Relations of behavior élasto-visco-plastic of Chaboche

Summary:

This document describes the integration of the model of behavior élasto-visco-plastic of Chaboche with nonlinear and isotropic kinematic work hardening, with taking into possible account of viscosity. The established model has one or two variable kinematics, and takes into account all the variations of the coefficients with the temperature, and has an effect of work hardening on the tensorial variables of recall. This version also makes it possible to model (in an optional way) the viscous character of the material (viscosity of Norton). It is integrated by the solution of only one scalar equation nonlinear. This model is available in 3D, plane deformation, axisymetry. Modeling in plane constraint uses a method of condensation static (of Borst). One gives also elements to identify the coefficients of the relation of behavior.

Contents

1 Models élasto-visco-plastics of Chaboche available in *Code_Aster*

For the structural analysis subjected to cyclic loadings, work hardenings isotropic (linear or not) and linear kinematics classics [R5.03.02] and [R5.03.16] are not sufficient any more. In particular, one cannot correctly describe the stabilized cycles obtained in experiments on a tensile specimen subjected to an alternated imposed deformation or a traction and compression.

If one seeks to precisely describe the effects of a cyclic loading, it is desirable to adopt modelings more sophisticated (but easy to use) such as the model of Saïd Taheri, for example, cf [R5.03.05], or if the number of cycles is limited the model of Jean-Louis Chaboche who is introduced here.

Actually, the model of Chaboche can be more or less sophisticated. Models developed in *Code_Aster* comprise is a kinematic variable (VMIS_CIN1_CHAB and VISC_CIN1_CHAB) that is to say two (VISC_CIN2_CHAB and VMIS_CIN2_CHAB) , and of isotropic work hardening.

The choice to use two variable kinematics complicates certainly the model, but makes it possible to correctly identify the uniaxial tests in a broader range of deformations [bib2], [bib7]. A certain number of identifications of the parameters of this model were carried out mainly for the stainless steels A316 and A304 ([bib7], [bib8]).

The models comprise 8 parameters (only one kinematic variable) or 10 (two variable kinematics), introduced into the order DEFI_MATERIAU :

```
CIN1_CHAB (CIN1_CHAB_FO)                = _F (
                            ♦ R_0 =        R_0,
                            ◊ R_I =        R_I,   (useless if B=0)
                            ◊ B =          B  ,   (defect: 0.)

                            ♦ C_I =        C_I,
                            ◊ K =          K  ,   (defect: 1.)
                            ◊ W =          W  ,   (defect: 0.)

                            ♦ G_0 =        G_0,
                            ◊ A_I =        A_I,   (defect: 0.)
                          )
CIN2_CHAB (CIN2_CHAB_FO)                = _F (
                            ♦ R_0 =        R_0,
                            ◊ R_I =        R_I,
                               useless if B=0 or if effect of memory)
                            ◊ B =          B  ,   (defect: 0.)

                            ♦ C1_I =       C1_I,
                            ♦ C2_I =       C2_I,
                            ◊ K =          K   ,  (defect: 1.)
                            ◊ W =          W   ,  (defect: 0.)

                            ♦ G1_0 =       G1_0,
                            ♦ G2_0 =       G2_0,
                            ◊ A_I =        A_I,   (defect: 0.)
                          )
```

The 8 or 10 parameters are real constants. All these parameters can depend on the temperature (keywords CIN1_CHAB_FO or CIN2_CHAB_FO) and the expected values are of type function.

If one wants to introduce besides viscosity (models VISC_CIN1_CHAB and VISC_CIN2_CHAB), it is also necessary to provide in the order DEFI_MATERIAU, under the keyword LEMAITRE (or LEMAITRE_FO) parameters NR and UN_SUR_K, which can depend on the temperature.

```
LEMAITRE (LEMAITRE_FO)              = _F (
                     ♦ NR =          N,
                     ♦ UN_SUR_K =    1/K
```

The parameter UN_SUR_M keyword LEMAITRE (respectively LEMAITRE_FO) must obligatorily be put at zero (respectively with the identically worthless function).

It is possible also to take into account an effect of memory of largest deformation plastic using the models (VISC_CIN2_ MEMO and VMIS_CIN2_MEMO) . The keywords to be informed are:

```
MEMO_ECRO (MEMO_ECRO_FO)                = _F (
                     ♦ Q_M =      Qm,
                     ♦ Q_0 =      Q0,
                     ♦ DRIVEN =   driven,
                     ♦ ETA =      eta, (defect: 0.5)
```

In the event of loading nonproportional, it is necessary to enrich the model, by the data of two additional parameters:

CIN2_NRAD = _F (

◊ DELTA1= δ_1 (défaut= 1.E+0),

◊ DELTA2= δ_2 (défaut= 1.E+0),

with $0\leq\delta_1\leq 1$, $0\leq\delta_2\leq 1$

The laws of behavior are accessible in all the orders using the keyword BEHAVIOR with the following relations :
VISC_CIN1_CHAB, VISC_CIN2_CHAB, VISC_CIN2_MEMO, VISC_CIN2_NRAD, VISC_MEMO_NRAD, VMIS_CIN1_CHAB, VMIS_CIN2_CHAB, VMIS_CIN2_MEMO, VMIS_CIN2_NRAD, VMIS_MEMO_NRAD.

Notice : the model VISCOCHAB [R5.03.14] also allows to represent the effects described in this document. It comprises moreover of the terms of additional restoration and work hardening. But its use in structural analyses is more expensive in time calculation (because one must solve either by the method of Runge-Kutta or by the method of Newton a system of 27 equations to 27 unknown factors). Moreover, it poses problems of robustness when the step of time is large, because the method of Newton can fail. That involves many subdivisions of the step of time.

The models described in this document are optimized, insofar as they result in solving only one scalar resolution, and the method of very robust resolution used (method of Brent or secant, cf [R5.03.14]); it is thus a model able to integrate quickly great steps of time.

In the continuation of this document, one describes the characteristics of the various models. One presents then the detail of their digital integration in link with the construction of the coherent tangent matrix. Lastly, one also gives some elements for the identification of the characteristics of material.

2 Description of the models

2.1 Description of the models

At any moment, the state of material is described by the deformation ε , the temperature T , plastic deformation ε^p , cumulated plastic deformation p and the tensor of recall X . The equations of state then define according to these variables of state the constraint $\sigma=\sigma^H \mathbf{Id}+\tilde{\sigma}$ (broken up into parts hydrostatic and deviatoric), the isotropic share of work hardening R and the kinematic share X :

$$\sigma^H=\frac{1}{3}tr(\sigma)=K\,tr(\varepsilon-\varepsilon^{\mathbf{th}}) \text{ with } \varepsilon^{\mathbf{th}}=\alpha(T\text{-}T^{ref})\mathbf{Id} \qquad \text{éq 2.1-1}$$

$$\tilde{\sigma}=\sigma-\sigma^H\mathbf{Id}=2\mu(\tilde{\varepsilon}-\varepsilon^p) \qquad \text{éq 2.1-2}$$

$$R=R(p) \qquad \text{éq 2.1-3}$$

$$X=X(p,\varepsilon^p)=X_1(p,\varepsilon^p)+X_2(p,\varepsilon^p) \qquad \text{éq 2.1-4}$$

where K,μ,α and coefficients of $X(p)$ and $R(p)$ are characteristics of material which can depend on the temperature. More precisely, they are respectively the modules of compressibility and shearing, the thermal dilation coefficient, the functions of isotropic and kinematic work hardening. As for $T^{réf}$, it is the temperature of reference, for which one regards the thermal deformation as being worthless.

Note:

For the model VISC_CIN1_CHAB only the only tensorial variable is considered $X_1(p)$ thus $X_2(p)=0$. This remains valid for all the continuation: one will describe the two models formally in the same way, the model VISC_CIN1_CHAB resulting from VISC_CIN2_CHAB while supposing $X_2(p)=0$.

The evolution of the plastic deformation is controlled by a normal law of flow to a criterion of plasticity of von Mises:

$$F(\sigma,R,X)=(\tilde{\sigma}-X_1-X_2)_{eq}-R(p) \text{ with } A_{eq}=\sqrt{\frac{3}{2}\tilde{A}:\tilde{A}} \qquad \text{éq 2.1-5}$$

$$\dot{\varepsilon}^p=\dot{\lambda}\frac{\partial F}{\partial\sigma}=\frac{3}{2}\dot{\lambda}\frac{\tilde{\sigma}-X_1-X_2}{(\tilde{\sigma}-X_1-X_2)_{eq}} \qquad \text{éq 2.1-6}$$

$$\dot{p}=\dot{\lambda}=\sqrt{\frac{2}{3}\dot{\varepsilon}^p:\dot{\varepsilon}^p} \qquad \text{éq 2.1-7}$$

As for the plastic multiplier $\dot{\lambda}$, it is obtained by the condition of coherence:

$$\begin{cases} \text{si } F<0 \text{ ou } \dot{F}<0 & \dot{\lambda}=0 \\ \text{si } F=0 \text{ et } \dot{F}=0 & \dot{\lambda}\geq 0 \end{cases} \qquad \text{éq 2.1-8}$$

Note:

Evolution of the variables $\mathbf{X_1}$ and $\mathbf{X_2}$ is given by:

$$\begin{aligned} X_1&=\frac{2}{3}C_1(p)\alpha_1 \\ X_2&=\frac{2}{3}C_2(p)\alpha_2 \\ \dot{\alpha}_1&=\dot{\varepsilon}^p-\gamma_1(p)\alpha_1\dot{p} \\ \dot{\alpha}_2&=\dot{\varepsilon}^p-\gamma_2(p)\alpha_2\dot{p} \end{aligned} \qquad \text{éq 2.1-9}$$

Functions $C(p)$, $\gamma(p)$ and $R(p)$ are defined, in accordance with [bib2] by:

$$R(p)=R_\infty+(R_0-R_\infty)e^{-bp}$$
$$C_1(p)=C_1^\infty(1+(k-1)e^{-wp})$$
$$C_2(p)=C_2^\infty(1+(k-1)e^{-wp})$$
$$\gamma_1(p)=\gamma_1^0(a_\infty+(1-a_\infty)e^{-bp})$$
$$\gamma_2(p)=\gamma_2^0(a_\infty+(1-a_\infty)e^{-bp})$$

The evolution of these coefficients makes it possible to represent work hardening in several ways: classical isotropic work hardening (monotonous or cyclic) by $R(p)$, "work hardening" of the coefficients relating to the kinematics terms by $C(p)$ and $\gamma(p)$. (cf [feeding-bottle11]). The expressions into exponential are similar to the definition of nonlinear kinematic work hardening (eq.2, 1.9), and (in their principle) represent a variation of the coefficients since the subscripted value by 0 (for $p=0$) up to the subscripted value by ∞ when p becomes large.
This implies that the coefficients b and w are supposed to be positive. In the contrary case, a message of alarm is transmitted, because the solution calculated risk to be not physics.

