

## SSNV168 – Triaxial compression test drained with a behavior DRUCK\_PRAGER polishing substance

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### Summary:

This case test makes it possible to simulate a triaxial compression test drained on four different modelings during a nonlinear calculation. That makes it possible to propose the effect of the type of work hardening negative, parabolic or linear, in the case of models `AXIS` or `3D`.

#### Modeling a:

- model of the type “ `DRUCK_PRAGER` ” with linear negative work hardening for a containment of  $2\text{ MPa}$  .
- model `AXIS` with meshes `QUAD4` .

#### Modeling b:

- model of the type “ `DRUCK_PRAGER` ” with parabolic negative work hardening for a containment of  $2\text{ MPa}$  .
- modeling `AXIS` with meshes `QUAD4` .

#### Modeling C:

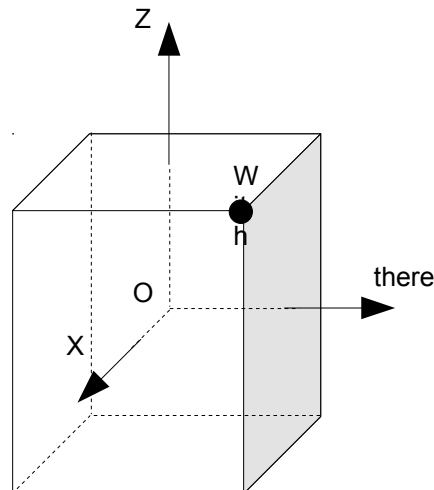
- model of the type “ `DRUCK_PRAGER` ” with linear negative work hardening for a containment of  $2\text{ MPa}$  .
- modeling `3D` with meshes `HEXA20` .

#### Modeling D:

- model of the type “ `DRUCK_PRAGER` ” with parabolic negative work hardening for a containment of  $2\text{ MPa}$  .
- modeling `3D` with meshes `HEXA20` .

## 1 Problem of reference

### 1.1 Geometry



- Dimension of the cube:  $1\text{m} \times 1\text{m} \times 1\text{m}$ .
- Center of the cube:  $O : (0., 0., 0.)$

### 1.2 Properties of material

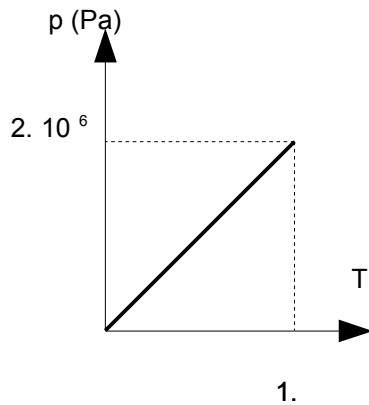
- Rubber band
  - $E = 5800.0 \text{ E6 Pa}$  Young modulus
  - $\nu = 0.3$  Poisson's ratio
- DRUCK\_PRAGER with linear negative work hardening
  - $\alpha = 0.33$  Coefficient of dependence in pressure
  - $p_{ultm} = 0.01$  Ultimate cumulated plastic deformation
  - $\sigma^Y = 2.57 \text{ E6 Pa}$  Plastic constraint
  - $h = -2.00 \text{ E8 Pa}$  Module of work hardening
- DRUCK\_PRAGER with parabolic negative work hardening
  - $\alpha = 0.33$  Coefficient of dependence in pressure
  - $p_{ultm} = 0.01$  Ultimate cumulated plastic deformation
  - $\sigma^Y = 2.57 \text{ E6 Pa}$  Plastic constraint
  - $\sigma_{ultm}^Y = 0.57 \text{ E6 Pa}$  Ultimate constraint

### 1.3 Boundary conditions and loadings

The boundary conditions and the loadings applied are the following:

- Stage  $A : t \in [0, 1.]$

A compression gradually is applied  $p = 2.10^6 \text{ Pa}$  on 3 faces of the cube (high, in front, right-hand side) according to the function presented on the figure below, and of the conditions of symmetry on the 3 others faces (low, behind, left).

