Law of damage to gradient ENDO_FISS_EXP

Summary:

This document describes the model of elastic behavior fragile ENDO_FISS_EXP, available in nonlocal modeling with gradient of damage GRAD_VARI. The damage is modelled there in an isotropic way, what preferentially implies to model the cracks in an individual way and not by homogenisation: the scale of discretization must be adapted to this choice. On the phenomenologic level, the criterion of damage is adapted to the modeling of the concretes, by in particular distinguishing the states from traction and compression. Moreover, a restoration of rigidity in compression is introduced to reflect the closing of the cracks. On the other hand, the model is not a priori adapted to model the damage of the concrete in compression. Lastly, it should be noted that the answer of the model tends towards that of a cohesive law when the characteristic length becomes small in front of the size of the structure. It results from it that the question of the identification this characteristic length does not arise for little that it is sufficiently small. And the model can give access to the sizes of interest of the cohesive laws, such as for example the opening of crack.

1 Scope of application

1.1 Finalities

The law of behavior ENDO_FISS_EXP , available in the order STAT_NON_LINE , aims to describe the quasi-fragile damage of the mechanical concrete under requests of traction. The damage is described there in an isotropic way, which positions the model to describe the cracks individually and not in a homogenized way. On the phenomenologic level, the model aims at reflecting more particularly:

- contrast enters the elastic limits in traction and compression, thus avoiding a premature damage in compression under prestressing, for example;
- the specific form of the surface of damage in multiaxial traction, near of a criterion to Rankine;
- impact on the rigidity of the closing of the cracks in the directions of compression, which also reinforces the robustness of simulations in the presence of emerging cracks.

In addition, the law ENDO_FISS_EXP enter within the framework as of nonlocal models to gradient of variable interns (modeling *_GRAD_VARI). The particular form of the law makes it possible to ensure a fundamental property: when the length characteristic of the model tends towards zero (in practice, when it becomes small in front of the size structural feature studied), the results tend towards those of a cohesive model, adapted well to predict the kinematics of propagation and the opening of crack. Consequently, except to seek information with scale sub-centimetric (what one disadvises), the characteristic length does not appear as a parameter to be readjusted but can be fixed directly starting from dimensions of the studied structure.

The interested reader will be able to find elements detailed on the formulation of the law and his physical validation in the reference [Lorentz, 2016].

1.2 Parameters materials

The law of behavior is based on eight internal parameters (which the constants of elasticity), as one will see it in the presentation of the equations in the following chapter. However, thanks to coherence with a cohesive law, one can reformulate these internal parameters in terms of sizes more accessible to the engineer. In addition to the Young modulus and the Poisson's ratio, one will inform resistances in traction and compression (the latter being less significant for the targeted problems), the energy of fracturing and two parameters (of which one is optional) to describe the form of the lenitive answer. Lastly, the characteristic length will be fixed on the basis of dimension of the studied work or the part. The order DEFI_MATER_GC deals with the transformation of these parameters of engineer into internal characteristics of the law and replaces the order DEFI_MATERIAU.

1.3 Internal variables

The degree of damage, understood enters 0 for a healthy material and 1 for a completely damaged material, constitutes a priori only the variable intern of interest for the user. She is stored in first place (V1) in the table of the internal variables stored by the code. As postprocessing, the V3 component corresponds to the impact of this level of damage on rigidity in traction, that is to say $A(a)$. Lastly, for information, of the questions of performance and digital robustness result in also filing in the field VARI_ELGA the state of the step of time running in the V2 component (0=elastic, 1=endommageant, 2=saturé) as well as the level of deformation reached at the end of the step of time in the V4 components in V9 (where the components of shearing are affected of a factor $\sqrt{(2)}$).

2 Continuous model

The whole of the equations of the model is joined together in tablwater 2-1. The link between the internal parameters of the model and the sizes more accessible to the engineer is also specified there. The choices of modeling call some remarks that one exposes below.