The presence of viscosity can model in a simple way [bib2] by replacing the condition of coherence [éq 2.1-8] by:

$$\dot{p}=\left(\frac{\langle F\rangle}{K}\right)^N \quad \textbf{éq 2.1-10}$$

$\langle F\rangle$ positive part of F (hooks of Macauley), K, N characteristics of viscosity (Norton) of material. Unchanged all the other equations of the model are left. It will be seen that such an introduction of viscosity involves only minor modifications of the implicit algorithm of integration of the law of behavior.

The effect of memory consists in replacing the evolution of isotropic work hardening by:

$$F(\sigma, R, X)=(\tilde{\sigma}-X_1-X_2)_{eq}-R_0-R(p)$$
$$\dot{R}=b(Q-R)\dot{p}$$
$$Q=Q_0+(Q_m-Q_0)(1-e^{-2\mu q})$$

$f(\varepsilon^p, \xi, q)=\frac{2}{3}J_2(\varepsilon^p-\xi)-q\le 0$ defining a field characterizing the maximum plastic deformations, of which q measurement the ray and ξ the center, calculated according to a law of normality i.e. with the law of evolution: $\dot{\xi}=\frac{1-\eta}{\eta}\dot{q}n^*$. The parameter η (which does not exist in the initial formulation [bib.2]), I makes it possible to partially take into account the effect of memory. If it is equal to 0.5, the initial formulation is found. If it is worth 1, q is equal to the standard of the greatest plastic deformation reached. If it is much lower than 0.5, the effect of memory is taken into account partly only.

Note:

- *The definition of* $\mathbf{X}_1$ *and* $\mathbf{X}_2$ *in the form [éq 2.1-9]:*
  - *allows to keep a formulation which takes into account the variations of the parameters with the temperature without introducing term in* $\dot{T}$ *as in [bib.4], in the same way that the viscoplastic model of Chaboche. These terms are necessary because their not taken into account would lead to inaccurate results [bib4].*
  - *allows to have a coherent writing with the thermodynamic expression of the plastic potential [bib2] (p.221).*

- *It is noted that the functions $C_1(p), \gamma_1(p), C_2(p), \gamma_2(p), R(p)$ intervening in the preceding equations make it possible all the three to model various effects of work hardening not linear. The introduction of work hardening, is on the level of the kinematic part, by $C(p)$, that is to say on the level of the term of recall, by the function $\gamma(p)$, the same effect on the classification tests [bib2] does not have. The use of a model with $\gamma(p)$ allows in particular to identify more easily of strong cyclic work hardenings. Several work of identification of the coefficients of the models of Chaboche was carried out besides on the basis of model with a work hardening represented by $\gamma(p)$ ([bib5], [bib6]), in particular for the stainless steels.*

2.2 Addition of the effect of memory

Implicit discretization of the problem with effect of memory led to a system of 20 equations to 20 unknown factors [7]:

6 éq: $\tilde{\sigma}=\frac{\mu}{\mu^-}\tilde{\sigma}^-+2\mu\left(\Delta\tilde{\varepsilon}-\Delta\varepsilon^p\right)$

1 eq :

$$\left(\tilde{\sigma}-\frac{2}{3}C_1\alpha_1^--\frac{2}{3}C_2\alpha_2^--\frac{2}{3}C_1\Delta\alpha_1(\Delta\varepsilon^p)-\frac{2}{3}C_2\Delta\alpha_2(\Delta\varepsilon^p)\right)_{eq}=R_0+R^-+\Delta R+K\left(\frac{\Delta p}{\Delta t}\right)^{1/N}$$

6 eq : $$\Delta\varepsilon^p=\frac{3}{2}\Delta p\,\frac{\tilde{\sigma}-\frac{2}{3}C_1\alpha_1^--\frac{2}{3}C_2\alpha_2^--\frac{2}{3}C_1\Delta\alpha_1-\frac{2}{3}C_2\Delta\alpha_2}{R_0+R(p)+K\left(\frac{\Delta p}{\Delta t}\right)^{1/N}}=\Delta p\; n$$

1 eq: $f(\varepsilon^p,\boldsymbol{\xi},q)=\frac{2}{3}J_2(\varepsilon^p-\xi)-q=\frac{2}{3}\sqrt{\frac{3}{2}(\varepsilon^p-\xi):(\varepsilon^p-\xi)}-q\le 0$

6 eq: $\Delta\xi=(1-\eta)\frac{\Delta q}{\eta}n^*$

with $\Delta R=b(Q-R)\Delta p$ $\quad Q=Q_0+(Q_m-Q_0)\left(1-e^{-2\mu(q^-+\Delta q)}\right)$

$\Delta q=\eta H(F)\langle\mathbf{n}:\mathbf{n}^*\rangle\Delta p$

$$\Delta\alpha_i=\frac{\Delta\varepsilon^p-\gamma_i\alpha_i^-\Delta p}{1+\gamma_i\Delta p}\qquad n^*=\frac{3}{2}\frac{\varepsilon^p-\xi}{J_2(\varepsilon^p-\xi)}\qquad n=\frac{3}{2}\frac{\tilde{\sigma}-\frac{2}{3}C_1\alpha_1-\frac{2}{3}C_2\alpha_2}{\left(\tilde{\sigma}-\frac{2}{3}C_1\alpha_1-\frac{2}{3}C_2\alpha_2\right)_{eq}}$$

the 20 unknown factors are: $\tilde{\sigma},\Delta\varepsilon^p,\Delta\xi,\Delta p,\Delta q$

2.3 Insertion of the effect of nonproportionality of the loading

In a way similar to the model `VISCOCHAB`, one can insert in `VISC_CIN2_CHAB/MEMO` equations translating the effect nonproportional. model obtained is here called `VISC/VMIS_CIN2_NRAD`, or `VISC/VMIS_MEMO_NRAD` (according to whether one takes into account or not the effect of memory).

$$\dot{\alpha}_1=\dot{\varepsilon}^p-\gamma_1(p)\alpha_1\dot{p} \qquad \dot{\boldsymbol{\alpha}}_1=\dot{\boldsymbol{\varepsilon}}^p-\gamma_1(p)\left(\delta_1\boldsymbol{\alpha}_1+(1-\delta_1)(\boldsymbol{\alpha}_1:\boldsymbol{n})\boldsymbol{n}\right)\dot{p}$$
becomes:
$$\dot{\alpha}_2=\dot{\varepsilon}^p-\gamma_2(p)\alpha_2\dot{p} \qquad \dot{\boldsymbol{\alpha}}_2=\dot{\boldsymbol{\varepsilon}}^p-\gamma_2(p)\left(\delta_2\boldsymbol{\alpha}_2+(1-\delta_2)(\boldsymbol{\alpha}_2:\boldsymbol{n})\boldsymbol{n}\right)\dot{p}$$

with $\boldsymbol{n}=\sqrt{\frac{3}{2}}\frac{\tilde{\boldsymbol{\sigma}}-\boldsymbol{X}_1-\boldsymbol{X}_2}{(\tilde{\boldsymbol{\sigma}}-\boldsymbol{X}_1-\boldsymbol{X}_2)_{eq}}$ thus $\boldsymbol{n}:\boldsymbol{n}=1$ and in particular $\dot{\boldsymbol{\varepsilon}}^p=\sqrt{\frac{3}{2}}\Delta p\boldsymbol{n}$

It is easy to check that this new expression of the evolution of the internal variables $\boldsymbol{\alpha}_i$ cost with the preceding expression if $\delta_i=1$, or in the event of **radial situation** , where one can pose $\boldsymbol{\alpha}_i=\xi\boldsymbol{n}$.

It comes then: $\dot{\boldsymbol{\alpha}}_i=\dot{\boldsymbol{\varepsilon}}^p-\gamma_i\dot{p}\left(\delta_i\xi\boldsymbol{n}+(1-\delta_i)\xi\boldsymbol{n}\right)=\dot{\boldsymbol{\varepsilon}}^p-\gamma_i\dot{p}\boldsymbol{\alpha}_i$.

3 Integration of the relations of behavior

To numerically carry out the integration of the law of behavior, one carries out a discretization in time and one adopts a diagram of implicit, famous Euler adapted for elastoplastic relations of behavior. Henceforth, the following notations will be employed: A^- , A and ΔA represent respectively the values of a quantity at the beginning and the step of time considered thus that its increment during the step. The problem is then the following: knowing the state at time t^- as well as the increments of deformation $\Delta\boldsymbol{\varepsilon}$ (resulting from the phase of prediction (cf reference material from STAT_NON_LINE [R5.03.01])) and of temperature ΔT , to determine the state of the internal variables at time t as well as the constraints $\boldsymbol{\sigma}$.

