Law of behavior very-rubber band: almost incompressible material

Summary:

One describes here the formulation adopted for a law of behavior very-rubber band of Signorini. This law is a generalized version of the laws of Mooney-Rivlin often adopted for elastomers. The parameters characterizing material are defined in DEFI_MATERIAU with the keyword ELAS_HYPER.

This model is selected in the order STAT_NON_LINE or DYNA_NON_LINE via the keyword RELATION = 'ELAS_HYPER' under the keywords BEHAVIOR. This relation extends to great transformations; this functionality is selected via the keyword DEFORMATION = 'GROT_GDEP'. It is available for the elements 3D, 3D_SI, C_PLAN and D_PLAN.

1 Potential of deformation

1.1 Kinematics

A solid is considered Ω subjected to great deformations. That is to say $\mathbf{F}$ the tensor of the gradient of the transformation making pass the initial configuration Ω_0 with the deformed current configuration Ω_t. One notes $\mathbf{X}$ the position of a point in Ω_0 and $\mathbf{x}$ the position of this same point after deformation in Ω_t. $\mathbf{u}$ is then the EPD.lacing enters the two configurations. One thus has:

$$\mathbf{x}=\mathbf{X}+\mathbf{u} \quad (1)$$

The tensor of the gradient of the transformation is written:

$$\mathbf{F}=\frac{\partial \mathbf{x}}{\partial \mathbf{X}}=\mathbf{I}+\nabla_{\mathbf{x}}\mathbf{u} \quad (2)$$

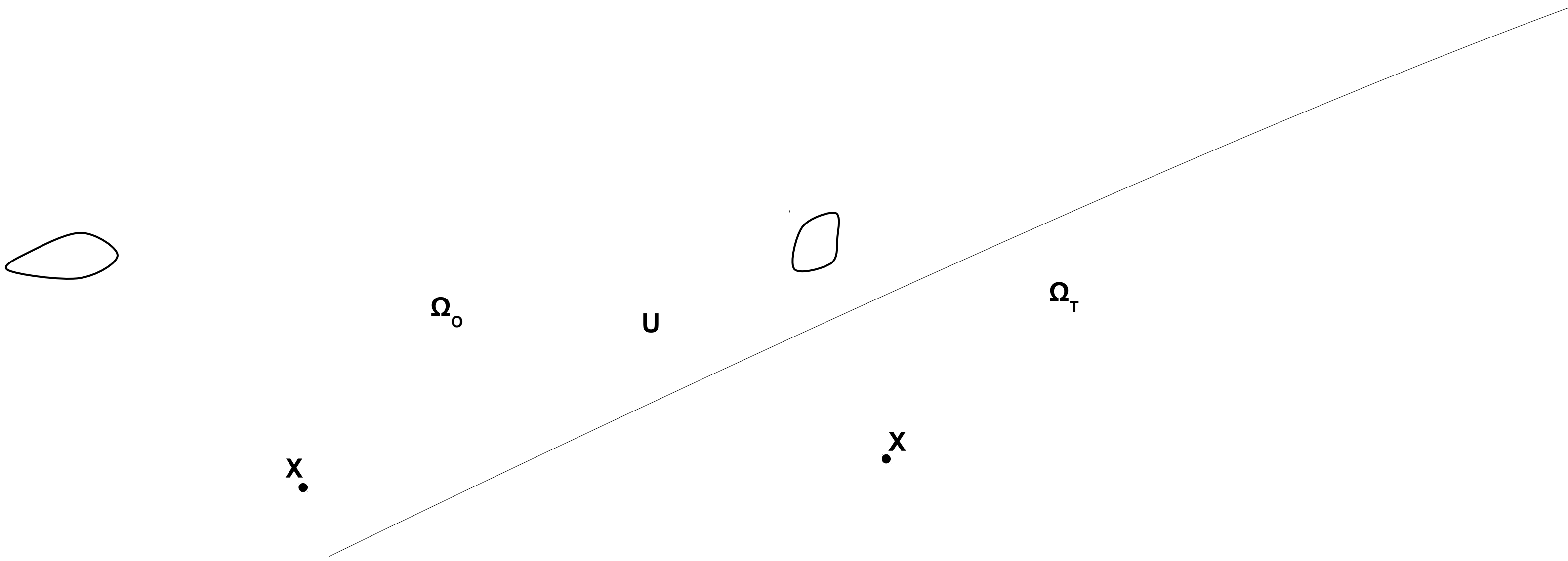

Figure 1.1-1: Transformation of the initial configuration with finale

This tensor is not the best candidate to describe the structural deformation. In particular, it is not worth zero for movements of rigid body and it describes all the transformation: change length of the infinitesimal elements but also their orientation. However a rotation movement pure does not generate constraints and it is thus preferable to use a measurement of the deformations which does not take into account this rigid rotation. Let us consider an element infinitesimal length noted $\mathrm{d}\mathbf{X}$ in the initial configuration and $\mathrm{d}\mathbf{x}$ in the final configuration. If the movement is a rigid rotation $\mathbf{R}$, one a:

$$\mathrm{d}\mathbf{x}=\mathbf{R}.\mathrm{d}\mathbf{X} \tag{3}$$

The standard of this vector after transformation is thus worth

$$d\mathbf{x}.d\mathbf{x}=\mathbf{R}.d\mathbf{X}.\mathbf{R}.d\mathbf{X}=\mathbf{R}^T.\mathbf{R}.d\mathbf{X}.d\mathbf{X} \tag{4}$$

As the transformation is purely rigid, one a:

$$\mathbf{R}^T.\mathbf{R}=\mathbf{I} \tag{5}$$

$\mathbf{R}$ is thus an orthogonal tensor. The tensor gradient of deformation can be written like the product of an orthogonal tensor and a positive definite tensor (polar factorization):

$$\mathbf{F}=\mathbf{R}.\mathbf{U} \tag{6}$$

The tensor $\mathbf{U}$ (called tensor of lengthening) is thus the first measurement of great deformations. On the other hand, it requires the polar factorization of $\mathbf{F}$, which is expensive operation. One thus prefers to use the tensor of the deformations of right Cauchy-Green:

$$\mathbf{C}=\mathbf{F}^{\mathrm{T}}.\mathbf{F}=\mathbf{U}^2 \tag{7}$$

This tensor is symmetrical. Three invariants of the tensor of Cauchy-Green $\mathbf{C}$ are given by[1] :

$$I_c=\mathrm{tr}(\mathbf{C}) \tag{8}$$

$$II_c=\frac{1}{2}.\left((\mathrm{tr}(\mathbf{C}))^2-\mathrm{tr}(\mathbf{C}^2)\right) \tag{9}$$

$$III_c=\det(\mathbf{C}) \tag{10}$$

The last invariant III_c described the change of volume, one can also write it:

$$III_c=(\det\mathbf{F})^2=J^2 \tag{11}$$

1.2 Potential of deformation – case compressible

A model very-rubber band supposes the existence of a potential density of energy internal Ψ, scalar function of the measurement of the deformations. If an isotropic behavior is considered, the function Ψ the being must too. It is shown that if the function Ψ depends only on the invariants of the tensor of the deformations of right Cauchy-Green $\mathbf{C}$, then it describes an isotropic behavior. The potential is thus a function of these three invariants:

$$\Psi=\Psi(I_c, II_c, III_c) \tag{12}$$

The second tensor of the constraints of Piola-Kirchhoff is obtained by derivation of this potential compared to the deformations (see [R5.03.20]):

$$\mathbf{S}=2.\frac{\partial\Psi}{\partial\mathbf{C}} \tag{13}$$

The most general form of a potential is polynomial. She is written according to the invariants of the tensor of right Cauchy-Green and the parameters materials C_{pqr} according to Rivlin:

$$\Psi=\sum_{(p,q,r)=0}^{\infty} C_{pqr}.(I_c-3)^p.(II_c-3)^q.(III_c-1)^r \tag{14}$$

with $C_{000}=0$.

There exist particular forms of this potential which are used very frequently, with $r=0$:

1 One notes $\mathrm{tr}(\mathbf{C})=C_{11}+C_{22}+C_{33}$ the trace of the tensor $\mathbf{C}$.

	p		q
	1	2	1
	C_{10}	C_{20}	C_{01}
Signorini	YES	YES	YES
Mooney-Rivlin	YES	NOT	YES
Néo-Hookéen	YES	NOT	NOT

1.3 Potential of deformation – case incompressible

1.3.1 Principle

Most materials very-rubber bands (as elastomers) are incompressible, i.e.:

$$\det \mathbf{F}=1 \quad (15)$$

And thus the third invariant III_c is thus worth:

$$III_c=1 \quad (16)$$

The potential very-rubber band in incompressible mode is thus rewritten:

$$\Psi=\sum_{(p,q)=0}^{\infty} C_{pq}.\left(I_c-3\right)^p.\left(II_c-3\right)^q \quad (17)$$

Unfortunately, such a writing leads to severe digital problems (except for the case of the plane constraints). We will thus propose a new writing which makes it possible to solve in a more effective way the case of the incompressibility and which has in more the good taste to be also valid in mode compressible, with a wise choice (and easy) of the coefficients.

