Elements MEMBRANE and GRILLE_MEMBRANE

Summary:

This document describes the formulation and the establishment in *Code_Aster* elements MEMBRANE and GRILLE_MEMBRANE. Elements MEMBRANE allow to model a linear behavior of unspecified membrane or a nonlinear behavior of isotropic membranes. Elements GRILLE_MEMBRANE are finite elements more specifically dedicated to the representation of steel reinforcements in a solid mass (for applications of standard Génie Civil reinforced concrete). The main features of these elements are the following ones:

- elements of membrane, without rigidity of torsion;
- pas de degrees of freedom of rotation, but on the other hand not of possibility of offsetting;
- geometrical support surface (triangle, quadrangle; linear or quadratic, and biquadratic only for MEMBRANE);

Elements GRILLE_MEMBRANE allow to model a non-linear behavior of the bars of reinforcement.

1 Introduction

Modeling MEMBRANE allows to represent the mechanical behavior of a possibly anisotropic membrane into linear and only isotropic into nonlinear. It makes it possible to model elements of structure whose rigidity of inflection is negligible.

Elements GRILLE_MEMBRANE allow to represent the possibly non-linear behavior of bars of reinforcement in a reinforced concrete structure. The principal constraint is that the bars of reinforcement must be periodically distributed on a surface, and directed all in the same direction. Let us specify however that bars of cross reinforcements can be modelled by superposition of two modelings GRILLE_MEMBRANE (see further).

Elements of the type GRILLE_MEMBRANE come to supplement the possibilities of modeling of reinforcement in *Code_Aster*, in complement of modeling GRILLE_EXCENTRE. One presents below the differences between modelings GRILLE_MEMBRANE and GRILLE_EXCENTRE.

It is pointed out that modeling GRILLE_EXCENTRE is based on a kinematics of hull DKT with only one sleep in the thickness [R3.07.03]. This base DKT implies the presence of degrees of freedom of rotation to the nodes of the elements GRILLE_EXCENTRE: if it allows the concept of offsetting, it is useless when one does not need offsetting (in this case it weighs down the useless model of way, because not only the degrees of freedom of rotation lengthen the vector of unknown factors, but it is necessary moreover block one considerable number of these degrees of freedom by double multiplier of Lagrange). Modeling GRILLE_MEMBRANE is a modeling based on a “surface” kinematics, it does not require other degrees of freedom that usual displacements (on the other hand, obviously, this modeling does not make it possible to use the concept of offsetting).

Modeling GRILLE_EXCENTRE, based on DKT, requires geometric standards of support of type triangle or linear quadrangle; modeling GRILLE_MEMBRANE is developed starting from the geometrical supports surface triangle or quadrangle, linear or quadratic.

For the two types of modeling, on the other hand, only a direction of reinforcement is available by finite elements. That makes it possible to model any type of reinforcement to several directions, by superimposing an element by direction; the cost of calculation generated by these duplications is weak: no the duplication of the degrees of freedom (thus constant cost of inversion of matrix), duplication of the elementary computation functions (but they remain simple, of reduced number in 3D – elements of surfaces against elements of volume – and elementary calculations for structures with great numbers of degrees of freedom are of weak cost compared to cost of inversion).

2 Formulation linear elements of MEMBRANE

For an element of membrane, the deformation energy can be put in the form:

$$\Phi=\frac{1}{2}\int \boldsymbol{\sigma}:\boldsymbol{\varepsilon}\, ds \tag{1}$$

with $\boldsymbol{\sigma}$ the membrane constraint and $\boldsymbol{\varepsilon}$ membrane deformation.

The only difficulty is to obtain an expression of the type $\boldsymbol{\varepsilon}=BU$ where one notes U nodal values of displacement.

For that, a little differential geometry should be used. EN noting a the natural base (nonorthogonal, only it third vector, normal on the surface, is normalized) plan of the reinforcement and g metric the contravariante associated with this base (cf [R3.07.04] .pour more details). One leaves the expression of the derivative contravariante:

$$\nabla u=\frac{\partial u^i}{\partial \xi_j}=u^i|_j\, a_i \otimes a^j \tag{2}$$

with $\{\xi_j\}$ an acceptable parameter setting of surface and $a_i=\frac{\partial x}{\partial \xi_i}$. By using the metric tensor g , one has then:

$$\nabla u=\frac{\partial u^i}{\partial \xi_j}=u^i|_j\, g^{jk}\, a_i \otimes a_k \tag{3}$$

One then defines the direction of reinforcement by the normalized vector e_1 that one supplements, for the facility of the talk in an orthonormal base $\{e_i\}$. ON calls R the operator of passage between this base and the natural base such as:

$$a_i=R_i^p\, e_p \tag{4}$$

One notes in Greek the indices taking only the values in $\{1,2\}$, and one obtains:

$$\varepsilon_{\alpha\beta}=(\nabla u)_{\alpha\beta}=\left(\frac{\partial u}{\partial \xi^j}\cdot a^i\right) R_i^{\alpha} R_k^{\beta} g^{jk} \tag{5}$$

By definition of R : $R_3^1=0$ and by definition of g : $g^{13}=g^{23}=0$. One thus obtains:

$$\varepsilon_{\alpha\beta}=(\nabla u)_{\alpha\beta}=\left(\frac{\partial u}{\partial \xi^{\delta}}\cdot a^{\gamma}\right) R_{\gamma}^{\alpha} R_{\theta}^{\beta} g^{\delta\theta} \tag{6}$$