Material parameters

Lamé coefficients	λ, μ
Tensile and compressive strengths	f_t, f_c
Fracture energy	G_F
Softening parameters	p, q
Nonlocal length scale	D

Stress – strain relation

The stress and strain eigenvectors are identical

The stress and strain eigenvalues are related through:

$$\sigma_i = \mathrm{A}(a)\left[\lambda \operatorname{tr}\varepsilon + 2\mu\,\varepsilon_i\right] + \left[1-\mathrm{A}(a)\right]\left[\frac{\lambda}{2}\mathrm{S}'(\operatorname{tr}\varepsilon) + \mu\,\mathrm{S}'(\varepsilon_i)\right]$$

$$\mathrm{A}(a) = \frac{(1-a)^2}{(1-a)^2 + \dfrac{3}{2}\dfrac{(\lambda+2\mu)G_F}{\sigma_c^2 D}\,a\left[1 + p\,a\exp\left(q^2a^2\right)\right]}$$

$$\mathrm{S}'(x) = \begin{cases} (2x - 1/\gamma)\exp\left(\dfrac{1}{\gamma x}\right) & \text{if } x \le 0 \\ 0 & \text{if } x > 0 \end{cases}$$

Damage evolution

Initial damage surface

$$\mathrm{f}_S(\boldsymbol{\sigma}) = \left\|\frac{\boldsymbol{\sigma}}{\sigma_0} + \left(\beta_0 - \frac{1}{3}\right)\operatorname{tr}\left(\frac{\boldsymbol{\sigma}}{\sigma_0}\right)\mathbf{Id}\right\| + \left\|\mathbf{exp}\left(\frac{\boldsymbol{\sigma}}{\sigma_0}\right)\right\| - \gamma_0$$

Evolution law

$$\mathrm{g}(\boldsymbol{\varepsilon},a) \le 0 \quad ; \quad \dot{a} \ge 0 \quad ; \quad \mathrm{g}(\boldsymbol{\varepsilon},a)\,\dot{a} = 0$$

$$\mathrm{g}(\boldsymbol{\varepsilon},a) = -\mathrm{A}'(a)\Gamma(\boldsymbol{\varepsilon}) + c\nabla^2 a - k$$

$$\Gamma(\boldsymbol{\varepsilon}) = \frac{1}{2}\frac{\sigma_c^2}{(\lambda+2\mu)}\chi(\boldsymbol{\varepsilon})^2$$

Implicit def. of $\chi(\boldsymbol{\varepsilon})$: $\mathrm{f}_S\left(\dfrac{\lambda(\operatorname{tr}\boldsymbol{\varepsilon})\mathbf{Id} + 2\mu\boldsymbol{\varepsilon}}{\chi(\boldsymbol{\varepsilon})}\right) = 0$

Boundary and interface conditions

$$\partial a/\partial n = 0$$

$$[\![a]\!] = 0 \quad \text{and} \quad [\![c\,\partial a/\partial n]\!] = 0$$

Internal parameters

Damage surface parameters σ_0, γ_0 and β_0 set through:

$$\mathrm{f}_S\left(\begin{bmatrix} f_t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = 0 \;;\; \mathrm{f}_S\left(-\begin{bmatrix} f_c & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = 0 \;;\; \beta_0 = 0.1$$

Confined peak stress σ_c

$$\mathrm{f}_S\left(\sigma_c\begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda/(\lambda+2\mu) & 0 \\ 0 & 0 & \lambda/(\lambda+2\mu) \end{bmatrix}\right) = 0$$

Evolution law parameters

$$k = \frac{3}{4}\frac{G_F}{D} \quad \text{and} \quad c = \frac{3}{8}D\,G_F$$

Compressive – tensile transition

$$\gamma = 10 \times \frac{E}{f_c}$$

Table 2-1: Equations of the continuous model

2.1 Relation stress-strains

The function $A(a)$ who intervenes in the relation stress-strain measures the impact of the level of damage on residual rigidity (in traction). Its particular form is guided by coherence with a cohesive law. She plays the same part as the more classical form in (1*has*) used in many models of the literature.

In addition, to take account of the restoration of rigidity in compression, the law of elasticity is not linear any more, even if it remains hyperelastic (*i.e.* it derives from an energy) in order to avoid any production or consumption parasites of energy. It is similar to the formulation adopted for the law `ENDO_ISOT_BETON` [R7.01.04]; she takes account not only sign of the clean deformations but also of that of the trace of the deformations. On the practical level, it thus requires to determine the eigenvalues of the tensor of deformation.

Lastly, to improve the robustness of the model and the convergence of calculation, the jump of rigidity on the way traction and compression was regularized, by means of the function $S(x)$ infinitely derivable. It ensures a progressive transition between a zero value and a quadratic function; the speed (or brutality) of the transition is controlled there by the parameter γ. A value by default is proposed in the order `DEFI_MATER_GC` who stipulates that 90 % of rigidity without regularization are found for a deformation (in compression) about fc/E.

