Contact in great slips with X-FEM

Summary:

This document presents a new approach to deal with the problems of contact in great slips with the eXtended Finite Element Method (X-FEM) [R7.02.12]. One considers the continuous hybrid formulation of problems of contact between solids [bib2] and the strategy of resolution is similar to that already implemented in Code_Aster for the framework classical finite elements [bib3]. The treatment of contact-friction in small slips is the object of the document [R5.03.54]. A new type of mixed element of contact is introduced, specific to framework X-FEM. The geometrical procedure of reactualization and the algorithm of pairing, new elements for the X-FEM, are presented in detail in this document as well as the matric terms resulting from the linearization of the weak formulation of the problem.

The approach is implemented in *Code_Aster* in 2D and 3D, and treats at the same time interfaces completely cut by a crack as well as interfaces with bottom of crack. It is usable with the order `STAT_NON_LINE` [U4.51.03]. The friction of the Coulomb type is taken into account.

Contents

1 Introduction

During the digital implementation of the X-FEM [bib1, biberon5], with the problem of the rubbing contact was dealt by the method continues, called also “continuous hybrid formulation” in the documentation of Code_Aster, under the assumption of the small disturbances (HP). The mathematical formulation of the continuous method [bib2] was thus adapted to the XFEM and one points out here the main features of the digital implementation in HP [R5.03.54]:

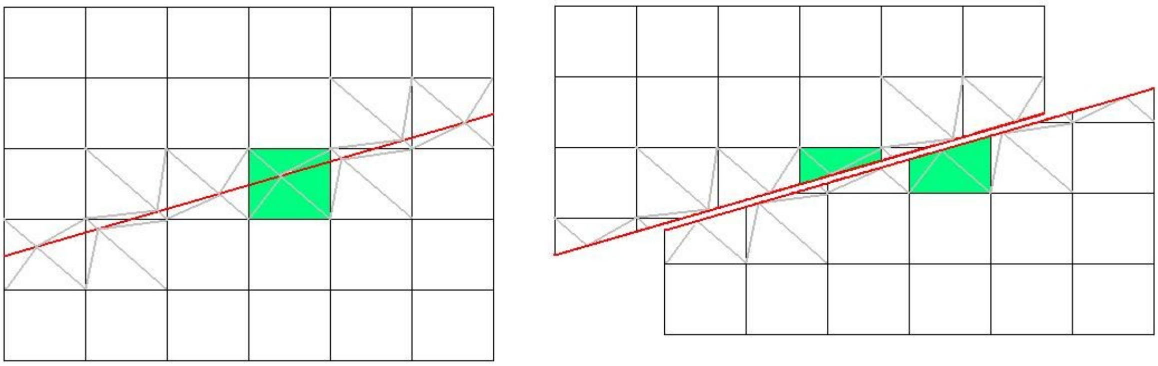
- the lips of the crack are treated like only one surfaces geometrical discontinuity;
- the geometrical reactualization of the surface of contact and master-slave pairing are not carried out (the notions of surface slave and main surface do not have a direction here);
- the jump of displacement is expressed according to the discontinuous degrees of freedom of enrichment introduced by X-FEM.

The calculation of the contributions of contact is thus carried out on the level of the finite elements crossed by the crack and one does not pass by mixed elements of contact, supported by late meshs, as it is the case for the classical approach of the continuous method.
In the following chapter one also recalls the weak formulation of the problem of contact with XFEM, solved by the method continues.

The treatment of the contact at the time of the modeling of the great slips with X-FEM required a new reflection for its implementation. Compared to the case HP (Figure 1a), the main difficulty was to make communicate the piece slave of an element fissured with the main piece of another fissured element (Figure 1b). Indeed, in this case two surfaces of contact must be declared and from the points of contact located inside a mesh cut by the crack will find themselves in opposite with points belonging to another mesh, so cut by the crack.

The difficulty is due to the inexistence of nodes on the interface of contact (the crack generates only points of intersection with the edges of the meshs). The late meshs of contact (segments slave and Master in the case 2D) generated starting from these nodes for grids FEM cannot be generated more here as it was the case for the classical formulation [bib3].

The found solution was the creation of new late meshs, of a higher order. These meshs are formed by the degrees of freedom of the mesh slave containing the point of contact and of the degrees of freedom of the mesh Master containing his project. For the case illustrated on the Figure 1b, instead of having late meshs SEG2-SEG2, as it was the case for the classical formulation, in X-FEM great slips one has late meshs QUAD4-QUAD4. The details on the creation and the characteristics of such late meshs, which represent the supports of the new element of contact, will be presented in the third chapter of this document.



a) b)

Figure 1. Grid X-FEM. a) Treatment HP; b) Treatment great slips.

Two other big steps must then be realized before the calculation of the contributions of contact: the geometrical reactualization of surfaces of contact and the pairing of the points of contact. Their description is the object of the first part of the 4$^{\text{ème}}$ chapter, named “Strategy of resolution “. In the second part of this chapter one linearizes the mixed variational formulation to extract the discrete formulation from the elementary terms of contact.

Implementation the data-processing of the approach great slips with X-FEM is described in the document [D9.05.06], for the actual position of the digital implementation.

