Viscoplastic relation of behavior of Taheri

Summary

One presents in this document the establishment of the relation of behavior of Taheri viscoplastic, available for the whole of the isoparametric elements (continuous medium 2D and 3D) except for the plane constraints. After a presentation of the equations of evolution of this law, one describes the system obtained by implicit discretization; it is shown in particular that he always admits a solution.

This model is well adapted to describe the answer of the austenitic steels under cyclic requests, and in particular the phenomenon of progressive deformation. On the other hand, because of its complexity (two surfaces of load, semi-discrete internal variable), it does not appear desirable to employ it for different applications (monotonous way of loading, for example).

Contents

1 Description of the model

The relation of behavior proposed by Taheri [bib5] makes it possible to describe the answer of the austenitic steels under cyclic requests: it is indeed well adapted to represent the phenomenon of progressive deformation. Before stating the equations themselves, one can specify that this model differs from classical plasticity (criterion of von Mises with kinematic and isotropic work hardening) by two characteristics, sources of difficulties in the digital formulation. On the one hand, the evolution of the dissipative variables rests on two criteria of load instead of one: the first, classic, condition the appearance of plastic deformation, the second makes it possible to keep a trace of the “maximum” work hardening reached by material to give an account of the phenomenon of ratchet. In addition, to represent the progressive deformation satisfactorily, a semi-discrete internal variable was introduced. Constant when the behavior is dissipative, it evolves only in the elastic mode of material. Of original appearance, this model does not rest any less on physical bases, always exposed in Taheri [bib5]. It is accessible, in a viscoplastic wide version (necessary to describe the behavior under high temperatures), by the order `STAT_NON_LINE` under the keyword `RELATION : VISC_TAHERI`.

1.1 Plastic behavior

A detailed description of the law of behavior is given in Taheri and al. [bib6]. Briefly, the state of material is described by its state of deformation, its temperature like four internal variables:

$\boldsymbol{\varepsilon}$	tensor of total deflection
T	temperature
p	cumulated plastic deformation
$\boldsymbol{\varepsilon}^p$	tensor of plastic deformation
σ^p	constraint of peak, memory of maximum work hardening
$\boldsymbol{\varepsilon}_n^p$	plastic tensor deformation due to the last discharge (variable semi-discrete).

The equations of state which express the thermodynamic forces associated according to the variables with state write:

$$\boldsymbol{\sigma}=K\,\mathrm{Tr}\left(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^{th}\right)\mathbf{Id}+2\mu\left(\tilde{\boldsymbol{\varepsilon}}-\boldsymbol{\varepsilon}^{p}\right) \qquad \boldsymbol{\varepsilon}^{th}=\alpha\left(T-T^{réf}\right)\mathbf{Id} \qquad \textbf{éq 1.1-1.1-1}$$

$$R=\mathrm{D}\left[R^0+\left(\frac{2}{3}\right)^a A\left(\boldsymbol{\varepsilon}^p-\boldsymbol{\varepsilon}_n^p\right)_{eq}^a\right] \qquad \mathrm{D}=1-m\,e^{-b\,p\left(1-\frac{\sigma^p}{S}\right)} \qquad \textbf{éq 1.1-1.1-2}$$

$$\mathbf{X}=\mathrm{C}\left[S\,\boldsymbol{\varepsilon}^p-\sigma^p\,\boldsymbol{\varepsilon}_n^p\right] \qquad \mathrm{C}=C_\infty+C_1 e^{-b\,p\left(1-\frac{\sigma^p}{S}\right)} \qquad \textbf{éq 1.1-1.1-3}$$

$\tilde{\mathbf{a}}$	deviatoric part of a tensor $\mathbf{a}$
R	isotropic variable of work hardening
$\mathbf{X}$	kinematic variable of work hardening
K , μ	modules of compressibility and shearing
α	thermal dilation coefficient
$T^{réf}$	temperature of reference
S	constraint of ratchet
$b, R^0, A, \mathbf{a}, m, C_\infty, C_1$	other characteristics of work hardening of material

Let us note that the moduli of elasticity and the thermal dilation coefficient are indicated by the user by the order DEFI_MATERIAU, keyword ELAS, while the characteristics of work hardening are fixed by the keyword TAHERI. These characteristics can depend on the temperature, by employing the keywords ELAS_FO and TAHERI_FO. Also let us specify that an example of identification of the characteristics of work hardening on uniaxial situations is given in Geyer [bib2].

The evolution of the internal variables is defined by two criteria. The first controls traditional with work hardenings kinematics and isotropic plasticity compounds:

$$\mathrm{F}=(\tilde{\boldsymbol{\sigma}}-\mathbf{X})_{eq}-R\leq 0 \text{ et } \boldsymbol{\sigma}^0=\frac{\tilde{\boldsymbol{\sigma}}-\mathbf{X}}{(\tilde{\boldsymbol{\sigma}}-\mathbf{X})_{eq}} \quad \text{éq 1.1-1.1-4}$$

$(\cdot)_{eq}$	equivalent standard: $\mathbf{a}_{eq}=\left(\frac{3}{2}\tilde{\mathbf{a}}:\tilde{\mathbf{a}}\right)^{\frac{1}{2}}$
F	criterion of plasticity
$\mathbf{s}^0$	normal external with the criterion F

This criterion is matched classical condition of load/discharge:

$$\begin{cases}\text{si } \mathrm{F}<0 \text{ ou } \dot{\boldsymbol{\sigma}}:\mathbf{s}^0\leq 0 & \dot{p}=0 & (\text{élasticité})\\ \text{si } \mathrm{F}=0 \text{ et } \dot{\boldsymbol{\sigma}}:\mathbf{s}^0>0 & \dot{p}\geq 0 \text{ tel que } \dot{\mathrm{F}}=0 & (\text{plasticité})\end{cases} \quad \text{éq 1.1-1.1-5}$$

