Finite elements treating the quasi-incompressibility

Summary:

In certain situations, the mechanical behavior of material imposes that voluminal dilation remains worthless, in other words that the deformation is done with constant volume: isotropic elasticity with Poisson's ratio equal to 0.5 , perfect plastic flows analyzes limit of it…

One proposes here to treat this condition of “incompressibility” or “quasi-incompressibility” by using a valid formulation as well in the compressible case as in the quasi-incompressible case. For that, one uses a variational formulation with 3 fields where the unknown factors are displacement, voluminal deformation and the multiplier of associated Lagrange (which would correspond to the pressure in the incompressible case). Two versions of this formulation are proposed: one for the small deformations, the other valid one in the presence of great deformations. In the situation of a bi-univocal relation between the pressure and swelling, case of the plasticity of Von Mises, it is possible to come to eliminate the unknown factor from swelling. There is then a formulation with two fields displacement/pressure.

After some recalls on the difficulties which raise the resolution of the incompressible problems, one describes the mixed finite elements established (in 3D and 2D, plan and axisymmetric into small and great deformations), and one also presents the broad outlines of integration in *Code_Aster* (modelings `INCO_UP`, `INCO_UPG`, `INCO_UPO`).

This modeling is necessary to practise the limiting analyses and to model elastic behaviors for Poisson's ratios close to 0.5 . It can also be useful in the case of modelings generating of strong plastic deformations and for which traditional modelings can be insufficient and generate oscillations of constraints.

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1 Difficulties related to the treatment of the incompressibility

In certain situations, the mechanical behavior of material imposes that the deformation is done with constant volume. The materials having this property of not-dilatancy are often qualified “incompressible” materials. We will see that these problems pose two types of difficulties. The first difficulty is related to the writing of the condition of incompressibility, second is related to the digital problems which this constraint generates. These difficulties are found when the material is quasi-incompressible.

One reasons here in small disturbances but the problem remains the same one within the framework as of finished transformations.

1.1 Incompressible” and “quasi-incompressible” behaviors “

Within the framework of the mechanics of the continuous mediums, deformation of an isochoric type is characterized by the fact that the gradient of the transformation $\boldsymbol{F}$ is such as $J=\det(\boldsymbol{F})=1$. If one places oneself within the framework as of small disturbances, the preceding condition is reduced to:

$$\operatorname{tr}(\boldsymbol{\varepsilon})=\operatorname{div} u=0$$

The tensor $\boldsymbol{\varepsilon}$ is thus only deviatoric: $\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}^{\boldsymbol{D}}$.

It results from it that in the case of isotropic materials, the invariant $\operatorname{tr}(\boldsymbol{\varepsilon})$ (or $\det(\boldsymbol{F})$) does not intervene in the expression of the density of free energy φ ; thus in the case of incompressible elasticity in HP, one has simply:

$$\varphi(\boldsymbol{\varepsilon})=\mu\, \boldsymbol{\varepsilon}^{\boldsymbol{D}}:\boldsymbol{\varepsilon}^{\boldsymbol{D}}$$

This density makes it possible to express only the deviatoric part of the tensor of the constraints:

$$\boldsymbol{\sigma}^{\boldsymbol{D}}=2\mu\, \boldsymbol{\varepsilon}^{\boldsymbol{D}}$$

In fact, the constraint is defined except for a constant p who is opposite average pressure:

$$\boldsymbol{\sigma}=2\mu\boldsymbol{\varepsilon}^{\boldsymbol{D}}+p\boldsymbol{I} \quad (1.1\text{-}1)$$

Note:

Incompressible isotropic elasticity is of course a borderline case of isotropic elasticity with a Poisson's ratio $\nu=\frac{E}{2\nu}-1$ *tending towards* 0.5 *.*

There is not that the elastic materials whose Poisson's ratio is equal or slightly lower than 0.5 *who utilize the condition of incompressibility. Thus, it intervenes also in the case of plastic rigid material* $\left(\frac{\partial\Phi}{\partial \operatorname{tr}\boldsymbol{\sigma}}=0\right)$. *Indeed, one has in this case:*

$$\dot{\boldsymbol{\varepsilon}}=\dot{\lambda}\frac{\partial\Phi}{\partial\boldsymbol{\sigma}} \;;\; \dot{\lambda}\geq 0 \;;\; \Phi\leq 0 \;;\; \dot{\lambda}\,\Phi=0$$

What leads to the condition of incompressibility $\operatorname{tr}(\dot{\boldsymbol{\varepsilon}})=0$ *.*

In addition, in the case of elastoplasticity, when the plastic deformations become largely higher than the elastic strain, one finds oneself in an almost incompressible case with $\operatorname{tr}(\boldsymbol{\varepsilon})\simeq 0$ *.*

Lastly, the materials checking a relation of behavior of the type NORTON-HOFF (law used for calculations of analysis limits [R7.07.01]) show also the characteristic of incompressibility:

$$\boldsymbol{\varepsilon}(\boldsymbol{v})=\alpha(\sigma_{\mathrm{eq}})^{n-1}\boldsymbol{\sigma}^{\boldsymbol{D}}\text{avec } \; n\geq 1 \;\text{ et }\; \alpha>0$$

where $\sigma_{eq}=\sqrt{\frac{3}{2}\boldsymbol{\sigma}^{D}:\boldsymbol{\sigma}^{D}}$ *is the equivalent constraint of Von Mises.*

1.2 Some possible digital solutions

If one wants to treat the condition of incompressibility exactly, we saw it, the constraint is not completely determined starting from the deformation (cf [éq 1.1-1]). It is thus necessary to use a mixed formulation, i.e. to introduce (at least) another unknown factor of the problem which will make it possible to determine the tensor of the constraints completely. Several alternatives are possible, simplest consisting in imposing the condition of incompressibility using a multiplier of Lagrange which is then the pressure p.