If one notes now $\hat{B}$ the derivative of the functions of form at the point of Gauss considered, it comes:

$$\varepsilon_{\alpha\beta}=R_{\gamma}^{\alpha} R_{\theta}^{\beta} g^{\delta\theta} \hat{B}_{\delta,n} (a^{\gamma})_i U_{i,n} \tag{7}$$

With n , the index of the node. D' where it B sought:

$$B_{i,n}=R_{\gamma}^{\alpha} R_{\theta}^{\beta} g^{\delta\theta} \hat{B}_{\delta,n} (a^{\gamma})_i \tag{8}$$

From B , there are then all the classical expressions of the deformation:

$$\boldsymbol{\varepsilon}=BU \tag{9}$$

nodal forces:

$$F=\int B^T \sigma \tag{10}$$

ET of the tangent matrix:

$$K=\int B^T \frac{\partial \sigma}{\partial \varepsilon} B \tag{11}$$

3 Nonlinear formulation of the elements of MEMBRANE

All the theoretical part as well as the development of the finite element of membrane are founded on the work of Anh the VAN, published in his work *Hulls and membranes, Bases of the nonlinear approach* [1] .
In this document, we will respect following conventions, except counter-indication:

- the letter Latin ∈ {1,2,3}
- Greek letters ∈ {1.2}
- one uses the convention of summation of Einstein
- one writes in capital letter the components referring to the initial configuration
- one writes into tiny the components referring to the deformed configuration
- one writes in fat the tensors of a nature equal to or higher than 1
- one will put an index 0 to mean that one is on the initial configuration
- accodances $\{\square\}$ indicate a vector column
- hooks $\langle\square\rangle$ indicate a vector line
- right hooks $[\square]$ indicate a matrix having more than one line and more than one column

The study of the membranes rests on many elements of differential geometry, us will thus expose of them the most important principles within the framework of our study. The differential geometry refers to the application of differential calculus to the geometry, it intervenes in particular in the problems utilizing the concept of curve. It will be useful for us to take account of the curved geometry of the membranes.

3.1 Differential geometry

One works in three-dimensional Euclidean space $\mathcal{E}$ provided with the usual scalar product $(a, b) \mapsto a.b$, euclidian norm $\|.\|$ and of an orthonormal total reference mark $(O;\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$.

The first element to be clarified is the local base (or natural covariante bases), this base neither orthogonal nor is normalized a priori. That is to say a surface $\mathbf{S_0}$, $\mathbf{P_0}$ a point pertaining to this surface and (ξ^1,ξ^2) a frame of reference of this surface. One defines the natural base covariante $(\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}_3)$ by (while noting $\times$ the vector product) :

$$\mathbf{A}_\alpha=\frac{\partial \mathbf{P_0}}{\partial \xi^\alpha} \quad ; \quad \mathbf{N}\equiv\mathbf{A}_3\equiv\frac{\mathbf{A_1}\times\mathbf{A_2}}{\|\mathbf{A_1}\times\mathbf{A_2}\|} \tag{12}$$

One will write then in the deformed configuration, while noting $\mathbf{S}$ surface and $\mathbf{P}$ the position of the point $\mathbf{P}_0$ after deformation:

$$\mathbf{a}_\alpha=\frac{\partial \mathbf{P}}{\partial \xi^\alpha} \quad ; \quad \mathbf{N}\equiv\mathbf{a}_3\equiv\frac{\mathbf{a_1}\times\mathbf{a_2}}{\|\mathbf{a_1}\times\mathbf{a_2}\|} \tag{13}$$

One can now define the first fundamental form:

$$A_{\alpha\beta}\equiv\mathbf{A}_\alpha.\mathbf{A}_\beta \tag{14}$$

one can then build the matrix :

$$[A..]\equiv\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{bmatrix} \tag{15}$$

this matrix is symmetrical and invertible (because $\mathbf{A}_1$ and $\mathbf{A}_2$ are free). His reverse is noted:

$$[\mathrm{A}^{\cdot\cdot}] \equiv [\mathrm{A}_{\cdot\cdot}]^{-1} \equiv \begin{bmatrix} \mathrm{A}^{11} & \mathrm{A}^{12} \\ \mathrm{A}^{21} & A^{22} \end{bmatrix} \quad (16)$$

the dual base then is defined $(\mathbf{A^1}, \mathbf{A^2})$ base $(\mathbf{A}_1, \mathbf{A_2})$ by:

$$\mathbf{A}^{\boldsymbol{\alpha}} \equiv \mathrm{A}^{\alpha\beta} \mathbf{A}_{\boldsymbol{\beta}} \quad (17)$$

That makes it possible to define the metric tensor:

$$\mathbf{A} \equiv \mathrm{A}_{\alpha\beta} \mathbf{A}^{\boldsymbol{\alpha}} \otimes \mathbf{A}^{\boldsymbol{\beta}} \quad (18)$$

In theory of the hulls or membranes, it is the tensor $\mathbf{A}$ who will intervene instead of tensor identity I in 3D.
One will write: $\|\mathrm{A}_1 \times \mathrm{A}_2\| = \sqrt{\mathrm{A}}$ where $A \equiv det(A_{\cdot\cdot})$, one calls A the jacobien of the transformation.
Finally, the tensor of curve is defined:

$$\mathbf{B} = \mathrm{B}_{\alpha\beta} \mathbf{A}^{\boldsymbol{\alpha}} \otimes \mathbf{A}^{\boldsymbol{\beta}} = (\mathbf{N}_{,\boldsymbol{\alpha}} . \mathbf{A}_{\boldsymbol{\beta}}) . \mathbf{A}^{\boldsymbol{\alpha}} \otimes \mathbf{A}^{\boldsymbol{\beta}} \quad (19)$$

$\mathbf{B}$ is by symmetrical definition, as its name indicates it is used to measure the curvature of a surface in a point.