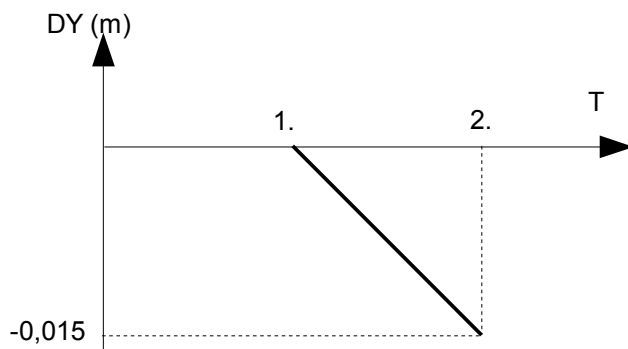


- Stage b:  $t \in ]1, 2.]$

From the state of stress at the moment  $t = 1.s$ , one applies to the faces of the cube the following conditions:

**Imposed displacements:**

- displacement varies gradually on the face of straight lines according to the function presented on the figure below:



- Conditions of symmetry on the 3 faces (low, behind, left).

**Imposed loadings:**

One applies a pressure of  $p = 2.10^6 Pa$  on the 2 others faces (in front of and high).

## 2 Reference solution

### 2.1 Method of calculating used for the reference solution

#### 2.1.1 Displacement $DY$

Displacement  $DY$  of reference to the point  $A$ , corresponds to imposed displacement.

$$DY = -0.015(t-1)$$

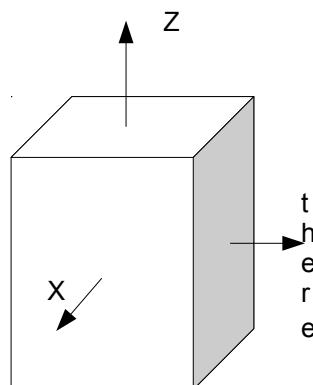
#### 2.1.2 Constraint $SIXX$

The constraint  $SIXX$  corresponds to the loading applied.

#### 2.1.3 Constraint $SIYY$ and cumulated plastic deformation $VI$

Triaxial calculation in conditions drained with the law of `DRUCK_PRAGER`.

Comparison with an analytical solution



$$\sigma_{eq} + \alpha I_1 - R(p) = 0$$

$$\sigma_{eq} + \alpha I_1 - \sigma^Y = 0 \text{ at the top} \quad (1)$$

One imposes  $\sigma_{xx} = \sigma_{zz} = -2 \text{ MPa} = \sigma^0$

$$\sigma_{eq} = \sqrt{\frac{3}{2}} S_{II}$$

$$S = \begin{pmatrix} \sigma_{xx} - \frac{1}{3} tr \sigma \\ \sigma_{yy} - \frac{1}{3} tr \sigma \\ \sigma_{zz} - \frac{1}{3} tr \sigma \end{pmatrix} \text{ with } tr \sigma = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = \sigma_{yy} + 2 \sigma^0$$

$$S = \begin{pmatrix} \sigma^0 - \frac{1}{3}\sigma_{yy} - \frac{2}{3}\sigma^0 \\ \sigma_{yy} - \frac{1}{3}\sigma_{yy} - \frac{2}{3}\sigma^0 \\ \sigma^0 - \frac{1}{3}\sigma_{yy} - \frac{2}{3}\sigma^0 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} -\sigma_{yy} + \sigma^0 \\ 2\sigma_{yy} - 2\sigma^0 \\ -\sigma_{yy} + \sigma^0 \end{pmatrix}$$

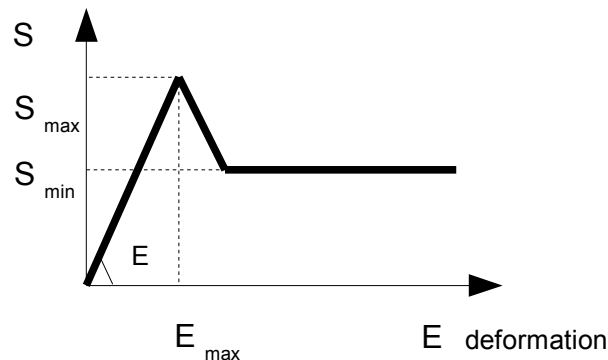
$$S_{II} = S \cdot S = \frac{1}{3} \sqrt{2(\sigma_{yy} - \sigma^0)^2 + (2\sigma_{yy} - 2\sigma^0)^2}$$

$$\sigma_{eq} = \sqrt{\left(\frac{3}{2}\right) S_{II}}$$

what gives us  $\sigma_{eq} = |(\sigma_{yy} - \sigma^0)|$  (2)