2.2 Evolution of the damage

The presence of one term in Laplacian of damage in the function threshold as well as a boundary condition on the field of damage of type normal gradient no one at the edge reflects the nonlocal character of the model. More precisely, that corresponds to the nonlocal formulation with gradient of internal variable as introduced into the booklet [R5.04.01]. On the digital level, it authorizes a treatment (almost) classical of the law of behavior. However, the choice of a surface of damage phénoménologiquement acceptable led to give up the existence of a single energy functional calculus to describe the whole of the equations of the problem. Here, the evolution of the damage and the balance of the structure result from minimization of two functional distinct energies. On the practical level, the tangent operator is not then symmetrical any more.

3 Digital integration

3.1 Discretized equations

The discretization in time of the equations of behavior is based on a diagram of implicit Euler, i.e. the various variables of the problem are expressed at the final moment of the step of time. The expression of the constraint at the end of the step of time obeys the same relation thus as that in the table, the deformation and the damage there being also expressed at the end of the step of time. Remain the delicate part which consists in calculating the evolution of the damage.

To integrate the equation of evolution of the damage is a problem certainly scalar but nonlinear. First of all, while following the approach of resolution suggested in [Lorentz and Godard, 2011] and taken again in the booklet [R5.04.01], a term closely connected in damage comes to be substituted at the end in Laplacian of the field of damage in the function threshold g :

$$\tilde{g}(a)=-A'(a)\Gamma(\boldsymbol{\varepsilon})+\lambda+r(\alpha-a)-k \qquad (1)$$

At the stage of integration of the law of behavior, sizes k and α are considered known (as well as the coefficient of penalization r positive), following the example of the deformations at the end of the step of time because they are expressed according to the nodal unknown factors of the problem. In the same way, the value of the function $\Gamma(\boldsymbol{\varepsilon})$ is also known, even if its calculation requires also the solution of a nonlinear equation which further one will reconsider. This is why the algorithmic function threshold above depends only on a. With final, the problem to be solved consists in finding $a\leq 1$ such as:

$$\Delta a\geq 0 \quad ; \quad \tilde{g}(a)\leq 0 \quad ; \quad \Delta a\,\tilde{g}(a)=0 \qquad (2)$$

where $\Delta a=a-a_n$ indicate the increment of damage during the step of time. And if there does not exist solution (lower than 1), then $a=1$ (saturated damage).

3.2 Resolution of the algorithmic equation of evolution

Like A is convex, $\Gamma(\boldsymbol{\varepsilon})$ positive and r positive, the algorithmic function threshold is decreasing. There are thus existence and unicity of the solution (important for the robustness of digital integration). In practice, the problem in three stages is solved:

- If $\tilde{g}(a_n)\leq 0$ then $a=a_n$
- If $\tilde{g}(1)\geq 0$ then $a=1$
- If not to solve $\tilde{g}(a)=0$ with $a_n\leq a\leq 1$ by a method of Newton on controlled terminals

3.3 Effective calculation of the function gamma

According to the table 2-1, the calculation of $\Gamma(\boldsymbol{\varepsilon})$ return to that of $\chi(\boldsymbol{\varepsilon})$ who is defined only in an implicit way: it is thus necessary to solve a nonlinear equation which one knows that it admits a single solution, cf [Lorentz, 2016]. For that, one starts by introducing some notations and a change of variable in which x is the new unknown factor:

$$\boldsymbol{\sigma}^e=\lambda\, tr(\boldsymbol{\varepsilon})\,\boldsymbol{Id}+2\mu\,\boldsymbol{\varepsilon} \text{ with } \boldsymbol{s}=\frac{\boldsymbol{\sigma}^e}{\|\boldsymbol{\sigma}^e\|} \quad (3)$$

The space of the tensor $\boldsymbol{s}$ is generated by:

$$span(\boldsymbol{s})=\{s_1,s_2,s_3\} \quad (4)$$

With:

$$s_\beta=\|\boldsymbol{s}+(\beta_0-\frac{1}{3})\,tr(\boldsymbol{s})\,\boldsymbol{Id}\| \text{ and } \chi=\frac{\|\boldsymbol{\sigma}^e\|}{x\,\sigma^0} \quad (5)$$

The equation to solve determine x and thus X is written as follows:

$$s_\beta x+\sqrt{\sum_{i=1}^{3}\exp(2s_i x)}-\gamma_0=0 \quad (6)$$

The first member is convex. One considers one raising x_0 solution then one solves the equation by a method of Newton from x_0 , which leads to a succession of decreasing values and which converge towards the solution of the equation because of announced convexity. It should be noted that raising it is obtained by one two following relations, where s_M indicate largest of the three eigenvalues of s :

$$s_\beta x+\sqrt{\sum_{i=1}^{3}\exp(2s_i x)}\geq s_\beta x \;\Rightarrow\; x\leq\frac{\gamma_0}{s_\beta} \quad (7)$$

Or:

$$s_\beta x+\sqrt{\sum_{i=1}^{3}\exp(2s_i x)}\geq\exp(s_M x) \;\Rightarrow\; x\leq\frac{\ln\gamma_0}{s_M} \text{ if } s_M>0 \quad (8)$$

One will take smallest of both raising by preoccupation with a performance (or the only one if $s_M\leq 0$).