3.2 Assumptions

To arrive at the equations characterizing the behavior of the membranes it is necessary to make several assumptions. The goal here is not redémontrer how one arrives to these assumptions, us will thus give them directly without going into the details, those are present in [1] .
The assumptions are:

- Displacements of the membrane are represented by displacements of average surface.
- One uses the kinematic assumption of Cosserat:
  - Any point defined along a fibre remains after deformation along this same fibre.
  - There is no change in form of fibre in the thickness.
- The thickness H of the membrane is very low, mathematically that results in the conditions $\mathrm{H}|\mathrm{tr}\,\mathrm{B}| \ll 1$ and $\mathrm{H}^2(\det \mathrm{B}) \ll 1$.
- One considers a state of plane stress $[\sigma] = \begin{bmatrix} \sigma^{11} & \sigma^{12} & 0 \\ \sigma^{21} & \sigma^{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$.
- The density is constant in the thickness.
- The constraints are constant in the thickness.
- The bending moment of order 2 is neglected.

3.3 Equations governing the movement of the membranes

3.3.1 Principle of the virtual powers

The tool used for the setting in equation of the membranes is the principle of the virtual powers (PPV), whose expression in Lagrangian variables is written: $\forall\ t$, $\forall$ field virtual speeds U^* ,

$$-\int_{\Omega_0} \boldsymbol{\Pi}^{\mathrm{T}} : \mathbf{grad}_{\mathbf{Q_0}} \mathbf{U}^* \mathrm{d}\Omega_0 + \int_{\Omega_0} \rho_0 . \mathbf{f} . \mathbf{U}^* \mathrm{d}\Omega_0 + \int_{\partial\Omega_0} \mathbf{T} . \mathbf{U}^* \mathrm{d}A_0 = \int_{\Omega_0} \rho_0 . \ddot{\mathbf{U}} . \mathbf{U}^* \mathrm{d}\Omega_0 \quad (20)$$

With:

- $\boldsymbol{\Pi}$: the tensor of Piola-Kirchhoff I
- $\mathbf{T} = \boldsymbol{\Pi} . \mathbf{n}$: the vector of nominal constraint (on the edge)

- ρ_0 : density with the state of reference
- $\mathbf{f}$: mass forces

One notes the derivative temporal with the notation " $\dot{}$ ". Ω_0 represent initial volume and one will note S_0 initial average surface. The border $\partial\Omega_0$ understands the faces higher and lower and the edge of the membrane. One will note H the thickness.

One recognizes in the PPV the following expressions:

- Virtual power of the quantities of acceleration

$$P^*(\rho_0 \ddot{\mathbf{U}}) = \int_{\Omega_0} \rho_0 . \ddot{\mathbf{U}} . \mathbf{U}^* \mathrm{d}\Omega_0 \qquad (21)$$

by using the assumptions small thickness and constant density in the thickness, one obtains:

$$P^*(\rho_0 \ddot{\mathbf{U}}) = \int_{S_0} \rho_0 \mathrm{H} . \ddot{\mathbf{U}} . \mathbf{U}^* \mathrm{d}S_0 \qquad (22)$$

- virtual power of the internal efforts

$$P^*_{int} = -\int_{\Omega_0} \mathbf{\Pi}^{\mathrm{T}} : \mathbf{grad}_{\mathbf{Q_0}} \mathbf{U}^* \mathrm{d}\Omega_0 \qquad (23)$$

to write the constraints more simply one will use the tensor of the constraints of Piola-Kirchhoff II, $\Sigma = \Sigma^{ij} \mathrm{A}_i \otimes \mathrm{A}_j$. Like the components of S constraints of Cauchy $\sigma^{\alpha\beta}$ are constant in the thickness and that the thickness H is very low one can affirm that the components of constraints of Piola-Kirchhoff II $\Sigma^{\alpha\beta}$ are also constant in the thickness. While supposing moreover one state of constraint one planes arrives at:

$$[\Sigma] = \begin{bmatrix} \Sigma^{11} & \Sigma^{12} & 0 \\ \Sigma^{21} & \Sigma^{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (24)$$

Thanks to these three assumptions one can deduce that:

$$P^*_{int} = -\int_{S_0} \mathrm{N}^{\alpha\beta} \mathbf{a}_{\alpha} . \mathbf{U}^*_{,\beta} \mathrm{d}S_0 \quad \text{with} \quad \mathrm{N}^{\alpha\beta} = \mathrm{H}\Sigma^{\alpha\beta} \qquad (25)$$

- Virtual power of the external efforts

$$P^*_{ext} = \int_{\Omega_0} \rho_0 . \mathbf{f} . \mathbf{U}^* \mathrm{d}\Omega_0 + \int_{\partial\Omega_0} \mathbf{T} . \mathbf{U}^* \mathrm{d}A_0 \qquad (26)$$

that one can rewrite in the form:

$$P^*_{ext} = \int_{S_0} \mathrm{p} . \mathbf{U}^* \mathrm{d}S_0 + \int_{S_0} \mathbf{c} . \mathbf{a}^*_3 \mathrm{d}S_0 + \int_{\partial S_0} (\mathbf{F} . \mathbf{U}^* + \mathbf{C} . \mathbf{a}^*_3) \mathrm{d}s_0 \qquad (27)$$

where $\mathbf{p}$ (resp. $\boldsymbol{F}$) is the surface force on S_0 (resp. linear on the edge ∂S_0) and $\mathbf{c}$ (resp. $\mathbf{C}$) is the surface couple on S_0 (resp. linear on the edge ∂S_0). One breaks up $\mathbf{p}$ and $\mathbf{F}$ in the current natural base: $p \equiv p^{\alpha} a_{\alpha} + p^3 a_3$ and $F \equiv F^{\alpha} a_{\alpha} + F^3 a_3$.

