Elastoplastic relation of behavior with linear and isotropic kinematic work hardening nonlinear. Plane modelings 3D and constraints

Summary:

This document describes an elastoplastic law of behaviour to work hardening mixed, kinematic linear and isotropic nonlinear. The equations to solve integrate this relation of behavior numerically are specified, as well as the coherent tangent matrix.

This behavior is usable for modelings of continuous mediums 3D, 2D (AXIS, C_PLAN, D_PLAN), and for modelings DKT, COQUE_3D and PIPE.

Contents

1 Introduction

When the way of loading is not monotonous any more, work hardenings isotropic and kinematic are not equivalent any more. In particular, one can expect to have simultaneously a kinematic share and an isotropic share. If one seeks to precisely describe the effects of a cyclic loading, it is desirable to adopt modelings sophisticated (but easy to use) such as the model of Taheri, for example, to see [R5.03.05]. On the other hand, for less complex ways of loading, one can wish to include only one linear kinematic work hardening, all nonthe linearities of work hardening being carried by the isotropic term. That makes it possible to follow a traction diagram precisely, while representing nevertheless phenomena such as the Bauschinger effect [bib1] (see for example it [Figure 5-a]).

The characteristics of work hardening are then given by a traction diagram and a constant, called of Prager, for the term of kinematic work hardening linear. They are introduced into the order `DEFI_MATERIAU`:

Linear isotropic work hardening	Nonlinear isotropic work hardening

`DEFI_MATERIAU`
`ECRO_LINE`
`SY:` *elastic limit*
`D_SIGM_EPSI:` *slope of the traction diagram*
`PRAGER: (C:` *constant of Prager* `)`
	`DEFI_MATERIAU(`
	`TRACTION: (SIGM:` *traction diagram*
	`PRAGER: (C:` *constant of Prager* `)`

These characteristics can also depend on the temperature, on condition that employing the keywords factors then `ECMI_LINE_FO` and `ECMI_TRAC_FO` instead of `ECRO_LINE` and `TRACTION`. The use of these laws of behavior is available in the orders `STAT_NON_LINE` or `DYNA_NON_LINE`:

Linear isotropic work hardening	Nonlinear isotropic work hardening

`STAT_NON_LINE`
`BEHAVIOR:`
`RELATION: 'VMIS_ECMI_LINE'`
	`STAT_NON_LINE`
	`BEHAVIOR:`
	`RELATION: 'VMIS_ECMI_TRAC'`

In the continuation of this document, one precisely describes the model of combined work hardening. One presents then the detail of his digital integration in link with the construction of the coherent tangent matrix. Lastly, a tensile test uniaxial pressing illustrates the identification of the characteristics of material.

2 Description of the model

At any moment, the state of material is described by the deformation ε, the temperature T, plastic deformation ε^{p} and cumulated plastic deformation p. The equations of state then define according to these variables of state the constraint $\sigma=\sigma^{H}\mathbf{Id}+\tilde{\sigma}$ (broken up into parts hydrostatic and deviatoric), the isotropic share of work hardening R and the kinematic share $\mathbf{X}$, also called forced recall:

$$\sigma^{H}=\frac{1}{3}tr(\sigma)=K\,tr(\varepsilon-\varepsilon^{\mathbf{th}})\,\text{avec}\,\varepsilon^{\mathbf{th}}=\alpha(T\text{-}T^{réf})\mathbf{Id} \qquad \text{éq 2-1}$$

$$\tilde{\sigma}=\sigma-\sigma^{H}\mathbf{Id}=2\mu(\tilde{\varepsilon}-\varepsilon^{p})\,\text{où}\,\tilde{\varepsilon}=\varepsilon-\frac{1}{3}tr(\varepsilon)\mathbf{Id} \qquad \text{éq 2-2}$$

$$R=\mathrm{R}(p) \qquad \text{éq 2-3}$$

$$\mathbf{X}=C\,\varepsilon^{\mathrm{p}} \qquad \text{éq 2-4}$$

where K,μ,α,R and C are characteristics of material which can depend on the temperature. More precisely, they are respectively the modules of compressibility and shearing, the thermal dilation coefficient average (see [R4.08.01]), the isotropic function of work hardening and the constant of Prager. As for $T^{réf}$, it is the temperature of reference, for which the thermal deformation is worthless.