One takes into account the variations of the characteristics compared to the temperature by noticing that:

$$\sigma^H=\frac{K}{K^-}\sigma^{H^-}+K\,\mathrm{tr}\left(\Delta\varepsilon-\Delta\varepsilon^{\mathbf{th}}\right) \qquad \text{éq 2.2-1}$$

$$\tilde{\sigma}=\frac{\mu}{\mu^-}\tilde{\sigma}^-+2\mu\left(\Delta\tilde{\varepsilon}-\Delta\varepsilon^p\right)=\tilde{\sigma}^\varepsilon-2\mu\Delta\varepsilon^p \qquad \text{éq 2.2-2}$$

with

$$\tilde{\sigma}^\varepsilon=\frac{\mu}{\mu^-}\tilde{\sigma}^-+2\mu\Delta\tilde{\varepsilon}$$

Within sight of the equation [éq 2.2-1], one notes that the hydrostatic behavior is purely elastic if K is constant. Only the treatment of the deviatoric component is delicate.

In the absence of viscous term, the relation of discretized coherence is:

Elastic mode: $F\le 0$ and $\Delta p=0$
Plastic mode: $F=0$ and $\Delta p\ge 0$

On the other hand, in the presence of viscosity, the condition of coherence is replaced by the equation [éq 2.1 - 10] which, discretized, is written:

$$\frac{\Delta p}{\Delta t}=\left(\frac{\langle F\rangle}{K}\right)^N\Leftrightarrow\langle F\rangle=K\left(\frac{\Delta p}{\Delta t}\right)^{1/N}$$

In other words, while posing:

$$\tilde{F}=F-K\left(\frac{\Delta p}{\Delta t}\right)^{1/N}$$

the viscoplastic increment of cumulated deformation is determined by:

$$\begin{array}{rl}\text{Régime élastique :} & \tilde{F}\le 0 \text{ et } \Delta p=0\\ \text{Régime viscoplastique :} & \tilde{F}=0 \text{ et } \Delta p\ge 0\end{array}$$ **éq 2.2-3**

Finally, by adopting an implicit discretization, the only difference between the laws in plastic and viscoplastic behavior lies in the form of the function of load F : one observes a complementary term in the event of viscosity there. In fact, incremental plasticity seems the borderline case of incremental viscoplasticity when K tends towards zero. This convergence was already described by J.L. Chaboche and G. Cailletaud in [bib3].

In the continuation of this paragraph, one will thus detail the integration of the viscoplastic law. To find the case of the plastic behavior, it is enough to take $K=0$ in the equations below (one recalls that the user to place itself in this case must obligatorily remove the keyword LEMAITRE or LEMAITRE_FO order DEFI_MATERIAU).

$$\tilde{\sigma}-X_1-X_2=\tilde{\sigma}^e-\frac{2}{3}C_1\alpha_1^- -\frac{2}{3}C_2\alpha_2^- -2\mu\Delta\varepsilon^p-\frac{2}{3}\left(C_1\Delta\alpha_1+C_2\Delta\alpha_2\right)$$

Equations of flow [éq 2.1-6] and [éq 2.1-7], once discretized, and the condition of coherence [éq 2.2-3] are written (by noticing that $p=\lambda$):

$$\Delta\varepsilon^p=\frac{3}{2}\Delta p\frac{\tilde{\sigma}^e-\frac{2}{3}C_1\alpha_1^- -\frac{2}{3}C_2\alpha_2^- -2\mu\Delta\varepsilon^p-\frac{2}{3}C_1\Delta\alpha_1-\frac{2}{3}C_2\Delta\alpha_2}{\left(\tilde{\sigma}^e-\frac{2}{3}C_1\alpha_1^- -\frac{2}{3}C_2\alpha_2^- -2\mu\Delta\varepsilon^p-\frac{2}{3}C_1\Delta\alpha_1-\frac{2}{3}C_2\Delta\alpha_2\right)_{eq}}$$ **éq 2.2-4**

$$\tilde{F}\le 0 \quad \Delta p\ge 0 \quad \tilde{F} \quad \Delta p=0$$ **éq 2.2-5**

The treatment of the condition of coherence (preceding equation) is classical. One starts with an elastic test ($\Delta p=0$) who is well the solution if the criterion of plasticity is not exceeded, i.e. if:

$$\left(\tilde{\sigma}^e-\frac{2}{3}C_1(p^-)\alpha_1^- -\frac{2}{3}C_2(p^-)\alpha_2^-\right)_{eq}-R(p^-)<0$$ **éq 2.2-6**

In the contrary case, the solution is plastic ($\Delta p>0$) and the condition of coherence is reduced to $\tilde{F}=0$. To solve it, it is shown that one can bring back oneself to a scalar problem while expressing $\Delta\varepsilon^p$ and $\Delta\alpha_1,\Delta\alpha_2$ according to Δp . By gathering the equations of the problem resulting from the implicit discretization, one obtains the system of equations:

$$\left(\tilde{\sigma}^e - \frac{2}{3}C_1\alpha_1^- - \frac{2}{3}C_2\alpha_2^- - 2\mu\Delta\varepsilon^p - \frac{2}{3}C_1\Delta\alpha_1 - \frac{2}{3}C_2\Delta\alpha_2\right)_{eq} = R(p) + K\left(\frac{\Delta p}{\Delta t}\right)^{1/N} \qquad \textbf{éq 2.2-7}$$

$$\Delta\varepsilon^p = \frac{3}{2}\Delta p \frac{\tilde{\sigma}^e - \frac{2}{3}C_1\alpha_1^- - \frac{2}{3}C_2\alpha_2^- - 2\mu\Delta\varepsilon^p - \frac{2}{3}C_1\Delta\alpha_1 - \frac{2}{3}C_2\Delta\alpha_2}{R(p) + K\left(\frac{\Delta p}{\Delta t}\right)^{1/N}} \qquad \textbf{éq 2.2-8}$$

$$\begin{aligned} \Delta\alpha_1 &= \Delta\varepsilon^p - \gamma_1\alpha_1\Delta p \\ \Delta\alpha_2 &= \Delta\varepsilon^p - \gamma_2\alpha_2\Delta p \end{aligned} \qquad \textbf{éq 2.2-9}$$

In this writing, it should well be noted that $p = p^- + \Delta p$ and $\alpha_i = \alpha_{i^-} + \Delta\alpha_i$ and that C_i, γ_i are functions of p. By considering the three last equations, this linear system in $\Delta\varepsilon^p$ and $\Delta\alpha_i$ can be solved to express these quantities according to Δp. Indeed, it is equivalent to:

$$\Delta\varepsilon^p\left(R(p) + 3\mu\Delta p + K\left(\frac{\Delta p}{\Delta t}\right)^{1/N}\right) = \Delta p\left(\frac{3}{2}\tilde{\sigma}^e - C_1\alpha_1^- - C_2\alpha_2^- - C_1\Delta\alpha_1 - C_2\Delta\alpha_2\right) \qquad \textbf{éq 2.2-10}$$

$$\begin{aligned} \Delta\alpha_1(1+\gamma_1\Delta p) &= \Delta\varepsilon^p - \gamma_1\alpha_1^-\Delta p \\ \Delta\alpha_2(1+\gamma_2\Delta p) &= \Delta\varepsilon^p - \gamma_2\alpha_2^- Dp \end{aligned} \qquad \textbf{éq 2.2-11}$$

While calculating $C_1\Delta\alpha_1$ and $C_2\Delta\alpha_2$ and by replacing them in the expression of $\Delta\varepsilon^p$ one obtains an expression of $\Delta\varepsilon^p$ according to Δp only:

$$\begin{aligned} C_1\Delta\alpha_1 &= \left(\frac{C_1}{1+\gamma_1\Delta p}\right)\Delta\varepsilon^p - \left(\frac{C_1\gamma_1\alpha_1^-\Delta p}{1+\gamma_1\Delta p}\right) = M_1(p)\Delta\varepsilon^p - M_1(p)\gamma_1\Delta p\,\alpha_1^- \\ C_2\Delta\alpha_2 &= \left(\frac{C_2}{1+\gamma_2\Delta p}\right)\Delta\varepsilon^p - \left(\frac{C_2\gamma_2\alpha_2^-\Delta p}{1+\gamma_2\Delta p}\right) = M_2(p)\Delta\varepsilon^p - M_2(p)\gamma_2\Delta p\,\alpha_2^- \\ \text{avec } M_i(p) &= \frac{C_i(p)}{1+\gamma_i(p)\Delta p} \end{aligned} \qquad \textbf{éq 2.2-12}$$

By deferring this expression in the expression of $\Delta\varepsilon^p$ one finds:

$$\Delta\varepsilon^p = \frac{1}{\left(R(p) + (3\mu + M_1 + M_2)\Delta p + K\left(\frac{\Delta p}{\Delta t}\right)^{1/N}\right)}\left(\frac{3}{2}\Delta p\,\tilde{\sigma}^e - \Delta p\left((C_1 - M_1\gamma_1\Delta p)\,\alpha_1^- + (C_2 - M_2\gamma_2\Delta p)\,\alpha_2^-\right)\right)$$

what is simplified in:

$$\Delta \varepsilon^p = \frac{1}{D(p)} \left(\frac{3}{2} \Delta p \tilde{\sigma}^e - \Delta p \left(M_1 \alpha_1^- + M_2 \alpha_2^- \right) \right) \quad \textbf{éq 2.2-13}$$

with:

$$D(p) = R(p) + \left(3\mu + M_1(p) + M_2(p) \right) \Delta p + K \left(\frac{\Delta p}{\Delta t} \right)^{1/N}$$