Note:

> *If one chooses a procedure of penalization, one is reduced to the quasi-incompressible case and thus to the difficulties evoked below.*

One can also, in particular in the case of linear elasticity, to choose to make material slightly compressible. In this way, the constraint is entirely defined starting from displacement and the use of a mixed formulation is not essential any more. On the other hand, the resolution of these problems with the classical finite elements in displacement, raises digital difficulties. Indeed, the kinematic constraint that a deformation with constant volume represents is very strong, even too strong if the degrees of freedom of the element are not important enough. Thus, the triangle with 3 nodes can present phenomena of blocking, i.e. the “grid” cannot become deformed. In a less extreme way, most classical elements, in particular linear, behaves in an abnormally rigid way. New elements must thus be used in order to “slacken” the system. These elements can be based on various types of formulation:

- only in displacement
- mixed: displacements/forced, displacements/pressures, deformations/forced, voluminal displacements/pressures/dilations,…

In all the cases, if one does not take there keeps, one can have digital difficulties. Several tracks are used to facilitate the deformation of the elements:

- to use under-integration makes it possible to improve the results but it presents a disadvantage: it can lead to the appearance of parasitic modes or hourglass. To solve this problem, one can is to enrich the matrix by rigidity thanks to matrices of stabilization which come to neutralize the hourglass modes, that is to say to use methods of projection which consist in projecting in a smaller space the condition of incompressibility in order to eliminate the phenomena of blocking. Most known is the method B-Bar [bib1],
- to enrich the element using additional degrees of freedom: one speaks then about methods with increased deformations, modes incompatible,… [bib2]

1.3 Option selected and frameworks of application

We chose here to choose a formulation which covers theincompressible one as well (until the incompressible one) that the compressible one. For that, the term $\mathrm{tr}(\boldsymbol{\varepsilon})$ is treated like an independent variable. With the multiplier of Lagrange associated, that led to a formulation with 3 or 2 fields. A version in great deformations was also developed on the same principle. In this case, the variable independent related to the condition of incompressibility is not any more $\mathrm{tr}(\varepsilon)$ but $J=\det(\boldsymbol{F})$.

The advantage of the formulation with 3 fields compared to the version with 2 fields is that it makes it possible to use in a transparent way all the elastoplastic laws of behavior available in Aster (not need to separate the deviatoric part and the spherical part of the tensor of the constraints). It is thus not limited to the elasticity or the elastoplasticity of Von Mises. On the other hand, it introduces a large number of additional degrees of freedom. Some is the formulation selected, one will not be able to treat the case where the Poisson's ratio is strictly equal to 0.5 , because one uses for the calculation of the elastic constraint the term $\frac{E\nu}{(1+\nu)(1-2\nu)}\mathrm{tr}(\varepsilon)$, of which the denominator is null when $\nu=0.5$.

Consequently, formulations incompressible **must be used** :

- to deal the limiting analysis problems for which one supposes that the flow is done with constant volume [R7.07.01],
- to deal with elastic problems whose Poisson's ratio is higher than 0.45 .

They can also be used:

- to deal with the problems where the plastic deformations are important, which generates oscillations on the level of the constraints (example: in the case of calculations on notched samples). Of course, this formulation being more expensive than the formulation in classical displacement, it is to be held for the case which poses problem and where one is interested in the values of the constraints (one can initially try to use under-integrated quadratic elements which improve already the solution).

2 Mixed variational formulation of the problem

2.1 Formulation within the framework of the small deformations

That is to say a solid Ω subjected to:

- a field of imposed displacement $\boldsymbol{u}=\boldsymbol{u}_0$ on Γ_u
- a stress field imposed $\boldsymbol{t}=\boldsymbol{\sigma}.\boldsymbol{n}=\boldsymbol{t}_0$ on Γ_t
- a voluminal field of effort $\boldsymbol{f}$ on Ω

In the classical case of the finite elements in displacement (modeling 3D or D_PLAN or AXIS in *Code_Aster*), when the problem derives from an energy, the solved problem is the following:

to find $\boldsymbol{u}\in V$ with $\boldsymbol{\sigma}$ checking the relation of behavior, which minimizes the potential energy:

$$\Pi(u)=\frac{1}{2}\int_{\Omega}\boldsymbol{\sigma}:\boldsymbol{\varepsilon}\,d\Omega-\int_{\Omega}\boldsymbol{f}.\boldsymbol{u}\,d\Omega-\int_{\Gamma_t}\boldsymbol{t}.\boldsymbol{u}\,d\Gamma$$

As we explained to [§1], this formulation is not appropriate when one seeks to approach the incompressible solution, i.e. of the condition $\operatorname{div}(\boldsymbol{u})=0$ or $\operatorname{tr}(\boldsymbol{\varepsilon})=0$. To circumvent this difficulty, a solution is to separately treat the spherical part of the tensor of the deformations (the part which poses digital problems) and its deviatoric part. One will thus have:

$$\boldsymbol{\varepsilon}(\boldsymbol{u},g)=\boldsymbol{\varepsilon}^{\boldsymbol{D}}(\boldsymbol{u})+\frac{g}{3}\boldsymbol{I} \text{ where } \boldsymbol{\varepsilon}^{\boldsymbol{D}}(\boldsymbol{u})=\boldsymbol{\varepsilon}(\boldsymbol{u})-\frac{1}{3}tr(\boldsymbol{\varepsilon}(\boldsymbol{u}))\boldsymbol{I} \text{ and } g=tr(\boldsymbol{\varepsilon}(\boldsymbol{u})) \quad (2.1\text{-}1)$$

The preceding problem is thus brought back to the resolution of a problem to 2 variables, $\boldsymbol{u}$ and g, under the constraint $g=tr(\boldsymbol{\varepsilon}(\boldsymbol{u}))$. It can be brought back to the resolution of an unconstrained problem by introducing a multiplier of Lagrange p ; he is written:

to find $\boldsymbol{u}\in V$, p and g solutions of the problem of point-saddles for the Lagrangian one:

$$\mathscr{L}(\boldsymbol{u},p,g)=\int_{\Omega}\left[\boldsymbol{\sigma}:\left(\boldsymbol{\varepsilon}^{\boldsymbol{D}}(\boldsymbol{u})+\frac{g}{3}\boldsymbol{I}\right)+p(\operatorname{div}(\boldsymbol{u})-g)\right]d\Omega-\int_{\Omega}\boldsymbol{f}.\boldsymbol{u}\,d\Omega-\int_{\Gamma_t}\boldsymbol{t}.\boldsymbol{u}\,d\Gamma \quad (2.1\text{-}2)$$