K,μ are connected to the Young modulus E and with the Poisson's ratio by:

$$3\mathrm{K}=3\lambda+2\mu=\frac{E}{1-2\nu}$$

$$2\mu=\frac{E}{1+\nu}$$

Note:

Concerning the kinematic share of work hardening [éq 2-4], one notes that it is linear in this model. In addition, it is necessary to take care of the fact that in certain references, one calls constant of Prager $2\mathrm{C}/3$ and not C. In the same way, for the isotropic function of work hardening, the elastic limit is included there by $R(0)=\sigma^{y}$, certain references treating it except for.

Evolution of the internal variables ε^{p} et p is controlled by a normal law of flow associated with a criterion of plasticity F:

$$\mathrm{F}(\sigma,R,\mathbf{X})=(\tilde{\sigma}-\mathbf{X})_{eq}-R \text{ with } \mathbf{A}_{eq}=\sqrt{\frac{3}{2}\tilde{\mathbf{A}}\cdot\tilde{\mathbf{A}}} \qquad \text{éq 2-5}$$

$$\dot{\varepsilon}^{p}=\dot{\lambda}\frac{\partial\mathrm{F}}{\partial\sigma}=\frac{3}{2}\dot{\lambda}\frac{\tilde{\sigma}-\mathbf{X}}{(\tilde{\sigma}-\mathbf{X})_{eq}} \qquad \text{éq 2-6}$$

$$\dot{p}=\dot{\lambda}=\sqrt{\frac{2}{3}\dot{\varepsilon}^{p}\cdot\dot{\varepsilon}^{\mathrm{p}}} \qquad \text{éq 2-7}$$

As for the plastic multiplier $\dot{\lambda}$, it is obtained by the condition of following coherence:

$$\begin{cases}\text{si}\,\mathrm{F}<0\,\text{ou}\,\dot{\mathrm{F}}<0 & \dot{\lambda}=0\\ \text{si}\,\mathrm{F}=0\,\text{et}\,\dot{\mathrm{F}}=0 & \dot{\lambda}>=0\end{cases} \qquad \text{éq 2-8}$$

3 Integration of the relation of behavior

To numerically carry out the integration of the law of behavior, one carries out a discretization in time and one adopts a diagram of implicit, famous Euler adapted for relations of behavior elastoplastic. Henceforth, the following notations will be employed: A^{-}, A et ΔA the values of a quantity represent respectively A at the beginning and the step of time considered thus that its increment during the step. The problem is then the following: knowing the state at time t^{-} as well as the increments of deformation $\Delta\varepsilon$ and of temperature ΔT , to determine the state at time t as well as the constraints σ .

Initially, one takes into account the variations of the characteristics compared to the temperature by noticing that:

$$\sigma^{H}=\frac{K}{K^{-}}\sigma^{H^{-}}+K\,\mathrm{tr}\left(\Delta\varepsilon-\Delta\varepsilon^{\mathbf{th}}\right) \qquad \text{éq 3-1}$$

$$\tilde{\sigma}=\frac{\mu}{\mu^{-}}\tilde{\sigma}^{-}+2\mu\left(\Delta\tilde{\varepsilon}-\Delta\varepsilon^{\mathrm{p}}\right) \qquad \text{éq 3-2}$$

$$\mathbf{X}=\frac{C}{C^{-}}\mathbf{X}^{-}+C\Delta\varepsilon^{\mathrm{p}} \qquad \text{éq 3-3}$$

Within sight of the equation [éq 3-1], one notes that the hydrostatic behavior is purely elastic. Only the treatment of the deviatoric component is delicate. To reduce the writings to come, one introduces $\tilde{\mathbf{s}}^{e}$ the difference $\tilde{\sigma}-\mathbf{X}$ in the absence of increment of plastic deformations, so that:

$$\tilde{\sigma}-\mathbf{X}=\underbrace{\frac{\mu}{\mu^{-}}\tilde{\sigma}^{-}-\frac{C}{C^{-}}\mathbf{X}^{-}+2\mu\Delta\tilde{e}}_{\tilde{s}^{e}}-\left(2\mu+C\right)\Delta\varepsilon^{p} \qquad \text{éq 3-4}$$

The equations of flow [éq 2-6] and [éq 2-7] and the condition of coherence [éq 2-8] are written once discretized and by noticing that $p=\lambda$:

$$\Delta\varepsilon^{p}=\frac{3}{2}\Delta p\frac{\tilde{\sigma}-\mathbf{X}}{\left(\tilde{\sigma}-\mathbf{X}\right)_{eq}} \qquad \text{éq 3-5}$$

$$\mathrm{F}<=0 \quad \Delta p>=0 \quad \mathrm{F}\Delta p=0 \qquad \text{éq 3-6}$$