1.3.2 Tensor of Cauchy-Green modified right

One starts by defining a new tensor of right Cauchy-Green $\mathbf{C}^*$ (pure or isochoric deviatoric dilations) such as:

$$\mathbf{C}^*=J^{-\frac{2}{3}}.\mathbf{C} \quad (18)$$

Indeed, $J^{\frac{1}{3}}.\mathbf{I}$ indicate the pure voluminal deformation. This tensor remains symmetrical. Its invariants are:

$$I_c^*=J^{-\frac{2}{3}}.I_c \quad (19)$$

$$II_c^*=J^{-\frac{4}{3}}.II_c \quad (20)$$

$$III_c^*=1 \quad (21)$$

Knowing that:

$$J=\left(C_{11}.C_{22}.C_{33}-C_{12}^2.C_{33}-C_{23}^2.C_{11}+2.C_{12}.C_{13}.C_{23}-C_{13}^2.C_{22}\right)^{\frac{1}{2}} \quad (22)$$

With:

$$I_c^*=\frac{C_{11}+C_{22}+C_{33}}{J^{2/3}} \quad (23)$$

And:

$$II_c^*=\frac{C_{22}.C_{33}+C_{11}.C_{33}+C_{11}.C_{22}}{J^{4/3}} \quad (24)$$

These invariants are also called invariants **reduced** of $\mathbf{C}$.

1.3.3 Potential of deformation modified

If one expresses the potential of deformation using the invariants reduced of $\mathbf{C}$, one can break up the potential into two parts:

$$\Psi=\Psi^{iso}+\Psi^{vol} \quad (25)$$

There is the part Ψ^{iso} who corresponds to the isochoric deformations ($J=1$):

$$\Psi^{\text{iso}}=\sum_{p,q=0}^{\infty} C_{pq}.\left(I_c^*-3\right)^p.\left(II_c^*-3\right)^q \quad (26)$$

And the part Ψ^{vol} who corresponds to the voluminal deformations ($J\neq 1$):

$$\Psi^{\text{vol}}=\frac{K}{2}.(J-1)^2 \quad (27)$$

K is the coefficient of compressibility. This formulation makes it possible to keep account of the effects incompressible and compressible:

1. In the incompressible case, and the framework of a formulation by finite elements the parameter K play the part of a coefficient of penalization of the condition of incompressibility;
2. In the compressible case this same coefficient translates a material property: hydrostatic compressibility.

If the model characterizing material is of Mooney-Rivlin type ($p=q=1$), K can be given by:

$$K=\frac{4\left(C_{01}+C_{10}\right)(1+\nu)}{3(1-2\nu)} \quad (28)$$

In the case of small deformations, $E=4(C_{01}+C_{10})(1+\nu)$ represent the Young modulus while $G=2(C_{01}+C_{10})$ represent the modulus of rigidity.

1.3.4 Tensor of the constraints of Piola-Kirchhoff 2

The tensor of constraints of Piola-Kirchhoff 2, representing the stresses measured in the initial configuration, is written:

$$\mathbf{S}=2.\frac{\partial\Psi}{\partial\mathbf{C}} \quad (29)$$

Factor 2 makes it possible to find the usual expression in small deformations. One can separate it in two parts:

$$\mathbf{S}=\mathbf{S}^{\text{iso}}+\mathbf{S}^{\text{vol}} \quad (30)$$

With:

$$\mathbf{S}^{\text{iso}}=2.\frac{\partial\Psi^{\text{iso}}}{\partial\mathbf{C}} \quad (31)$$

And:

$$\mathbf{S}^{\text{vol}}=2.\frac{\partial\Psi^{\text{vol}}}{\partial\mathbf{C}} \quad (32)$$

$\mathbf{S}^{\text{iso}}$ is the tensor of the isochoric constraints and $\mathbf{S}^{\text{vol}}$ is that of the voluminal or hydrostatic constraints.