2 Problem of contact with X-FEM

The problem of contact treated in this document relates to the cracks modelled by X-FEM: only one field is thus considered Ω for the field of displacements. On part of its border, Γ_u , one considers a set of conditions of Dirichlet and on another part, Γ_t , one considers a set of conditions of Neumann (Figure 2.a). The efforts of contact will appear on noted internal discontinuity Γ_c .
One breaks up the density the effort of contact r in a normal part λ , which indicates the normal pressure, and another tangential r_τ :

$$r=\lambda n+r_\tau \quad (1)$$

where n represent the vector of the normal entering to ω_2 .

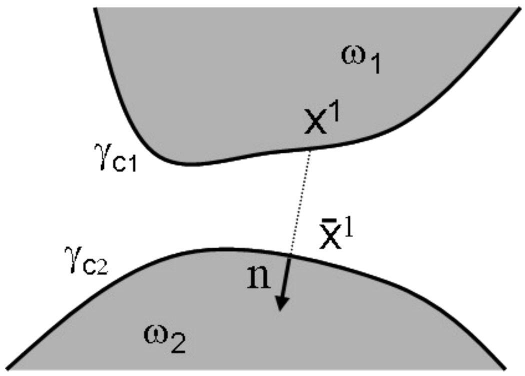


Figure 2. NotationS problem of contact

For a problem of contact, illustrated on Figure 2, let us consider the field u belonging to the unit V_0 fields of displacements kinematically acceptable:

$$V_0=\{v\in H^1, v \text{ discontinu à travers } \Gamma_c, v=0 \text{ sur } \Gamma_u\} .$$

By considering the notations introduced previously, the strong formulation of the local equations of balance defined on the initial configuration, supplemented initial conditions and boundary conditions of the problem considered, is:

$$\begin{aligned} &\operatorname{div}\Pi+f=0 && \text{dans } \Omega, \\ &u=\bar{u} && \text{sur } \Gamma_u \\ &\Pi\cdot N=t && \text{sur } \Gamma_t \\ &\Pi\cdot N=r && \text{sur } \Gamma_c \end{aligned} \quad (2)$$

where Π represent the first tensor of the constraints of Piola-Kirchhoff, f and t are the densities of the internal and surface efforts, respectively, and $\bar{u}$ represent the conditions of Dirichlet.
Let us consider a zoom on the deformed configuration (Figure 2.b) with ω_1 the part slave and ω_2 the corresponding part Master and borders, γ_{c1} and γ_{c2} , potentially in contact. The principal point for the problem of contact is then the evaluation of the game between a point of contact x^1 considered on the border slave and his project $\bar{x}^1$:

$$d_n=(x^1-\bar{x}^1)\cdot n \quad (3)$$

The project of the point of contact is calculated according to the principle of the minimal distance, representative thus the orthogonal projection of the point of contact x^1 on the border Master γ_{c2} .

For the contact, the laws of Signorini are written then:

$$d_n \leq 0 \text{ , } \lambda \leq 0 \text{ , } \lambda d_n = 0 \quad (4)$$

To make the equations (4) ready for the weak formulation, one transforms them into only one strictly equivalent equation according to [3] given by the formula (5) :

$$\lambda - \chi(g_n) g_n = 0 \quad (5)$$

In (5),

- χ is the indicating function of $\mathbb{R}^-$ ($\chi=1$ if contact and $\chi=0$ if not contact),
- $g_n = \lambda - \rho_n d_n$ is the multiplier of increased contact, with ρ_n one strictly positive reality.

An alternative consists in adopting a strategy of penalization, in which case the laws (4) are written:

$$\lambda + \chi(\lambda - \rho_n d_n) \kappa_n d_n = 0 \quad (6)$$

In (6),

- χ is the indicating function of $\mathbb{R}^-$ ($\chi=1$ if contact and $\chi=0$ if not contact),
- κ_n is a large coefficient of penalization in front of the rigidity of the structure.

For the phenomena of friction, one uses the laws of Coulomb which are written as follows:

$$\begin{cases} \|r_\tau\| \leq \mu |\lambda| \\ \|r_\tau\| < \mu |\lambda| \Rightarrow v_\tau = 0 \\ \|r_\tau\| = \mu |\lambda| \Rightarrow \exists \alpha \geq 0 \text{ ; } v_\tau = -\alpha \cdot r_\tau \end{cases} \quad (7)$$

In (7),

- μ is the coefficient of friction of Coulomb,
- v_τ is tangent relative speed.

As for the laws of contact, one can write the law of friction (7) as follows in an equivalent way:

$$\begin{gathered} r_\tau = \mu \lambda \Lambda \\ \Lambda - P_{B(0,1)}(g_\tau) = 0 \end{gathered} \quad (8)$$

In (8),

- Λ is the semi-multiplier of friction,
- $P_{B(0,1)}$ is projection on the ball unit,
- $g_\tau = \Lambda + \rho_\tau v_\tau$ is the semi-multiplier (vectorial) of increased friction,
- ρ_τ is a strictly positive reality.