The treatment of the condition of coherence [éq 3-6] is classical. One starts with an elastic test ($\Delta p=0$) who is well the solution if the criterion of plasticity is not exceeded, i.e. if:

$$\mathrm{F}=s_{eq}^{e}-\mathrm{R}\left(p^{-}\right)<=0 \qquad \text{éq 3-7}$$

In the contrary case, the solution is plastic ($\Delta p>0$) and the condition of coherence [éq 3-6] is reduced to $\mathrm{F}=0$. To solve it, one starts by showing that one can bring back oneself to a scalar problem while eliminating $\Delta\varepsilon^{p}$. Indeed, by taking account of [éq 3-4] and [éq 3-5], one notes that $\Delta\varepsilon^{p}$ is colinéaire with $\tilde{\mathbf{s}}^{e}$ because:

$$\Delta\varepsilon^{\mathrm{p}}=\frac{3}{2}\frac{\Delta p}{\left(\tilde{\sigma}-\mathbf{X}\right)_{eq}}\left[\tilde{s}^{e}-\left(2\mu+C\right)\Delta\varepsilon^{p}\right] \qquad \text{éq 3-8}$$

In addition, according to [éq 3-5], the standard of $\Delta\epsilon^{p}$ is worth:

$$\left(\Delta\varepsilon^{\mathrm{p}}\right)_{eq}=\frac{3}{2}\Delta p \qquad \text{éq 3-9}$$

One from of thus deduced immediately the expression from $\Delta\varepsilon^{\mathrm{p}}$ according to Δp :

$$\Delta \varepsilon^p = \frac{3}{2} \Delta p \frac{\tilde{\mathbf{s}}^e}{s^e_{eq}} \qquad \text{éq 3-10}$$

It now only remains to replace $\Delta \varepsilon^{\mathbf{p}}$ by his expression [éq 3-10] in the equation [éq 3-4] one obtains:

$$\tilde{\sigma} - \mathbf{X} = \tilde{s}^e \left[1 - \frac{\frac{3}{2}(2\mu + C) \Delta p}{s^e_{eq}} \right]$$

while deferring $\tilde{\sigma} - \mathbf{X}$ in the equation $\mathrm{F} = 0$, one brings back oneself to a scalar equation in Δp to solve, namely:

$$|s^e_{eq} - \frac{3}{2}(2\mu + C) \Delta p| - \mathrm{R}(p^- + \Delta p) = 0 \qquad \text{éq 3-11}$$

Insofar as the function R is positive, which one will admit henceforth, there exists a solution Δp with this equation, characterized by:

$$\frac{3}{2}(2\mu + C) \Delta p + \mathrm{R}(p^- + \Delta p) = s^e_{eq} \quad \text{where} \quad 0 < \Delta p < \frac{2}{3} \frac{s^e_{eq}}{2\mu + C} \qquad \text{éq 3-12}$$

Let us note that in the interval specified in [éq 3-12], the solution is single. For details as for the solution of this equation, one will refer to [R5.03.02].

The typical case of the plane constraints is studied with [§6].

4 Calculation of tangent rigidity

In order to allow a resolution of the total problem (equilibrium equations) by a method of Newton, it is necessary to determine the coherent tangent matrix of the incremental problem. For that, one once more adopts the convention of writing of the symmetrical tensors of order 2 in the form of vectors with 6 components. Thus, for a tensor $\mathbf{a}$:

$$\mathbf{a} = {}^t\left[a_{xx} \quad a_{yy} \quad a_{zz} \sqrt{2} a_{xy} \quad \sqrt{2} a_{xz} \quad \sqrt{2} a_{yz} \right] \qquad \text{éq 4-1}$$

If moreover the hydrostatic vector is introduced $\mathbf{1}$ and stamps it deviatoric projection $\mathbf{P}$:

$$\mathbf{1} = {}^t[1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0] \qquad \text{éq 4-2}$$

$$\mathbf{P} = \mathbf{Id} - \frac{1}{3} \mathbf{1} \otimes \mathbf{1} \qquad \text{éq 4-3}$$