This problem can be solved, by writing the conditions of optimality:

$$\begin{cases}\dfrac{\partial\mathscr{L}}{\partial\boldsymbol{u}}=\displaystyle\int_{\Omega}(\boldsymbol{\sigma}^{\boldsymbol{D}}+p\boldsymbol{I}):\delta\boldsymbol{\varepsilon}\,d\Omega-\int_{\Omega}\boldsymbol{f}.\delta\boldsymbol{u}\,d\Omega-\int_{\Gamma_t}\boldsymbol{t}.\delta\boldsymbol{u}\,d\Gamma=0\\ \dfrac{\partial\mathscr{L}}{\partial p}=\displaystyle\int_{\Omega}(\operatorname{div}(\boldsymbol{u})-g)\delta p\,d\Omega=0\\ \dfrac{\partial\mathscr{L}}{\partial g}=\displaystyle\int_{\Omega}\left(\frac{1}{3}\operatorname{tr}(\boldsymbol{\sigma})-p\right)\delta g\,d\Omega=0\end{cases} \quad (2.1\text{-}3)$$

Note:

> *the first equation corresponds to the equilibrium equation, the second equation translates the kinematic relation binding* g *with* $\boldsymbol{u}$ *, the third equation gives the expression of the*

multiplier of Lagrange p, when the problem does not derive from an energy, one can directly use the system of equations [éq 2.1-3].

If there exists a bi-univocal relation between the pressure and swelling such as for example for an elastoplastic material with a criterion of plasticity of the type von Mises, it is possible to clarify swelling and thus to remove the third equation of the système [éq 2.1-3]. One then obtains the system of two equations to two unknown factors which follows:

$$\begin{cases} \int_{\Omega} (\boldsymbol{\sigma}^{\boldsymbol{D}} + p\,\boldsymbol{I}) : \delta\,\boldsymbol{\varepsilon}\, d\Omega - \int_{\Omega} \boldsymbol{f} . \delta\,\boldsymbol{u}\, d\Omega - \int_{\Gamma_t} \boldsymbol{t} . \delta\,\boldsymbol{u}\, d\Gamma = 0 \\ \int_{\Omega} (\mathrm{div}(\boldsymbol{u}) - \frac{p}{\kappa})\,\delta\, p\, d\Omega = 0 \end{cases} \quad (2.1\text{-}4)$$

Where κ is the module of compressibility.

2.2 Formulation in great deformations

As for the small deformations, it is possible to propose a variational formulation valid for the great deformations. The principle is identical, but one is based in this case on the decomposition of the tensor gradient of the transformation $\boldsymbol{F} = \boldsymbol{I} + \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}}$ proposed by Flory [bib3]:

$$\boldsymbol{F} = \boldsymbol{F}^s\, \bar{\boldsymbol{F}} \text{ with } \boldsymbol{F}^s = J^{1/3}\, \boldsymbol{I} \quad \text{et} \quad \bar{\boldsymbol{F}} = J^{-1/3}\, \boldsymbol{F} \quad \text{et} \quad J = \det(\boldsymbol{F})$$

The idea there still, is to enrich kinematics by the means of a variable by swelling g, a priori independent of displacements, and slightly related to the variation of volume by a weak relation:

$$B(J) \underset{faible}{\approx} B \circ A(g) = B(A(g))$$

Several relations were tested:

$$\begin{cases} J = 1 + g \\ J^2 = 1 + g \\ \ln(J) = g \\ J = \exp(g) \end{cases}$$

For certain simulations, small differences were observed. For the elements `INCO_UPG` with deformation `SIMO_MIEHE`, it is finally the linear relation which was established in the code: therefore, in version 9, $B(J) = J$ and $A(g) = 1 + g$. For the elements `INCO_UPG` with deformation `GDEF_LOG`, it is the relation in logarithm which was retained.

Nevertheless, this choice not being inevitably final, one proposes to write the problem in the case general. One thus introduces a gradient of deformation enriched:

$$\tilde{\boldsymbol{F}} = \left(\frac{A(g)}{J}\right)^{\frac{1}{3}} \boldsymbol{F} \quad (2.2\text{-}1)$$

The weak formulation of the problem is based on the research of the point saddles the Lagrangian one $\mathscr{L}$, in which the multiplier of Lagrange p and a third field g, independent of both others, ensuring in a weak way that the relation enters J and g is checked:

$$\mathscr{L}(\boldsymbol{u}, g, p) = \int_{\Omega_0} \psi(\tilde{\boldsymbol{F}})\, d\Omega_0 - W_{ext}(\boldsymbol{u}) + \int_{\Omega_0} p\left[B(J) - B \circ A(g)\right] d\Omega_0 \quad (2.2\text{-}2)$$

where W_{ext} represent the potential of the external efforts and $\psi(\tilde{F})$ the deformation energy.
This problem can be solved as in small deformations by writing the conditions of optimality. The variation of Lagrangian is written:

$$\delta\mathscr{L}=\int_{\Omega_0}\left[\boldsymbol{P}:\delta\tilde{\boldsymbol{F}}+p\left(\frac{\partial B(J)}{\partial J}J\frac{\delta J}{J}-\frac{\partial B\circ A(g)}{\partial g}\delta g\right)+\delta p\left(B(J)-B\circ A(g)\right)\right]d\Omega_0-\delta W_{ext}(\boldsymbol{u}) \quad (2.2\text{-}3)$$

with $\boldsymbol{P}$ the first tensor of the constraints of Piola-Kirchhoff:

By injecting the variation of the transformation enriched and the expression by the constraint of Kirchhoff $\boldsymbol{\tau}=\boldsymbol{P}\,\tilde{\boldsymbol{F}}^{T}$ [1], one obtains the following form for the variation of the Lagrangian one:

$$\begin{aligned}\delta\mathscr{L} &= \int_{\Omega_0}\left(\boldsymbol{\tau}^{d}+p\frac{\partial B(J)}{\partial J}J\boldsymbol{I}\right):\delta L\,d\Omega_0\\ &+ \int_{\Omega_0}\left(\frac{\operatorname{tr}(\boldsymbol{\tau})}{3}\frac{\partial A(g)/\partial g}{A(g)}-p\frac{\partial B\circ A(g)}{\partial g}\right)\delta g\,d\Omega_0+\int_{\Omega_0}\left(B(J)-B\circ A(g)\right)\delta p\,d\Omega_0\\ &- \delta W_{ext}(\boldsymbol{u})\end{aligned} \quad (2.2\text{-}4)$$

where one introduced the gradient eulérien displacement ($\boldsymbol{x}$ represent the vector position at the end of the increment):

$$\delta L=\frac{\partial\delta\boldsymbol{u}}{\partial\boldsymbol{x}}=\delta\boldsymbol{F}.\boldsymbol{F}^{-1}$$

