one moment t depends by no means on the previous moments: all occurs as if several independent simulations were carried out, each one with a value of different loading. The gap with the analytical solution widens because the law of behavior is used `GROT_GDEP` who utilizes great displacements.

4 Modeling B

4.1 Characteristics of modeling

Modeling is identical to modeling A, but in this case one will work in plane deformations (D_PLAN) and, like already mentioned above, the Poisson's ratios do not intervene any more in the solution.
The external crown defines main surface.

4.2 Characteristics of the grid

Idem modeling A.

4.3 Sizes tested and results

One calculates the contact pressure (LAGS_C) for the node A coordinates $(0.6,0.0)$, that which at the initial moment is more on the right of the interface between the two crowns. For each step of load, one compares the computed value with that given by equation 1.3. The tolerance is fixed at 2% compared to the analytical value.

Identification	Reference	Aster	tolerance
LAGS_C with the node A	$p(t)$	Analytical	2.10^{-2}

4.4 Comments

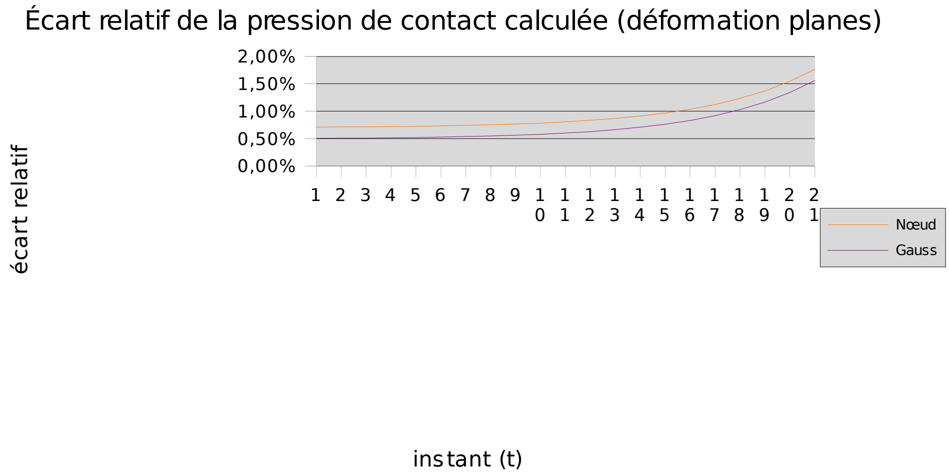

Figure 4.4-1: evolution of the variation enters the analytical solution and the solution given by Code_Aster for an integration to Nœuds and at the points of Gauss.

Idem modeling A.

5 Modeling C

5.1 Characteristics of modeling

Thereafter, one will work only with modeling in plane constraints.

The Young moduli and the Poisson's ratios remain the same ones. One fixes the value of the pressure on the edge of the crown external with ($p=1.0E7$) and one imposes a rotation of the interior crown in accordance with equation 1.6 for a number $N=100$ of step of time.
The external crown defines main surface.

5.2 Characteristics of the grid

Idem modeling A.

5.3 Sizes tested and results

One calculates the contact pressure (LAGS_C) for the node A coordinates $(0.6, 0.0)$, that which at the initial moment is more on the right of the interface between the two crowns. The computed values are compared with the value obtained according to equation 1.5 for an external pressure of $p=1.0E7$. The rotation of the interior crown is applied. One looks at the variations of the external pressure during this rotation. The tolerance is fixed at 4% compared to the analytical value. One tests when the grids are again in opposite.

Identification	Reference	Aster	tolerance
LAGS_C with the node A	$\lambda=9.26E6$	Analytical	4.10^{-2}

5.4 Comments

In addition to the calculation of the contact pressure, one is also interested by his variation in the course of time. The incompatibility of the meshs Masters and slaves induced of the fluctuations on the value this pressure. Figure 5.4-1 shows (in a a little coarse way) this effect. A way of attenuating it consists in using either a finer grid, or of the elements of a higher nature.