The field of sign is also introduced $S_f = I_{B(0,1)}(g_\tau)$. We have $S_f = 1$ for an adherent point, and $S_f = 0$ for a slipping point.

One can also choose a method penalized to write this law:

$$\Lambda - P_{B(0,1)}(\kappa_\tau v_\tau) = 0 \quad (9)$$

In (9),

- κ_τ is the coefficient of penalization in friction.

In 3D, Λ is a vector of the tangent plan on the surface of the crack and it is necessary to define a base covariante tangent plan in which it could be expressed as follows:

$$\Lambda = \Lambda^1 \tau_1 + \Lambda^2 \tau_2 \quad (10)$$

There exist an infinity of couples (τ_1, τ_2) being able to form this base.

For the method of contact X-FEM HP, one uses the gradients of the level sets to define this base (see the semi-multiplier part of friction of chapter 4 of [bib1]), which ensures the continuity of the base (τ_1, τ_2) from one node to another of the grid.

In great slips one cannot any more define it thus since it must be reactualized with each geometrical iteration (object of chapter 4 of this document) and depends then on the current geometry of the facets of contact Masters.

For the method of contact FEM in great slips, the selected base is that directed by the current geometry of the surface meshs of contact.

One makes in the same way for X-FEM great slips by replacing the concept of mesh surface by that of a facet of contact. The problem is that the continuity of the base (τ_1, τ_2) is not ensured any more as for contact X-FEM HP.

For contact X-FEM in great slips, one thus decides to reorientate the tangents by using a vector fixes in the total base, so as to reduce the discontinuity of the field (τ_1, τ_2) on the way from one element to another.

The normal vector first of all is calculated n who is single, one projects then the fixed vector e_1 total base in the plan of normal n to build τ_1 , one builds then τ_2 who is the vector product of n and τ_1 :

$$\tau_1 = \frac{P_\tau . e_1}{\|P_\tau . e_1\|} ; \tau_2 = n \wedge \tau_1 \qquad (11)$$

In (11),

- n is the directing vector of the tangent plan,
- $P_\tau = (I_d - n \otimes n)$ is the operator of projection on the tangent level with n ,
- $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is the direction x in the total reference mark,
- (τ_1, τ_2) is the base chosen to write the equation (8)

Attention, the case $\|P_\tau . e_1\| = 0$ is not treated. It causes an error. It is necessary thus to pay attention so that the plan of contact is not directed perpendicularly with e_1 .

Spaces of the unknown factors of contact are the following:

$$H = \left\{ \lambda \in H^{-1/2}(\Gamma_c), \lambda \le 0 \text{ sur } \Gamma_c \right\},$$
$$\mathbf{H} = \left\{ r_\tau \in H^{-1/2}(\Gamma_c), \|r_\tau\| \le \mu \lambda_n \text{ sur } \Gamma_c \right\},$$

The weak formulation with three fields exit of [2] is written then:

To find $(u, \lambda, \Lambda) \in V_0 \times H \times \mathbf{H} \quad \forall (u^*, \lambda^*, \Lambda^*) \in V_0 \times H \times \mathbf{H}$ such that (12-14) are checked:

Equilibrium equation:
$$\int_M tr(\Pi(\nabla_p(u))(\nabla_p(u^*)))\, dM - \int_{\Gamma_c} \chi(g_n) g_n n \cdot [u^*]\, d\Gamma$$
$$- \int_{\Gamma_c} \mu \chi(g_n) \lambda P_{B(0,1)}(g_\tau)(I_d - n \otimes n)[u^*]\, d\Gamma = L_{meca}(u^*)$$

(12)

Law of contact:

$$\int_{\Gamma_c} -\frac{1}{\rho_n}\left(\lambda-\chi(g_n)g_n\right)\lambda^* d\,\Gamma=0 \quad (13)$$

Law of friction:

$$\int_{\Gamma_c} \frac{\mu\lambda\chi(g_n)}{\rho_\tau}\left(\Lambda-P_{B(0,1)}(g_\tau)\right)\Lambda^* d\,\Gamma+\int_{\Gamma_c}\left(1-\chi(g_n)\right)\Lambda\,\Lambda^* d\,\Gamma=0 \quad (14)$$

Where:

- $tr(\cdot)$ is the operator traces of a tensor,
- Π is the first tensor of constraints of Piola-Kirchoff,
- $\nabla_p(u^*)$ is the gradient of u^* compared to the coordinates p ,
- $\otimes$ is the operator of the tensorial product,
- Id is the second tensor identity,
- $L_{meca}(u^*)$ is the virtual work of the external forces.