It now only remains to replace $\Delta \varepsilon^p$ in the expressions of $C_1 \Delta \alpha_1$ and $C_2 \Delta \alpha_2$ to express this term according to Δp by:

$$C_1 \Delta \alpha_1 = \frac{M_1}{D} \left(\frac{3}{2} \Delta p \tilde{\sigma}^e - \Delta p \left(M_1 \alpha_1^- + M_2 \alpha_2^- \right) \right) - M_1 \gamma_1 \Delta p \alpha_1^-$$
$$C_2 \Delta \alpha_2 = \frac{M_2}{D} \left(\frac{3}{2} \Delta p \tilde{\sigma}^e - \Delta p \left(M_1 \alpha_1^- + M_2 \alpha_2^- \right) \right) - M_2 \gamma_2 \Delta p \alpha_2^-$$

then to substitute the expression obtained thus that $\Delta \varepsilon^p$ according to Δp in the equation $\tilde{F} = 0$, and one obtains a scalar equation in Δp to solve, namely:

$$\tilde{F}(p) = \left(\tilde{\sigma}^e - \frac{2}{3} C_1 \alpha_1^- - \frac{2}{3} C_2 \alpha_2^- - 2\mu \Delta \varepsilon^p - \frac{2}{3} C_1 \Delta \alpha_1 - \frac{2}{3} C_2 \Delta \alpha_2 \right)_{eq} - R(p) - K \left(\frac{\Delta p}{\Delta t} \right)^{1/N} = 0$$

what is simplified in:

$$\tilde{F}(p) = \frac{R(p) + K \left(\frac{\Delta p}{\Delta t} \right)^{1/N}}{D(p)} \left(\tilde{\sigma}^e - \frac{2}{3} M_1 \alpha_1^- - \frac{2}{3} M_2 \alpha_2^- \right)_{eq} - R(p) - K \left(\frac{\Delta p}{\Delta t} \right)^{1/N} = 0 \quad \textbf{éq 2.2-14}$$

This scalar equation in Δp is solved numerically, by a research method of zero of function (method of secants which one briefly describes in appendix 2).

It is normalized in the following way:

$$\tilde{\tilde{F}}(p) = 1 - \frac{D(p)}{\left(\tilde{\sigma}^e - \frac{2}{3} M_1 \alpha_1^- - \frac{2}{3} M_2 \alpha_2^- \right)_{eq}} = 0 \quad \text{éq 2.2-15}$$

Once determined Δp, one can calculate $\Delta \varepsilon^p$ using the equation [éq 2.2-13] then $\Delta \alpha_1$ and $\Delta \alpha_2$ using the equations [éq 2.2-11]. It any more but does not remain to calculate the tensor of the constraints, by the equations [éq 2.2-1] and [éq 2.2-2], and to bring up to date the internal variables α_1 and α_2.

Note:

- *an interesting borderline case (for the validation of this model) arises while posing $\gamma_i = 0$. One finds oneself then exactly in the situation of linear kinematic work hardening (if $R(p) = \sigma_y$, [R5.03.02]) or of mixed work hardening for $R(p)$ unspecified (cf [R5.03.16]),*
- *these models are also available in plane constraints, by a global method (static condensation due to R. of Borst) [R5.03.03].*

3.1 Integration of the terms taking of account it not radiality

The discretization leads to: $\Delta\boldsymbol{\alpha}_i=\Delta\boldsymbol{\varepsilon}^p-\gamma_i\Delta p\left[\delta_i(\boldsymbol{\alpha}_i^-+\Delta\boldsymbol{\alpha}_i)+(1-\delta_i)((\boldsymbol{\alpha}_i^-+\Delta\boldsymbol{\alpha}_i):\boldsymbol{n})\boldsymbol{n}\right]$

Let us calculate $\Delta\boldsymbol{\alpha}_i:\boldsymbol{n}=\sqrt{\frac{3}{2}}\Delta p-\gamma_i\Delta p\left(\delta_i\sqrt{\frac{3}{2}}\beta_i+\delta_i\Delta\boldsymbol{\alpha}_i:\boldsymbol{n}+(1-\delta_i)\sqrt{\frac{3}{2}}\beta_i+(1-\delta_i)(\Delta\boldsymbol{\alpha}_i:\boldsymbol{n})\right)$

while having posed $\boldsymbol{\alpha}_i^-:\boldsymbol{n}=\sqrt{\frac{3}{2}}\beta_i$. O N can thus express $\Delta\boldsymbol{\alpha}_i:\boldsymbol{n}$ according to Δp and β_i

$\Delta\boldsymbol{\alpha}_i:\boldsymbol{n}(1+\gamma_i\Delta p)=\sqrt{\frac{3}{2}}\Delta p(1-\gamma_i\beta_i)$ that is to say $\Delta\boldsymbol{\alpha}_i:\boldsymbol{n}=\dfrac{\sqrt{\frac{3}{2}}\Delta p(1-\gamma_i\beta_i)}{(1+\gamma_i\Delta p)}$

One can thus express $\Delta\boldsymbol{\alpha}_i$ only according to Δp and $\beta_i=\sqrt{\frac{2}{3}}\boldsymbol{\alpha}_i^-:\boldsymbol{n}$ and to propagate these modifications in method of resolution used previously:

$$\Delta\boldsymbol{\alpha}_i(1+\gamma_i\delta_i\Delta p)=\Delta\boldsymbol{\varepsilon}^p-\gamma_i\Delta p\,\delta_i\boldsymbol{\alpha}_i^--\gamma_i\Delta p(1-\delta_i)(\boldsymbol{\alpha}_i^-:\boldsymbol{n})\boldsymbol{n}-\gamma_i\Delta p(1-\delta_i)(\Delta\boldsymbol{\alpha}_i:\boldsymbol{n})\boldsymbol{n}$$

By using the expression of $\Delta\boldsymbol{\alpha}_i:\boldsymbol{n}$ in fonction of Δp and β_i ,

$$\Delta\boldsymbol{\alpha}_i(1+\gamma_i\delta_i\Delta p)=\Delta\boldsymbol{\varepsilon}^p-\gamma_i\Delta p\,\delta_i\boldsymbol{\alpha}_i^--\gamma_i\Delta p(1-\delta_i)\sqrt{\frac{3}{2}}\beta_i\boldsymbol{n}-\gamma_i\Delta p(1-\delta_i)\frac{\sqrt{\frac{3}{2}}\Delta p(1-\gamma_i\beta_i)}{(1+\gamma_i\Delta p)}\boldsymbol{n}$$

$$\Delta\boldsymbol{\alpha}_i(1+\gamma_i\delta_i\Delta p)=\Delta\boldsymbol{\varepsilon}^p-\gamma_i\Delta p\,\delta_i\boldsymbol{\alpha}_i^--\gamma_i(1-\delta_i)\frac{\beta_i+\Delta p}{1+\gamma_i\Delta p}\Delta\boldsymbol{\varepsilon}^p$$

$\Delta\boldsymbol{\alpha}_i(1+\gamma_i\delta_i\Delta p)=\Delta\boldsymbol{\varepsilon}^p N_i(\Delta p,\beta_i)-\gamma_i\Delta p\,\delta_i\boldsymbol{\alpha}_i^-$ with

$$N_i(\Delta p,\beta_i)=\frac{1+\gamma_i\Delta p\,\delta_i-\gamma_i(1-\delta_i)\beta_i}{1+\gamma_i\Delta p}$$

There still, one can check that if $\delta_i=1$, one finds the equations without effect of nonradiality.

To continue to solve, it is necessary to calculate:

$C_i\Delta\boldsymbol{\alpha}_i=M_iN_i\Delta\boldsymbol{\varepsilon}^p-\gamma_i\Delta p\,\delta_iM_i\boldsymbol{\alpha}_i^-$ with $M_i=\dfrac{C_i}{(1+\gamma_i\delta_i\Delta p)}$

S I although the calculation of the increase in plastic deformation is similar to the classical case:

$$\Delta\boldsymbol{\varepsilon}^p=\sqrt{\frac{3}{2}}\Delta p\boldsymbol{n} \text{ with } \boldsymbol{n}=\sqrt{\frac{3}{2}}\frac{\tilde{\boldsymbol{\sigma}}-\frac{2}{3}C_1\boldsymbol{\alpha}_1-\frac{2}{3}C_2\boldsymbol{\alpha}_2}{\left(\tilde{\boldsymbol{\sigma}}-\frac{2}{3}C_1\boldsymbol{\alpha}_1-\frac{2}{3}C_2\boldsymbol{\alpha}_2\right)_{eq}}.$$

By using the expressions calculated previously as well as the expression of the criterion:

$$\left(\tilde{\sigma}^{e}-\frac{2}{3}C_1\alpha_1^{-}-\frac{2}{3}C_2\alpha_2^{-}-2\mu\,\Delta\,\varepsilon^{p}-\frac{2}{3}C_1\Delta\,\alpha_1-\frac{2}{3}C_2\Delta\,\alpha_2\right)_{eq}=R(p)+K\left(\frac{\Delta\,p}{\Delta\,t}\right)^{1/N}$$ it comes:

$$\Delta\,\boldsymbol{\varepsilon}^{p}\left(R(p)+3K\left(\frac{\Delta\,p}{\Delta\,t}\right)^{1/N}+\Delta\,p(3\mu+M_1N_1+M_2N_2)\right)=\frac{3}{2}\Delta\,p\left(\tilde{\boldsymbol{\sigma}}^{e}-\frac{2}{3}M_1\boldsymbol{\alpha}_1^{-}-\frac{2}{3}M_2\boldsymbol{\alpha}_2^{-}\right)$$

thus $\boldsymbol{n}=\sqrt{\frac{3}{2}}\;\dfrac{\tilde{\boldsymbol{\sigma}}^{e}-\frac{2}{3}M_1\boldsymbol{\alpha}_1^{-}-\frac{2}{3}M_2\boldsymbol{\alpha}_2^{-}}{D}$ with

$$D(\Delta\,p;\beta_1;\beta_2)=R(p)+K\left(\frac{\Delta\,p}{\Delta\,t}\right)^{1/N}+\Delta\,p(3\mu+M_1N_1+M_2N_2)$$

Notice : L with still, O N can check that if one does not take account of the nonradial effect, $\delta_i=1$, which involves $N=1$. One finds well the classical expression of the normal $\boldsymbol{n}$.