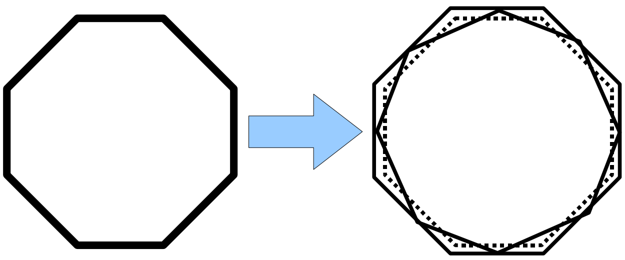

Figure 5.4-1: Fluctuation of the contact pressure due to the incompatibility of the meshs

One gives below Figure 5.4-2 and Figure 5.4-3 evolutions of the contact pressure calculated for 100 pas de time corresponding to the rotation of an element along circumference for $R=R_2$. The results presented here are got for $p=1.0E4$. To consider most correctly possible the contact pressure, during rotation, when surfaces of main contact and slave are not compatible any more, it is necessary to use the diagrams of integration of nature the highest possible with a substantial refinement of the grid compared to the situation with compatible grids of modelings A and B (in this case it is necessary to refine 10 times more on the circumference and radially and to use a diagram of Gauss of order 10 or Simpson with order 4 to bring back the error in pressure to 15% to semi course). The result is very clearly improved by the recourse to the quadratic elements (cf modeling G with an error of about 4%).

Rigidification of the interior crown (see Figure 5.4-4) do not seem to have of effect on quality of the solution, contrary to what can occur for a quadratic grid (see Figure 9.3-2).

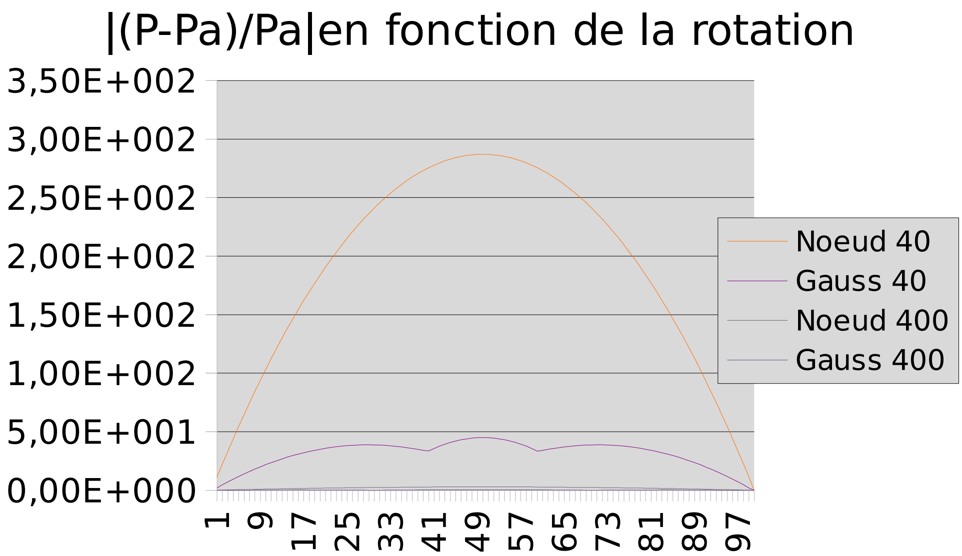

Figure 5.4-2: Evolution of the contact pressure during rotation for 160 and 1600 elements on the circumference $R=R2$. With the grid with 160 elements one reaches an error of 30000% with integration to the nodes and 5000% with integration at 2 points of Gauss for an error lower than 4% without rotation with compatibility of the grids of surfaces Master and slave.