1.3.5 Lagrangian tensor of elasticity

The elastic tensor of stiffness (“tangent” matrix for the problem of Newton) is given by the double derivation of the potential:

$$\mathbf{K}=4.\frac{\partial^2\Psi}{\partial\mathbf{C}.\partial\mathbf{C}}=4.\frac{\partial^2\Psi^{\text{iso}}}{\partial\mathbf{C}.\partial\mathbf{C}}+4.\frac{\partial^2\Psi^{\text{vol}}}{\partial\mathbf{C}.\partial\mathbf{C}} \quad (33)$$

2 Analytical expressions

2.1 Case of the constraints

We will detail the analytical expression of the constraints of Piola-Kirchhoff for the potential very-rubber band of Signorini ($p=2$ and $q=1$) in the incompressible case. There is thus the tensor of constraints of Piola-Kirchhoff 2, representing the stresses measured in the initial configuration which is written:

$$\mathbf{S}=2.\frac{\partial\Psi^{\mathrm{iso}}}{\partial\mathbf{C}}+2.\frac{\partial\Psi^{\mathrm{vol}}}{\partial\mathbf{C}} \tag{34}$$

With the two potentials:

$$\begin{gathered}\Psi^{\mathrm{iso}}=C_{10}.\left(I_c^*-3\right)+C_{01}.\left(II_c^*-3\right)+C_{20}.\left(I_c^*-3\right)^2\\ \Psi^{\mathrm{vol}}=\frac{K}{2}.\left(J-1\right)^2\end{gathered} \tag{35}$$

To obtain the constraints, the potential should be derived:

$$\begin{gathered}\mathbf{S}_{ij}^{iso}=2\frac{\partial\Psi^{iso}}{\partial I_c^*}.\frac{\partial I_c^*}{\partial\mathbf{C}_{ij}}+2\frac{\partial\Psi^{iso}}{\partial II_c^*}.\frac{\partial II_c^*}{\partial\mathbf{C}_{ij}}\\ \mathbf{S}_{ij}^{vol}=2\frac{\partial\Psi^{vol}}{\partial J}.\frac{\partial J}{\partial\mathbf{C}_{ij}}\end{gathered} \tag{36}$$

With:

$$\frac{\partial\Psi^{iso}}{\partial I_c^*}=C_{10}+2.\mathrm{C}_{20}.\left(I_c^*-3\right)\quad\frac{\partial\Psi^{iso}}{\partial II_c^*}=C_{01}\quad\frac{\partial\Psi^{vol}}{\partial J}=K.\left(J-1\right) \tag{37}$$

As well as the derivative of the reduced invariants (*cf.* page 26 of [5] for the derivative of the invariants of a tensor):

$$\frac{\partial I_c^*}{\partial\mathbf{C}_{ij}}=III_c^{-\frac{1}{3}}.\left(\delta_{ij}-\frac{1}{3}.\mathbf{C}_{ij}^{-1}.I_c\right) \tag{38}$$

$$\frac{\partial II_c^*}{\partial\mathbf{C}_{ij}}=III_c^{-\frac{2}{3}}.\left(I_c.\delta_{ij}-\mathbf{C}_{ij}-\frac{2}{3}.\mathbf{C}_{ij}^{-1}.II_c\right) \tag{39}$$

$$\frac{\partial J}{\partial\mathbf{C}_{ij}}=\frac{1}{2}.III_c^{\frac{1}{2}}.\mathbf{C}_{ij}^{-1} \tag{40}$$

Here thus the analytical expression of the voluminal constraints:

$$\mathbf{S}_{ij}^{vol}=K.\left(J-1\right).J.\mathbf{C}_{ij}^{-1} \tag{41}$$

And of the isochoric constraints:

$$\mathbf{S}_{ij}^{iso}=2\left(\left(C_{10}+2.\mathrm{C}_{20}.\left(I_c^*-3\right)\right).J^{-\frac{2}{3}}.\left(\delta_{ij}-\frac{1}{3}.\mathbf{C}_{ij}^{-1}.I_c\right)+C_{01}.J^{-\frac{4}{3}}.\left(I_c.\delta_{ij}-\mathbf{C}_{ij}-\frac{2}{3}.\mathbf{C}_{ij}^{-1}.II_c\right)\right) \tag{42}$$

2.2 Case of the elastic matrix

We will detail the analytical expression of the elastic matrix for the potential very-rubber band of Signorini ($p=2$ and $q=1$) in the incompressible case. One thus has:

$$\mathbf{K}=4.\frac{\partial^2\Psi^{\text{iso}}}{\partial\mathbf{C}.\partial\mathbf{C}}+4.\frac{\partial^2\Psi^{\text{vol}}}{\partial\mathbf{C}.\partial\mathbf{C}} \quad (43)$$