In this case; there are 3 scalar unknown factors: $\Delta\,p$, β_1 , β_2 . In fact, it is possible to express β_1 and β_2 according to $\Delta\,p$ by noticing that :

$$\boldsymbol{n}=\sqrt{\left(\frac{3}{2}\right)}\frac{\tilde{\boldsymbol{\sigma}}^{e}-\frac{2}{3}M_1\boldsymbol{\alpha}_1^{-}-\frac{2}{3}M_2\boldsymbol{\alpha}_2^{-}}{\left(\tilde{\boldsymbol{\sigma}}^{e}-\frac{2}{3}M_1\boldsymbol{\alpha}_1^{-}-\frac{2}{3}M_2\boldsymbol{\alpha}_2^{-}\right)_{eq}}$$

. One can thus determine $\boldsymbol{n}$ according to $\Delta\,p$ only, then to calculate directly $\beta_i=\sqrt{\frac{2}{3}}\boldsymbol{\alpha}_i^{-}:\boldsymbol{n}$, which becomes explicit functions then of $\Delta\,p$. To solve, it is enough to replace the expressions above in the criterion (what returns to to write $\boldsymbol{n}:\boldsymbol{n}=1$):

$$\tilde{\tilde{F}}(p)=\left(\tilde{\boldsymbol{\sigma}}^{e}-\frac{2}{3}M_1\boldsymbol{\alpha}_1^{-}-\frac{2}{3}M_2\boldsymbol{\alpha}_2^{-}\right)_{eq}-D\left(\Delta\,p;\beta_1(\Delta\,p);\beta_2(\Delta\,p)\right)=0$$

3.2 Integration of the effect of memory

In the case of the effect of memory, the function $R(p)$ is not known any more explicitly, but via the system of equations:

1 eq : $f(\varepsilon^{p},\xi,q)=\frac{2}{3}J_2(\varepsilon^{p}-\xi)-q=\frac{2}{3}\sqrt{\frac{3}{2}(\varepsilon^{p}-\xi):(\varepsilon^{p}-\xi)}-q\le 0$

6 eq: $\Delta\,\xi=(1-\eta)\,H(F)\langle\mathbf{n}:\mathbf{n}^{*}\rangle\Delta\,p\,n^{*}=(1-\eta)\dfrac{\Delta\,q}{\eta}n^{*}$

With

$$\Delta\,R=b(Q-R)\Delta\,p \qquad Q=Q_0+(Q_m-Q_0)\left(1-e^{-2\mu(q^{-}+\Delta\,q)}\right) \qquad n^{*}=\frac{3}{2}\frac{\varepsilon^{p}-\xi}{J_2(\varepsilon^{p}-\xi)}$$

Knowing Δp , one starts by calculating $f(\varepsilon^p, \xi^-, q^-)$.

If this quantity is negative, then the solution of the system managing the effect of memory is: $\Delta q = 0, \Delta\xi = 0$.

In the contrary case, knowing Δp , it is necessary to find Δq and $\Delta\xi$ such as:

$$f(\varepsilon^p, \xi, q) = \frac{2}{3} J_2(\varepsilon^p - \xi^- - \Delta\xi) - q^- - \Delta q = 0$$

$$\Delta\xi = \frac{(1-\eta)}{\eta} \Delta q\, n^* = \frac{(1-\eta)}{\eta} \Delta q \frac{3}{2} \frac{\varepsilon^{p-} + \Delta\varepsilon^p - \xi^- - \Delta\xi}{\frac{3}{2}(q^- + \Delta q)}$$

Because $\Delta\boldsymbol{\varepsilon}^{\mathbf{p}} = \frac{1}{D(p)}\left(\frac{3}{2}\Delta p\,\tilde{\boldsymbol{\sigma}}^e - \Delta p\left(M_1 \boldsymbol{\alpha}_1^- + M_2\,\boldsymbol{\alpha}_2^-\right)\right)$ can be calculated explicitly from Δp .

It remains:

$$\Delta\xi\left(1 + \frac{(1-\eta)\Delta q}{\eta(q^- + \Delta q)}\right) = \frac{(1-\eta)}{\eta} \Delta q \frac{\varepsilon^{p-} + \Delta\varepsilon^p - \xi^-}{(q^- + \Delta q)} \Rightarrow$$

$$\Delta\xi(\eta q^- + \Delta q) = (1-\eta)\Delta q(\varepsilon^p - \xi^-) \Rightarrow \Delta\xi = \frac{(1-\eta)\Delta q(\varepsilon^p - \xi^-)}{\eta q^- + \Delta q}$$

while deferring in the equation of surface threshold: $f(\varepsilon^p, \xi, q) = 0$

$$\frac{2}{3} J_2(\varepsilon^p - \xi^- - \Delta\xi) - q^- - \Delta q = 0 = \frac{2}{3} J_2(\varepsilon^p - \xi^-)\left|\left(1 - \frac{(1-\eta)\Delta q}{\eta q^- + \Delta q}\right)\right| - q^- - \Delta q = 0$$

$$\Leftrightarrow \frac{2}{3} J_2(\varepsilon^p - \xi^-)|\eta(q^- + \Delta q)| - (q^- + \Delta q)(\eta q^- + \Delta q) = 0 \ si \ \ \eta q^- + \Delta q > 0$$

what makes it possible to calculate explicitly Δq from Δp :

$$\boxed{\Delta q = \eta \frac{2}{3} J_2(\boldsymbol{\varepsilon}^{\mathbf{p}} - \boldsymbol{\xi}^-) - \eta q^-}$$

It then remains to modify the function of isotropic work hardening while calculating:

$$Q = Q_0 + (Q_m - Q_0)\left(1 - e^{-2\mu(q^- + \Delta q)}\right) \qquad \text{then } \Delta R = b(Q - R)\Delta p$$

One can thus use the resolution of the scalar equation in Δp (éq 2.2-14) by using the expressions above.

Note:

- In [bib2] one also finds the expression *:* $dq = \eta H(f)\langle \mathbf{n:n}^* \rangle dp$.

This last equation results from the expression of speed of the multiplier. In the implicit discretization carried out here, it is not used for the resolution (since then the system would comprise more equations than unknown factors). Moreover, the 3 equations given in [bib2] are redundant: indeed, knowing $\Delta\varepsilon^p$ should be determined a tensorial variable $\Delta\xi$ and a scalar variable. Δq However we have a tensorial equation and two scalar equations.

This is due to the fact that the equation $dq=\eta H(f)\langle \mathbf{n{:}n}^{*}\rangle dp$ is resulting from the condition of coherence $df=0$ (what is specified in [bib2]) but is not used for the implicit resolution of the problem.

$$df(\varepsilon^p,\xi,q)=\frac{\varepsilon^p-\xi}{J_2(\varepsilon^p-\xi)}d\varepsilon^p-\frac{\varepsilon^p-\xi}{J_2(\varepsilon^p-\xi)}d\xi-dq=n:n^* dp-n^*:n^* dq-dq=n:n^* dp-2dq=0$$

It would be useful for an explicit resolution, by expressing the derivative compared to the time of all the sought variables.

- an interesting criterion, given in [bib2] makes it possible to adjust the parameters of the effect of memory. Indeed, by considering a simple loading of traction and compression, one must find $q=\frac{1}{2}\Delta\varepsilon^{p_{\max}}$ (while choosing $\eta=\frac{1}{2}$). For a material point in uniaxial load, the fields (uniform) have as components:

$$\boldsymbol{\sigma}=\sigma\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}\quad \boldsymbol{\varepsilon}^p=p\begin{pmatrix}1&0&0\\0&-\frac{1}{2}&0\\0&0&-\frac{1}{2}\end{pmatrix}$$

In this case, at the time of the first uniaxial load in the direction x :

$$\xi^-=0$$
$$q^-=0$$
$$\Delta q=\eta\varepsilon_x^p$$

In this case, $q=\frac{1}{2}\Delta\varepsilon^{p_{\max}}$, implies that

$\eta=\frac{1}{2}$ and $\Delta\xi=\frac{1}{2}(\varepsilon^p)$

Moreover, in the case of a cycle of symmetrical traction compression (in plastic deformation), one obtains, during the first symmetrical discharge (with $\eta=\frac{1}{2}$):

$$\xi^-=\frac{1}{2}\varepsilon^{p_{\max}}$$
$$q^-=\frac{1}{2}\varepsilon_{xx}^{p_{\max}}$$
$$\Delta q=\eta\left(\frac{2}{3}J_2(\varepsilon^p)-q^-\right)=\eta\left(|\varepsilon_{xx}^{p_{\min}}-\xi^-|-\frac{1}{2}\varepsilon_{xx}^{p_{\max}}\right)=\frac{1}{2}|\varepsilon_{xx}^{p_{\min}}|$$
$$q=q^-+\Delta q=\varepsilon_{xx\max}^p=\frac{1}{2}\Delta\varepsilon_{xx}^p$$
$$\Delta\xi=\frac{(1-\eta)\Delta q(\varepsilon^p-\xi^-)}{\eta q^-+\Delta q}=-\frac{1}{2}\Delta\varepsilon_{xx\max}^p$$
$$\xi=\xi^0+\Delta\xi=0$$

what corresponds well to the expected result (cf [bib2]): field $F=0$ centered on the origin, and of ray the half-amplitude of plastic deformation.