And the law of flow associated with the criterion F is:

$$\dot{\boldsymbol{\varepsilon}}^p=\frac{3}{2}\dot{p}\,\mathbf{s}^0 \text{ and thus } \dot{p}=\frac{2}{3}\dot{\varepsilon}^p_{\mathrm{eq}} \quad \text{éq 1.1-1.1-6}$$

The second criterion controls the evolution of the constraint of peak. Geometrically within the space of diverters of the constraints, it translates the fact that the first surface of load $(\mathrm{F}=0)$, represented by a sphere of center $\mathbf{X}$ and of ray R , inside a sphere of centre the origin remains and σ^p . He is written simply:

$$\mathrm{G}=X_{eq}+R-\sigma^p\leq 0 \quad \text{éq 1.1-1.1-7}$$

G criterion of maximum work hardening

According to the preceding geometrical considerations, the evolution of the constraint of peak is:

$$\begin{cases} \text{si } G<0 \text{ ou } \dot{X}_{eq}+\dot{R}\le 0 & \dot{\sigma}^p=0 \\ \text{si } G=0 \text{ et } \dot{X}_{eq}+\dot{R}>0 & \dot{\sigma}^p\ge 0 \text{ tel que } \dot{G}=0 \end{cases}$$ **éq 1.1-1.1-8**

It should be noticed that in the natural state of material, the constraint of peak is not worthless but is worth the initial elastic limit, namely:

$$\sigma^p(\text{initial})=(1-m)R^0$$

Until now, we did not evoke the evolution of the semi-discrete internal variable $\boldsymbol{\varepsilon}_n^p$. In fact, it evolves only in elastic mode. More exactly, this variable takes account of the state of plastic deformation during the last discharge; in other words, at the beginning of each discharge, this variable should take the value of the current plastic deformation instantaneously. However, to preserve a continuous behavior, one regularizes the evolution of $\boldsymbol{\varepsilon}_n^p$ in the following way:

In elastic mode:

$$\dot{\boldsymbol{\varepsilon}}_n^p=\dot{\xi}(\boldsymbol{\varepsilon}_n^p-\boldsymbol{\varepsilon}^p)\begin{cases} \text{si } \boldsymbol{\varepsilon}_n^p=\boldsymbol{\varepsilon}^p & \dot{\xi}=0 & (\text{élasticité classique}) \\ \text{si } \boldsymbol{\varepsilon}_n^p\ge\boldsymbol{\varepsilon}^p & \dot{\xi}\le 0 \text{ tq } \dot{F}=0 & (\text{pseudo-décharge}) \end{cases}$$ **éq 1.1-1.1-9**

In plastic mode:

$$\dot{\boldsymbol{\varepsilon}}_n^p=0$$

The behavior is thus completely given. Before passing to the introduction of viscosity, the observation of two surfaces of load calls an important remark. One could think that surface $G=0$ is actually activated only in plastic mode. In practice, it of it is nothing. One can for example quote the case of a thermal loading: a cooling involves (generally) a dilation of the surface of load $F=0$, so that the constraint of peak is brought to evolve to preserve $G\le 0$, and this same in elastic mode.

1.2 Taking into account of viscosity

To model the behavior of the stainless steels under cyclic loading when the temperature is about $550\,°C$, it is not possible any more to neglect the terms of creep. To give an account of these effects of viscosity while preserving the properties of the preceding model, a simple method consists in making viscous the evolution of the plastic deformation. In other words, viscosity intervenes only in plastic mode: no direct influence on the semi-discrete internal variable nor on the surface of load $G=0$. For that, while following Lemaitre and Chaboche [bib3], one replaces the condition of coherence [éq 1.1-5] by:

$$\dot{p}=\left(\frac{\langle F\rangle}{K\,p^{1/M}}\right)^N$$ **éq 1.2-1.2-1**

$\langle F\rangle$	positive part of F (hooks of Macauley)
K, N, M	characteristics of viscosity of material

The characteristics of viscosity of material are indicated in the order `DEFI_MATERIAU`, that is to say by the keyword `LEMAITRE` if they do not depend on the temperature, that is to say by the keyword `LEMAITRE_FO` in the contrary case. In the absence of one of these keywords, the behavior is supposed plastic.

Unchanged all the other equations of the model are left. It will be seen that such an introduction of viscosity involves only minor modifications of the implicit algorithm of integration of the law of behavior.

1.3 Description of the internal variables calculated by *Code_Aster*

Internal variables calculated by *Code_Aster* are 9. They are arranged in the following order:

1	p	cumulated plastic deformation
2	σ^p	constraint of peak
3 to 8	$\boldsymbol{\varepsilon}_n^p$	plastic tensor of deformation due to the last discharge (arranged in the order xx, yy, zz, xy, xz, yz)
9	χ	discharge/loadmeter (cf [§2.3]) 0 elastic discharge 1 classical plastic load 2 charges plastic on two surfaces 3 pseudo-discharge

As for the tensor of the viscoplastic deformations, it is not arranged among the internal variables but can be calculated in postprocessing via the order `CALC_CHAMP`, options '`EPSP_ELGA`' or '`EPSP_ELNO`', (cf [U4.61.02]).