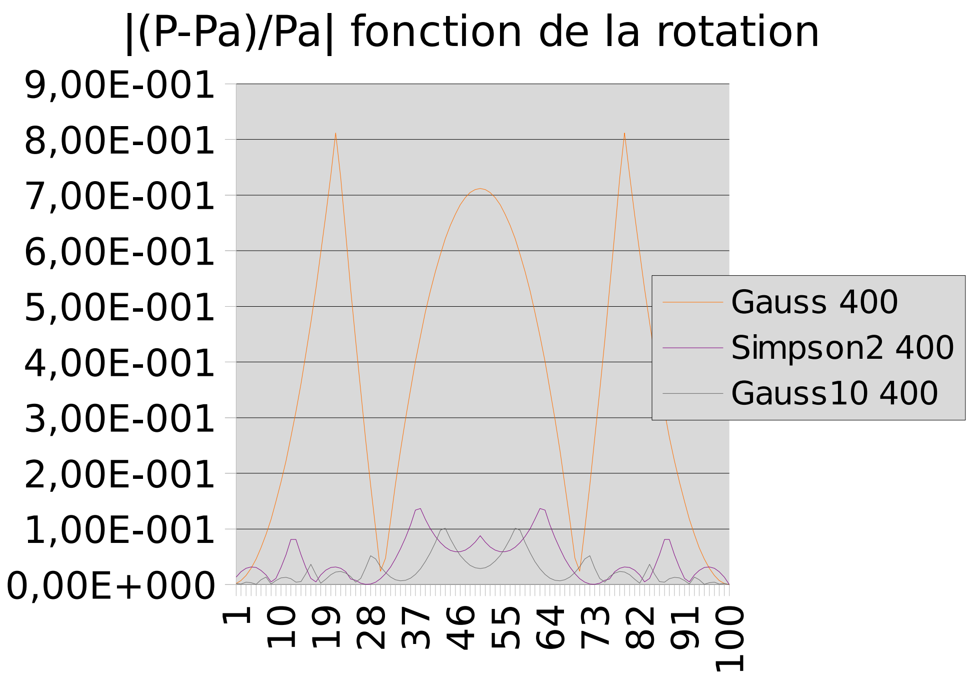

Figure 5.4-3: Evolution of the contact pressure during rotation for 1600 elements on the circumference $R=R2$ for a rigid interior crown. An integration with 2 points of Gauss leads to a maximum error of 80%. Only integrations of Gauss to 10 and Simpson to order 4 make it possible to bring back the maximum error to the neighbourhoods of 15% for incompatible grids.

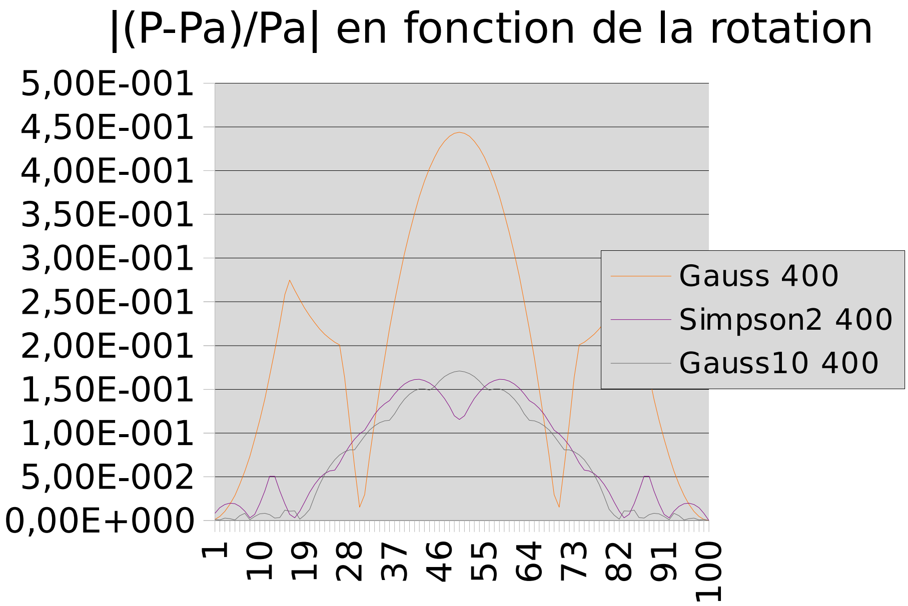

Figure 5.4-4: Evolution of the contact pressure during rotation for 1600 elements on the circumference $R=R2$. An integration with 2 points of Gauss leads to a maximum error of 45%. Only integrations of Gauss to 10 and Simpson to order 4 make it possible to bring back the maximum error to the neighbourhoods of 15% for incompatible grids.