In short, the system to be solved is the following:

$$\begin{aligned}\frac{\partial\mathscr{L}}{\partial\boldsymbol{u}}&=\int_{\Omega_0}\delta L:\left(\boldsymbol{\tau}^{d}+p\frac{\partial B(J)}{\partial J}J\boldsymbol{I}\right)d\Omega_0-\delta W_{ext}=0\\ \frac{\partial\mathscr{L}}{\partial g}&=\int_{\Omega_0}\delta g\left(\frac{\operatorname{tr}(\tau)}{3}\frac{\partial A(g)/\partial g}{A(g)}-p\frac{\partial B\circ A(g)}{\partial g}\right)d\Omega_0=0\\ \frac{\partial\mathscr{L}}{\partial p}&=\int_{\Omega_0}\delta p\left(B(J)-B\circ A(g)\right)d\Omega_0=0\end{aligned} \quad (2.2\text{-}5)$$

Notice *:*

The constraint of Kirchhoff resulting from the law of behavior, is thus written $\boldsymbol{\tau}=\boldsymbol{\tau}^{D}+p\frac{\partial B(J)}{\partial J}J\boldsymbol{I}$

1 It is a choice and one could just as easily have transported $\boldsymbol{P}$ by the compatible deformation $\boldsymbol{F}$ instead of the enriched deformation $\tilde{\boldsymbol{F}}$. But the latter has the advantage of preserving the symmetry of the formulation in the elastic case. Moreover, it lends itself better to the architecture of a computer code in which the integration of the laws of behavior is well differentiated from calculation from the terms specific to the finite elements (the routines of the behavior do not need to know the existence of two measurements of deformation) but only the enriched deformation).

With regard to obtaining the tangent matrix, she of course asks a little more calculation than in small deformations, and has the characteristic not to be symmetrical in the case general. She is in the code in the following form:

$$\boldsymbol{K}=\begin{bmatrix} K_{UU} & K_{UG} & K_{UP} \\ K_{GU} & K_{GG} & K_{GP} \\ K_{PU} & K_{PG} & K_{PP} \end{bmatrix}$$

Calculations are not here detailed. The reader will be able to refer to the reading of [8].

Notice *:*

This formulation makes it possible to regularize with lower costs, the models of ductile damage where the variable of damage is directly connected to the variation of volume.
Indeed, to control the localization of the damage and the deformation, the idea is to penalize the strong gradients of damage. As in this formulation with 3 fields, local swelling is treated like a nodal variable, its gradient is easily accessible numerically (subject to an at least linear interpolation).

In the spirit of the formulations with second gradient of displacement ([9], [10]), it is enriched by a quadratic term in gradient by swelling. The variation of Lagrangian is written then:

$$\begin{aligned}\delta\mathscr{L} &= \int_{\Omega_0}\left(\boldsymbol{\tau}^d+p\frac{\partial B(J)}{\partial J}J\boldsymbol{I}\right):\delta L+\left(\frac{\mathrm{tr}(\boldsymbol{\tau})}{3}\frac{\partial A(g)/\partial g}{A(g)}-p\frac{\partial B\circ A(g)}{\partial g}\right)\delta g\, d\Omega_0 \\ &+\int_{\Omega_0}\left[\left(B(J)-B\circ A(g)\right)\delta p+c\nabla g\cdot\nabla\delta g\right]d\Omega_0-\delta W_{ext}(\boldsymbol{u})\end{aligned} \quad (2.2\text{-}6)$$

c is a parameter to be determined and homogeneous with a force. This parameter introduces to some extent an internal length of coupling between the points materials. The term added here is isotropic: it is considered that the length interns to introduce is identical in all the directions. For the application to the ductile damage of steels, this assumption seems completely admissible. This formulation is usable for the model of Rousselier, [R5.03.07], with the help of the definition of the keyword `C_CARA` under the operand `NON_LOCAL` of `DEFI_MATERIAU` (see test Code_Aster ssnp122a)

Obtaining the formulation with two fields in great deformations follows the same principle as With regard to obtaining the tangent matrix, she of course asks a little more calculation than in small deformations, and has the characteristic not to be symmetrical in the case general. It is in the code in the following form:

3 Discretization by mixed finite elements

3.1 Choice of the discretization

When a mixed formulation is used, it is necessary to discretize at the same time the space of displacements, the multiplier of Lagrange p and "swelling" g. The experience gained on the mixed elements, in particular 2 fields for the incompressible elements, makes it possible to know that the discretization of these fields cannot be unspecified, under penalty of obtaining phenomena of oscillations (in particular on the level of the pressures) or phenomena of blocking (elements not being able to become deformed or too rigid). Thus it is necessary to have a number of points of sufficiently important Gauss of pressure to check the condition of incompressibility almost everywhere and a number of points of sufficiently low Gauss of pressure to have more degrees of freedom to calculate than constraints to be checked. One of the requirements to get satisfactory results is the checking by the finite element considered condition LBB (LADYJENSKAIA, BREZZI, BABUSKA). One can find in [bib5] and [bib6] of the examples of elements satisfying condition LBB.

Here the problem is a little different when the formulation 3 fields is retained.

In the actual position, the discretizations used are not the same ones in the version HP and the version great deformations.