3.3 Calculation of tangent rigidity

In order to allow a resolution of the total problem (equilibrium equations) by a method of Newton [R5.03.01], it is necessary to determine the coherent tangent matrix of the incremental problem.

This matrix is composed classically of an elastic contribution and a plastic contribution:

$$\frac{\delta\sigma}{\delta\varepsilon}=\frac{\delta\sigma^e}{\delta\varepsilon}-2\mu\frac{\delta\Delta\varepsilon^p}{\delta\varepsilon} \qquad \textbf{éq 2.3-1}$$

with $\sigma^e=\sigma+2\mu\Delta\varepsilon^p$, which gives again in particular $\tilde{\sigma}^e=\frac{\mu}{\mu^-}\tilde{\sigma}^-+2\mu\Delta\tilde{\varepsilon}$

One from of deduced immediately that in elastic mode (classical or pseudo-discharge), the tangent matrix is reduced to the elastic matrix:

$$\frac{\delta\sigma}{\delta\varepsilon}=\frac{\delta\sigma^e}{\delta\varepsilon} \qquad \textbf{éq 2.3-2}$$

For that, one once more adopts the convention of writing of the symmetrical tensors of order 2 in the form of vectors with 6 components. Thus, for a tensor a :

$$a={}^t\left[a_{xx} \quad a_{yy} \quad a_{zz} \; \sqrt{2}a_{xy} \quad \sqrt{2}a_{xz} \quad \sqrt{2}a_{yz}\right] \qquad \textbf{éq 2.3-3}$$

If moreover the hydrostatic vector is introduced 1 and stamps it deviatoric projection P :

$$1={}^t\left[1 \quad 1 \quad 1 0 \quad 0 \quad 0\right] \qquad \textbf{éq 2.3-4}$$

$$P=\mathbf{Id}-\frac{1}{3}1\otimes 1 \qquad \textbf{éq 2.3-5}$$

where $\otimes$ is the tensorial product

Then the matrix of coherent tangent rigidity is written for an elastic behavior:

$$\frac{\partial\sigma^e}{\partial\Delta\varepsilon}=K\,1\otimes 1+2\mu\,P \qquad \textbf{éq 2.3-6}$$

On the other hand, in plastic mode, the variation of the plastic deformation is not worthless any more.

One derives compared to $\tilde{\sigma}^e$, knowing that one a:

$$\frac{\delta\Delta\varepsilon^p}{\delta\varepsilon}=\frac{\delta\Delta\varepsilon^p}{\delta\tilde{\sigma}^e}.\frac{\delta\tilde{\sigma}^e}{\delta\varepsilon}=2\mu\frac{\delta\Delta\varepsilon^p}{\delta\tilde{\sigma}^e}.P \qquad \textbf{éq 2.3-7}$$

s space of the symmetrical tensors
P projector on the diverters

To calculate $\frac{\delta\Delta\varepsilon^p}{\delta\tilde{\sigma}^e}$, one uses the expression of $\Delta\varepsilon^p$ according to $\tilde{\sigma}_e$ and p :

$$\Delta \varepsilon^p = \frac{1}{D(p)} \left(\frac{3}{2} \Delta p \tilde{\sigma}_e - \Delta p \left(M_1 \alpha_1^- + M_2 \alpha_2^- \right) \right)$$

what is written in the form:

$$\Delta \varepsilon^p = A(p) \tilde{\sigma}_e + B_1(p) \alpha_1^- + B_2(p) \alpha_2^-$$

Thus:

$$\frac{\delta \Delta \varepsilon^p}{\delta \tilde{\sigma}^e} = A(p) \mathbf{Id} + \tilde{\sigma}^e \otimes \frac{\delta A(p)}{\delta \tilde{\sigma}^e} + \frac{\delta B_1(p)}{\delta \tilde{\sigma}^e} \otimes \alpha_1^- + \frac{\delta B_2(p)}{\delta \tilde{\sigma}^e} \otimes \alpha_2^-$$

Quantities of the type $\frac{\delta A(p)}{\delta \tilde{\sigma}^e}$ are calculated using: $\frac{\delta A(p)}{\delta \tilde{\sigma}^e} = \frac{\delta A(p)}{\delta p} \frac{\delta p}{\delta \tilde{\sigma}^e}$

Finally, it any more but does not remain to calculate the variation of p : $\frac{\delta p}{\delta \tilde{\sigma}^e}$

One uses for that: $\tilde{F}(p, \tilde{\sigma}^e) = 0$

$$\tilde{F}(p, \tilde{\sigma}_e) = \frac{R(p) + K\left(\frac{\Delta p}{\Delta t}\right)^{1/N}}{D(p)} \left(\tilde{\sigma}^e - \frac{2}{3} M_1 \alpha_1^- - \frac{2}{3} M_2 \alpha_2^- \right)_{eq} - R(p) - K \left(\frac{\Delta p}{\Delta t} \right)^{1/N} = 0$$

$$\tilde{F}_{,p}(p, \tilde{\sigma}^e) \delta p = - \tilde{F}_{,\tilde{\sigma}^e}(p, \tilde{\sigma}^e) \delta \tilde{\sigma}^e \Rightarrow \frac{\delta p}{\delta \tilde{\sigma}^e} = - \frac{\tilde{F}_{,\tilde{\sigma}^e}(p, \tilde{\sigma}^e)}{\tilde{F}_{,p}(p, \tilde{\sigma}^e)}$$ **éq 2.3-8**

The detail of calculations is given in appendix 1.

The initial tangent matrix, used by the option RIGI_MECA_TANG is obtained by adopting the behavior of the preceding step (elastic or plastic, meant by internal variable ξ being worth 0 or 1) and while making tend Δp towards zero in the preceding equations.

3.4 Significance of the internal variables

Internal variables of the two models at the points of Gauss (VELGA) are:

- V1 = p : cumulated plastic deformation (positive or worthless)
- V2 = ξ : being worth n (iteration count internal) if the point of Gauss plasticized during the increment or 0 if not.

The following internal variables are, for modeling 3D:

- For the model VMIS/VISC_CIN1_CHAB
  - V3 = α_{1xx}
  - V4 = α_{1yy}
  - V5 = α_{1zz}
  - V6 = α_{1xy}
  - V7 = α_{1xz}

  - `V8 =` $\alpha_{1\,yz}$

- For the model `VMIS/VISC_CIN2_CHAB`
  - `V3 =` $\alpha_{1\,xx}$
  - `V4 =` $\alpha_{1\,yy}$
  - `V5 =` $\alpha_{1\,zz}$
  - `V6 =` $\alpha_{1\,xy}$
  - `V7 =` $\alpha_{1\,xz}$
  - `V8 =` $\alpha_{1\,yz}$
  - `V9 =` $\alpha_{2\,xx}$
  - `V10 =` $\alpha_{2\,yy}$
  - `V11 =` $\alpha_{2\,zz}$
  - `V12 =` $\alpha_{2\,xy}$
  - `V13 =` $\alpha_{2\,xz}$
  - `V14 =` $\alpha_{2\,yz}$
  -

  For modelings `C_PLAN`, `D_PLAN`, and `AXIS` :

  - `V7 =` 0
  - `V8 =` 0
  - `V13 =` 0
  - `V14 =` 0

- For the model `VMIS/VISC_CIN2_MEMO`
  - `V3 =` $\alpha_{1\,xx}$
  - `V4 =` $\alpha_{1\,yy}$
  - `V5 =` $\alpha_{1\,zz}$
  - `V6 =` $\alpha_{1\,xy}$
  - `V7 =` $\alpha_{1\,xz}$
  - `V8 =` $\alpha_{1\,yz}$
  - `V9 =` $\alpha_{2\,xx}$
  - `V10 =` $\alpha_{2\,yy}$
  - `V11 =` $\alpha_{2\,zz}$
  - `V12 =` $\alpha_{2\,xy}$
  - `V13 =` $\alpha_{2\,xz}$
  - `V14 =` $\alpha_{2\,yz}$
  - `V15 =` $R(p)$
  - `V16 =` q
  - `V17 =` ξ_{xx}
  - `V18 =` ξ_{yy}
  - `V19 =` ξ_{zz}
  - `V20 =` ξ_{xy}

- V21 = ξ_{xz}
- V22 = ξ_{yz}
-
- V23 = $\varepsilon^{p}{}_{xx}$
- V24 = $\varepsilon^{p}{}_{yy}$
- V25 = $\varepsilon^{p}{}_{zz}$
- V26 = $\varepsilon^{p}{}_{xy}$
- V27 = $\varepsilon^{p}{}_{xz}$
- V28 = $\varepsilon^{p}{}_{yz}$

4 Principle of the identification of the parameters of the model.

In the simplest case (only one kinematic variable, $\gamma_1 = cste, C_1 = cste \quad , R(p) = \sigma_y$) coefficients of the model γ_1, C_1 can be identified on a simple tensile test uniaxial, or of course a cyclic curve of work hardening.