While introducing (2) in (1) one obtains

$$|(\sigma_{yy} - \sigma^0)| + \alpha(\sigma_{yy} + 2\sigma^0) - \sigma^Y = 0$$



$$\epsilon_{max} \Rightarrow \epsilon_{max} = \frac{\sigma_{max}}{E} = \frac{\sigma_{xx} - \sigma_{yy}}{E} > 0 \Rightarrow \sigma_{xx} > \sigma_{yy} \\ \Rightarrow \sigma_{yy} < \sigma^0$$

from where (2)  $\Rightarrow (-\sigma_{yy} + \sigma^0) + \alpha(\sigma_{yy} + 2\sigma^0) - \sigma^Y = 0$

$$\sigma_{max} \Rightarrow \sigma_{yy}^{max} = \frac{\sigma^Y - \sigma^0(2\alpha + 1)}{\alpha - 1}$$

from where  $\epsilon_{yy}^{max} = \frac{\sigma^0 - \sigma_{yy}^{max}}{E}$ , one replaces  $\sigma_{yy}^{max}$  by his value and one obtains

$$\epsilon_{max}^{yy} = \frac{3\alpha\sigma^0 - \sigma^Y}{E(\alpha - 1)}$$

$$\sigma_{min} \Rightarrow \sigma_{eq} + \alpha I_1 - R(p_{ultm}) = 0$$

$$\sigma_{yy}(\alpha - 1) + \sigma^0(2\alpha + 1) - R(p_{ultm}) = 0$$

$$\sigma_{yy}^{min} = \frac{R(p_{ultm}) - \sigma^0(2\alpha + 1)}{\alpha - 1} \quad (3)$$

$$\begin{cases} \epsilon - \epsilon^p = \frac{1+\nu}{E} \sigma - \frac{\nu}{E} (\text{tr } \sigma) \cdot I \\ \sigma_{eq} + \alpha I_1 - R(p) = 0 \end{cases}$$

$$\dot{\epsilon}^p = \lambda \left( \frac{3}{2} \frac{S}{\sigma_{eq}} + \alpha I \right) \text{ with } \lambda = \dot{p}$$

$$\text{however } S = \frac{1}{3} \begin{pmatrix} \sigma^0 - \sigma_{yy} \\ 2\sigma_{yy} - 2\sigma^0 \\ \sigma^0 - \sigma_{yy} \end{pmatrix} \quad \sigma_{eq} = |\sigma_{yy} - \sigma^0| = \sigma^0 - \sigma_{yy}$$

$$\text{from where } \frac{S}{\sigma_{eq}} = \frac{1}{3} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$\dot{\epsilon}_x^p = \dot{\epsilon}_z^p = \dot{p} \left( \frac{3}{2} \frac{1}{3} + \alpha \right) = \dot{p} \left( \alpha + \frac{1}{2} \right)$$

$$\dot{\epsilon}_y^p = \dot{p} \left( \frac{3}{2} \left( \frac{-2}{3} \right) + \alpha \right) = \dot{p} (\alpha - 1)$$

$$\Rightarrow \epsilon_y^p = p(\alpha - 1) + cste$$

$$\epsilon_y - \epsilon_y^p = \frac{1+\nu}{E} \sigma^y - \frac{\nu}{E} (\sigma^y + 2\sigma^0)$$

$$\epsilon_y^{max} - cste = \frac{1+\nu}{E} \sigma_{yy}^{max} - \frac{\nu}{E} (\sigma_{yy}^{max} + 2\sigma^0) = \frac{\sigma_{yy}^{max}}{E} - \frac{2\nu}{E} \sigma^0$$

$$\frac{3\alpha\sigma^0 - \sigma^Y}{E(\alpha-1)} - cste = \frac{1}{E} \left( \frac{\sigma^Y - \sigma^0(2\alpha+1)}{\alpha-1} \right) - \frac{2\nu}{E} \sigma^0$$

$$cste = \frac{\sigma^0[(5-2\nu)\alpha + (1+2\nu)] - 2\sigma^Y}{E(\alpha-1)} \quad (4)$$

and

$$\epsilon_y^p = p(\alpha-1) + cste$$

from where

$$\epsilon_y - \epsilon_y^p = \frac{1+\nu}{E} \sigma_{yy} - \frac{\nu}{E} (\sigma_{yy} + 2\sigma^0)$$

$$\epsilon_y - p(\alpha-1) - cste = \frac{1}{E} \sigma_{yy} - \frac{2\nu}{E} \sigma^0 \quad (5)$$