6 Modeling D

6.1 Characteristics of modeling

Idem modeling C but one plays with the Young moduli and the Poisson's ratios:

External crown: $E_1=1.0E9$, $\nu_1=0.3$

Interior crown: $E_2=1.0E8$, $\nu_2=0.2$

The external crown defines main surface.

6.2 Characteristics of the grid

Idem modeling A.

6.3 Sizes tested and results

One calculates the contact pressure (LAGS_C) for the node A coordinates $(0.6, 0.0)$, that which at the initial moment is more on the right of the interface between the two crowns. The computed values are compared with the value obtained according to equation 1.5 for an external pressure of $p=1.0E7$. The rotation of the interior crown is applied. One looks at the variations of the external pressure during this rotation. The tolerance is fixed at 4% compared to the analytical value. One tests when the grids are again in opposite.

Identification	Reference	Aster	tolerance
LAGS_C with the node A	$\lambda=2.418E6$	Analytical	4.10^{-2}

6.4 Comments

When displacements become too important due to a low rigidity of the crowns, the computed value deviates from that analytically calculated, once this solution was developed on the assumption of small deformations and that simulation was made in great displacements.

Other with dimensions, when the rigidity of the crowns is too important, the fluctuation of the contact pressure increases appreciably. This comes owing to the fact that when one imposes a displacement on a structure, the constraints to which this one is subjected can be too high (see infinite) in order to be compatible with the laws of mechanics. For a very rigid structure, a small displacement is possible only with considerable constraints.

7 Modeling E

7.1 Characteristics of modeling

Idem modeling C but the interior crown defines main surface in order to satisfy condition LBB ($P1$ in contact and $P2$ in displacement).

7.2 Characteristics of the grid

The grid comprises:

- 80 meshs of the type SEG2;
- 80 meshs of the type SEG3;
- 120 meshs of the type QUAD4 on the external crown;
- 120 meshs of the type QUAD8 on the interior crown.

7.3 Sizes tested and results

One calculates the contact pressure (LAGS_C) for the node A coordinates $(0.6,0.0)$, that which at the initial moment is more on the right of the interface between the two crowns. The computed values are compared with the value obtained according to equation 1.5 for an external pressure of $p=1.0E7$. The rotation of the interior crown is applied. One looks at the variations of the external pressure during this rotation. The tolerance is fixed at 4% compared to the analytical value. One tests when the grids have a maximum shift of a surface half-element.

Identification	Reference	Aster	tolerance
LAGS_C with the node A	$\lambda=9.26E6$	Analytical	4.10^{-2}

8 Modeling F

8.1 Characteristics of modeling

Idem modeling C.

The external crown defines main surface in order to satisfy condition LBB ($P1$ in contact and $P2$ in displacement).

8.2 Characteristics of the grid

We will work with elements of order 2 in order to observe the effects of their uses during calculations. For modeling E, the interior crown will comprise elements of order 2 and the interior one of the elements of order 1. For modeling F, one reverses. For modeling G, all the grid will have elements of order 2.

The grid comprises:
- 80 meshs of the type SEG2 ;
- 80 meshs of the type SEG3 ;
- 120 meshs of the type QUAD4 on the interior crown ;
- 120 meshs of the type QUAD8 on the external crown.

8.3 Sizes tested and results

One calculates the contact pressure (LAGS_C) for the node A coordinates $(0.6, 0.0)$, that which at the initial moment is more on the right of the interface between the two crowns. The computed values are compared with the value obtained according to equation 1.5 for an external pressure of $p=1.0\text{E}7$. The rotation of the interior crown is applied. One looks at the variations of the external pressure during this rotation. The tolerance is fixed at 4% compared to the analytical value. One tests when the grids have a maximum shift of a surface half-element.

Identification	Reference	Aster	tolerance
LAGS_C with the node A	$\lambda=9.26\text{E}6$	Analytical	4.10^{-2}

9 Modeling G

9.1 Characteristics of modeling

Idem modeling C.
The external crown defines main surface.

9.2 Characteristics of the grid

The quadratic grid comprises:

- 160 meshs of the type SEG3;
- 240 meshs of the type QUAD8.