Indeed in the uniaxial case, the model is reduced in 1D to [bib2]:

$$dX_1 = C_1 d\varepsilon^p - \gamma_1 X_1 \xi d\varepsilon^p, \xi = \pm 1$$
$$|\sigma - X_1| = \sigma_y$$

that one can integrate (in monotonous loading) in the following way:

$$X_1 = \xi \frac{C_1}{\gamma_1} + \left(X_1^0 - \xi \frac{C_1}{\gamma_1} \right) \exp\left(-\xi \gamma_1 \left(\varepsilon^p - \varepsilon_0^p \right) \right), \xi = \pm 1$$
$$\sigma = \xi \sigma_y + X_1$$

whose asymptote of the traction diagram makes it possible to obtain $\frac{C_1}{\gamma_1}$ by:

$\varepsilon^p \to \infty \quad X_1 \to \xi \frac{C_1}{\gamma_1}$ thus $\sigma \to \xi \left(\sigma_y + \frac{C_1}{\gamma_1} \right)$

and whose slope in the beginning provides C_1 (if $X_1^0 = 0$) :

$$\varepsilon^p \to 0 \quad \dot{X}_1 \to C_1 - y_1 X_1^0 \xi \quad X_1^0 = C_1 - y_1 X_1 \xi$$

For a model has two variable kinematics, without isotropic work hardening, a traction diagram still allows to find these relations:

$\varepsilon^p \to \infty \quad \sigma \to \xi \left(\sigma_y + \left(\frac{C_1}{\gamma_1} + \frac{C_2}{\gamma_2} \right) \right)$ and the slope in the beginning is worth $C_1 + C_2$

But apart from these simple cases a digital identification is necessary to obtain the parameters. One will be able to make this identification for example on tensile tests compression to imposed deformation. (cf. 10).

5 Elements of validation.

The tests allowing the elementary validation of these behaviors are:

test	title	behavior (S)	
	elementary tests of robustness		
comp001f	test of robustness law of behavior 3D VMIS_CIN1_CHAB	VMIS_CIN1_CHAB	
comp001g	test of robustness law of behavior 3D VMIS_CIN2_CHAB	VMIS_CIN2_CHAB	
comp002b	test of robustness law of behavior 3D VISC_CIN1_CHAB	VISC_CIN1_CHAB	
comp002c	test of robustness law of behavior 3D VISC_CIN2_CHAB	VISC_CIN2_CHAB	
comp002h	test of robustness law of behavior 3D VISC_CIN2_MEMO	VISC_CIN2_MEMO	
comp008g	variation temperature in the behavior VMIS_CIN1_CHAB	VMIS_CIN1_CHAB	
comp008h	variation temperature in the behavior VMIS_CIN2_CHAB	VMIS_CIN2_CHAB	
comp008j	variation temperature in the behavior VISC_CIN1_CHAB	VISC_CIN1_CHAB	
comp008k	variation temperature in the behavior VISC_CIN2_CHAB	VISC_CIN2_CHAB	
comp008i	variation temperature in the behavior VMIS_CIN2_MEMO	VMIS_CIN2_MEMO	
comp008l	variation temperature in the behavior VISC_CIN2_MEMO	VISC_CIN2_MEMO	
	thermoplastic tests of the IPSI		
hsnv124c	test phi2as number 1	VMIS_CIN1_CHAB	VMIS_CIN2_CHAB
hsnv124d	test phi2as number 1	VMIS_CIN1_CHAB	VMIS_CIN2_CHAB
hsnv125c	test phi2as number 2: traction, shearing, temperature variables	VMIS_CIN1_CHAB	VMIS_CIN2_CHAB
hsnv125e	test phi2as number 2: traction, shearing, temperature variables	VMIS_CIN2_MEMO	
	retiming		
ssna109a	model VISC_CIN2_CHAB with 550 degrees, prevalent viscosity	VISC_CIN2_CHAB	
ssna110a	model retiming VISC_CIN2_CHAB on 4 traction diagrams	VISC_CIN2_CHAB	
	effect of memory		
ssnd105a	tensile test with maximum memory of work hardening	VMIS_CIN2_MEMO	
ssnd105b	tensile test with maximum memory of work hardening	VISC_CIN2_MEMO	
ssnd105c	traction with maximum memory of work hardening axis	VISC_CIN2_MEMO	
ssnd111a	validation effect of memory VISC_CIN2_MEMO	VISC_CIN2_MEMO	
	Traction-shearing		
ssnv101b	test of traction-shearing in plane constraints (chaboche)	VMIS_CIN1_CHAB	VMIS_CIN2_CHAB
ssnv101c	test of traction-shearing 3D (chaboche)	VMIS_CIN1_CHAB	VMIS_CIN2_CHAB
ssnv101d	test of traction-shearing in plane deformations (chaboche)	VMIS_CIN1_CHAB	VMIS_CIN2_CHAB
ssnv118d	tensile test shearing in 3D (viscochab/ VISC_CIN2_MEMO)	VISC_CIN1_CHAB	VISC_CIN2_CHAB
	great deformations		
ssnd107b	multiple traction-rotations gdef_log in kinematic 3D	VMIS_CIN2_CHAB	VMIS_CIN2_MEMO
	Effect of nonproportionality		
ssnd105d	tensile test with maximum memory of work hardening and not radiality	VMIS_CIN2_NRAD	VISC_CIN2_NRAD
ssnd115a	test of traction-torsion with loading nonproportional	VMIS_CIN2_NRAD	

A validation compared to experimental results was carried out in (cf. 10), on traction-torsion and tensile tests compression. It makes it possible to highlight the effect of memory and nonproportionality.

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7 Description of the versions of the document

Version Aster	Author (S) or contributor (S), organization	Description of the modifications
5	P.Schoenberger EDF/R & D /MMN	Initial text, law of Chaboche
7	E.Lorentz, J.M.Proix EDF/R & D /AMA	Addition of the laws VMIS_CIN1_CHAB, VMIS_CIN2_CHAB
8	P. of Bonnières, J.M.Proix EDF/R & D /AMA	Addition of viscosity: laws VISC_CIN1_CHAB and VISC_CIN2_CHAB, and suppression of the law CHABOCHE.
9.3	J.M.Proix EDF/R & D /AMA	Addition of the law VMIS/VISC_CIN2_MEMO, fascinating of account the effect of memory of maximum work hardening.
11 3	J.M.Proix EDF/R & D /AMA	Addition of the law VMIS/VISC_CIN2_NRAD, fascinating of account the effect of nonproportionality of the loading.
12.1	J.M.Proix	Addition of the remark on the positivity of the coefficients K and W,

	EDF/R & D /AMA	card 21019

Annexe 1 Tangent matrix of behavior

To obtain the tangent behavior in the elastoplastic case, it is necessary to calculate $\frac{d\Delta \boldsymbol{\varepsilon}^p}{d\tilde{\boldsymbol{\sigma}}^e}$ [éq 2.3-7].

One uses for that the expression of $\Delta \boldsymbol{\varepsilon}^p$ according to $\tilde{\boldsymbol{\sigma}}^e$ and p, which is written in the form:

$$\Delta \varepsilon^p = \frac{3\Delta p}{2\mathrm{D}(p)} \tilde{\sigma}_e + B_1^*(p)\alpha_1^- + B_2^*(p)\alpha_2^-$$

with

$$B_i^*(p) = -\Delta p \frac{M_i(p)}{D(p)}$$

$$M_i(p) = \frac{C_i(p)}{1+\delta_i \gamma_i(p)\Delta p}$$

$$D(p) = R(p) + \left(3\mu + M_1(p) N_1(p,\beta_1) + M_2(p) N_2(p,\beta_2)\right)\Delta p + K\left(\frac{\Delta p}{\Delta t}\right)^{1/N}$$

The following definitions are pointed out:

$$R(p) = R_\infty + (R_0 - R_\infty) e^{-bp}$$

$$C_i(p) = C_i^\infty \left(1 + (k-1) e^{-wp}\right)$$

$$\gamma_i(p) = \gamma_i^0 \left(a_\infty + (1-a_\infty) e^{-bp}\right)$$

thus:

$$\frac{\delta \Delta \varepsilon^p}{\delta \tilde{\sigma}^e} = \frac{3\Delta p}{2\mathrm{D}(p)} \mathbf{Id} + \frac{\delta\left(\frac{3\Delta p}{2\mathrm{D}(p)}\right)}{\delta \tilde{\sigma}^e} \otimes \tilde{\sigma}^e + \frac{\delta B_1^*(p)}{\delta \tilde{\sigma}^e} \otimes \alpha_1^- + \frac{\delta B_2^*(p)}{\delta \tilde{\sigma}^e} \otimes \alpha_2^-$$

Quantities of the type $\frac{\delta A(p)}{\delta \tilde{\sigma}^e}$ are calculated using: $\frac{\delta A(p)}{\delta \tilde{\sigma}^e} = \frac{\delta A(p)}{\delta p} \frac{\delta p}{\delta \tilde{\sigma}^e}$

These various terms are expressed by:

- $\frac{\delta\left(\frac{3\Delta p}{2\mathrm{D}(p)}\right)}{\delta p} = \frac{3}{2} I(p)$ with $I(p) = \frac{1}{D(p)} - \frac{D'(p)}{D^2(p)} \Delta p$
- $\frac{\delta B_i^*(p)}{\delta p} = -\frac{M_i'(p)}{D(p)} \Delta p - M_i(p) . I(p) = H_i(p)$

Let us detail the calculation of D' :

- In the case of the effect of memory, it is enough to modify the term $R'(p)$.