One can finally write the PPV under form: $\forall\ t$, $\forall\ U^*$, $\forall\ a_{3^*}$

$$-\int_{S_0} \mathrm{N}^{\alpha\beta} \mathbf{a}_{\alpha} . \mathbf{U}^*_{,\beta} \mathrm{d}S_0 + \int_{S_0} \mathbf{p} . \mathbf{U}^* \mathrm{d}S_0 + \int_{S_0} \mathbf{c} . \mathbf{a}^*_3 \mathrm{d}S_0 + \int_{\partial S_0} (\mathbf{F} . \mathbf{U}^* + \mathbf{C} . \mathbf{a}^*_3) \mathrm{d}s_0 = \int_{S_0} \rho_0 \mathrm{H} . \ddot{\mathbf{U}} . \mathbf{U}^* \mathrm{d}S_0 \qquad (28)$$

3.3.2 Equation local of the movement and boundary conditions

One can now exploit the PPV to obtain the local equations of the dynamics of the membranes and the boundary conditions sthenic. These equations will not be used to code the finite element but the boundary conditions sthenic are necessary to understand the behavior of the membranes.

The first thing to be made is to carry out integrations by parts using the theorem of divergence in order to eliminate the derivative $U^*_{,\beta}$ and to reveal U^*. One obtains:

$$\int_{S_0} \frac{1}{\sqrt{A}} \left(N^{\alpha\beta} \mathbf{a_\alpha} \sqrt{A}\right)_{,\beta} \mathbf{U}^* dS_0 - \int_{\partial S_0} N^{\alpha\beta} \mathbf{a_\alpha} . \mathbf{U}^* \upsilon_0 . A_\beta ds_0 + \int_{S_0} \mathbf{p} . \mathbf{U}^* dS_0 + \int_{S_0} \mathbf{c} . \mathbf{a_3^*} dS_0 + \int_{\partial S_0} \left(\mathbf{F} . \mathbf{U}^* + \mathbf{C} . \mathbf{a_3^*}\right) ds_0 = \int_{S_0} \rho_0 H . \ddot{\mathbf{U}} . \mathbf{U}^* dS_0 \quad (29)$$

With υ_0 the unit normal external with ∂S_0 and located in the tangent plan at S_0.

By using the fact that virtual fields $\mathbf{U}^*$ and $\mathbf{a_3}^*$ are independent and arbitrary one obtains:

- Requirements on the external loading:
  - The surface couple must be null: $\mathbf{c} = \mathbf{0}$.
  - The linear force on the edge of the membrane should not have of component transverse according to $\mathbf{a_3}$: $F^3 = 0$.
  - The linear couple on the edge must be null: $\mathbf{C} = \mathbf{0}$.
- Local equation of dynamics: $\forall t$, $\forall$ $\mathbf{P_0} \in S_0$

$$\mathbf{a_{\alpha,\beta}} N^{\alpha\beta} + \mathbf{a_\alpha} \frac{1}{\sqrt{A}} \left(N^{\alpha\beta} \sqrt{A}\right)_{,\beta} + \mathbf{p} = \rho_0 H \ddot{\mathbf{U}} \quad (30)$$

- Boundary condition: $\forall$ t , $\forall$ $\mathbf{P_0} \in \partial S_0$

$$\mathbf{a_\alpha} N^{\alpha\beta} \boldsymbol{\upsilon_0} \mathbf{A_\alpha} = \mathbf{F} \iff \begin{Bmatrix} F^\alpha = N^{\alpha\beta} (\boldsymbol{\upsilon_0} \mathbf{A_\beta}) \\ F^3 = 0 \end{Bmatrix} \quad (31)$$

It is noticed that the requirements on the external loading relate to the nullity of certain terms. That could be intuity knowing that a membrane does not have rigidity in inflection. Indeed, to apply a couple to a membrane would involve a movement of rigid body. Moreover, a pressure applied to a plane membrane goes too to involve a rigid movement of body as long as the latter is not deformed and did not acquire a rigidity known as “geometrical”. To cure this problem, it will be necessary to create a “geometrical” rigidity artificial at the beginning of calculation by imposing a prestressing which one will be able to remove thereafter.

3.3.3 Laws of behaviors

To bind the constraints of material to its displacements, it is necessary to use laws of behavior. In the case of the membranes one uses laws of behavior known as “hyperelastic”.

We will be satisfied with two hyperelastic laws of behavior among all those which exist: the law of Coming-Kirchhoff Saint and the law néo-Hookéenne. We will characterize them by their voluminal deformation energy *W* function of the tensor of deformation of Green-Lagrange $\mathbf{E}$ (such as $\mathbf{E} = E_{ij} A^i \otimes A^j$ with $E_{ij} = \frac{1}{2}\left(a_{ij} - A_{ij}\right)$) or of the tensor of expansion $\mathbf{C}$ (such as $\mathbf{C} = C_{ij} \mathbf{A^i} \otimes \mathbf{A^j}$ with $C_{ij} = a_{ij}$).