Then the matrix of coherent tangent rigidity is written for an elastic behavior:

$$\frac{\partial s}{\partial \Delta \varepsilon} = K\, \mathbf{1} \otimes \mathbf{1} + 2\mu\, \mathbf{P} \qquad \text{éq 4-4}$$

and for a plastic behavior:

$$\frac{\partial s}{\partial \Delta \varepsilon} = K\, \mathbf{1} \otimes \mathbf{1} + 2\mu \left(1 - \frac{3\mu \Delta p}{s^e_{eq}} \right) \mathbf{P} + 9\mu^2 \left(\frac{\Delta p}{s^e_{eq}} - \frac{1}{\mathrm{R}'(p) + \frac{3}{2}(2\mu + C)} \right) \left(\frac{\tilde{\mathbf{s}}^e}{s^e_{eq}} \otimes \frac{\tilde{\mathbf{s}}^e}{s^e_{eq}} \right) \qquad \text{éq 4-5}$$

The initial tangent matrix, used by the option RIGI_MECA_TANG is obtained by adopting the behavior of the preceding step (elastic or plastic, meant by internal variable x being worth 0 or 1) and while taking $\Delta p=0$ in the equation [éq 4-5].

Note:

> `RIGI_MECA_TANG` *is the operator linearized by report* ***at time*** *(cf [R5.03.01], [R5.03.05]) and to the problem of speed corresponds what is called; in this case, the linearization compared to* Δu *, in* $\Delta u=0$ *, provides the same expression.*

One now proposes to show the expression [éq 4-5]. By differentiating them [éq 2-1] and [éq 2-2] at fixed temperature, one obtains immediately:

$$\delta\sigma=\left[K\,\mathbf{1}\otimes\mathbf{1}+2\mu\,\mathbf{P}\right]\delta\varepsilon-2\mu\,\delta\varepsilon^{p} \qquad \text{éq 4-6}$$

If the mode of behavior is plastic, the incremental law of flow [éq 3-10] provides then:

$$\delta\varepsilon^{p}=\frac{3}{2}\delta p\frac{\tilde{\mathbf{s}}^{\mathbf{e}}}{s_{eq}^{e}}+\frac{3}{2}\Delta p\,\delta\left(\frac{\tilde{\mathbf{s}}^{\mathbf{e}}}{s_{eq}^{e}}\right) \qquad \text{éq 4-7}$$

As for dp, it is obtained by differentiating the implicit equation [éq 3-12]:

$$\left[\frac{3}{2}(2\mu+C)+\mathrm{R}'(p)\right]\delta p=\delta s_{eq}^{e} \qquad \text{éq 4-8}$$

Lastly, it any more but does not remain to provide the variations of $\tilde{\mathbf{s}}^{\mathbf{e}}$:

$$\delta\tilde{\mathbf{s}}^{\mathbf{e}}=2\mu\,\delta\tilde{\varepsilon}\,ds_{eq}^{e}=3\mu\frac{\tilde{\mathbf{s}}^{\mathbf{e}}}{s_{eq}^{e}}\cdot\delta\tilde{\varepsilon}\,\delta\left(\frac{\tilde{\mathbf{s}}^{\mathbf{e}}}{s_{eq}^{e}}\right)=\frac{1}{s_{eq}^{e}}\left(2\mu-3\mu\frac{\tilde{\mathbf{s}}^{\mathbf{e}}}{s_{eq}^{e}}\otimes\frac{\tilde{\mathbf{s}}^{\mathbf{e}}}{s_{eq}^{e}}\right)\cdot\delta\tilde{\varepsilon} \qquad \text{éq 4-9}$$

While replacing then [éq 4-7], [éq 4-8] and [éq 4-9] in [éq 4-6], one obtains well the expression [éq 4-5].

This expression is formally identical to that defined in R5.03.02: [éq 4-3] and is written:

$$\frac{\partial\sigma}{\partial\Delta\varepsilon}=K\,\mathbf{1}\otimes\mathbf{1}+2\mu\left(1-\frac{3\mu\xi\Delta p}{s_{eq}^{e}}\right)\left(\mathbf{Id}-\frac{1}{3}\mathbf{1}\otimes\mathbf{1}\right)+9\mu^{2}\xi\left(\frac{\Delta p}{s_{eq}^{e}}-\frac{1}{\mathrm{R}'+\frac{3}{2}(2\mu+C)}\right)\left(\frac{\tilde{\mathbf{s}}^{\mathbf{e}}}{s_{eq}^{e}}\otimes\frac{\tilde{\mathbf{s}}^{\mathbf{e}}}{s_{eq}^{e}}\right)$$

with $\xi=1$ if $\Delta\varepsilon$ conduit with a plasticization, and $\xi=0$ if not.