It is thus necessary to derive (twice) the potential:

$$\mathbf{K}_{ijkl}^{iso}=\frac{\partial^2\Psi^{iso}}{\partial^2 I_c^*}.\frac{\partial^2 I_c^*}{\partial\mathbf{C}_{ij}.\partial\mathbf{C}_{kl}}+\frac{\partial^2\Psi^{iso}}{\partial^2 II_c^*}.\frac{\partial^2 II_c^*}{\partial\mathbf{C}_{ij}.\partial\mathbf{C}_{kl}}$$
$$\mathbf{K}_{ijkl}^{vol}=\frac{\partial^2\Psi^{vol}}{\partial^2 J}.\frac{\partial^2 J}{\partial\mathbf{C}_{ij}.\partial\mathbf{C}_{kl}} \quad (44)$$

Constant the materials are supposed to be constant. One thus has:

$$\frac{\partial^2\Psi^{iso}}{\partial^2 I_c^*}=2.\mathrm{C}_{20}\quad\frac{\partial^2\Psi^{iso}}{\partial^2 II_c^*}=0\quad\frac{\partial^2\Psi^{vol}}{\partial^2 J}=K \quad (45)$$

It is seen that the coefficient K is well a coefficient of penalization and that its choice impacts the conditioning of the matrix. The derivative of the reduced invariants:

$$\frac{\partial^2 I_c^*}{\partial\mathbf{C}_{ij}.\partial\mathbf{C}_{kl}}=III_c^{-\frac{1}{3}}.\left(\mathbf{C}_{ki}^{-1}.\mathbf{C}_{lj}^{-1}.I_c-\mathbf{C}_{ij}^{-1}.\delta_{kl}-\mathbf{C}_{kl}^{-1}.\delta_{ij}+\frac{1}{3}.\mathbf{C}_{kl}^{-1}.\mathbf{C}_{ij}^{-1}.I_c\right) \quad (46)$$

$$\frac{\partial^2 II_c^*}{\partial\mathbf{C}_{ij}.\partial\mathbf{C}_{kl}}=\begin{array}{c}-\frac{2}{3}.III_c^{-\frac{2}{3}}.\mathbf{C}_{kl}^{-1}.\left(I_c.\delta_{ij}-\mathbf{C}_{ij}-\frac{2}{3}.\mathbf{C}_{ij}^{-1}.II_c\right)+\\ III_c^{-\frac{2}{3}}.\left(\delta_{kl}.\delta_{ij}-\delta_{ik}.\delta_{jl}+\frac{2}{3}.\mathbf{C}_{ki}^{-1}.\mathbf{C}_{lj}^{-1}.II_c-\frac{2}{3}.\mathbf{C}_{ij}^{-1}.\left(I_c.\delta_{kl}-\mathbf{C}_{kl}\right)\right)\end{array} \quad (47)$$

$$\frac{\partial^2 J}{\partial\mathbf{C}_{ij}.\partial\mathbf{C}_{kl}}=\frac{1}{4}.III_c^{\frac{1}{4}}.\left(\mathbf{C}_{kl}^{-1}.\mathbf{C}_{ij}^{-1}-2.\mathbf{C}_{ki}^{-1}.\mathbf{C}_{lj}^{-1}\right) \quad (48)$$

3 Bibliographical references

[1] G.A. Holzapfel – “Nonlinear solid mechanics” – Wiley – 2001.

[2] J. Bonnet – “ Nonlinear continuum mechanics for finite element analysis ” – Cambridge University Close – 1997.

[3] Mr. A. Crisfield – “ Nonlinear finite element analysis of solid and structures ” – Wiley – 1991.

[4] A. Delalleau – “Analysis of the mechanical behavior of the skin *in vivo* ” – Doctorate University Jean Monnet de Saint Étienne – 2007.

[5] C. Truesdell, W. Nool - The Non-Linear Field Theories of Mechanics, vol. 3, Springer, 2004.

4 Description of the versions of the document

Version Aster	Author (S) Organization (S)	Description of the modifications
8.4	Mr. Abbas, T. Baranger EDF-R&D/AMA, UCBL	text inital
8.5	M.Abbas, EDF-R&D/AMA	Correction page 2, cf drives REX 11026
10.1	J.M.Proix EDF-R&D/AMA	Replacement of GREEN by GROT_DEP
10.2	Mr. Abbas, EDF-R&D/AMA	Complete rewriting, analytical expressions of the constraints and the tangent matrix