3.1.1 Small deformations

For the small deformations, we took as a starting point the classical uses of the mixed formulations (e.g. [bib7]), by using an element of the type $P2/P1/P1$ for the formulation with 3 fields. In other words, displacement is quadratic, the pressure and swelling is all the two linear ones. The finite elements used for the formulation with 3 fields are thus the following:

in 2D:	u	triangle with 6 nodes	quadrilateral with 8 nodes	
	p, g	triangle with 3 nodes	quadrilateral with 4 nodes	
in 3D:	u	tetrahedron with 10 nodes	hexahedron with 20 nodes	pentahedron with 15 nodes
	p, g	tetrahedron with 4 nodes	hexahedron with 8 nodes	pentahedron with 6 nodes

For each type of element, one uses only one family of points of Gauss:

- 3 points for the triangles
- 4 points for the quadrilaterals
- 4 points for the tetrahedrons
- 8 points for the hexahedrons
- 21 points for the pentahedrons

For the formulation with two fields, an element of the type $P2/P1$ was introduced. Displacement thus has a quadratic interpolation while the pressure is interpolated linearly. Into the case of the use of a discretization in triangles or linear tetrahedrons, two methods of stabilization were introduced. The first corresponds to the stabilized finite element $P1+/P1$. $+$ corresponds to the introduction of an additional degree of freedom to the center of the element into the interpolation of displacements. This additional degree is commonly called “bubble”. This method of stabilization functions only on elements simplexes (triangle in 2D and tetrahedron in 3D). It has the advantage of using very few degrees of freedom. The second method of stabilization corresponds to the method Orthogonal Sub-Grid Scale (OSGS) [bib11]. The advantage of this method is to function for all topologies of elements. Its principal disadvantage is to introduce a third unknown factor (and thus additional degrees of freedom) corresponding to the field of pressure project π on orthogonal space with the fields of displacement.

The finite elements used for the formulation with 2 fields are thus the following:

Interpolation		P1+/P1	P1/P1 OSGS	P2/P1	P1/P1 OSGS	P2/P1	P1/P1 OSGS	P2/P1
in 2D:	u	triangle with 3 nodes	triangle with 3 nodes	triangle with 6 nodes	quadrilateral with 4 nodes	quadrilateral with 8 nodes		
	p	triangle with 3 nodes	triangle with 3 nodes	triangle with 3 nodes	quadrilateral with 4 nodes	quadrilateral with 4 nodes		
	π		triangle with 3 nodes		quadrilateral with 4 nodes			
in	u	tetrahedro	tetrahedron	tetrahedron	cubic with 8	cubic with	pentahedro	pentahedro

		n with 4 nodes + bubble	with 4 nodes	with 10 nodes	nodes	20 nodes	n with 6 nodes	n with 15 nodes
3D:	p	tetrahedron with 4 nodes	tetrahedron with 4 nodes	tetrahedron with 4 nodes	cubic with 8 nodes	cubic with 8 nodes	pentahedron with 6 nodes	pentahedron with 6 nodes
	π		tetrahedron with 4 nodes		cubic with 8 nodes		pentahedron with 6 nodes	

The families of points of Gauss used are the same ones as those of the formulation with 3 fields. One will note that for the elements $P1+/P1$, one uses one point of Gauss for integration.

3.1.2 Great deformations

From version 11, the choice of the interpolations in great deformations is identical to that of the small deformations. The elements are of the type $P2/P1/P1$ for the formulations with 3 fields and $P2/P1$ for the formulation with 2 fields. The finite elements used for the formulation with 3 fields are thus the following:

in 2D:	u	triangle with 6 nodes	quadrilateral with 8 nodes	
	p, g	triangle with 3 nodes	quadrilateral with 4 nodes	
in 3D:	u	tetrahedron with 10 nodes	hexahedron with 20 nodes	pentahedron with 15 nodes
	p, g	tetrahedron with 4 nodes	hexahedron with 8 nodes	pentahedron with 6 nodes

They are the same families of points of Gauss as those of the small deformations which were used.

3.2 Writing of the discrete problem

One approaches here initially, the writing of the discrete problem within the framework of the formulation with 3 fields. That is to say $\boldsymbol{u}^e$, p^e and g^e, vectors of the elementary nodal unknown factors (respectively displacement, pressure and swelling). If $\boldsymbol{N}^{\boldsymbol{u}}$, $\boldsymbol{N}^{\boldsymbol{p}}$ and $\boldsymbol{N}^{\boldsymbol{g}}$ are the functions of forms (respectively interpolations of displacement, pressure and swelling) associated with the finite element considered:

$$\boldsymbol{u}=\boldsymbol{N}^{\boldsymbol{u}}\boldsymbol{u}^{\boldsymbol{e}}$$
$$p=\boldsymbol{N}^{\boldsymbol{p}}p^e$$
$$g=\boldsymbol{N}^{\boldsymbol{g}}g^e$$

3.2.1 Writing in small deformations

$\boldsymbol{B}$ is the classical matrix of derivation allowing to pass from $\boldsymbol{u}^e$ with $\boldsymbol{\varepsilon}$:

$$\boldsymbol{\varepsilon}=\boldsymbol{B}\boldsymbol{u}^{\boldsymbol{e}}$$

In the formulation, one distinguishes e_{dev} and e_{dil} , which leads us to define the operators B_{dev} and B_{dil} such as: $\varepsilon^D=B_{dev}U^e$ and $\frac{\operatorname{tr}\,\varepsilon}{3}=B_{dil}U^e$

The discretized form of the equations of the problem with 3 fields [éq 2.1-3] is written:

$$\boldsymbol{F}_u=\int_\Omega \boldsymbol{B}^T(\boldsymbol{\sigma}^D+p\boldsymbol{I})\,d\Omega=\boldsymbol{F}_{ext}$$
$$\boldsymbol{F}_p=\int_\Omega (\boldsymbol{N}^p)^T(\boldsymbol{B}_{dil}\boldsymbol{u}-g)\,d\Omega=0$$
$$\boldsymbol{F}_g=\int_\Omega (\boldsymbol{N}^g)^T\left(\frac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})-p\right)d\Omega=0$$