Within the framework of a penalized formulation, contact pressures and shear stresses due to friction are explicit according to displacement. However and as explained in documentation [R7.02.12], it is necessary to utilize λ in the equilibrium equation for a rigorous satisfaction of condition LBB. On the other hand, the equation of friction does not intervene in the resolution, it has only one role of postprocessing. The three preceding equations (12-14) become then:
Equilibrium equation:

$$\begin{aligned}&\int_\Omega tr(\Pi(\nabla_p(u))(\nabla_p(u^*)))\,d\,\Omega-\int_{\Gamma_c}\chi\lambda n\cdot[u^*]d\Gamma\\&-\int_{\Gamma_c}\mu\chi\lambda P_{B(0,1)}(\kappa_\tau v_\tau)(I_d-n\otimes n)[u^*]d\Gamma=L_{meca}(u^*)\end{aligned} \quad (15)$$

Law of contact:

$$\int_{\Gamma_c} -\frac{1}{\kappa_n}\left(\lambda+\chi\kappa_n d_n\right)\lambda^* d\,\Gamma=0 \quad (16)$$

Law of friction:

$$\int_{\Gamma_c}\chi\left(\Lambda-P_{B(0,1)}(\kappa_\tau v_\tau)\right)\Lambda^* d\,\Gamma+\int_{\Gamma_c}(1-\chi)\Lambda\,\Lambda^* d\,\Gamma=0 \quad (17)$$

The linearization as well as the discretization of this formulation are presented in the fourth chapter for the hybrid elements of contact developed in order to implement in Code_Aster the approach great slips with XFEM. Previously, we will introduce into the chapter according to the new hybrid elements of contact.

3 Hybrid element X-FEM of contact for the approach great slips

The concept of element hybrid of contact, also called mixed element of contact, was for the first time introduced into Code_Aster for the formulation of hybrid contact continuous [bib3]. It is about a couple formed by a point of integration of contact, located on surface slave, and of the main element containing the project of the point of contact on main surface. The support of such an element, which has geometrical degrees of freedom and of degrees of freedom of contact (multipliers of contact-friction) is named late mesh (denomination used in Code_Aster). A late mesh is not part of the grid of the starting model. It is formed by the association of the mesh slave containing the point of integration of contact and of the main mesh nearest containing its project. Taking into account the definition of the element of contact, recalled above, for each point of integration of contact a new element of contact will be generated. On Figure 3, one shows a typical 2D example formation of a hybrid element of contact.

Following pairing, one finds for the point of contact PC who belongs to the mesh slave $N_1 N_2$, the main mesh $N_3 N_4$. The projection of the point of contact on this mesh will give the project what is called PR. The late mesh thus created will be of type SEG2-SEG2 and one will name it $N_1 N_2 N_3 N_4$ for this example. The degrees of freedom of displacement will be stored in these 4 nodes, which are also nodes constitutive for meshs QUAD4 of the model. The degrees of freedom of contact-friction are stored, by convention, only with the nodes of the mesh slave $N_1 N_2$.

For the approach great slips with X-FEM, the approach described above cannot be applied any more because the interface of contact, resulting from a crack X-FEM, does not have any more nodes but only points of intersection which cannot store the degrees of freedom of displacements or contact-friction.

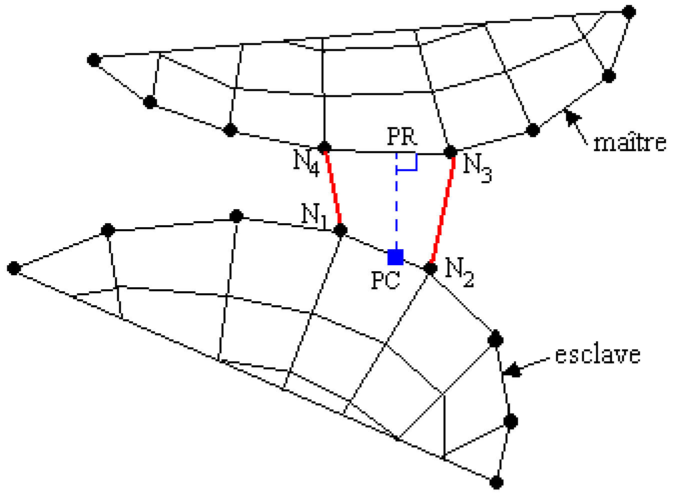


Figure 3. Illustration of the creation of a classical late mesh for a model 2D.

Since it is not possible to define meshs on the potential border in contact (segments for modeling 2D or meshs surface for that 3D), the found solution was to use the meshs cut by the crack to create late meshs of a new type. One makes use of it then to define the hybrid element of contact X-FEM great slips by considering the degrees of freedom of displacement and contact-friction as well as fields of digital integration to calculate the contributions of rubbing contact. In addition for the method of contact X-FEM, the choice was to make carry the degrees of freedom of contact by the mesh slave, as for the method of classical contact FEM.

3.1 Hybrid element of contact X-FEM

The degrees of freedom of contact are introduced only with the nodes carrying the geometrical degrees of freedom already, in accordance with the algorithm of satisfaction of condition LBB suggested in [bib9]. Thus on Figure 4, defining mesh late (it is considered that the mesh slave is that located in lower part and the mesh Master that above), the mesh slave is a QUAD4 ($N_1 N_2 N_3 N_4$) all the nodes store at the same time the degrees of freedom of contact and the geometrical degrees of freedom. The mesh Master is also a QUAD4 ($N_9 N_{10} N_{11} N_{12}$): its nodes store only geometrical degrees of freedom.

