Law of behavior in great rotations and small deformations

Summary:

One describes here the formulation adopted to treat great rotations and small deformations. This formulation is valid for all the laws of behavior defined under BEHAVIOR order STAT_NON_LINE and provided with modelings three-dimensional (3D), axisymmetric (AXIS), in plane deformations (D_PLAN) and in plane constraints (C_PLAN).

This functionality is selected via the keyword DEFORMATION = 'GROT_GDEP' under BEHAVIOR.

Contents

1 Some definitions

One points out here some definitions of tensors related to the great deformations.
One calls tensor gradient of the transformation $\mathbf{F}$, the tensor which makes pass from the initial configuration Ω_0 with the deformed current configuration $\Omega(t)$.

$$\mathbf{F}=\frac{\partial \hat{\mathbf{x}}}{\partial \mathbf{X}}=\mathbf{Id}+\nabla_{\mathbf{X}}\mathbf{u} \text{ with } \mathbf{x}=\hat{\mathbf{x}}(\mathbf{X},t)=\mathbf{X}+\mathbf{u} \quad \textbf{éq 1-1}$$

where $\mathbf{X}$ is the position of a point in Ω_0 , $\mathbf{x}$ the position of this same point after deformation in $\Omega(t)$ and $\mathbf{u}$ displacement.
Various tensors of deformations can be obtained by eliminating rotation in the local transformation. This can be done in two manners, either by using the polar theorem of decomposition, or by directly calculating the variations length and angle (variation of the scalar product).
Of Lagrangian description is obtained (i.e. on the initial configuration):

- By the polar decomposition:

$$\mathbf{F}=\mathbf{R}\,\mathbf{U} \quad \textbf{éq 1-2}$$

where $\mathbf{R}$ is the tensor of rotation (orthogonal) and $\mathbf{U}$ the tensor of pure deformations right (symmetrical and definite positive).

- By a direct calculation of the deformations:

$$\mathbf{E}=\frac{1}{2}(\mathbf{C}-\mathbf{Id}) \text{ with } \mathbf{C}=\mathbf{F}^T\mathbf{F} \quad \textbf{éq 1-3}$$

where $\mathbf{E}$ is the tensor of deformation of Green-Lagrange and $\mathbf{C}$ the tensor of right Cauchy-Green.
Tensors $\mathbf{U}$ and $\mathbf{C}$ are connected by the following relation:

$$\mathbf{C}=\mathbf{U}^2 \quad \textbf{éq 1-4}$$

2 Assumption of the small deformations and great rotations

When the deformations are small, there are no fundamental difficulties to write the laws of behavior: the various models “great deformations” lead to the same model “small deformations”, and this as well for isotropic behaviors as anisotropic. Only the difficulty of a geometrical nature related to finished rotation remains.

To write the model in great rotations and small deformations, one leaves the polar decomposition $\mathbf{F}$ that is to say $\mathbf{F}=\mathbf{R}\,\mathbf{U}$. Like the tensor $\mathbf{U}$ is a tensor of deformation pure and in addition small, one can calculate, by a law of behavior small deformations, the tensor of the constraints $\boldsymbol{\sigma}^*$ associated with this history in deformation $\mathbf{U}$. It is then enough to subject to this tensor $\boldsymbol{\sigma}^*$, rotation $\mathbf{R}$ to obtain the tensor of the constraints $\boldsymbol{\sigma}$ associated with the history in deformation $\mathbf{F}$, as follows:

$$\boldsymbol{\sigma}=\mathbf{R}\,\boldsymbol{\sigma}^*\mathbf{R}^T \quad \textbf{éq 2-1}$$

One can summarize this diagram as follows:

$$\mathbf{F} \rightarrow \mathbf{U} = \mathbf{Id} + \boldsymbol{\varepsilon} \overset{ldc\,HPP}{\rightarrow} \boldsymbol{\sigma}^* \rightarrow \boldsymbol{\sigma} = \mathbf{R}\,\boldsymbol{\sigma}^*\,\mathbf{R}^T \qquad \textbf{éq 2-2}$$

The disadvantage of this computation channel is that it requires the polar decomposition of $\mathbf{F}$. Two assumptions are made then to avoid it.
On the one hand, to avoid the calculation of $\mathbf{U}$, one can approach the deformation HP $\boldsymbol{\varepsilon}$, by the deformation of Green $\mathbf{E}$, by benefiting owing to the fact that the deformations are small:

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{Id}) = \frac{1}{2}(\mathbf{U} - \mathbf{Id})(\mathbf{U} + \mathbf{Id}) = \boldsymbol{\varepsilon} + \frac{1}{2}\boldsymbol{\varepsilon}^2 \approx \boldsymbol{\varepsilon} \qquad \textbf{éq 2-3}$$

One from of deduced then $\boldsymbol{\sigma}^*$ by the law of behavior “small deformations”.
In addition, in the same manner to avoid the calculation of $\mathbf{R}$, one can approach the tensor of the constraints HP $\boldsymbol{\sigma}^*$ by the second tensor of Piola-Kirchhoff $\mathbf{S}$:

$$\mathbf{S} = J\,\mathbf{F}^{-1}\boldsymbol{\sigma}\,\mathbf{F}^{-T} = \mathrm{Det}(\mathbf{U})\,\mathbf{U}^{-1}\boldsymbol{\sigma}^*\mathbf{U}^{-1} = \boldsymbol{\sigma}^* + \boldsymbol{\sigma}^*\,\mathrm{O}(\boldsymbol{\varepsilon}) \approx \boldsymbol{\sigma}^* \qquad \textbf{éq 2-4}$$

One from of deduced then $\boldsymbol{\sigma}$ by:

$$\boldsymbol{\sigma} = \frac{1}{J}\mathbf{F}\mathbf{S}\mathbf{F}^T \qquad \textbf{éq 2-5}$$

Finally, in the presence of great rotations and to small deformations, it is enough to write the law of behavior “small deformations” with, as starter, the history of the deformations of Green $\mathbf{E}$, and at exit, history of the constraints of Piola-Kirchhoff $\mathbf{S}$. This approach is valid as well for isotropic laws of behavior as anisotropic.
As for the adapted variational formulation, it is about that adopted in very-elasticity (behavior `ELAS`, `ELAS_VMIS_XXX` under `BEHAVIOR` with the deformations of the type `GROT_GDEP)`. For more details, one will refer to the associated reference document [R5.03.20]. It is necessary however to be sure that the problem studied induced many small deformations because if not one cannot make any more simplifications [éq 2-3] and [éq 2-4]. Without this assumption, the variation with a plastic behavior increases quickly with the intensity of the deformations.

3 Bibliography

1. CANO V., LORENTZ E., “Introduction into *Code_Aster* of a model of behavior in great elastoplastic deformations with isotropic work hardening”, intern EDF DER, HI-74/98/006/0, 1998 Notes

4 Description of the versions of the document

Version Aster	Author (S) Organization (S)	Description of the modifications
6	V.Cano EDF-R&D/AMA	Initial text
10,1	J.M.Proix EDF-R&D/AMA	Change of GREEN in GROT_GDEP