The tangent matrix of the problem is symmetrical and is based on the following terms:

$$\boldsymbol{K}_{uu}=\frac{\partial\boldsymbol{F}_u}{\partial\boldsymbol{u}^e}=\int_\Omega \boldsymbol{B}_{dev}^T\boldsymbol{D}\,\boldsymbol{B}_{dev}\,d\Omega$$
$$\boldsymbol{K}_{up}=\frac{\partial\boldsymbol{F}_u}{\partial p^e}=\int_\Omega \boldsymbol{B}_{dil}^T\boldsymbol{N}^p\,d\Omega$$
$$\boldsymbol{K}_{ug}=\frac{\partial\boldsymbol{F}_u}{\partial g^e}=\frac{1}{3}\int_\Omega \mathrm{tr}(\boldsymbol{B}_{dev}^T\boldsymbol{D})\boldsymbol{N}^g\,d\Omega$$
$$\boldsymbol{K}_{pp}=\frac{\partial\boldsymbol{F}_p}{\partial p^e}=\boldsymbol{0}$$
$$\boldsymbol{K}_{pg}=\frac{\partial\boldsymbol{F}_p}{\partial g^e}=-\int_\Omega (\boldsymbol{N}^p)^T\boldsymbol{N}^g\,d\Omega$$
$$\boldsymbol{K}_{gg}=\frac{\partial\boldsymbol{F}_g}{\partial g^e}=\frac{1}{9}\int_\Omega (\boldsymbol{N}^g)^T\mathrm{tr}(\boldsymbol{D})\boldsymbol{N}^g\,d\Omega$$

With regard to the formulation with 2 fields, she results easily from the preceding one. The forms discretized of the equations give us:

$$\boldsymbol{F}_u=\int_\Omega \boldsymbol{B}^T(\boldsymbol{\sigma}^D+p\boldsymbol{I})\,d\Omega=\boldsymbol{F}_{ext}$$
$$\boldsymbol{F}_p=\int_\Omega (\boldsymbol{N}^p)^T\left(\boldsymbol{B}_{dil}\boldsymbol{u}-\frac{\boldsymbol{N}^p}{\kappa}\right)d\Omega=0$$

The tangent matrix of the problem is symmetrical and is based on the following terms:

$$\boldsymbol{K}_{uu}=\frac{\partial\boldsymbol{F}_u}{\partial\boldsymbol{u}^e}=\int_\Omega \boldsymbol{B}_{dev}^T\boldsymbol{D}\,\boldsymbol{B}_{dev}\,d\Omega$$
$$\boldsymbol{K}_{up}=\frac{\partial\boldsymbol{F}_u}{\partial p^e}=\int_\Omega \boldsymbol{B}_{dil}^T\boldsymbol{N}^p\,d\Omega$$
$$\boldsymbol{K}_{pp}=\frac{\partial\boldsymbol{F}_p}{\partial p^e}=-\frac{1}{\kappa}\int_\Omega (\boldsymbol{N}^p)^T\boldsymbol{N}^p\,d\Omega$$

3.2.2 Writing in great transformations

The writing being a little tiresome, the reader will be able to refer to the reading of [8] to have more information.

4 Integration in *Code_Aster* incompressible finite elements

4.1 General presentation of the incompressible element in small deformations

The finite elements are integrated in *Code_Aster* in 2D plane deformations, in 2D axisymmetric and 3D. 3 modelings are accessible by using the following options for `AFFE_MODELE` :

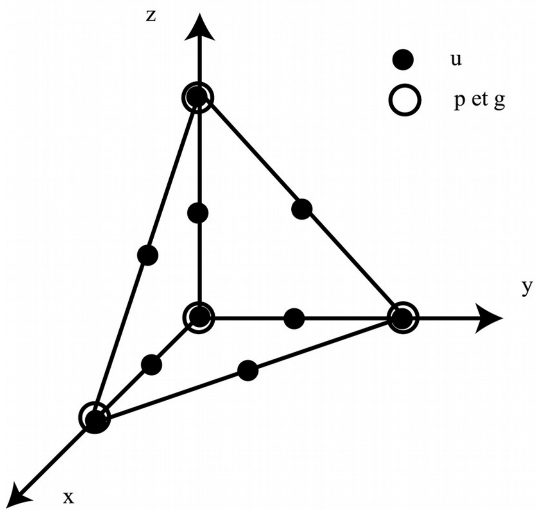
- `'3D_INCO_UPG'`, `'3D_INCO_UP'` or `'3D_INCO_UPO'` for the 3D and respectively for the formulation with 3, 2 fields and 2 fields stabilized with method OSGS,
- `'D_PLAN_INCO_UPG'`, `'D_PLAN_INCO_UP'` or `'D_PLAN_INCO_UPO'` for the 2D in plane deformations and respectively for the formulation with 3, 2 fields and 2 fields stabilized with method OSGS,
- `'AXIS_INCO_UPG'`, `'AXIS_INCO_UP'` or `'AXIS_INCO_UPO'` for the axisymmetric 2D and respectively for the formulation with 3, 2 fields and 2 fields stabilized with method OSGS.

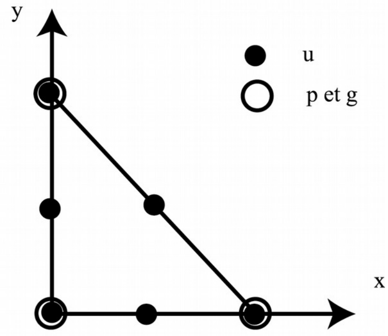
In the catalogue of the elements, the incompressible elements can apply to the meshs:

Meshs	Formulation	Many nodes in displacements	Many nodes in pressure or swelling	Many nodes of gradient of pressure project
TRIA3	2 fields	3	3	
TRIA3	2 fields OSGS	3	3	3
TRIA6	2 and 3 fields	6	3	
QUAD4	2 fields OSGS	4	4	4
QUAD8	2 and 3 fields	8	4	
HEXA20	2 and 3 fields	20	8	
TETRA4	2 fields	4	4	
TETRA4	2 fields OSGS	4	4	4
TETRA10	2 and 3 fields	10	4	
PENTA6	2 fields OSGS	6	6	6
PENTA15	2 and 3 fields	15	6	

In the routines of initializations of the incompressible elements, one defines:
- 1 only family of points of GAUSS (cf §3.1),
- 2 families of functions of forms respectively associated with displacements (functions of forms of degree 2) and under the terms with pressure and swelling (of degree 1) if one is in formulation 3 fields.

Let us take as example the tetrahedral element with 10 nodes: the degrees of freedom in displacement are carried by all the nodes, on the other hand, only the 4 nodes tops have the degrees of freedom p and g.





Accessible components for the field DEPL are thus

- displacements: DX, DY and DZ in 3D with all the nodes,
- pressure: NEAR for the nodes tops,
- swelling (formulation with 3 fields): GONF for the nodes tops.