The expression below is a direct application of the expression (5)

$$\epsilon_y^{min} = p_{ultm}(\alpha-1) + cste + \frac{\sigma_{yy}^{min}}{E} - 2\frac{\nu}{E} \sigma^0 \quad (6)$$

By introducing the expressions (3) and (4) in (6) one obtains:

$$\epsilon_y^{min} = p_{ultm} \left[ \frac{E(\alpha-1)^2 + h}{E(\alpha-1)} \right] + \frac{\sigma^Y}{E(1-\alpha)} + \sigma^0 \left[ \frac{3\alpha + 4\nu(1-\alpha)}{E(\alpha-1)} \right]$$

The expression below is a direct application of the expression (5)

$$p = \frac{-\sigma_{yy}}{E(\alpha-1)} + \frac{2\nu\sigma^0}{E(\alpha-1)} + \frac{\epsilon_y}{(\alpha-1)} - \frac{cste}{(\alpha-1)} \quad (7)$$

For reasons of simplification one poses

$$p = A\sigma_{yy} + B$$

with

$$\begin{cases} A = \frac{1}{E(1-\alpha)} \\ B = \frac{2\nu\sigma^0}{E(\alpha-1)} + \frac{\epsilon_y}{\alpha-1} - \frac{cste}{\alpha-1} \end{cases}$$

on the basis of the equation  $\sigma_{eq} + \alpha I_1 - R(p) = 0$

with in the case:

- linear  $R(p) = hp + \sigma^Y$
- parabolic  $R(p) = \frac{6c f(p) \cos(\phi)}{3 - \sin(\phi)}$

$$\text{with } f(p) = \begin{cases} \left( 1 - \left( 1 - \sqrt{\frac{\sigma_{ultm}^Y}{\sigma^Y}} \frac{p}{p_{ultm}} \right)^2 \right) & 0 < p < p_{ultm} \\ \frac{\sigma_{ultm}^Y}{\sigma^Y} & p_{ultm} < p \end{cases} \text{ if}$$

### Linear case

$$\sigma_{eq} + \alpha I_1 - R(p) = 0 \quad \text{case where } \underline{0 < p < p_{ultm}}$$

$$\sigma_{yy}(\alpha - 1) + \sigma^0(2\alpha + 1) - hp - \sigma^Y = 0 \quad (8)$$

By introducing the expression of p (7) in (8) the following expression is obtained

$$\sigma_{yy} = \epsilon_y \left( \frac{Eh}{h + E(\alpha - 1)^2} \right) + \sigma^Y \left( \frac{E(\alpha - 1)}{h + E(\alpha - 1)^2} \right) + \sigma^0 \left( \frac{2\sqrt{h} - (2\alpha + 1)E(\alpha - 1)}{h + E(\alpha - 1)^2} \right) + \frac{hE.cste}{h + E(\alpha - 1)^2}$$

Parabolic case where  $\underline{0 < p < p_{ultm}}$

$$\sigma_{yy}(\alpha - 1) + \sigma^0(2\alpha + 1) - R(p) = 0$$

$$R(p) = \left( 1 - \left( 1 - \alpha_2 \right) \frac{p}{p_{ultm}} \right)^2 \frac{6c \cos(\phi)}{3 - \sin(\phi)}$$

$$R(p) = \sigma^Y \left( 1 - \left( 1 - \sqrt{\frac{\sigma_{ultm}^Y}{\sigma^Y}} \frac{p}{p_{ultm}} \right)^2 \right)$$