9.3 Sizes tested and results

One calculates the contact pressure (LAGS_C) for the node A coordinates $(0.6, 0.0)$, that which at the initial moment is more on the right of the interface between the two crowns. The computed values are compared with the value obtained according to equation 1.5 for an external pressure of $p=1.0E7$. The rotation of the interior crown is applied. One looks at the variations of the external pressure during this rotation. The tolerance is fixed at 4% compared to the analytical value. One tests when the grids are again in opposite.

Identification	Reference	Aster	tolerance
LAGS_C with the node A	$\lambda=9.26E6$	Analytical	4.10^{-2}

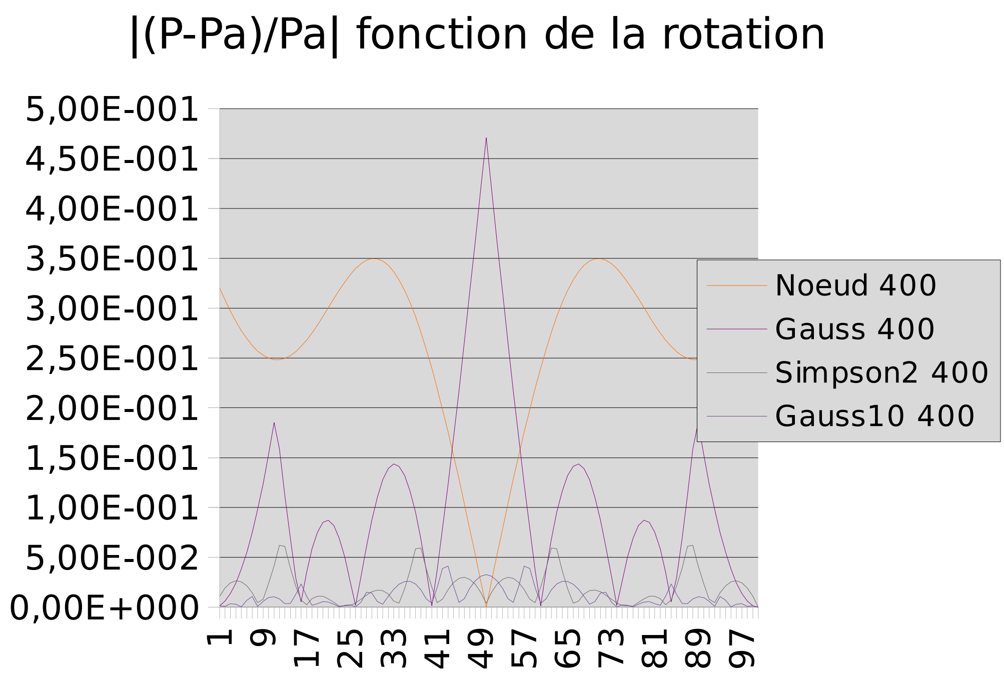

Figure 9.3-1: Evolution of the contact pressure during rotation for 1600 elements on the circumference $R=R2$. An integration with 3 points of Gauss leads to a maximum error of 45%. A nodal integration does not give correct results when the grids Master and slave are compatible. Only integrations of Gauss to order 10 and Simpson to order 4 make it possible to bring back the maximum error to the neighbourhoods of 5% for incompatible grids.

One gives above the evolutions of the contact pressure calculated for 100 pas de time corresponding to the rotation of an element along circumference for $R=R_2$. The results presented here are got for $p=1.0\text{E}4$. To consider most correctly possible the contact pressure, during rotation, when surfaces of main contact and slave are not compatible any more, it is necessary to use the diagrams of integration of nature the highest possible with a substantial refinement of the grid compared to the situation with compatible grids of modelings A and B (in this case it is necessary to refine 10 times more and to use diagrams of Gauss of order 10 or Simpson to order 4 to bring back the error in pressure to 5%). The result is very clearly improved compared to that obtained with linear elements (on the same refined grid, the diagrams of order 10 of Gauss and order 4 of Simpson lead to errors in pressure of 15%).