It is then a question of pairing two meshs crossed by the crack, that slave containing the point of contact PC (on the segment $P_1^e P_2^e$) and that main containing the project of the point of contact PR (on the segment $P_1^m P_2^m$). For the hybrid element of contact shown on Figure 4, the segment of contact is $P_1^e P_2^e$, formed starting from the 2 points of intersection of the lip slave. The area of reference for the functions of form is illustrated on the right of the image.

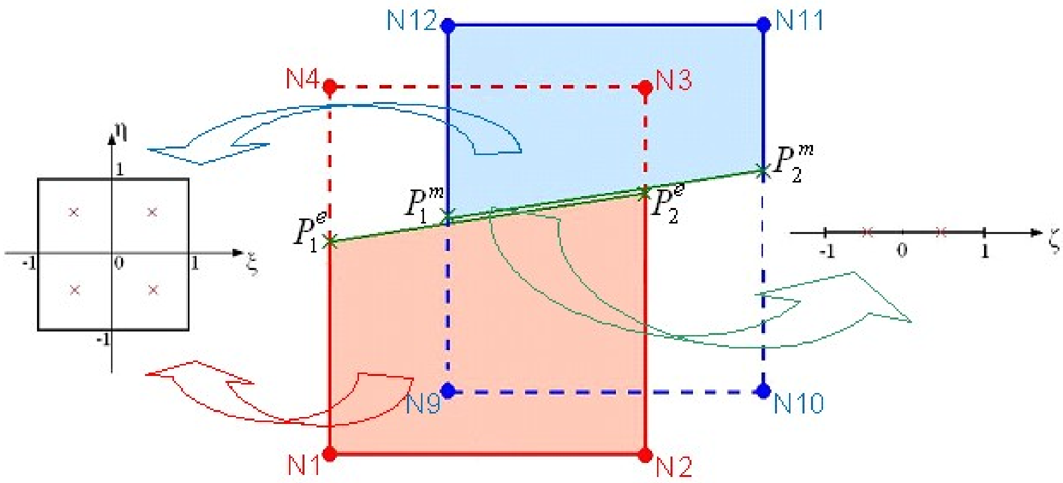


Figure 4. Hybrid example of element of contact great slips X-FEM.

Being given the characteristic of the method X-FEM, which is to consider the geometrical degrees of freedom of two types, classics a_i and nouveau riches b_i (for more details on the characteristics of elements X-FEM one can refer to [bib1]), it results a hybrid element X-FEM from it of the type QUAD4-QUAD4 of which the degrees of freedom for each node are presented in Table 2.

Node	*Degree of freedom*													
$N1$	DX	DY	H1X	H1Y	H2X	H2Y	H3X	H3Y	H4X	H4Y	E1X	E1Y	λ	Λ
$N2$	DX	DY	H1X	H1Y	H2X	H2Y	H3X	H3Y	H4X	H4Y	E1X	E1Y	λ	Λ
$N3$	DX	DY	H1X	H1Y	H2X	H2Y	H3X	H3Y	H4X	H4Y	E1X	E1Y	λ	Λ
$N4$	DX	DY	H1X	H1Y	H2X	H2Y	H3X	H3Y	H4X	H4Y	E1X	E1Y	λ	Λ
$N9$	DX	DY	H1X	H1Y	H2X	H2Y	H3X	H3Y	H4X	H4Y	E1X	E1Y		
$N10$	DX	DY	H1X	H1Y	H2X	H2Y	H3X	H3Y	H4X	H4Y	E1X	E1Y		
$N11$	DX	DY	H1X	H1Y	H2X	H2Y	H3X	H3Y	H4X	H4Y	E1X	E1Y		
$N12$	DX	DY	H1X	H1Y	H2X	H2Y	H3X	H3Y	H4X	H4Y	E1X	E1Y		

Table 2. The table of the degrees of freedom for a hybrid element of contact, QUAD4-QUAD4.

One is interested now in the approximation of displacements for the points located on the lips of the crack. For a point located on the segment of contact slave, therefore pertaining to the part full with the mesh slave, this approximation is written:

$$u^e(x)=\sum_{i=1}^{nnes} a_i \varphi_i^e(x)+\sum_{i=1}^{nnes}\sum_{j=1}^{nfhe} He_i^{j,e} b_i^j \varphi_i^e(x)+\sum_{i=1}^{nnes} He_i^{1,e}\sqrt{r_e}\, c_i^1 \varphi_i^e(x) \,, \quad (18)$$

Where:

- $nnes$ indicate the number of nodes slaves tops,
- $nfhe$ indicate the number of degrees of freedom Heaviside present in the element slave (1 in the classical case, 4 at the most in the muti-Heaviside meshs),
- a_i and b_i c_i^1 are respectively the degrees of freedom classical, Heaviside nouveau riches and Ace-tip (the first correspondent with functions of nonworthless forms on the lip of the crack),
- $He_i^{j,e}$ corresponds to the value of the function characteristic of the field side slave, associated with the node i for the degree of Heaviside freedom j. The function characteristic of field is worth "0" if the point of gauss does not belong to the corresponding field.
- $r_e=\sum_{i=1}^{nnes} |lst_i| \phi_i^e(x)$ is the distance from the point slave at the bottom of crack,
- the term $\sum_{i=1}^{nnes} \sqrt{r_e}\, c_i^1 \phi_i^e(x)$ is present only if the element slave has degrees of freedom Ace Tip i.e the element slave is of type Heaviside Ace-tip, (see § 3.2 for the types of elements considered),
- ϕ_i^e are the functions of form of the element relative (quadrilateral with 4 nodes for the example set on Figure 4 where one shows on the left image the area of reference).