By replacing this new expression of p (7) in the preceding equation

$$R(p) = \sigma^Y \left[ 1 - \left( 1 - \sqrt{\frac{\sigma_{ultm}^Y}{\sigma^Y}} \frac{A\sigma_{yy} + B}{p_{ultm}} \right)^2 \right]$$

while developing one obtains the following expression:

$$R(p) = a\sigma_{yy}^2 + b\sigma_{yy} + c$$



$$\text{avec } \begin{cases} a = \frac{A^2}{p_{ultm}^2} \left( 1 - \sqrt{\frac{\sigma_{ultm}^Y}{\sigma^Y}} \right)^2 \sigma^Y \\ b = -2 \sigma^Y \left[ 1 - \left( 1 - \sqrt{\frac{\sigma_{ultm}^Y}{\sigma^Y}} \right) \frac{B}{p_{ultm}} \right] \left[ \frac{A}{p_{ultm}} \left( 1 - \sqrt{\frac{\sigma_{ultm}^Y}{\sigma^Y}} \right) \right] \\ c = \left[ 1 - \left( 1 - \sqrt{\frac{\sigma_{ultm}^Y}{\sigma^Y}} \right) \frac{B}{p_{ultm}} \right]^2 \end{cases}$$

One finds after simplification:

$$\sigma_{yy}(\alpha - 1) + \sigma^0(2\alpha + 1) - a\sigma_{yy}^2 - b\sigma_{yy} - c = 0$$

that is to say

$$a\sigma_{yy}^2 + (1 + b - \alpha)\sigma_{yy} + (c - \sigma^0(2\alpha + 1)) = 0$$

Resolution of the polynomial of order 2:

$$\Delta = (1 + b - \alpha)^2 - 4a(c - \sigma^0(2\alpha + 1))$$

$$\begin{cases} \sigma_1 = \frac{-(1 + b - \alpha) - \sqrt{\Delta}}{2a} \\ \sigma_2 = \frac{-(1 + b - \alpha) + \sqrt{\Delta}}{2a} \end{cases}$$

## 2.2 Reference variables

- Constraint  $SIXX$  with the node  $A$
- Constraint  $SIYY$  with the node  $A$
- Cumulated plastic deformation  $VI$  with the node  $A$
- Displacement  $DY$  with the node  $A$

## 2.3 Result of reference

Size	Not	Inst	Référence*	Reference **
$SIXX (N/m^2)$	With	2.0	$-2.0 E6$	$-2.0 E6$
$SIYY (N/m^2)$	With	1.07	$-8.09 E6$	$-8.09 E6$
		1.16	$-8.20 E6$	$-8.01 E6$
		1.34	$-6.89 E6$	$-6.63 E6$
		1.53	$-5.80 E6$	$-5.81 E6$
$VI$	With	1.07	0	0
		1.16	$1.99 E-3$	$2.04 E-3$
		1.34	$6.35 E-3$	$6.42 E-3$
		1.53	$1.09 E-2$	$1.09 E-2$
$DY (m)$	With	1.07	$-1.05 E-3$	$-1.05 E-3$
		1.16	$-2.40 E-3$	$-2.40 E-3$
		1.34	$-5.10 E-3$	$-5.10 E-3$
		1.53	$-7.95 E-3$	$-7.95 E-3$

\* linear work hardening \*\* parabolic work hardening

## 2.4 Uncertainty on the solution

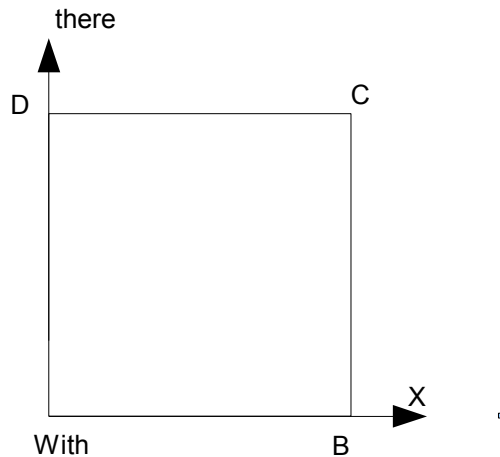
Analytical solution

## 3 Modeling A

### 3.1 Characteristics of modeling A

Modeling AXIS .

Model of DRUCK\_PRAGER with linear negative work hardening.



Many nodes	4	
Many meshes	5	That is to say:
		SEG2 4
		QUAD4 1

The square is in space  $[0., 1.] \times [0., 1.]$  .