While using [éq 3-12], one finds:

$$\frac{\partial s}{\partial\Delta\varepsilon}=\lambda^{*}\vec{\mathbf{1}}\otimes\vec{\mathbf{1}}+2\mu^{*}\mathbf{Id}-\xi\frac{9\mu^{2}}{H(p)}\left(1-\frac{\mathrm{R}'.\Delta p}{\mathrm{R}(p)}\right)\frac{1}{\mathrm{R}'+\frac{3}{2}(2\mu+C)}\left(\frac{\sigma^{\mathbf{dev}}}{R(p)}\otimes\frac{\sigma^{\mathbf{dev}}}{R(p)}\right)$$

with $\lambda^{*}=K-\frac{2\mu}{3}\frac{G(\Delta p)}{H(\Delta p)}2\mu^{*}=2\mu\frac{G(\Delta p)}{H(\Delta p)}$

for the option `FULL_MECA` : $\boldsymbol{\sigma}^{\mathbf{dev}}=\tilde{\boldsymbol{\sigma}}-\mathbf{X}$

for the option `RIGI_MECA_TANG` : $\boldsymbol{\sigma}^{\mathbf{dev}}=\tilde{\boldsymbol{\sigma}}^{-}-\mathbf{X}^{-}$

with $H(\Delta p)=1+\frac{\frac{3}{2}(2\mu+C)\xi\Delta p}{R(p)}$

and $G(\Delta p)=1+\frac{3}{2}C\xi\frac{\Delta p}{R(p)}$

5 Identification of the characteristics of material

Let us consider a tensile test uniaxial pressing, [Figure 5-a]. One proposes to show how it allows to identify the constant of Prager and the isotropic function of work hardening. In such a test, the various tensors are with fixed directions, i.e.:

$$\tilde{\sigma}=\sigma\,\Delta\mathbf{X}=\mathbf{X}\,\Delta\,\varepsilon^p=\frac{3}{2}\varepsilon^p\,\mathbf{D}\ \text{ with }\ \mathbf{D}=\begin{bmatrix}2/3 & & \\ & -1/3 & \\ & & -1/3\end{bmatrix} \qquad \text{éq. 5-1}$$

As long as the loading is monotonous, therefore in phase of traction, one obtains the following relations immediately:

$$p=\varepsilon^p \quad X=\frac{3}{2}C\,\varepsilon^p \quad s^t=\frac{3}{2}C\,\varepsilon^p+\mathrm{R}(\varepsilon^p) \qquad \text{éq. 5-2}$$

The constant of Prager is determined by the first plasticization in compression, since one a:

$$\begin{cases}\sigma_A^t=\frac{3}{2}C\,\varepsilon_A^p+R(\varepsilon_A^p)\\ \sigma_A^c=\frac{3}{2}C\,\varepsilon_A^p-R(\varepsilon_A^p)\end{cases}\Rightarrow C=\frac{\sigma_A^t+\sigma_A^c}{3\,\varepsilon_A^p} \qquad \text{éq 5-3}$$

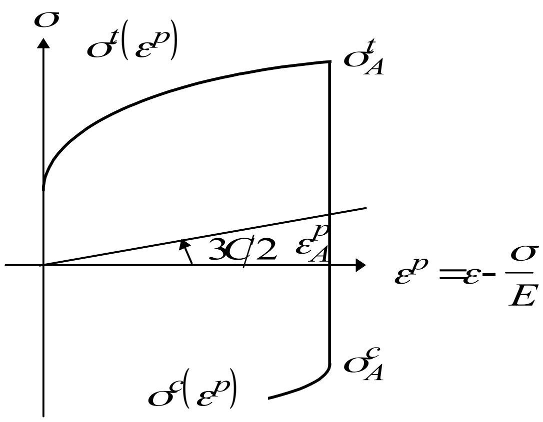

Figure 5-a: Tensile test uniaxial pressing

The curve of work hardening $\sigma^t=F(\varepsilon^p)$ is deduced traction diagram $\sigma^t=F(\varepsilon)$ provided by the user under the keywords `ECRO_LINE` ((`SY` and `D_SIGM_EPSI` (linear work hardening)) or `TRACTION` (unspecified work hardening). It finally makes it possible to obtain the isotropic function of work hardening by [éq 5-2]:

$$\mathrm{R}(\varepsilon^p)=s^t(\varepsilon^p)-\frac{3}{2}C\,\varepsilon^p\ .$$

For the effective calculation of R (p), according to the R5.03.02 document, one titrates party of the linearity (`ECMI_LINE`) or of the linearity per pieces of the traction diagram (`ECMI_TRAC`):