2 Digital formulation of the relation of behavior

In order to be able to treat within the same framework plasticity and viscoplasticity, one chooses to proceed to an implicit discretization of the relations of behavior, (cf [R5-03-02]). Let us note moreover that an explicit procedure of integration is delicate for two reasons: on the one hand, the treatment of the semi-discrete variable is necessarily implicit and can lead to digital oscillations (one pseudo - discharge, therefore one solves $\mathrm{F}=0$, and as a result, F can be (very weak but) higher than zero, from where load with the step following instead of continuing the discharge), and in addition, the equation [éq 1.1-2] is not derivable when $\boldsymbol{\varepsilon}^p=\boldsymbol{\varepsilon}_n^{p^-}$

2.1 Implicit discretization of the equations of behavior

Henceforth, one adopts the convention of following notation. If u indicate a quantity, then:

u^-	quantity u at the beginning of the step of time
Δu	increment of the quantity u during the step of time
u	quantity u at the end of the step of the time (not of exhibitor $+$)

Let us start by introducing the elastic constraint, i.e. the constraint in the absence of increment of plastic deformation. One can notice besides that only the term deviatoric cheek a role in the nonlinear part of the behavior:

$$\boldsymbol{\sigma}^e=K\,\mathrm{tr}\left(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^{\mathrm{th}}\right)\mathbf{Id}+\underbrace{2\mu\left(\tilde{\boldsymbol{\varepsilon}}-\boldsymbol{\varepsilon}^{p^-}\right)}_{\tilde{\boldsymbol{\sigma}}^e}\ \text{ et }\ \tilde{\boldsymbol{\sigma}}=\tilde{\boldsymbol{\sigma}}^e-2\mu\,\Delta\,\boldsymbol{\varepsilon}^p \quad \textbf{éq 2.1-2.1-1}$$

By taking account of the equations of state [éq 1.1-1] and [éq 1.1-3] and of the law of flow [éq 1.1-6], one a:

$$\mathbf{s}\overset{\text{déf}}{=}\tilde{\boldsymbol{\sigma}}-\mathbf{X}=\tilde{\boldsymbol{\sigma}}^e-\mathrm{C}\left(S\,\boldsymbol{\varepsilon}^{p^-}-\sigma^p\,\boldsymbol{\varepsilon}_n^p\right)-\frac{3}{2}\left(2\mu+\mathrm{C}S\right)\Delta\,p\,\mathbf{s}^0 \quad \textbf{éq 2.1-2.1-2}$$

By noting that $\mathbf{s}^0$ is not other than $\mathbf{s}$ normalized, one from of deduced immediately:

$$\left[s_{eq}+\frac{3}{2}(2\mu+CS)\Delta p\right]\mathbf{s}^0=\underbrace{\tilde{\boldsymbol{\sigma}}^e-C\left(S\,\boldsymbol{\varepsilon}^{p-}-\sigma^p\,\boldsymbol{\varepsilon}_n^p\right)}_{\mathbf{s}^e} \quad \text{éq 2.1-2.1-3}$$

Consequently, $\mathbf{s}$ is entirely determined by:

$$\mathbf{s}=s_{eq}\mathbf{s}^0 \quad \text{avec} \quad \mathbf{s}^0=\frac{\mathbf{s}^e}{s^e_{eq}} \quad \text{et} \quad s_{eq}=s^e_{eq}-\frac{3}{2}(2\mu+CS)\Delta p \quad \text{éq 2.1-2.1-4}$$

Finally, the functions of load are:

$$F=s_{eq}-D\left[R^0+\left(\frac{2}{3}\right)^a A\left(\boldsymbol{\varepsilon}^{p-}-\boldsymbol{\varepsilon}_n^p+\frac{3}{2}\Delta p\,\mathbf{s}^0\right)_{eq}^a\right] \quad \text{éq 2.1-2.1-5}$$

$$G=C\left[S\,\boldsymbol{\varepsilon}^{p-}-\sigma^p\,\boldsymbol{\varepsilon}_n^p+\frac{3}{2}S\Delta p\,\mathbf{s}^0\right]_{eq}+D\left[R^0+\left(\frac{2}{3}\right)^a A\left(\boldsymbol{\varepsilon}^{p-}-\boldsymbol{\varepsilon}_n^p+\frac{3}{2}\Delta p\,\mathbf{s}^0\right)_{eq}^a\right]-\sigma^p \quad \text{éq 2.1-2.1-6}$$

2.2 Taking into account of the viscous terms

In the absence of viscous terms, the relation of discretized coherence is:

$$\begin{array}{ll}\text{Régime élastique :} & F\leq 0 \text{ et } \Delta p=0\\ \text{Régime plastique :} & F=0 \text{ et } \Delta p\geq 0\end{array} \quad \text{éq 2.2-2.2-1}$$

On the other hand, in the presence of viscosity, the condition of coherence is replaced by the equation [éq 1.2 - 1] which, discretized, is written:

$$\frac{\Delta p}{\Delta t}=\left(\frac{\langle F\rangle}{K\,p^{1/M}}\right)^N \Leftrightarrow \langle F\rangle=K\,p^{1/M}\left(\frac{\Delta p}{\Delta t}\right)^{\frac{1}{N}} \quad \text{éq 2.2-2.2-2}$$

In other words, while posing:

$$\tilde{F}=F-K\,p^{1/M}\left(\frac{\Delta p}{\Delta t}\right)^{\frac{1}{N}} \quad \text{éq 2.2-2.2-3}$$

the viscoplastic increment of cumulated deformation is determined by:

$$\begin{array}{ll}\text{Régime élastique :} & \tilde{F}\leq 0 \text{ et } \Delta p=0\\ \text{Régime viscoplastique :} & \tilde{F}=0 \text{ et } \Delta p\geq 0\end{array} \quad \text{éq 2.2-2.2-4}$$

Finally, by adopting an implicit discretization, the only difference between the laws in plastic and viscoplastic behavior lies in the form of the function of load F : one observes a complementary term in the event of viscosity there. In fact, incremental plasticity seems the borderline case (without associated digital difficulty) of incremental viscoplasticity when viscosity K tends towards zero. Let us note that this remark was already mentioned by Chaboche and al. [bib1].