The constraint of Piola-Kirchhoff II, $\boldsymbol{\Sigma}$, is then connected to $\mathbf{E}$ or $\mathbf{C}$ by:

$$\Sigma = \mathrm{Sym} \frac{\partial w}{\partial \mathbf{E}} = 2\, \mathrm{Sym} \frac{\partial w}{\partial \mathbf{C}} \quad (32)$$

with

$$\mathrm{Sym}\frac{\partial w}{\partial \mathbf{E}}=\frac{1}{2}\left(\frac{\partial w}{\partial \mathbf{E}}+\left(\frac{\partial w}{\partial \mathbf{E}}\right)^{T}\right) \quad (33)$$

3.3.3.1 Law of Coming-Kirchhoff Saint

The standard material hyperelastic is characterized by the law of Coming-Kirchhoff Saint. Its voluminal deformation energy is expressed by:

$$w(E)=\mathbf{\Sigma_0}:\mathbf{E}+\frac{1}{2}\mathbf{E}:\mathbf{D}:\mathbf{E} \quad (34)$$

with $\boldsymbol{D}$ a tensor of a nature 4 called tensor of elasticity.

3.3.3.2 Law Néo-Hookéenne

The law néo-Hookéenne is another law of classical behavior hyperelastic, it is in particular used for the incompressible material study. This law gives better results when the body undergoes great deformations. Its voluminal deformation energy is expressed by:

$$w(\mathbf{C})=\frac{\mu}{2}(\mathrm{tr}\,\mathbf{C}-3)-\mu.\ln\mathbf{J}+\frac{\lambda}{2}(\ln\mathbf{J})^2 \quad (35)$$

μ and λ are the coefficients of Lamé, they are defined by:

$$\lambda=\frac{E\nu}{(1+\nu)(1-2\nu)} \quad \text{and} \quad \mu=\frac{E}{2(1+\nu)} \quad (36)$$

with E (resp. ν) the Young modulus (resp. the Poisson's ratio).

3.4 Discretization finite elements

One presented in the preceding paragraphs the equations being used for the resolution of the behavior of the membranes. One reformulates in order to be able to code them in code_aster.

3.4.1 Interpolation of the geometry

One will note indifferently $\mathbf{P_0}$ or $\mathbf{X}$ the initial position of a current particle located on initial average surface S_0 . Coordinates of the point $\mathbf{X}$ in a fixed orthonormal reference mark $(\mathbf{O}\ ;\ \mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3)$ are noted $(X_1,\ X_2,\ X_3)$.

The element of reference of a finite element *E* surface S_0 is noted e_ξ . Coordinates of the element of reference are noted ($\xi^1,\ \xi^2)$. Geometry of the element e is represented by the interpolation:

$$\begin{array}{c} e_\xi \rightarrow e \\ (\xi^1,\ \xi^2)\mapsto \mathbf{X}\equiv\mathbf{P_0} \quad \text{avec} \quad \forall i\in\{1,2,3\}, \quad X_i=\langle N(\xi^1,\ \xi^2)\rangle\{X_i\}^e \end{array} \quad (37)$$

where

- The vector line $\langle N\rangle$ is the geometrical function of interpolation of the element e : $\langle N\rangle=\langle N_1,N_2,\ldots,N_{nne}\rangle$ with nne the number of nodes of the element.
- The vector column $\{X_i\}^e$ contains them nne coordinates X_i nodes of the element e .

In a point $\mathbf{X}\equiv\mathbf{P_0}$ element e , vectors $\mathbf{A}_\alpha=\frac{\partial \mathbf{P_0}}{\partial \xi^\alpha}$ natural base are calculated by

$$\forall \alpha \in \{1,2\},\ \forall i \in \{1,2,3\},\ (\mathbf{A}_{\boldsymbol{\alpha}})_i = \langle N_{,\alpha} \rangle \{X_i\}^e \tag{38}$$

The element of surface dS_0 is defined by $dS_0 = \left\| \frac{\partial X}{\partial \xi^1} \times \frac{\partial X}{\partial \xi^2} \right\| d\xi^1 d\xi^2 = \|\mathbf{A_1} \times \mathbf{A_2}\| d\xi^1 d\xi^2$

The reasoning is strictly identical for the vectors of edge, one changes just the notation. Geometry of the element e' is represented by the interpolation:

$$\begin{array}{lll} e'_\xi & \rightarrow e' & \\ \xi & \mapsto X & \text{avec } \forall i \in \{1,2,3\},\ X_i = \langle N'(\xi) \rangle \{X_i\}^{e'} \end{array} \tag{39}$$

3.4.2 Interpolation of the field of displacement

The vector displacement is broken up $U = U(\xi^1,\ \xi^2, t)$ average surface in the fixed orthonormal base $(\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3)$:

$$\mathbf{U} = U^i \mathbf{e_i} \tag{40}$$

Since the base $(\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3)$ is orthonormal, the components covariantes and contravariantes in this base is confused. Consequently, the high or low position of the indices i in the preceding relation is without importance.

One works with the isoparametric finite elements, D years a finite element e surface S_0 , each component U^i displacement is thus interpolated by:

$$\forall i \in \{1,2,3\},\quad U_i = \langle N(\xi^1,\ \xi^2) \rangle \{U_i(t)\}^e = N_a U_a^{ie} \quad (\text{sommation sur } a \in [1, nne]) \tag{41}$$

where

- Functions of interpolation N_a are the same ones as previously
- The vector column $\{U_i\}^e$ contains them nne components of displacement $U_i(t)$ nodes of the element e .