3.4 Tangent matrix

To determine the coherent tangent matrix with the selected diagram of integration does not raise a particular difficulty (derivation of functions made up and derivation of implicit functions) with this close the eigenvalues of the deformation intervene in the calculation of $\Gamma(\boldsymbol{\varepsilon})$ and in that of the elastic law at the time to distinguish traction from compression. We examine each one of these points here more precisely, in optics to free itself from the derivative of the associated clean vectors which, one points out it, are not defined when the eigenvalues are not distinct.

For the derivative of $\Gamma(\boldsymbol{\varepsilon})$, only the term which with x associate the standard of $\mathbf{exp}(x\,\boldsymbol{s})$ require a particular treatment. However:

$$\delta\|\mathbf{exp}(x\,\boldsymbol{s})\| = \frac{\delta\,\mathbf{exp}(x\,\boldsymbol{s}) : \mathbf{exp}(x\,\boldsymbol{s})}{\|\mathbf{exp}(x\,\boldsymbol{s})\|} = \|\mathbf{exp}(x\,\boldsymbol{s})\|\,\delta x + \frac{\mathbf{exp}(x\,\boldsymbol{s})}{\|\mathbf{exp}(x\,\boldsymbol{s})\|} : \mathbf{exp}'(x\,\boldsymbol{s}) : (x\,\delta\,\boldsymbol{s}) \qquad (9)$$

This stage, even if the derivative of exponential (tensor of order four) appears formally in the expression, it is not useful to proceed indeed to the derivative. Indeed, the derivative of exponential is symmetrical because it is the derivative of the following isotropic scalar function (cf [Silhavy, 2000] for the properties of derivability):

$$f(\boldsymbol{S}) = \sum_{i=1}^{3} e^{A_i} \;;\quad \mathbf{exp}(\boldsymbol{A}) = \frac{\partial f}{\partial \boldsymbol{A}} \;;\quad \mathbf{exp}'(\boldsymbol{A}) = \frac{\partial^2 f}{\partial \boldsymbol{A}\,\partial \boldsymbol{A}} \qquad (10)$$

And thus:

$$\mathbf{exp}(\boldsymbol{A}) : \mathbf{exp}'(\boldsymbol{A}) = \mathbf{exp}'(\boldsymbol{A}) : \mathbf{exp}(\boldsymbol{A}) = \mathbf{exp}(2\,\boldsymbol{A}) \qquad (11)$$

This last result is shown easily by noticing that $\boldsymbol{A}$ and $\mathbf{exp}(\boldsymbol{A})$ commutate. With final, the calculation of the tangent matrix will pass by that of exponential tensorial, according to the relation above, without needing to derive the clean vectors. And for the calculation of exponential tensorial, one will be based on the method known as of "scaling and squaring", to see for example [Higham, 2005], which is freed from calculation from the clean vectors.

Concerning the relation stress-strains, it displays a term of trace which does not pose a particular problem of derivation and an infinitely derivable functional relation between the clean constraint σ_i and clean deformation ε_i. For this one, one enters within the framework of the derivative of tensorial functions based on a scalar function of the eigenvalues: the article [Carlson and Hoger, 1986] provides the expressions of the derivative then (tensorial) that the eigenvalues are distinct or not.

4 Description of the versions of the document

Version Aster	Author (S) Organization (S)	Description of the modifications
13.2	E.LORENTZ, EDF-R&D/AMA	Initial text

5 Bibliography

- Carlson D.E., Hoger A. (1986) The derivative of has tensor-valued function of has tensor. Quarter. Appl. Maths. 44,409-423.
- Higham NR. J. (2005) The scaling and squaring method for the matrix exponential revisited. SIAM J. Matrix Anal. Appl. 26,1179-1193.
- Lorentz E. (2016) A nonlocal ramming model for lime pit concrete consist with cohesive fracture. Submitted.
- Lorentz E., Godard V. (2011). Gradient ramming models: toward full-scale computations. Comput. Methods Appl. Mech. Engrg. 200,1927-1944.
- Silhavy Mr. (2000) Differentiability properties of isotropic functions. Duke Mathematical Journal 104,367-373.