2.3 Discretization of the conditions of coherence

Before discretizing the conditions of coherence and describing the various possible modes of behavior, a remark is essential as for the treatment of the semi-discrete variable. Like ξ only intervenes "to control" $\boldsymbol{\varepsilon}_n^p$, one can always be reduced during a step of time to:

$$\boldsymbol{\varepsilon}_n^p=\xi\,\boldsymbol{\varepsilon}_n^{p-}+(1-\xi)\,\boldsymbol{\varepsilon}^{p-} \qquad 0\leq\xi\leq 1 \qquad \textbf{éq 2.3-2.3-1}$$

The value of ξ is then fixed by the conditions of coherence, which translates the equation of evolution [éq 1.1-9] on the continuous level. Such a parameter setting with each step of time makes it possible to be freed from storage from ξ , in condition well-sure of preserving the values of $\boldsymbol{\varepsilon}_n^p$.

After this opening remark, one can be interested in the conditions of coherence. For the criterion G who controls the evolution of the constraint of peak, the discretized form of the condition of coherence is:

$$\mathrm{G}(\Delta p,\Delta\sigma^p,\xi)\leq 0 \quad \Delta\sigma^p\geq 0 \quad \Delta\sigma^p\,\mathrm{G}(\Delta p,\Delta\sigma^p,\xi)=0 \qquad \textbf{éq 2.3-2.3-2}$$

The condition of coherence relating to F is more delicate insofar as it controls the evolution of the plastic deformation in plastic mode of load and the evolution of ξ in mode of discharge. Once discretized, she is written:

In plastic mode of load $(\xi=1)$:

$$\mathrm{F}(\Delta p,\Delta\sigma^p,\xi=1)=0 \qquad \Delta p\geq 0 \qquad \Delta p\,\mathrm{F}(\Delta p,\Delta\sigma^p,\xi=1)=0 \qquad \textbf{éq 2.3-2.3-3}$$

In mode of discharge $(\Delta p=0)$:

$$\mathrm{F}(\Delta p=0,\Delta\sigma^p,\xi)=0 \qquad 0\leq\xi\leq 1\mathrm{x} \qquad \xi\,\mathrm{F}(\Delta p=0,\Delta\sigma^p,\xi)=0 \qquad \textbf{éq 2.3-2.3-4}$$

To be able to select the mode of behavior of material, and thus the equations to be solved, the first question is:

Do we Somme in plastic or elastic situation?

In fact, there exists a solution in elastic mode (pseudo-discharge $\xi>0$ or classical elasticity $\xi=0$) if one can find an increment of constraint of peak such as:

Incremental condition of discharge (equation scalar in $\Delta\sigma^p$):

$$\begin{aligned}&\mathrm{F}(\Delta p=0,\Delta\sigma^p,\xi=1)\leq 0\\&\mathrm{G}(\Delta p=0,\Delta\sigma^p,\xi=1)\leq 0 \qquad \Delta\sigma^p\geq 0 \qquad \Delta\sigma^p\,\mathrm{G}(\Delta p=0,\Delta\sigma^p,\xi=1)=0\end{aligned} \qquad \textbf{éq 2.3-2.3-5}$$

In the event of plastic load, i.e. when there does not exist $\Delta\sigma^p$ satisfying [éq 2.3-5], one has then to solve the nonlinear system in Δp and $\Delta\sigma^p$ according to:

Plastic mode (nonlinear system in Δp and $\Delta\sigma^p$):

$$\mathrm{F}\left(\Delta p, \Delta\sigma^{p}, \xi=1\right)=0 \quad \Delta p \geq 0$$
$$\mathrm{G}\left(\Delta p, \Delta\sigma^{p}, \xi=1\right) \leq 0 \quad \Delta\sigma^{p} \geq 0 \quad \Delta\sigma^{p}\mathrm{G}\left(\Delta p, \Delta\sigma^{p}, \xi=1\right)=0$$
éq 2.3-2.3-6

On the other hand, in elastic situation, two choices are still possible: pseudo-discharge $(\xi>0)$ or classical elasticity $(\xi=0)$. The second case being more favorable, one starts by examining whether it is not realizable, i.e. if there exists an increment of constraint of peak such as:

Incremental condition of classical elastic mode (scalar equation in $\Delta\sigma^{p}$):

$$\mathrm{F}\left(\Delta p=0, \Delta\sigma^{p}, \xi=0\right) \leq 0$$
$$\mathrm{G}\left(\Delta p=0, \Delta\sigma^{p}, \xi=0\right) \leq 0 \quad \Delta\sigma^{p} \geq 0 \quad \Delta\sigma^{p}\mathrm{G}\left(\Delta p=0, \Delta\sigma^{p}, \xi=0\right)=0$$
éq 2.3-2.3-7

Lastly, if it were to be a question of a discharge pseudo-rubber band, it remains to solve the nonlinear system in ξ and $\Delta\sigma^{p}$ according to:

Discharge pseudo-rubber band (system nonlinear in ξ and $\Delta\sigma^{p}$):

$$\mathrm{F}\left(\Delta p=0, \Delta\sigma^{p}, \xi\right)=0 \quad 0 \leq \xi \leq 1$$
$$\mathrm{G}\left(\Delta p=0, \Delta\sigma^{p}, \xi\right) \leq 0 \quad \Delta\sigma^{p} \geq 0 \quad \Delta\sigma^{p}\mathrm{G}\left(\Delta p=0, \Delta\sigma^{p}, \xi\right)=0$$
éq 2.3-2.3-8

Let us note as of now that the nonlinear systems [éq 2.3-6] and [éq 2.3-8] can be reduced to the solution of a simple scalar equation if $\Delta\sigma^{p}=0$ allows to obtain a solution.