`ECMI_LINE` :

$$\sigma^t = F(\varepsilon^p) = \sigma_y + \frac{E.E_T}{E-E_T}\, p$$
$$R(p) = \sigma_y + \left(\frac{E.E_T}{E-E_T} - \frac{3}{2}C\right) p = \sigma_y + R'.p \qquad \text{éq 5-4}$$

The equation [éq 3-12] becomes then:

$$\frac{3}{2}(2\mu + C)\Delta p + \sigma_y + R'.(p + \Delta p) = s^e_{eq} \qquad \text{éq 5-5}$$

`ECMI_TRAC`:

$$\sigma^t = F(\varepsilon^p) = \sigma_i + \frac{\sigma_{i+1} - s_i}{p_{i-1} - p_i}(p - p_i),\ \text{pour}\ p_i <= p <= p_{i+1}$$
$$R(p) = \sigma_i + \frac{\sigma_{i+1} - \sigma_i}{p_{i-1} - p_i}(p - p_i) - \frac{3}{2}Cp = \sigma_i - \frac{\sigma_{i+1} - \sigma_i}{p_{i-1} - p_i} p_i + R'.p \qquad \text{éq 5-6}$$

Note:

For the use: the correspondence enters the model of behavior `VMIS_CINE_LINE` and the behavior `VMIS_ECMI_LINE` is the following one:
With `VMIS_CINE_LINE` , it is necessary to introduce into `DEFI_MATERIAU` a linear work hardening of slope And by:
`D_SIGM_EPSI : And`
With `VMIS_ECMI_LINE` , to reproduce same behaviour with linear kinematic work hardening, it is necessary to give in `DEFI_MATERIAU` .

- *a linear work hardening of slope E_T : `D_SIGM_EPSI : And`*
- *The constant of Prager C : `PRAGER : C`*

C is determined by: $C = \frac{2}{3}\frac{EE_T}{E-E_T}$

It should well be noticed that the identification of C and of $R(\varepsilon^p)$ have directions only in one limited field of deformations (small deformations). In particular, if $\sigma^t(\varepsilon^p)$ present an asymptote σ^t_{max} for ε^p sufficient large, then the kinematic contribution of work hardening does not have any more meaning. It is thus advised to restrict itself with the field where work hardening is strictly positive.

6 Typical case of the plane constraints: calculation of Δ p

It is necessary to add to the equations [éq 3-1] with [éq 3-4] the condition of plane constraints $s_{33}=0$, which adds an unknown factor (corresponding deformation):

$$\sigma^H = \frac{K}{K^-} s^{H^-} + K\, \mathrm{tr}(\Delta\varepsilon - \Delta\varepsilon^{\mathbf{th}}) \qquad \text{éq 6-1}$$

$$\tilde{\sigma} = \frac{\mu}{\mu^-}\tilde{\sigma}^- + 2\mu(\Delta\tilde{\varepsilon} - \Delta\varepsilon^p) \qquad \text{éq 6-2}$$

$$\mathbf{X}=\frac{C}{C^{-}}\mathbf{X}^{-}+C\Delta\varepsilon^{p} \qquad \text{éq 6-3}$$

$$\sigma_{33}=0 \qquad \text{éq 6-4}$$

Then, the equation [éq 3-4] becomes:

$$\tilde{\sigma}-X=\frac{m}{m^{-}}\tilde{\sigma}^{-}-\frac{C}{C^{-}}X^{-}+2\mathrm{mD}\tilde{\varepsilon}^{c}-(2\mu+C)\Delta\varepsilon^{p}+2\mu\Delta\tilde{e}^{y}=\tilde{\sigma}^{e}-(2\mu+C)\Delta\varepsilon^{p}+2\mu\Delta\tilde{\varepsilon}^{y} \qquad \text{éq 6-5}$$

where $\Delta\tilde{e}^{c}$ is entirely determined by the elastic behavior:

$$\Delta\tilde{\varepsilon}_{33}^{c}=\frac{-\nu}{1-\nu}\left(\Delta\tilde{\varepsilon}_{11}^{c}+\Delta\tilde{\varepsilon}_{22}^{c}\right),\ \Delta\tilde{\varepsilon}_{11}^{c}=\Delta\tilde{\varepsilon}_{11},\ \Delta\tilde{\varepsilon}_{22}^{c}=\Delta\tilde{\varepsilon}_{22}$$

and $\Delta\tilde{\varepsilon}^{y}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&\Delta Y\end{bmatrix}$ is unknown. It is also supposed that $\sigma_{13}=\sigma_{23}=\varepsilon_{13}=\varepsilon_{23}=0$.