Like $R = R^- + \Delta R = R^- + b\frac{\left(Q(\Delta p) - R^-\right)}{1+b\Delta p} \Delta p = R^- + \frac{b\Delta p}{1+b\Delta p}\left(Q_M + (Q_0 - Q_M) e^{-2\mu q} - R^-\right)$

$$R'(p) = \frac{b}{1+b\Delta p}\left(\frac{Q-R^-}{1+b\Delta p} - 2\mu\Delta p (Q - Q_M) \frac{\partial \Delta q}{\partial \Delta p}\right) = \frac{b}{1+b\Delta p}\left(\frac{Q-R^-}{1+b\Delta p} - 2\mu\Delta \mathrm{p} (Q_0 - Q_M) \frac{\partial \Delta q}{\partial \Delta p}\right)$$

however $\Delta q=\eta\left(\frac{2}{3}J_2\left(\varepsilon^p-\xi^-\right)-q^-\right)$ thus $\frac{\partial\Delta q}{\partial\Delta p}=\eta\frac{\varepsilon^p-\xi^-}{J_2\left(\varepsilon^p-\xi^-\right)}\frac{\partial\varepsilon^p}{\partial\Delta p}$

and $\frac{\delta\Delta\varepsilon^p}{\delta\Delta p}=\frac{\delta\left(\frac{3\Delta p}{2\mathrm{D}(p)}\right)}{\delta\Delta p}\tilde\sigma^e+\frac{\delta B_1^*(p)}{\delta\Delta p}\alpha_1^-+\frac{\delta B_2^*(p)}{\delta\Delta p}\alpha_2^-=\frac{3}{2}I(\Delta p)\tilde\sigma^e+H_1^*(\Delta p)\alpha_1^-+H_2^*(\Delta p)\alpha_2^-$

- In the case of not proportionality ($\delta_1\neq 1$ or $\delta_2\neq 1$), some derivative are modified:

$$M_i'(p)=\frac{C_i'(p)}{1+\delta_i\gamma_i(p)\Delta p}-\frac{C_i(p)}{(1+\delta_i\gamma_i(p)\Delta p)^2}\left(\gamma_i'\delta_i\Delta p+\gamma_i\delta_i\right)$$

$$D'=R'+\frac{K}{N\Delta t}\left(\frac{\Delta p}{\Delta t}\right)^{\frac{1}{N}-1}+3\mu+M_1N_1+M_2N_2+\Delta p(M_1'N_1+M_2'N_2+M_1N_1'+M_2N_2')$$

with $N_i'=\frac{1+\gamma_i'(\delta_i\Delta p+(\delta_i-1)\beta_i)+\gamma_i(\delta_i+(\delta_i-1)\beta_i')-N_i(\gamma_i+\gamma_i'\Delta p)}{1+\gamma_i\Delta p}$

It remains to calculate: $\frac{\delta p}{\delta\tilde\sigma^e}$

One thus uses, following [éq 2.3-8]: $\frac{\delta p}{\delta\tilde\sigma^e}=-\frac{\tilde F_{,\tilde\sigma^e}(p,\tilde\sigma^e)}{\tilde F_{,p}(p,\tilde\sigma^e)}$

$$\tilde F(p,\tilde\sigma^e)=S_{eq}(p,\tilde\sigma^e)-R(p)-K\left(\frac{\Delta p}{\Delta t}\right)^{1/N}=G(p,\tilde\sigma^e)-R(p)-K\left(\frac{\Delta p}{\Delta t}\right)^{1/N}$$

with

$$S=A\tilde\sigma^e+B_1\alpha_1^-+B_2\alpha_2^-\qquad A=\frac{R(p)+K\left(\frac{\Delta p}{\Delta t}\right)^{1/N}}{D(p)}\quad B_i=-\frac{2}{3}\frac{M_i(p)\left(R(p)+K\left(\frac{\Delta p}{\Delta t}\right)^{1/N}\right)}{D(p)}$$

Then, while posing $R_v(p)=R(p)+K\left(\frac{\Delta p}{\Delta t}\right)^{1/N}$:

$$\frac{\delta p}{\delta\tilde\sigma^e}=-\frac{G_{,\tilde\sigma^e}(p,\tilde\sigma^e)}{G_{,p}(p,\tilde\sigma^e)-R_v'(p)}=-\frac{\frac{3}{2}\frac{R_v(p)}{D(p)}\frac{S}{S_{eq}}}{\frac{3}{2}\frac{S}{S_{eq}}:S_{,p}-R_v'(p)}=-\frac{3}{2}\frac{\frac{R_v}{DS_{eq}}\left(A\tilde\sigma^e+B_1\alpha_1^-+B_2\alpha_2^-\right)}{\frac{3}{2}\frac{S}{S_{eq}}:S_{,p}-R_v'(p)}$$

$$=-\frac{3}{2}\frac{L_1(p).\tilde\sigma^e+L_{21}(p)\alpha_1^-+L_{22}(p)\alpha_2^-}{L_3(p)}$$

¿
with

$$L_1(p)=\frac{R_v^2(p)}{D^2(p)\times S_{eq}}=\frac{A^2(p)}{S_{eq}}$$

$$L_{21}(p)=\frac{R_v(p)}{D(p)}B_1(p)\frac{1}{S_{eq}}\qquad L_{22}(p)=\frac{R_v(p)}{D(p)}B_2(p)\frac{1}{S_{eq}}$$ ¿

$$L_3(p)=\frac{3}{2}\frac{S}{S_{eq}}:\left(A'(p)\tilde{\sigma}^e+B_1'(p)\alpha_1^-+B_2'(p)\alpha_2^-\right)-R'(p)-\frac{K}{N\Delta t}\left(\frac{\Delta p}{\Delta t}\right)^{\frac{1}{N}-1}$$

Finally, $\frac{\delta\Delta\varepsilon^p}{\delta\tilde{\sigma}^e}$ puts itself in the form:

$$\frac{\delta\Delta\varepsilon^p}{\delta\tilde{\sigma}^e}=\frac{3}{2}\frac{\Delta p}{D(p)}\mathbf{Id}+\frac{3}{2}\left(I_S(p)\tilde{\sigma}^e+I_{a1}(p)\alpha_1^-+I_{a2}(p)\alpha_2^-\right)\otimes\tilde{\sigma}^e$$
$$+\left(H_s^1\tilde{\sigma}^e+H_{a1}^1\alpha_1^-+H_{a2}^1\alpha_2^-\right)\otimes\alpha_1^-$$
$$+\left(H_s^2\tilde{\sigma}^e+H_{a1}^2\alpha_1^-+H_{a2}^2\alpha_2^-\right)\otimes\alpha_2^-$$

with:

$$I_S(p)=-\frac{3}{2}I(p).\frac{L_1(p)}{L_3(p)}\qquad I_{a1}(p)=-\frac{3}{2}\frac{I(p)L_{21}(p)}{L_3(p)}$$
$$I_{a2}(p)=-\frac{3}{2}\frac{I(p)L_{22}(p)}{L_3(p)}$$

$$H_s^1(p)=-\frac{3}{2}\frac{H_1(p).L_1(p)}{L_3(p)}\quad H_{a1}^1(p)=-\frac{3}{2}\frac{H_1(p)L_{21}(p)}{L_3(p)}\quad H_{a2}^1(p)=-\frac{3}{2}\frac{H_1(p)L_{22}(p)}{L_3(p)}$$
$$H_s^2(p)=-\frac{3}{2}\frac{H_2(p).L_1(p)}{L_3(p)}\quad H_{a1}^2(p)=-\frac{3}{2}\frac{H_2(p)L_{21}(p)}{L_3(p)}\quad H_{a2}^2(p)=-\frac{3}{2}\frac{H_2(p)L_{22}(p)}{L_3(p)}$$

Annexe 2 Resolution of the equation F (△ p) = 0

It is a question of solving a nonlinear scalar equation by seeking the solution in a confidence interval. For that, one proposes to couple a method of secant with a control of the interval of research. That is to say the following equation to solve:

$$f(x)=0\text{ , }x\in[a,b]\text{ , }f(a)<0\text{ , }f(b)>0 \qquad \textbf{éq A2-1}$$

The method of the secant consists in building a succession of points x^n who converges towards the solution. It is defined by recurrence (linear approximation of the function by its cord):

$$x^{n+1}=x^{n-1}-f(x^{n-1})\frac{x^n-x^{n-1}}{f(x^n)-f(x^{n-1})} \qquad \textbf{éq A2-2}$$

In addition, if x^{n+1} was to leave the interval, then one replaces it by the terminal of the interval in question:

$$\begin{cases}\text{si } x^{n+1}<a \text{ alors } x^{n+1}:=a\\ \text{si } x^{n+1}>b \text{ alors } x^{n+1}:=b\end{cases} \qquad \textbf{éq A2-3}$$

On the other hand, if x^{n+1} is in the interval running, then one reactualizes the interval:

$$\begin{cases}\text{si } x^{n+1}\in[a,b] \text{ et } f(x^{n+1})<0 \text{ alors } a=x^{n+1}\\ \text{si } x^{n+1}\in[a,b] \text{ et } f(x^{n+1})>0 \text{ alors } b=x^{n+1}\end{cases} \qquad \textbf{éq A2-4}$$

One considers to have converged when f is sufficiently close to 0 (tolerance to be informed). As for the first two leader characters, one can choose the terminals of the interval, or, if one has an estimate of the solution, one can adopt this estimate and one of the terminals of the interval.

Note:

This method functions well if there is only one solution in the interval $[a,b]$. Without that being formally shown, one can note that $f(0)>0$.
One seeks then b such as $f(b)<0$.
One leaves for that $b=\frac{\left(\tilde{s}^e\frac{2}{3}C_1\alpha_1^- -\frac{2}{3}C_2\alpha_2^-\right)_{eq}-R(p^-)}{3\mathrm{m}}$
If $f(b)>0$, one multiplies b by 10 and one tests if $f(b)>0$, and so on, until finding a value b such as $f(b)<0$.
One is sure that there is then at least a solution on $[a,b]$.