One can summarize the algorithm of choice of the equations to be solved by the decision tree below.

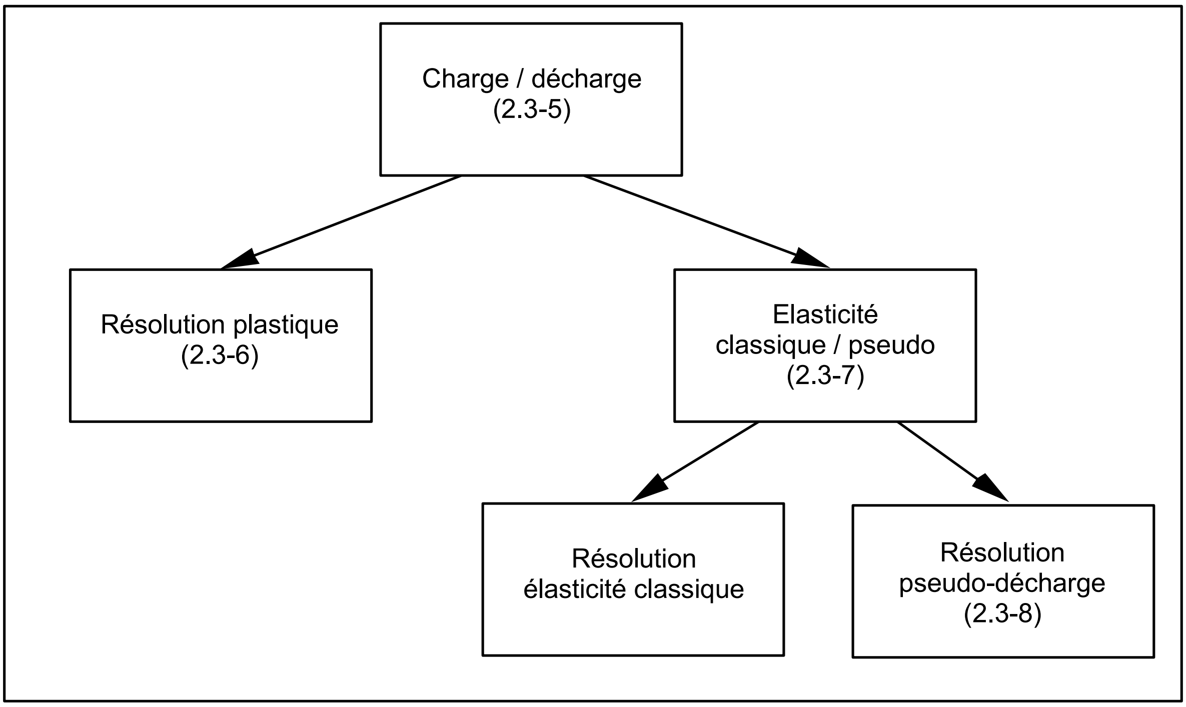

Figure 2.3-2.3-a : Decision tree to choose the mode of behavior

2.4 Framing of the solutions

With the reading of the preceding paragraph, one could note the need for solving (numerically) a certain number of scalar equations or of nonlinear systems. For that, it is always interesting to have an interval on which to seek the solution. On the one hand, a framing of the solution shows its existence (what strongly reinforces the chances of success of an algorithm of resolution!), and on the other hand, it allows a suitable digital processing, therefore surer.

Concerning $\Delta\sigma^p$, one undervaluing is of course 0. In addition, the constraint of ratchet S represent a limit beyond which the model does not have any more a direction. In fact, with the examination of constant materials obtained by identification, cf Taheri and al. [bib6], G becomes indeed negative when $\sigma^p=S$ if the difference between the plastic deformation and the plastic deformation due to the last discharge is not too important (a few %):

$$\frac{\mathrm{G}(\Delta p,\sigma^p=S,\xi)}{S}=\underbrace{(C_\infty+C_1)}_{\approx 13}\underbrace{(\boldsymbol{\varepsilon}^p-\boldsymbol{\varepsilon}_n^p)_{eq}}_{\leq 3\,\%}+\underbrace{(1-m)}_{\approx 0,75}\left(\underbrace{\frac{R^0}{S}}_{\approx 20\,\%}+\underbrace{\left(\frac{2}{3}\right)^a}_{\approx 0,5}\frac{A}{S}\underbrace{(\boldsymbol{\varepsilon}^p-\boldsymbol{\varepsilon}_n^p)_{eq}^a}_{\approx 0,7}\right)-1\leq 0 \quad \text{éq 2.4-2.4-1}$$

One can also seek one raising for Δp. By examining the expression of F :

$$\begin{aligned}\mathrm{F}(\Delta p,\Delta\sigma^p,\xi)&\leq s_{eq}-\mathrm{D}R^0\\&\leq s_{eq}^e-\frac{3}{2}(2\mu+\mathrm{C}S)\Delta p-\mathrm{D}R^0\\&\leq s_{\max}^e-\frac{3}{2}(2\mu+C_\infty S)\Delta p-\mathrm{D}(p^-)R^0\end{aligned} \quad \text{éq 2.4-2.4-2}$$