One writes:

$$\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \left[N_1 N_1 N_1 \middle| N_2 N_2 N_2 \middle| \cdots \right] \begin{Bmatrix} U^1 \\ (U^2)^1 \\ U^3 \\ U^1 \\ (U^2)^2 \\ U^3 \\ \vdots \end{Bmatrix}^e \tag{42}$$

What one shortens in:

$$\{\vec{U}\} = [N] \{\mathbf{U}\}^e \tag{43}$$

3.4.3 Interpolation and discretization of the field of displacement

One breaks up the tensor gradient of the field of displacement in the base $\mathbf{A_i} \otimes \mathbf{A_j}$:

$$\mathbf{H} \equiv \text{grad}_{Q_0} \mathbf{U}(\mathbf{Q_0}, t)|_{Q_0 = P_0} = H_{ij}\, \mathbf{A^i} \otimes \mathbf{A^j} \tag{44}$$

Only components $H_{i\beta}$ we interest, one rewrites this tensor in the form of vector column:

$$(\tilde{\mathbf{H}}) \equiv \begin{pmatrix} H_1 \\ H_2 \\ H_3 \\ H_4 \\ H_5 \\ H_6 \end{pmatrix} = \begin{pmatrix} H_{11} \\ H_{21} \\ H_{31} \\ H_{12} \\ H_{22} \\ H_{32} \end{pmatrix} \quad (45)$$

There is then the relation:

$$\{\tilde{\mathbf{H}}\} = [G]\{\mathbf{U}\}^e \quad (46)$$

with $[G]$ a matrix of dimension $6x3nne$ such as:

$$[G] \equiv \begin{bmatrix} N_{1,1}\begin{bmatrix} (\mathbf{A_1})_1 & (\mathbf{A_1})_2 & (\mathbf{A_1})_3 \\ (\mathbf{A_2})_1 & (\mathbf{A_2})_2 & (\mathbf{A_2})_3 \\ (\mathbf{A_3})_1 & (\mathbf{A_3})_2 & (\mathbf{A_3})_3 \end{bmatrix} & N_{2,1}\begin{bmatrix} (\mathbf{A_1})_1 & (\mathbf{A_1})_2 & (\mathbf{A_1})_3 \\ (\mathbf{A_2})_1 & (\mathbf{A_2})_2 & (\mathbf{A_2})_3 \\ (\mathbf{A_3})_1 & (\mathbf{A_3})_2 & (\mathbf{A_3})_3 \end{bmatrix} & \cdots \\ N_{1,2}\begin{bmatrix} (\mathbf{A_1})_1 & (\mathbf{A_1})_2 & (\mathbf{A_1})_3 \\ (\mathbf{A_2})_1 & (\mathbf{A_2})_2 & (\mathbf{A_2})_3 \\ (\mathbf{A_3})_1 & (\mathbf{A_3})_2 & (\mathbf{A_3})_3 \end{bmatrix} & \cdots & \cdots \end{bmatrix} \quad (47)$$

Virtual fields, indicated by “*”, will be discretized in an identical way.

3.4.4 Discretization of the principle of the virtual powers

Now that one has our functions of interpolation, one can write the principle of the virtual powers in matric form:

- Virtual power of the quantities of acceleration:

$$P^*(\rho_0 \ddot{\mathbf{U}}) = \int_{S_0} \rho_0 H . \ddot{\mathbf{U}} . \mathbf{U}^* d S_0 = \sum_e \langle \mathbf{U}^* \rangle^e [\mathbf{M}]^e \{\ddot{\mathbf{U}}\}^e \quad (48)$$

with $[\mathbf{M}]$ the matrix masses elementary:

$$[\mathbf{M}]^e \equiv \int_e \rho_0 [N]^T [N] H dS_0 \quad (49)$$

- Virtual power of the internal efforts:

$$P^*_{int} = -\int_{\Omega_0} \mathbf{\Pi}^T : \mathbf{grad}_{Q_0} \mathbf{U}^* d\Omega_0 = -\sum_e \langle \mathbf{U}^* \rangle^e \{\mathbf{\Psi}\}^e \quad (50)$$

with:

$$\{\mathbf{\Psi}\}^e \equiv \int_e [G]^T \{\tilde{\mathbf{\Pi}}\} H dS_0 \quad (51)$$

Same manner as previously, only components $\Pi^{i\beta}$ we interest:

$$\{\tilde{\mathbf{\Pi}}\} \equiv \begin{Bmatrix} \Pi_1 \\ \Pi_2 \\ \Pi_3 \\ \Pi_4 \\ \Pi_5 \\ \Pi_6 \end{Bmatrix} = \begin{Bmatrix} \Pi^{11} \\ \Pi^{21} \\ \Pi^{31} \\ \Pi^{21} \\ \Pi^{22} \\ \Pi^{32} \end{Bmatrix} \quad (52)$$

R emarque: has U place to use $\{\tilde{\mathbf{\Pi}}\}$, one will express thereafter $\{\mathbf{\Psi}\}^e$ according to Piola Kirchhoff II $(\mathbf{\Sigma})$ in order to be able to use the laws of behavior of the paragraph 3.3.3 .

- Virtual power of the external efforts:

$$P^*_{ext}=\int_{S_0} \mathrm{p}.\mathbf{U}^* \mathrm{d}S_0+\int_{\partial S_0} \mathbf{F}.\mathbf{U}^* \mathrm{d}s_0=\sum_e \langle \mathbf{U}^* \rangle^e \{\mathbf{\Phi}\}^e+\sum_{e'} \langle \mathbf{U}^* \rangle^{e'} \{\mathbf{\Phi}'\}^{e'} \quad (53)$$

with

$$\{\mathbf{\Phi}\}^e \equiv \int_e [N]^T \{\mathbf{p}\} \mathrm{d}S_0 \quad \text{and} \quad \{\mathbf{\Phi}'\}^{e'} \equiv \int_{e'} [N]^T \{\mathbf{F}\} \mathrm{d}s_0 \quad (54)$$

One notes $\{\mathbf{p}\}$ (resp. $\{\mathbf{F}\}$) the vector containing them three components of the vector forces surface $\mathbf{p}$ (resp. linear force $\mathbf{F}$) in the orthonormal base $(\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3)$.