One always has:

$$\Delta\varepsilon^{\mathrm{p}}=\frac{3}{2}\Delta p\frac{\tilde{\sigma}-\mathbf{X}}{\left(\tilde{\sigma}-\mathbf{X}\right)_{eq}} \qquad \text{éq 6-6}$$

$$\mathrm{F}=\sigma_{eq}-R(p)<=\ 0\Delta p^{3}0\quad F\Delta p=0 \qquad \text{éq 6-7}$$

Elastic test:

- If

$$\mathrm{F}=s_{eq}^{e}-R\left(p^{-}\right)<=0 \qquad \text{éq 6-8}$$

  then

$$\tilde{\sigma}=\tilde{\mathbf{s}}_{e}\ ,\ \Delta p=0\ ,\ \Delta Y=0 \qquad \text{éq 6-9}$$

$$\sigma^{H}=\frac{K}{K^{-}}\sigma^{H^{-}}+K\,\mathrm{tr}\left(\Delta\varepsilon^{c}-\Delta\varepsilon^{\mathbf{th}}\right) \qquad \text{éq 6-10}$$

- If not, the solution is plastic: $\Delta p>0$, $\Delta Y\neq0$. One can still bring back oneself to a scalar problem in Δp .

By taking account of [éq 6-5] and [éq 6-6], one notes that $\tilde{\sigma}-X$ is colinéaire with $\tilde{\mathbf{s}}^{\mathrm{e}}+2\mu\Delta\tilde{\varepsilon}^{y}$ because:

$$\left(\tilde{\sigma}-\mathbf{X}\right)\left(1+\frac{\frac{3}{2}(2\mu+C)\Delta p}{R(p)}\right)=\left(\tilde{\sigma}-\mathbf{X}\right)H(\Delta p)=\left[\tilde{\mathbf{s}}^{e}+2\mu\Delta\tilde{\varepsilon}^{y}\right] \qquad \text{éq 6-11}$$

Thus:

$$\left(\tilde{\sigma}_{33}-X_{33}\right)H(\Delta p)=\left[\tilde{s}_{33}^{e}+\frac{4}{3}\mu\Delta Y\right] \qquad \text{éq 6-12}$$

We will express the equation [éq 6-12] according to Δp only. According to [éq 6-4]:

$$\sigma_{33}=0=\tilde{\sigma}_{33}+\sigma^{H}=\tilde{\sigma}_{33}+\sigma_{e}^{H}+K.\Delta Y\ \text{, with } \sigma_{e}^{H}=\frac{K}{K^{-}}\sigma^{H^{-}}+K\,\mathrm{tr}\left(\Delta\varepsilon^{c}-\Delta\varepsilon^{\mathbf{th}}\right) \qquad \text{éq 6-13}$$

While using [éq 6-5], [éq 6-6] and the incompressibility of the plastic deformations, one can show that:

$$\tilde{s}^e_{33}+\sigma^H_e=-\frac{C}{C^-}X^-_{33}$$ éq 6-14

Then:

$$\tilde{\sigma}_{33}=\tilde{s}^e_{33}-K.\Delta Y+\frac{C}{C^-}X^-_{33}$$ éq 6-15

As according to [éq 6-3]:

$$X_{33}=\frac{C}{C^-}X^-_{33}+C.\Delta\varepsilon^p_{33}=\frac{C}{C^-}X^-_{33}+C.\frac{3}{2}\Delta p\frac{\tilde{\sigma}_{33}-X_{33}}{R(p)}$$ éq 6-16

$$X_{33}.G(\Delta p)=\frac{C}{C^-}X^-_{33}+\frac{3}{2}C\Delta p\frac{\tilde{\sigma}_{33}}{R(p)}\text{ , with }G(\Delta p)=1+\frac{3}{2}C\frac{\Delta p}{R(p)}$$ éq 6-17