One from of deduced one raising for Δp, such as $\mathrm{F}(\Delta p,\Delta\sigma^p,\xi)\leq 0$:

$$\Delta p\leq\frac{s_{\max}^e(\sigma^p)-\mathrm{D}(p^-,\sigma^p)R^0}{\frac{3}{2}(2\mu+C_\infty S)} \quad \text{éq 2.4-2.4-3}$$

$$s_{\max}^e(\sigma^p)=\max\begin{cases}\left[\tilde{\boldsymbol{\sigma}}^e-\mathrm{C}(p^-,\sigma^p)(S\,\boldsymbol{\varepsilon}^{p-}-\sigma^p\boldsymbol{\varepsilon}_n^p)\right]_{eq}\\\left[\tilde{\boldsymbol{\sigma}}^e-C_\infty(S\,\boldsymbol{\varepsilon}^{p-}-\sigma^p\boldsymbol{\varepsilon}_n^p)\right]_{eq}\end{cases} \quad \text{éq 2.4-2.4-4}$$

In particular, one can give one raising (coarse) Δp independent of $\Delta\sigma^p$:

$$\Delta p_{max}=\frac{s_{max}^{e\,max}-(1-m)R^0}{\frac{3}{2}(2\mu+C_\infty S)} \quad \text{éq 2.4-2.4-5}$$

$$s_{max}^{emax}=\sigma_{eq}^e+(C_1+C_\infty)\max\begin{cases}\left[S\,\boldsymbol{\varepsilon}^{p-}-\sigma^{p-}\boldsymbol{\varepsilon}_n^p\right]_{eq}\\\left[S\,\boldsymbol{\varepsilon}^{p-}-S\,\boldsymbol{\varepsilon}_n^p\right]_{eq}\end{cases} \quad \text{éq 2.4-2.4-6}$$

One can then notice that the systems [éq 2.3-6] and [éq 2.3-8] always admit a solution. Indeed, so for each system, one writes respectively $\Delta\sigma^p(\Delta p)$ et $\Delta\sigma^p(\xi)$ solutions of $\mathrm{G}=0$, then one a:

- Nonlinear system of plastic load:

$$F\left(\Delta p=0, \Delta \sigma^{p}\left(\Delta p=0\right), \xi=1\right) \geq 0 \text{ et } F\left(\Delta p_{\max}, \Delta \sigma^{p}\left(\Delta p_{\max}\right), \xi=1\right) \leq 0$$ **éq 2.4-2.4-7**

- Nonlinear system of pseudo-discharge:

$$F\left(\Delta p=0, \Delta \sigma^{p}\left(\xi=1\right), \xi=1\right) \leq 0 \text{ et } F\left(\Delta p=0, \Delta \sigma^{p}\left(\xi=0\right), \xi=0\right) \geq 0$$ **éq 2.4-2.4-8**

3 Methods of digital resolution

The resolution of the incremental equations confronts us either with a nonlinear scalar equation, or with a nonlinear system with two unknown factors. Below the digital methods employed are exposed. One also examines the calculation of the tangent matrix, possibly used by the total algorithm of STAT_NON_LINE, (cf [R5.03.01]).

3.1 Scalar equation: method of secants

It is a question of solving a nonlinear scalar equation by seeking the solution in a confidence interval. For that, one proposes to couple a method of secant with a control of the interval of research. That is to say the following equation to solve:

$$\mathrm{f}(x)=0 \quad x\in[a,b] \quad \mathrm{f}(a)<0 \quad \mathrm{f}(b)>0 \qquad \textbf{éq 3.1-3.1-1}$$

The method of the secant consists in building a succession of points x^n who converges towards the solution. It is defined by recurrence (linear approximation of the function by its cord):

$$x^{n+1}=x^{n-1}-\mathrm{f}(x^{n-1})\frac{x^n-x^{n-1}}{\mathrm{f}(x^n)-\mathrm{f}(x^{n-1})} \qquad \textbf{éq 3.1-3.1-2}$$

In addition, if x^{n+1} was to leave the interval, then one replaces it by the terminal of the interval in question:

$$\begin{cases}\text{si } x^{n+1}<a \text{ alors } x^{n+1}:=a\\ \text{si } x^{n+1}>b \text{ alors } x^{n+1}:=b\end{cases} \qquad \textbf{éq 3.1-3.1-3}$$

On the other hand, if x^{n+1} is in the interval running, then one reactualizes the interval:

$$\begin{cases}\text{si } x^{n+1}\in[a,b] \text{ et } \mathrm{f}(x^{n+1})<0 \text{ alors } a:=x^{n+1}\\ \text{si } x^{n+1}\in[a,b] \text{ et } \mathrm{f}(x^{n+1})>0 \text{ alors } b:=x^{n+1}\end{cases} \qquad \textbf{éq 3.1-3.1-4}$$

One considers to have converged when f is sufficiently close to 0 (tolerance to be informed). As for the first two leader characters, one can choose the terminals of the interval, or, if one has an estimate of the solution, one can adopt this estimate and one of the terminals of the interval.

3.2 Nonlinear systems: method of Newton and linear research

One presents here a method of Newton with which one associated a linear technique of research and a control of the direction of descent not to leave the field of research (terminals on the unknown factors).

That is to say the following nonlinear system:

$$\begin{cases}\mathrm{F}(x,y)=0\\ \mathrm{G}(x,y)=0\end{cases} \text{ avec } \begin{cases}x_{\min}\leq x\leq x_{\max}\\ y_{\min}\leq y\leq y_{\max}\end{cases} \qquad \textbf{éq 3.2-3.2-1}$$

If (x,y) is a point of the field of research, then one builds a succession of points (x^n,y^n) who converges towards a solution (or at least, it is hoped for) by the following process.