4.2 General presentation of the incompressible element in great deformations

The finite elements are integrated in *Code_Aster* in 2D plane deformations, in 2D axisymmetric and 3D. 3 modelings being based on a formulation with 3 fields are accessible by using the following options for AFFE_MODELE :

- '3D_INCO_UPG' for the 3D,
- 'D_PLAN_INCO_UPG' for the 2D in plane deformations,
- 'AXIS_INCO_UPG' for the axisymmetric 2D.

3 modelings using the formalism of great deformations of GDEF_LOG and being based on a formulation with 2 fields are accessible by using the following options for AFFE_MODELE :

- '3D_INCO_UP' for the 3D,
- 'D_PLAN_INCO_UP' for the 2D in plane deformations,
- 'AXIS_INCO_UP' for the axisymmetric 2D.

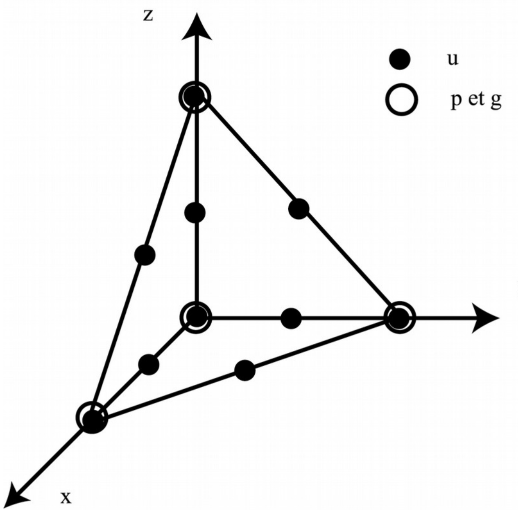
In the catalogue of the elements, the incompressible elements can apply to the meshs:

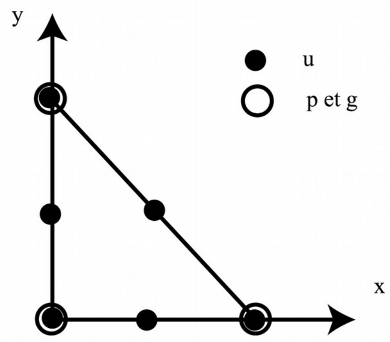
Meshs	Many nodes in displacements	Many nodes in pressure (and swelling)
TRIA6	6	3
QUAD8	8	4
HEXA20	20	8
TETRA10	10	4
PENTA15	15	6

In the routines of initialization of the incompressible elements, one defines:

- 1 only family of points of GAUSS (cf §3.1),
- 2 families of functions of forms respectively associated with displacements and the pressure (functions of forms of degree 2) and under the terms with swelling (functions of forms of degree 1).

Let us take as example the tetrahedral element with 10 nodes: the degrees of freedom in displacement and pressure are carried by all the nodes, on the other hand, only the 4 nodes tops have the degrees of freedom of swelling.





Accessible components for the field DEPL are thus

- displacements: DX, DY and DZ in 3D with all the nodes,
- pressure: NEAR for the nodes top,
- swelling: GONF for the nodes top.

4.3 Use of modeling

Modelings INCO_UP, INCO_UPG and INCO_UPO can be used with the non-linear operators of mechanics STAT_NON_LINE and DYNA_NON_LINE. It is also possible to use the linear operator of mechanics MECA_STATIQUE however this is strongly disadvised because the got results can be of poor quality. The version small deformations is accessible while using DEFORMATION=' PETIT' under BEHAVIOR, the version great deformations while using DEFORMATION=' SIMO_MIEHE' or DEFORMATION=' GDEF_LOG'. The relations of behavior usable are those available respectively in small deformations and great deformations SIMO_MIEHE or GDEF_LOG for modeling INCO_UPG. Modelings INCO_UP and INCO_UPO are currently limited to the relations ELAS and VMIS_ISOT_XXX.

It is thus not possible to use modelings with the orders:

- CALC_MATR_ELEM/CALC_VECT_ELEM/ASSE_MATRICE/ASSE_VECTEUR/RESOUDRE

Being given the shape of the tangent matrix for the formulations to 3 fields (INCO_UPG), it is often necessary to use solvor MUMPS to solve the linear systems.

It is advised to use the convergence criteria by constraint of reference RESI_REFE_RELA except for the elements INCO_UPO because the option is not available.

4.4 Formulation of the elementary terms of the second member

The loads can be gravity, of the surface forces distributed, the pressures. The elementary terms are calculated in a classical way for the degrees of freedom of displacement and one affects the zero value for the degrees of freedom of pressure and swelling.

4.5 Calculation of the strains and the stresses

In this formulation, it is advisable to distinguish the stress field resulting from the law from behavior $\boldsymbol{\sigma}_{ldc}$, stress field which checks balance and which is defined by the relation $\boldsymbol{\sigma}=\boldsymbol{\sigma}_{ldc}^{\boldsymbol{D}}+p\boldsymbol{I}$.

It is the latter field which is stored in SIEF_ELGA as well as the relation binding the multiplier p and $\boldsymbol{\sigma}_{ldc}$. In small deformations, components of SIEF_ELGA are:

- SIXX, SIYY, SIZZ, SIXY in 2D like SIXZ and SIYZ in 3D: components of the tensor $\boldsymbol{\sigma}=\boldsymbol{\sigma}_{ldc}^{\boldsymbol{D}}+p\boldsymbol{I}$,
- SIP who is equal to $\left(\frac{1}{3}\mathrm{tr}(\boldsymbol{\sigma}_{ldc})-p\right)$,

In great deformations, components of SIEF_ELGA are:

- SIXX, SIYY, SIZZ, SIXY in 2D like SIXZ and SIYZ in 3D: components of the tensor $\boldsymbol{\sigma}=\left(\frac{\boldsymbol{\tau}^d}{J}+p\frac{\partial B(J)}{\partial J}\boldsymbol{I}\right)$,
- SIP who is equal to $\left(\frac{\mathrm{tr}(\boldsymbol{\tau})}{3\mathrm{J}}-p\frac{\partial B(J)}{\partial J}\right)$,

It is also possible to recompute EPSI_ELGA, which is the field of deformation to the classical direction.

One can also carry out a calculation of load limits with POST_ELEM.