Coordinates of the points ( $m$ ) :

$A : (0., 0.)$   
 $B : (1., 0.)$   
 $C : (1., 1.)$   
 $D : (0., 1.)$

Meshs:

$M1$  : surface  $ABDC$   
 $M2$  : segment  $AB$   
 $M3$  : segment  $BC$   
 $M4$  : segment  $CD$   
 $M5$  : segment  $DA$

Groups of nodes:

$A, B$



## 3.2 Sizes tested and results

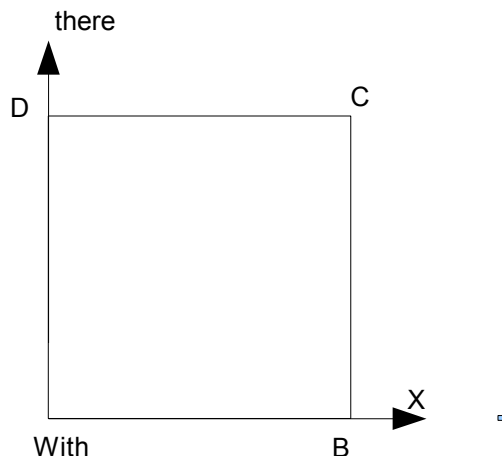
Size	Not	Inst	Reference	Tolerance (%)
$S_{IXX}$ (Pa)	C	2.0	$-2.0 E6$	0.1
$S_{IYY}$ (Pa)	C	1.07	$-8.09 E6$	0.1
		1.16	$-8.20 E6$	0.1
		1.34	$-6.89 E6$	0.1
		1.53	$-5.80 E6$	0.1
$V_I$	C	1.07	0	0.1
		1.16	$1.99 E-3$	0.1
		1.34	$6.35 E-3$	0.1
		1.53	$1.09 E-2$	0.1
$DY$ (m)	C	1.07	$-1.05 E-3$	0.1
		1.16	$-2.40 E-3$	0.1
		1.34	$-5.10 E-3$	0.1
		1.53	$-7.95 E-3$	0.1

## 4 Modeling B

### 4.1 Characteristics of modeling B

Modeling AXIS.

Model of DRUCK\_PRAGER with parabolic negative work hardening.



Many nodes	4	
Many meshes	5	That is to say:
		SEG2 4
		QUAD4 1

The square is in space  $[0.,1.] \times [0.,1.]$ .

Coordinates of the points ( $m$ ) :

$A:(0.,0.)$   
 $B:(1.,0.)$   
 $C:(1.,1.)$   
 $D:(0.,1.)$

Meshs:

$M1$  : surface  $ABDC$   
 $M2$  : segment  $AB$   
 $M3$  : segment  $BC$   
 $M4$  : segment  $CD$   
 $M5$  : segment  $DA$

Groups of nodes:

$A, B$



## 4.2 Sizes tested and results

Size	Not	Inst	Reference	Tolerance (%)
$S_{IXX} (Pa)$	C	2.0	$-2.0 E6$	0.1
$S_{IYY} (Pa)$	C	1.07	$-8.09 E6$	0.1
		1.16	$-8.01 E6$	0.1
		1.34	$-6.63 E6$	0.1
		1.53	$-5.81 E6$	0.1
$V_I$	C	1.07	0	0.1
		1.16	$2.04 E-3$	0.1
		1.34	$6.42 E-3$	0.1
		1.53	$1.09 E-2$	0.1
$DY (m)$	C	1.07	$-1.05 E-3$	0.1
		1.16	$-2.40 E-3$	0.1
		1.34	$-5.10 E-3$	0.1
		1.53	$-7.95 E-3$	0.1

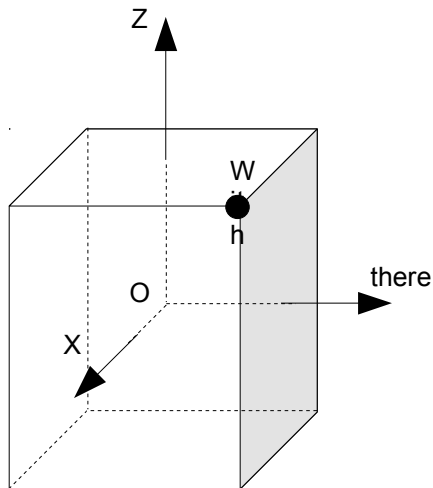


## 5 Modeling C

### 5.1 Characteristics of modeling C

Modeling 3D.

Model of DRUCK\_PRAGER with linear negative work hardening.