Same manner, for a point located on the mesh Master, one will have:

$$u^m(x)=\sum_{i=1}^{nnm} a_i \varphi_i^m(x)+\sum_{i=1}^{nnm}\sum_{j=1}^{nfhm} He_i^{j,m} b_i^j \varphi_i^m(x)+\sum_{i=1}^{nnm} He_i^{1,m}\sqrt{r_m}\, c_i^1 \varphi_i^m(x) \,, \quad (19)$$

Where:

- nnm indicate the number of main nodes,
- $nfhm$ indicate the number of degrees of freedom Heaviside present in the main element (1 in the classical case, 4 at the most in the muti-Heaviside meshs).
- $He_i^{j,m}$ corresponds to the value of the function characteristic of field of main side, associated with the node i for the degree of Heaviside freedom j. The function characteristic of field is worth "0" if the point of gauss does not belong to the corresponding field.
- $r_m=\sum_{i=1}^{nnm} |lst_i| \phi_i^m(x)$ is the distance from the main point at the bottom of crack,
- the term $\sum_{i=1}^{nnes} \sqrt{r_m}\, c_i^1 \phi_i^e(x)$ is present only if the main element has degrees of freedom Ace Tip i.e the main element is of type Heaviside Ace-tip (see § 3.2 for the types of elements considered),
- ϕ_i^m are the functions of form of the element relative.

While using (18) and (19), one can write the discretized relation of the game between the point of contact (PC) and its project (PR):

$$d_n=\left[\sum_{i=1}^{nnes}\left(a_i+\sum_{j=1}^{nfhe}(He_i^{j,e} b_i)+He_i^{1,e}\sqrt{r_e}\, c_i^1\right)\phi_i^e(x_{CP})-\sum_{i=1}^{nnm}\left(a_i+\sum_{j=1}^{nfhm}(He_i^{j,m} b_i)+He_i^{1,m}\sqrt{r_m}\, c_i^1\right)\phi_i^m(x_{PR})\right]\cdot n_{PR} \quad (20)$$

The approximation of the unknown factors of contact utilizes them ϕ_i^e (functions of form of the element relative slave trained by $nnes$ nodes slaves tops), so that:

$$\lambda(x)=\sum_{k=1}^{nnes} \lambda_k\, \phi_k(x) \,, \quad \Lambda(x)=\sum_{k=1}^{nnes} \Lambda_k\, \phi_k(x) \quad (21\ (a))$$

In other words, the functions of forms of contact are the same ones as those of displacements in the element slave.

Another innovation is the possibility of taking into account funds of crack with contact in great slips: therefore one represented in table 2 besides the geometrical degrees of freedom classical (D) and Heaviside (H1), the degrees of freedom ace-tips (E1 only because it is only whose singular function is nonworthless on discontinuity and who intervenes for the contact). Nevertheless these degrees of freedom ace-tips are optional, whether it is for the main element or the element slave. Thus in the table, the degrees of freedom in blue appear only if the element slave is Heaviside-ace-tip, and the degrees of freedom in green appear only if the main element is Heaviside-ace-tip. It is then possible to have 4 types of late elements different for the same types of meshs:

- Heaviside – Heaviside,
- Heaviside – Heaviside Ace-tip,
- Heaviside Ace-tip – Heaviside,
- Heaviside Ace-tip – Heaviside Ace-tip,

On the element containing the point (Ace-tip), the interface does not undergo great slips, It is thus not considered to be useful to consider a D-pairing for this mesh. One thus does not take counts the following elements of them:

- Heaviside Ace-tip – Ace-tip,
- Ace-tip – Heaviside Ace-tip,
- Ace-tip – Ace-tip.

The integration of the terms of contact on the element Ace-tip will be treated same manner as in small slips.

The taking into account of the multi-Heaviside elements is carried out with the addition of the degrees of additional Heaviside freedom ($H2, H3, H4$). Thus in the same way as for the taking into account of the degrees of freedom ace-tips, these additional degrees of freedom are optional, than it is for the main element or the element slave. Thus in table 2, the degrees of freedom in red can appear only if the element slave is multi-Heaviside and the degrees of freedom in magenta can appear only if the main element is multi-Heaviside. It is then possible to have 15 types of additional late elements for each type of meshs:

- $H1-H2$, $H1-H3$, $H1-H4$,
- $H2-H1$, $H2-H2$, $H2-H3$, $H2-H4$,
- $H3-H1$, $H3-H2$, $H3-H3$, $H3-H4$,
- $H4-H1$, $H4-H2$, $H4-H3$, $H4-H4$,

For example for the late mesh QUAD4-QUAD4, the element $H4-H2$ have the ddl of table 3.