- Determination of the direction of descent

  A direction of descent $(\delta x, \delta y)$ is given by the resolution of the linear system 2 X 2:

$$\begin{bmatrix} F^n_{,x} & F^n_{,y} \\ G^n_{,x} & G^n_{,y} \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \end{bmatrix} = -\begin{bmatrix} F^n \\ G^n \end{bmatrix} \quad \textbf{éq 3.2-3.2-2}$$

- Correction of the direction of descent

  The direction of descent is corrected $(\delta x, \delta y)$ so that the points considered are in the field of research (with ρ_{max} the maximum length which one is authorized to describe along the direction of descent):

$$\begin{cases} \text{si} \quad x+\rho_{\max}\delta x < x_{\min} \quad \delta x := \dfrac{x_{\min}-x}{\rho_{\max}} \quad \text{si} \quad x+\rho_{\max}\delta x > x_{\max} \quad \delta x := \dfrac{x_{\max}-x}{\rho_{\max}} \\ \text{si} \quad y+\rho_{\max}\delta y < y_{\min} \quad \delta y := \dfrac{y_{\min}-y}{\rho_{\max}} \quad \text{si} \quad y+\rho_{\max}\delta y > y_{\max} \quad \delta y := \dfrac{y_{\max}-y}{\rho_{\max}} \end{cases} \quad \textbf{éq 3.2-3.2-3}$$

- Linear research

  It any more but does not remain to minimize the quantity $\mathrm{E}=(\mathrm{F}^2+\mathrm{G}^2)/2$ in the direction of descent. Let us note that the standard E that one minimizes thus is a measurement of the mistake made in the resolution of the system: it is worthless when (x, y) is solution of the system [éq 3.2-1]. To minimize E, one simply will seek to cancel his derivative, i.e. to solve the scalar equation:

$$\underbrace{\frac{\partial}{\partial \rho}\left[\mathrm{E}(x+\rho\,\delta x, y+\rho\,\delta y)\right]}_{(\mathrm{FF}_{,x}+\mathrm{GG}_{,x})\delta x + (\mathrm{FF}_{,y}+\mathrm{GG}_{,y})\delta y} = 0 \quad \text{et} \quad 0 \le \rho \le \rho_{\max} \quad \textbf{éq 3.2-3.2-4}$$

- Convergence criteria

  One considers to have converged when the error E is lower than a prescribed size. In addition, if the standard of the direction of descent becomes too weak (another size to be informed), one can think that the algorithm does not manage to converge.

3.3 Criteria of stop

Until now, the values of stop and the iteration counts maximum of the preceding methods of resolution were not specified. Two cases should be distinguished.

- When one seeks to check the conditions of coherence (scalar equation or nonlinear system following the situation), one expects precise results, of which the relative tolerance η is fixed by the user in the order `STAT_NON_LINE` under the keyword `RESI_INTE_RELA`, (cf [U4.32.01]). According to whether one seeks to solve $\mathrm{F}=0, \mathrm{G}=0$ or simultaneously $\mathrm{F}=\mathrm{G}=0$, the criterion of stop is expressed respectively:

$$\left|\frac{\mathrm{F}}{R^0}\right| \le \eta \quad \text{ou} \quad \left|\frac{\mathrm{G}}{R^0}\right| \le \eta \quad \text{ou} \quad \frac{1}{R^0}\sqrt{\frac{\mathrm{F}^2+\mathrm{G}^2}{2}} \le \eta$$

R^0 initial elastic limit, provided by the user, cf [§ 1.1].

In addition, the user always specifies a maximum iteration count in the order `STAT_NON_LINE` under the keyword `ITER_INTE_MAXI`, (cf [U4.32.01]).

- When one carries out iterations of linear research, one seeks to obtain a faster convergence (or at least sourer). One should not therefore devoting an excessive time to it. This is why one fixed once for a a whole iteration count maximum equal to 3, a maximum terminal ρ_{max} equalize to 2 and one criterion of relative stop of 1 %:

$$\frac{\partial}{\partial \rho}\left[\mathrm{E}\left(x+\rho\,\delta x,\, y+\rho\,\delta y\right)\right]\Big|_{\rho} \leq 10^{-2}\frac{\partial}{\partial \rho}\left[\mathrm{E}\left(x+\rho\,\delta x,\, y+\rho\,\delta y\right)\right]\Big|_{\rho=0}$$

3.4 Tangent matrix

In the optics of a resolution of the equilibrium equations (total) by a method of Newton, it is essential to determine the consistent matrix of the tangent behavior, (cf Simo and al. [bib4]). This matrix is composed classically of an elastic contribution and a plastic contribution:

$$\frac{\delta \boldsymbol{\sigma}}{\delta \boldsymbol{\varepsilon}}=\frac{\delta \boldsymbol{\sigma}^e}{\delta \boldsymbol{\varepsilon}}-2\mu\frac{\delta \Delta \boldsymbol{\varepsilon}^p}{\delta \boldsymbol{\varepsilon}} \quad \textbf{éq 3.4-3.4-1}$$

One from of deduced immediately that in elastic mode (classical or pseudo-discharge), the tangent matrix is reduced to the elastic matrix:

Elastic mode:

$$\frac{\delta \boldsymbol{\sigma}}{\delta \boldsymbol{\varepsilon}}=\frac{\delta \boldsymbol{\sigma}^e}{\delta \boldsymbol{\varepsilon}} \quad \textbf{éq 3.4-3.4-2}$$