Many nodes	20	
Many meshes	7	That is to say:
		QUAD8 6
		HEXA20 1

Geometry of the cube ( $m$ ) :

Center  $O(0.,0.,0.)$   
Side  $C=1 m$

Groups of meshes:

<i>BAS</i> :	surface of the cube pertaining to the plan	$Z = -0.5$
<i>HAUT</i> :	surface of the cube pertaining to the plan	$Z = +0.5$
<i>DROITE</i> :	surface of the cube pertaining to the plan	$Y = +0.5$
<i>GAUCHE</i> :	surface of the cube pertaining to the plan	$Y = -0.5$
<i>DERRIERE</i> :	surface of the cube pertaining to the plan	$X = -0.5$
<i>DEVANT</i> :	surface of the cube pertaining to the plan	$X = +0.5$

## 5.2 Sizes tested and results

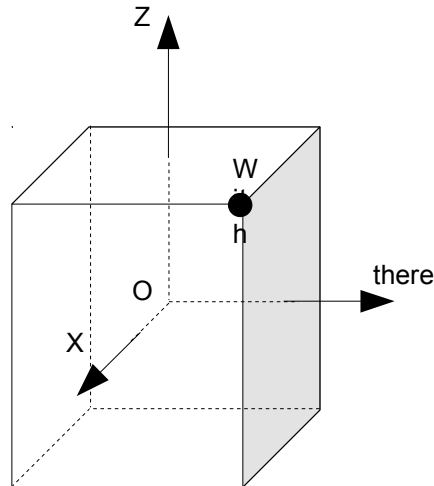
Size	Not	Inst	Reference	Tolerance (%)
<i>SIXX (Pa)</i>	With	2.0	$-2.0 E6$	0.1
<i>SIZZ (Pa)</i>	With	1.07	$-8.09 E6$	0.1
		1.16	$-8.20 E6$	0.1
		1.34	$-6.89 E6$	0.1
		1.53	$-5.80 E6$	0.1
<i>VI</i>	With	1.07	0	0.1
		1.16	$1.99 E-3$	0.1
		1.34	$6.35 E-3$	0.1
		1.53	$1.09 E-2$	0.1
<i>DZ (m)</i>	With	1.07	$-1.05 E-3$	0.1
		1.16	$-2.40 E-3$	0.1
		1.34	$-5.10 E-3$	0.1
		1.53	$-7.95 E-3$	0.1

## 6 Modeling D

### 6.1 Characteristics of modeling D

Modeling 3D.

Model of DRUCK\_PRAGER with parabolic negative work hardening



Many nodes	20	
Many meshes	7	That is to say:
		QUAD8      6
		HEXA20     1

Geometry of the cube ( $m$ ):

Center  $O(0.,0.,0.)$   
Side  $C=1 m$

Groups of meshes:

<i>BAS</i> :	surface of the cube pertaining to the plan	$Z = -0.5$
<i>HAUT</i> :	surface of the cube pertaining to the plan	$Z = +0.5$
<i>DROITE</i> :	surface of the cube pertaining to the plan	$Y = +0.5$
<i>GAUCHE</i> :	surface of the cube pertaining to the plan	$Y = -0.5$
<i>DERRIERE</i> :	surface of the cube pertaining to the plan	$X = -0.5$
<i>DEVANT</i> :	surface of the cube pertaining to the plan	$X = +0.5$

## 6.2 Sizes tested and results

Size	Not	Inst	Reference	Tolerance ( % )
<i>SIXX (Pa)</i>	With	2.0	$-2.0 E6$	0.1
<i>SIZZ (Pa)</i>	With	1.07	$-8.09 E6$	0.1
		1.16	$-8.01 E6$	0.1
		1.34	$-6.63 E6$	0.1
		1.53	$-5.81 E6$	0.1
<i>VI</i>	With	1.07	0	0.1
		1.16	$2.04 E-3$	0.1
		1.34	$6.42 E-3$	0.1
		1.53	$1.09 E-2$	0.1
<i>DZ (m)</i>	With	1.07	$-1.05 E-3$	0.1
		1.16	$-2.40 E-3$	0.1
		1.34	$-5.10 E-3$	0.1
		1.53	$-7.95 E-3$	0.1

## 7 Summary of the results

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The law of behavior of the type `DRUCK_PRAGER` with a linear negative work hardening and a parabolic negative work hardening gives satisfactory results with modelings `AXIS` and `3D`.