Node	Degree of freedom											
$N1$	DX	DY	H1X	H1Y	H2X	H2Y	H3X	H3Y	H4X	H4X	λ	Λ
$N2$	DX	DY	H1X	H1Y	H2X	H2Y	H3X	H3Y	H4X	H4X	λ	Λ
$N3$	DX	DY	H1X	H1Y	H2X	H2Y	H3X	H3Y	H4X	H4X	λ	Λ
$N4$	DX	DY	H1X	H1Y	H2X	H2Y	H3X	H3Y	H4X	H4X	λ	Λ
$N9$	DX	DY	H1X	H1Y	H2X	H2Y						
$N10$	DX	DY	H1X	H1Y	H2X	H2Y						
$N11$	DX	DY	H1X	H1Y	H2X	H2Y						
$N12$	DX	DY	H1X	H1Y	H2X	H2Y						

Table 3. The table of the degrees of freedom for the hybrid element of contact H4-H2.

For the moment an element X-FEM cannot be multi-Heaviside and ace-tip at the same time. That supposes that a bottom of crack must be relatively distant from a junction (spacing of at least two meshs). One thus did not consider pairing between a multi-Heaviside element and an element ace-tip.

Once the characteristics of hybrid element X-FEM of contact defined, one can pass to the presentation of the strategy of resolution of the problem of contact in great slips, strategy which will be the object of the following chapter.

4 Strategy of resolution

The principal stages of the resolution of a problem of contact in great slips with the continuous hybrid formulation can be presented as 4 loops imbricated for each step of time as it follows:

- reactualization of the geometry of surfaces of contact and launching of the algorithm of pairing;
- buckle on the thresholds of friction (method of point-fixed);
- buckle on the statutes of contact (method of the active constraints);
- buckle Newton generalized.

In method of contact XFEM small slips, the first stage is not present for the approach HP: she was added for the approach great slips. Its presentation is the object of the first part of this chapter.

For the large ones slips with XFEM, one decides to free oneself from the second phase. Rather than a fixed point on the threshold of friction, one prefers to treat non-linearity by implementing the terms linearized in the tangent matrix. This been part of one of the many changes which appeared during the linearization of the variational mixed formulation following the introduction of the new hybrid element of contact. Thus the second part of this chapter is dedicated to the determination of the new discrete expressions for the terms corresponding to the contributions of contact.

4.1 Geometrical reactualization and pairing

For the geometrical reactualization of surfaces of contact resulting from a crack modelled by the X-FEM the first operation to make is the duplication of the lips of the crack. This operation makes then possible, at the beginning of each step of time, the pairing of the points of contact located on the surface designated as slave. In a preoccupation with a general information, one presents the principles of the duplication of the lips of the crack using an example 3D shown on Figure 5.

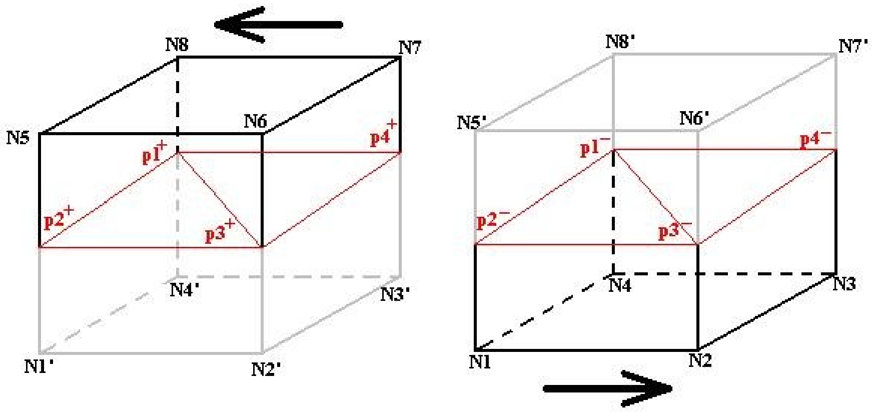


Figure 5. Illustration of the duplication of the facets of contact.

Following the duplication of the points of intersection between the edges of the mesh and the crack one obtains, for each fissured element, two series of facets of contact (facets which are formed with these points): facets slaves which will be attached to the part of the mesh located below the crack (Heaviside function associated with the crack which generates the facet $H(x,y)=-1$), and main facets, located above the crack (Heaviside function associated with the crack which generates the facet $H(x,y)=+1$). For example, for a mesh HEXA8 fissured in great slips (Figure 5), there will be the facets slaves $p1^- p2^- p3^-$ and $p1^- p3^- p4^-$, as well as the main facets $p1^+ p2^+ p3^+$ and $p1^+ p3^+ p4^+$. The geometrical reactualization relates to the calculation of the geometry of these facets before each new pairing. For each facet of contact, geometrical coordinates of points of intersection are calculated by adding with the initial coordinates the displacements carried out since the initial configuration.

The algorithm of pairing which follows the geometrical reactualization has like objective to find for each point of contact considered on surface slave the facet in opposite located on main surface. To illustrate this algorithm, which is based on the principles of the similar algorithm developed for the formulation continues classical, one uses the example shown on Figure 6.