3.4.5 Linearization and form of the matrix tangent stiffness

In the diagram of Newton-Raphson, the problem will be to calculate the term $\frac{\partial\{R\}}{\partial\{U\}}$, that one calls matrix tangent stiffness, of dimension $3nn\,x\,3nn$ with nn the total number of nodes of the element . One calls $\{R\}=\{\mathbf{\Phi}\}-\{\mathbf{\Psi}\}$ the residue. One writes:

$$[\mathbf{K}] \equiv \frac{\partial\{\mathbf{R}\}}{\partial\{\mathbf{U}\}}=\frac{\partial\{\mathbf{\Psi}\}}{\partial\{\mathbf{U}\}}-\frac{\partial\{\mathbf{\Phi}\}}{\partial\{\mathbf{U}\}} \equiv [\mathbf{K}_{\mathbf{\Psi}}]+[\mathbf{K}_{\mathbf{\Phi}}] \quad (55)$$

One has at the elementary level:

$$[\mathbf{K}]^e \equiv \frac{\partial\{\mathbf{\Psi}\}^e}{\partial\{\mathbf{U}\}^e}-\frac{\partial\{\mathbf{\Phi}\}^e}{\partial\{\mathbf{U}\}^e} \equiv [\mathbf{K}_{\mathbf{\Psi}}]^e+[\mathbf{K}_{\mathbf{\Phi}}]^e \quad (56)$$

One calls $[\mathbf{K}_{\Psi}]^e$ the elementary tangent matrix due to the internal efforts and $[\mathbf{K}_{\Phi}]^e$ the elementary tangent matrix due to the external efforts.
If there exist also following loadings applied to the edge of the membrane, it appears also the matrix elementary tangent stiffness due to the external efforts on the edge e' :

$$[\mathbf{K}']^{e'} \equiv \frac{-\partial\{\mathbf{\Phi}'\}^{e'}}{\partial\{\mathbf{U}\}^{e'}} \quad (57)$$

It will be noted that the following loadings exist only in great deformations. Indeed, as in small deformations one makes the assumption of a fixed geometry to carry out calculations, the loadings which depend on displacement do not take place to be.

3.4.6 Calculation of the constraints in the membrane

An important remark, although obvious taking into consideration equation, is that one works on the not deformed configuration of the membrane. That has as a consequence which one will be able to calculate the constraints of Piola-Kirchhoff II and not those of Cauchy. However, one does not have equation relating to the evolution thickness, which in fact also an unknown factor. One can show that the constraints of Cauchy are related to those of Piola-Kirchhoff by the equation:

$$\mathrm{h}\,\sigma^{\alpha\beta}=\frac{\mathrm{H}\Sigma^{\alpha\beta}}{\dfrac{\|\mathbf{a}_1\times\mathbf{a}_2\|}{\|\mathbf{A}_1\times\mathbf{A}_2\|}} \quad (58)$$

what gives two unknown factors for an equation. One will be able to thus have access only to the constraints of Cauchy integrated on the thickness, so called “generalized efforts”, expressing oneself by “ $\mathrm{h}\boldsymbol{\sigma}$ ”.

3.5 Expression of the elements coded in code_aster

One gives in this paragraph the form of the various tensors such as they are written in code_aster.

One has for the law of behavior of Coming saint Kirchhoff:

$$\Sigma^{\alpha\beta}=\Sigma_0^{\alpha\beta}+\frac{E}{1-\nu^2}\left[\frac{1}{2}(1-\nu)\left(A^{\alpha\gamma}A^{\beta\delta}+A^{\alpha\delta}A^{\beta\gamma}\right)+A^{\alpha\beta}A^{\gamma\delta}\right]\frac{1}{2}\left(a_{\delta\gamma}-A_{\delta\gamma}\right) \quad (59)$$

And thus:

$$\frac{\partial\Sigma^{\alpha\beta}}{\partial E_{\delta\gamma}}=\mu\left(A^{\alpha\gamma}A^{\beta\delta}+A^{\alpha\delta}A^{\beta\gamma}\right)+\frac{2\lambda\mu}{\lambda+2\mu}A^{\alpha\beta}A^{\gamma\delta} \quad (60)$$

And for the law néo Hookéenne:

$$\Sigma^{\alpha\beta}=\Sigma_0^{\alpha\beta}+\mu\left(A^{\alpha\beta}-a^{\alpha\beta}\right)+\frac{\lambda}{2}\ln\left(\frac{\lambda q}{2\mu}W\left(\frac{2\mu}{\lambda q}\frac{e^{2\mu}}{\lambda}\right)\right)a^{\alpha\beta} \quad (61)$$

ET docc:

$$\frac{1}{2}\frac{\partial\Sigma^{\alpha\beta}}{\partial E_{\delta\gamma}}=\frac{\mu C_{33}}{1+2\mu\frac{C_{33}}{\lambda}}a^{\alpha\beta}a^{\gamma\delta}+\frac{1}{2}\left(\mu-\frac{\lambda}{2}\ln\det\mathbf{C}\right)\left(a^{\alpha\delta}a^{\gamma\beta}+a^{\alpha\gamma}a^{\delta\beta}\right) \quad (62)$$

With $C_{33}=\frac{\lambda}{2\mu}W\left(\frac{2\mu}{\lambda q}e^{\frac{2\mu}{\lambda}}\right)$, $q=\frac{\det[a_{\alpha\beta}]}{\det[A_{\alpha\beta}]}$, and W the function of Lambert $w(z)e^{w(z)}=z$ ($z\in C$).