So finally one rigidifies the interior crown for the same pressure $p=1.0\text{E}4$, the following result is got. The whole of the diagrams of integration behaves correctly with an error lower than 1% except the diagram with the node which gives a constant error to 33.4%.

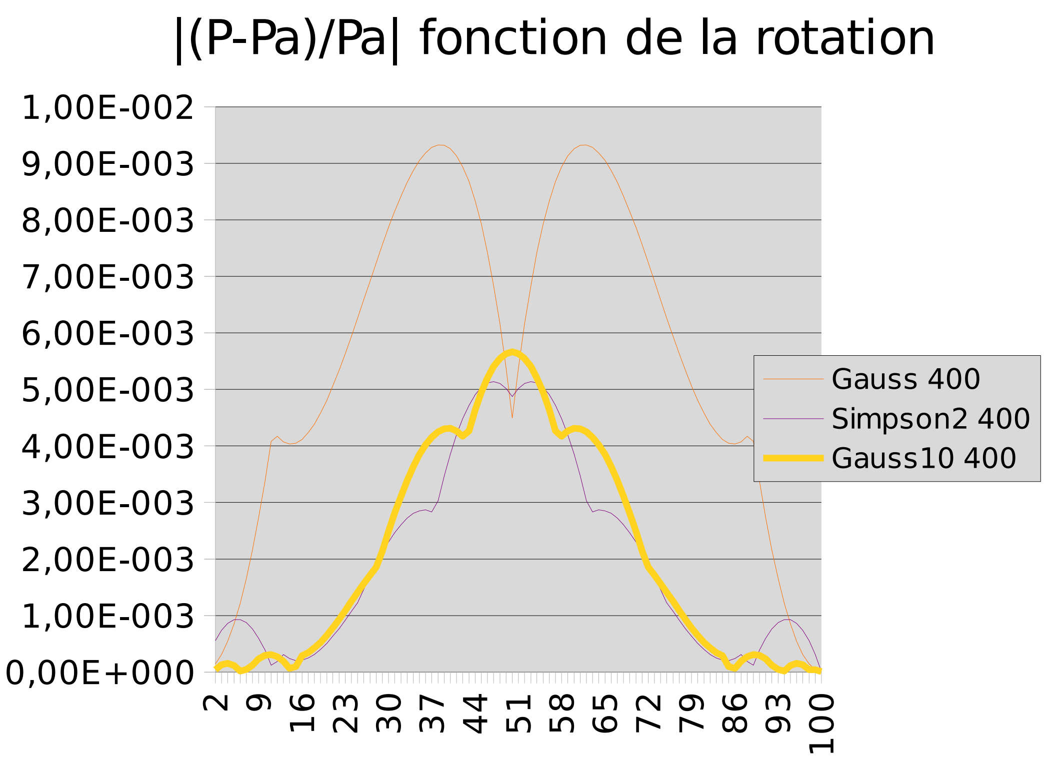

Figure 9.3-2: Evolution of the contact pressure during rotation for 1600 elements on the circumference $R=R2$ for a rigid interior disc. The whole of the diagrams of Gauss to order 3, Gauss with order 10 and Simpson with order 4 gives an error lower than 1%. A diagram with the node gives a constant error on the whole of the rotation of 33.4% not represented here.

10 Modeling H

10.1 Characteristics of modeling

Idem modeling G except that one uses an under-integrated modeling, 'C_PLAN_SI'.
The external crown defines main surface.

10.2 Characteristics of the grid

The quadratic grid comprises:

- 160 meshs of the type SEG3;
- 240 meshs of the type QUAD8.

10.3 Sizes tested and results

One calculates the contact pressure (LAGS_C) for the node A coordinates $(0.6, 0.0)$, that which at the initial moment is more on the right of the interface between the two crowns. The computed values are compared with the value obtained according to equation 1.5 for an external pressure of $p=1.0E7$. The rotation of the interior crown is applied. One looks at the variations of the external pressure during this rotation. The tolerance is fixed at 2% compared to the analytical value. One tests when the grids are again in opposite.

Identification	Reference	Aster	tolerance
LAGS_C with the node A	$\lambda=9.26E6$	Analytical	2.10^{-2}