On the other hand, in plastic mode, the variation of the plastic deformation is not worthless any more. The rules of made up derivation make it possible to obtain:

$$\frac{\delta \Delta \boldsymbol{\varepsilon}^p}{\delta \tilde{\boldsymbol{\sigma}}^e}=\frac{3}{2}\left[\mathbf{s}^0\otimes\frac{\delta p}{\delta \tilde{\boldsymbol{\sigma}}^e}+\Delta p\frac{\delta \mathbf{s}^0}{\delta \tilde{\boldsymbol{\sigma}}^e}\right]=\frac{3}{2}\left[\mathbf{s}^0\otimes\frac{\delta p}{\delta \boldsymbol{\sigma}^e}+\frac{\Delta p}{s^e_{eq}}\left(\mathbf{Id}-\frac{3}{2}\mathbf{s}^0\otimes\mathbf{s}^0\right)\right] \quad \textbf{éq 3.4-3.4-3}$$

$\otimes$ tensorial product

One can note that one preferred to derive compared to $\tilde{\boldsymbol{\sigma}}^{\mathbf{e}}$, knowing that one a:

$$\frac{\delta \Delta \boldsymbol{\varepsilon}^p}{\delta \boldsymbol{\varepsilon}}=\frac{\delta \Delta \boldsymbol{\varepsilon}^p}{\delta \boldsymbol{\sigma}^e}.\frac{\delta \boldsymbol{\sigma}^e}{\delta \boldsymbol{\varepsilon}}=2\mu\frac{\delta \Delta \boldsymbol{\varepsilon}^p}{\delta \tilde{\boldsymbol{\sigma}}^e}.\mathbf{P} \quad \text{avec} \quad \mathbf{P}:\begin{cases} S\rightarrow S \\ \boldsymbol{\varepsilon}\rightarrow\tilde{\boldsymbol{\varepsilon}}\end{cases} \quad \textbf{éq 3.4-3.4-4}$$

S space of the symmetrical tensors
$\mathbf{P}$ projector on the diverters

Finally, it any more but does not remain to calculate the variation of p. For that, it is necessary to distinguish if it is about a classical mode of plasticity $\left(\Delta \sigma^p=0\right)$ or of plasticity on two surfaces. As follows:

Classical plasticity: $\mathrm{F}\left(p,\tilde{\boldsymbol{\sigma}}^e\right)=0$

$$\mathrm{F}_{,p}\left(p,\tilde{\boldsymbol{\sigma}}^e\right)\delta p=-\mathrm{F}_{,\tilde{\boldsymbol{\sigma}}^e}\left(p,\tilde{\boldsymbol{\sigma}}^e\right)\delta\tilde{\boldsymbol{\sigma}}^e \quad\Rightarrow\quad \frac{\delta p}{\delta\tilde{\boldsymbol{\sigma}}^e}=-\frac{\mathrm{F}_{,\tilde{\boldsymbol{\sigma}}^e}\left(p,\tilde{\boldsymbol{\sigma}}^e\right)}{\mathrm{F}_{,p}\left(p,\tilde{\boldsymbol{\sigma}}^e\right)}$$ **éq 3.4-3.4-5**

Plasticity on two surfaces: $\mathrm{F}\left(p,\sigma^p,\tilde{\boldsymbol{\sigma}}^e\right)=0$ et $\mathrm{G}\left(p,\sigma^p,\tilde{\boldsymbol{\sigma}}^e\right)=0$

$$\begin{bmatrix}\mathrm{F}_{,p} & \mathrm{F}_{,\sigma^p}\\ \mathrm{G}_{,p} & \mathrm{G}_{,\sigma^p}\end{bmatrix}\begin{bmatrix}\delta p\\ \delta\sigma^p\end{bmatrix}=-\begin{bmatrix}\mathrm{F}_{,\tilde{\boldsymbol{\sigma}}^e}\\ \mathrm{G}_{,\tilde{\boldsymbol{\sigma}}^e}\end{bmatrix}.\delta\tilde{\boldsymbol{\sigma}}^e \quad\Rightarrow\quad \frac{\delta p}{\delta\tilde{\boldsymbol{\sigma}}^e}=\frac{\begin{vmatrix}\mathrm{F}_{,\sigma^p} & \mathrm{F}_{,\tilde{\boldsymbol{\sigma}}^e}\\ \mathrm{G}_{,\sigma^p} & \mathrm{G}_{,\tilde{\boldsymbol{\sigma}}^e}\end{vmatrix}}{\begin{vmatrix}\mathrm{F}_{,p} & \mathrm{F}_{,\sigma^p}\\ \mathrm{G}_{,p} & \mathrm{G}_{,\sigma^p}\end{vmatrix}}$$ **éq 3.4-3.4-6**

An attentive examination of the expressions [éq 2.1-5] and [éq 2.1-6] makes it possible to note that the variations of F and G compared to $\tilde{\boldsymbol{\sigma}}^e$ are not necessarily colinéaires with $\mathbf{s}^0$. By keeping account of [éq 3.4 - 3], one from of deduced whereas the tangent matrix is in general not symmetrical in plastic mode. Rather than to impose the use of a nonsymmetrical solvor, much more expensive in time calculation, one prefers to symmetrize this matrix.

3.5 Plane constraints

The treatment of the plane constraints adds a nonlinear equation to solve, coupled with the systems [éq 2.3 - 6] and [éq 2.3-8], (cf [R5-03-02]). In front of this considerable difficulty and the absence of expressed need, one preferred not to give the opportunity of forcing a state of plane stresses on the level of the law of behavior. In other words, modeling `C_PLAN` is not available for the law of behavior `VISC_TAHERI`.

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5 Description of the versions of the document

Version Aster	**Author (S) Organization (S)**	**Description of the modifications**
4	E.Lorentz EDF-R&D/AMA	Initial text