In the next expressions one will note δ the symbol of Kronecker and one will have:

$$i=3(a-1)+p \quad \text{and} \quad j=3(b-1)+q \quad (63)$$

With vec $i,\ j\in(1,3\times nne)$, $a,\ b\in(1,nne)$ and $p,\ q\in(1,2,3)$.
The matrix masses is written:

$$[\mathbf{M}]_{ij}^e\equiv\int_e\delta_{pq}\rho_0 N_a N_b H\, dS_0 \quad (64)$$

The elementary tangent matrix due to the internal efforts:

$$[\mathbf{K}_\Psi]_{ij}^e=\int_e N_{a,\alpha}N_{b,\beta}\left(\delta_{pq}\Sigma^{\beta\alpha}+x_{p,\gamma}x_{q,\delta}\frac{1}{2}\left[\left(\frac{\partial\boldsymbol{\Sigma}}{\partial\mathbf{E}}\right)^{\gamma\alpha\beta\delta}+\left(\frac{\partial\boldsymbol{\Sigma}}{\partial\mathbf{E}}\right)^{\gamma\alpha\delta\beta}\right]\right)H\, dS_0 \quad (65)$$

And the vector forces intern elementary:

$$\{\boldsymbol{\Psi}\}_i^e=\int_e N_{a,\alpha}x_{p,\gamma}\Sigma^{\gamma\alpha}H\, dS_0 \quad (66)$$

4 Formulation of the elements of GRILLE_MEMBRANE

For a tablecloth of uniaxial reinforcement, the deformation energy can be put in the form:

$$\Phi=\frac{1}{2}\int S\,\sigma\,\varepsilon\,ds \quad (67)$$

with S the section of reinforcement per unit of length, σ the constraint (scalar) and ε deformation (scalar). One seeks to obtain an expression of the type $\varepsilon=BU$ where one notes U nodal values of displacement. By taking again the approach of the preceding section, one shows this time that:

$$\varepsilon=(\nabla u)_{11}=\left(\frac{\partial u}{\partial \xi^{\beta}}\cdot a^{\alpha}\right)R_{\alpha}^{1}R_{\gamma}^{1}g^{\beta\gamma} \quad (68)$$

By introducing the derivative $\hat{B}$ functions of form at the point of Gauss considered, it comes:

$$\varepsilon=R_{\alpha}^{1}R_{\gamma}^{1}g^{\beta\gamma}\hat{B}_{\beta,n}(a^{\alpha})_{i}U_{i,n} \quad (69)$$

D' where it B sought. It will be noted that it has the shape of a vector, due to the scalar nature of the required deformation:

$$B_{i,n}=R_{\alpha}^{1}R_{\gamma}^{1}g^{\beta\gamma}\hat{B}_{\beta,n}(a^{\alpha})_{i} \quad (70)$$

From B , one finds all the classical expressions of the deformation, the nodal forces and the tangent matrix:

$$\begin{aligned}\varepsilon&=BU\\ F&=\int B^{T}\sigma\\ K&=\int B^{T}\frac{\partial\sigma}{\partial\varepsilon}B\end{aligned} \quad (71)$$

It will be noted that they are the laws of behavior unidimensional who are used to obtain the constraint starting from the deformation. All laws of behavior available in unidimensional are usable. Failing this, one can also use the laws three-dimensional, thanks to the method De Borst.

5 Matrix of mass

For the elements MEMBRANE and GRILLE_MEMBRANE, the terms of the matrix of mass are obtained after discretization of the following variational formulation:

$$\delta W_{mass}^{ac} = \int_{-h/2}^{+h/2} \int_S \rho \ddot{\boldsymbol{u}} \delta \boldsymbol{u} \, dz \, dS = \int_S \rho_m \left(\ddot{u} \delta u + \ddot{v} \delta v + \ddot{w} \delta w \right) dS \quad (72)$$

Withvec $\rho_m = \int_{-h/2}^{+h/2} \rho \, dz$. Discretization of displacement (for N nodes) for this isoparametric element is:

$$\mathbf{u} = \sum_{k=1}^{N} N_k \begin{pmatrix} u_k \\ v_k \\ w_k \end{pmatrix} \Bigg\} k = 1, \cdots, N \quad (73)$$

The matrix of mass, in the base where the degrees of freedom are gathered according to the directions of translation, has then as an expression:

$$\mathbf{M} = \begin{pmatrix} \mathbf{M}_m & 0 & 0 \\ 0 & \mathbf{M}_m & 0 \\ 0 & 0 & \mathbf{M}_m \end{pmatrix} \quad (74)$$

With vec: $\mathbf{M}_m = \int_S \rho_m \mathbf{N}^T \mathbf{N} \, dS$ and $\mathbf{N} = \left(N_1 \cdots N_k \right)$.

6 Bibliography

[1] **The VAN, Anh** . *Hulls and membranes, Bases of the nonlinear approach* . 2014.

7 Description of the versions of the document

Version Aster	Author (S) Organization (S)	Description of the modifications
7.4	P.Badel EDF-R&D/AMA	Initial text
9.5	J.M.Proix EDF-R&D/AMA	Modification of GRID in GRILLE_EXCENTRE
11.3	Mr. David EDF-R&D/AMA	Addition of modeling MEMBRANE and minor modifications
13.1	S. Michel-Ponnelle EDF-R&D/AMA	Addition stamps of mass
13.3	NR. Lauzeral Central school of Nantes	Formulation of the element MEMBRANE in great transformations