5 Validation

5.1 Incompressible elastic case

Test SSLV130 (cf [V3.04.130]) makes it possible to check the validity of modeling in the case of an incompressible elastic cylinder subjected to an internal pressure. Its equivalent in great deformations also exist: test SSNV112 (cf [V6.04.112]).

5.2 Elastoplastic case

The goal of this example is to illustrate the contribution of incompressible modeling if the plastic deformations are important compared to the elastic strain. One studies for that a notched sample into axisymmetric, subjected to an imposed displacement. The geometry and the loading are represented on the figure below. The grid consists of 548 TRI6.

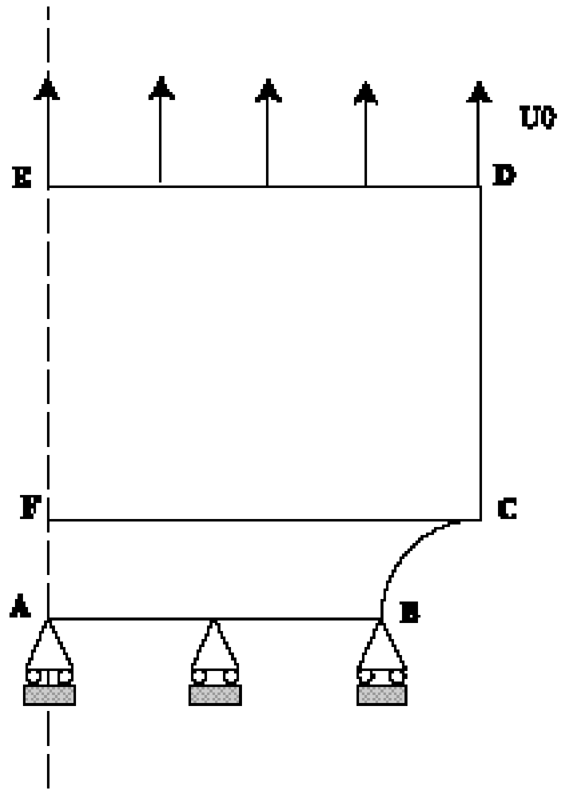


Figure 5.2-a: Geometry and boundary conditions

The behavior of material is of elastoplastic type with linear isotropic work hardening (`VMIS_ISOT_LINE`). The parameters are the following:

- $E=200\,000\,MPa$
- $\nu=0.3$
- $\sigma_y=200\,MPa$
- $E_T=1000\,MPa$

On the figure [Figure 5.2-b], one compares the constraint σ_{yy} obtained on the way FC (cf [Figure 5.2-a]) with classical modeling AXIS and modeling AXIS_INCO_UPG.

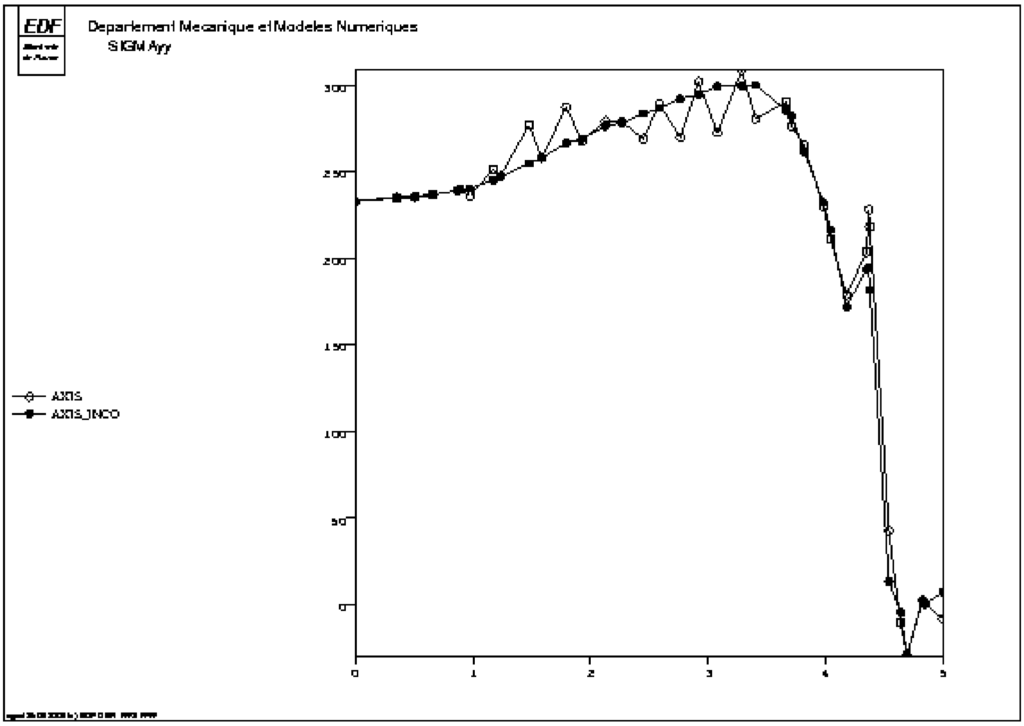


Figure 5.2-b: σ_{yy} along the line FC

It is seen very clearly that the solution obtained with the incompressible formulation makes it possible to be freed from the parasitic oscillations.

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7 Description of the versions

Version Aster	Author (S) Organization (S)	Description of the modifications
6.2	S.MICHEL-PONNELLE, E.LORENTZ EDF R & D AMA	Initial version
6.4	S.MICHEL-PONNELLE, E.LORENTZ EDF R & D AMA	Light update for version 6.4
7.2	S.MICHEL-PONNELLE, E.LORENTZ EDF R & D AMA	Addition of the formulation in great transformations
7.4	S.MICHEL-PONNELLE, E.LORENTZ EDF R & D AMA	Addition of the pentahedral elements
9.4	S.MICHEL-PONNELLE EDF R & D AMA E.LORENTZ EDF R & D SINETICS	New quasi-incompressible elements formulation in great transformations
10.3	S.FAYOLLE	New formulation with 2 fields in small

	EDF R & D AMA	deformation $U-P$
11.2	S.FAYOLLE EDF R & D AMA	New formulations OSGS and INCO_LOG
11.3	S.FAYOLLE EDF R & D AMA	New formulations INCO_LUP
12.1	S.FAYOLLE EDF R & D AMA	Renaming of the formulations (cards 16002.21945 and 21921)