From [éq 6-12], [éq 6-15], [éq 6-17], one obtains an equation flexible Δp and ΔY :

$$\Delta Y.\left(\frac{4}{3}\mu+K\frac{H(\Delta p)}{G(\Delta p)}\right)=\left[\tilde{\mathbf{s}}^{\mathbf{e}}_{33}\left(\frac{H(\Delta p)}{G(\Delta p)}-1\right)\right]$$ éq 6-18

The equation [éq 6-11] makes it possible to obtain the scalar equation in Δp to solve, namely:

$$(\tilde{\sigma}-\mathbf{X})_{eq}H(\Delta p)=R(p^-+\Delta p)H(\Delta p)=\left[\tilde{\mathbf{s}}^{\mathbf{e}}+2\mu\Delta\tilde{\varepsilon}^y\right]_{eq}$$ éq 6-19

Equation in which ΔY is function of Δp by the equation [éq 6-18].

As in the case of isotropic work hardening, one obtains a scalar equation in Δp , always nonlinear, which is solved by a method of secant.

Once Δp known, the calculation of $\tilde{\sigma}$ and X be carried out starting from the expression of ΔY, therefore of $\Delta\varepsilon$ entirely known, by a approach similar to that of the equation [éq 3-10].

$$\Delta\varepsilon^{\mathrm{p}}=\frac{3}{2}\Delta p\frac{\tilde{\mathbf{s}}^{\mathbf{e}}+2\mu\Delta\tilde{\varepsilon}^y}{\left(\tilde{\mathbf{s}}^{\mathbf{e}}+2\mu\Delta\tilde{\varepsilon}^y\right)_{eq}}=\frac{3}{2}\Delta p\frac{\tilde{\sigma}-\mathbf{X}}{H(\Delta p)\left(\tilde{\mathbf{s}}^{\mathbf{e}}+2\mu\Delta\tilde{\varepsilon}^y\right)_{eq}}$$ éq 6-20

$$\tilde{\sigma}=\frac{\mu}{\mu^-}\tilde{\sigma}^-+2\mu\left(\Delta\tilde{\varepsilon}-\Delta\varepsilon^p\right)$$ éq 6-21

One obtains while eliminating $\Delta\varepsilon^p$ from [éq 6-6], [éq 6-3] and [éq 6-2]:

$$\tilde{\sigma}=\left(\frac{\mu}{\mu^-}\tilde{\sigma}^-+2\mu\Delta\tilde{\varepsilon}\right)\frac{G(\Delta p)}{H(\Delta p)}+\frac{3}{2}2\mu\frac{\Delta p}{R(p)H(\Delta p)}\frac{C}{C^-}\mathbf{X}^-$$ éq 6-22

$$\mathbf{X}=\frac{3}{2}C\frac{\Delta p}{R(p)H(\Delta p)}\left(\frac{\mu}{\mu^-}\tilde{\sigma}^-+2\mu\Delta\tilde{e}\right)+\left(1-\frac{3}{2}C\frac{\Delta p}{R(p)H(\Delta p)}\right)\frac{C}{C^-}\mathbf{X}^-$$ éq 6-23

7 Significance of the internal variables

Internal variables of the model at the points of Gauss (`VARI_ELGA`) are for all modelings:

- `VARI_1` = p : cumulated plastic deformation (positive or worthless)
- `VARI_2` = ξ : being worth 1 if the point of Gauss plasticized during the increment or 0 if not.

X : tensor of recall:

For modeling 3D :

- VARI_3 = X_{11}
- VARI_4 = X_{22}
- VARI_5 = X_{33}
- VARI_6 = X_{12}
- VARI_7 = X_{13}
- VARI_8 = X_{23}

For modelings D_PLAN, C_PLAN, AXIS

- VARI_3 = X_{11}
- VARI_4 = X_{22}
- VARI_5 = X_{33}
- VARI_6 = X_{12}

For modelings of hulls (DKT, COQUE_3D), in local reference mark and each point of integration of each layer:

- VARI_3 = X_{11}
- VARI_4 = X_{22}
- VARI_5 = X_{33}
- VARI_6 = X_{12}

8 Bibliography

1) J. LEMAITRE, J.L. CHABOCHE: “Mechanical of solid materials”. Dunod 1992

Description of the versions of the document

Version Aster	Author (S) Organization (S)	Description of the modifications
5.	J.M Proix, E.Lorentz EDF-R&D/AMA	Initial version
8.5	J.M.Proix, EDF-R&D/AMA	Change of notation of the module of compressibility, cf drives REX 10218