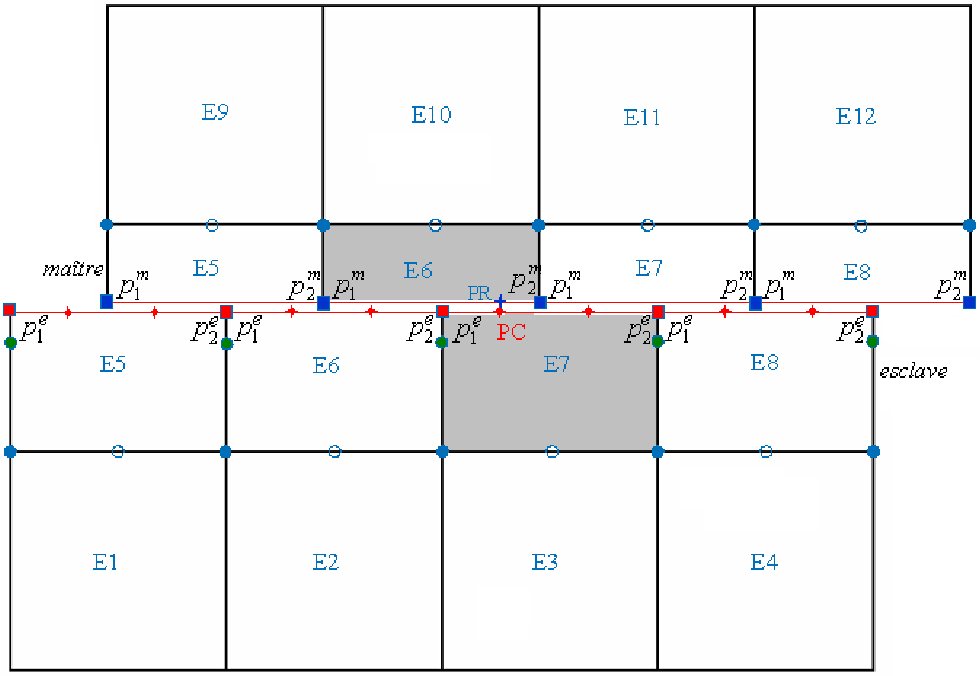


Figure 6. Illustration of the algorithm of pairing.

Thus let us consider the situation illustrated on Figure 6, a grid 2D containing meshs QUAD4, with 4 meshs crossed by a crack X-FEM, meshs which can also be made quadratic for the moment in their adding nodes mediums carrying of the multipliers of Lagrange of contact-friction. After the geometrical reactualization, one seeks to build the late mesh corresponding to the point of contact PC located on the segment of contact $p_1^e p_2^e$ mesh slave $E7$. With this intention, one buckles on all the meshs of contact (the fissured meshs) and then, inside each mesh, one buckles on the points of intersection with the crack (main side). The point of intersection nearest to the point of contact is retained (P_2^m of $E6$ on the figure in our case). One makes then the projection of the point of contact on the facets Masters of the meshs which are connected to the point of intersection selected. In the example, one thus projects on the facets of the meshs $E6$ and $E7$ and one checks on whom projection is interior. One finds thus that the mesh Master $E6$ will be paired with the mesh slave $E7$ for the point of contact PC .

Below one presents the principal stages of this algorithm of pairing:

- buckle on the meshs of contact slaves
  - ○ buckles on the facets of contact slaves
    - ■ buckles on the points of integration of contact slaves
      - calculation of the real coordinates of the point of contact
      - □ buckles on the meshs of contact Masters
        - ◘ buckles on the main points of intersection
          - calculation of the distance compared to the points of intersection
          - choice of the main point of intersection
        - ◘ fine of loop on the main points of intersection
      - □ fine of loop on the meshs of contact Masters
      - determination of the meshs connected to the main point of intersection
      - □boucle on the meshs of contact connected to the main point of intersection

- ◘ buckles on the facets of contact Masters
  - projection of the point of contact on the facet Master
  - calculation of the game enters the point of contact slave and the facet Master
  - if the game is smallest: information storage on pairing
- ◘ fine buckles on the facets of contact Masters
- ■ fine of loop on the points of contact
- ○ fine of loop on the facets of contact slaves
- • end of loop on the meshs of contact

Following pairing, a map of contact is filled for each point with contact. This map contains the necessary information for the calculation of the contributions of contact at the elementary level. The details on the composition of this map are given in [bib8].

Let us note BIEN that the geometrical algorithm of reactualization fixes a maximum value at the coefficients of penalization of contact and friction. To understand it, let us consider elements parents of characteristic size h , a contact pressure whose order of magnitude is σ , a coefficient of penalization of contact κ_n leading to a typical interpenetration $\delta=\sigma/\kappa_n$. Convergence is regarded as attack when the relative residue is lower than η .

The geometrical reactualization implies the calculation of functions of form of the elements Master and slaves at the points of contact in the current configuration. The geometrical reactualization makes it possible to calculate δ with a precision $\epsilon_{\mathrm{machine}} h$. In addition, when convergence is reached, the precision on this same quantity must be lower than $\eta\frac{\sigma}{\kappa_n}$. We thus have:

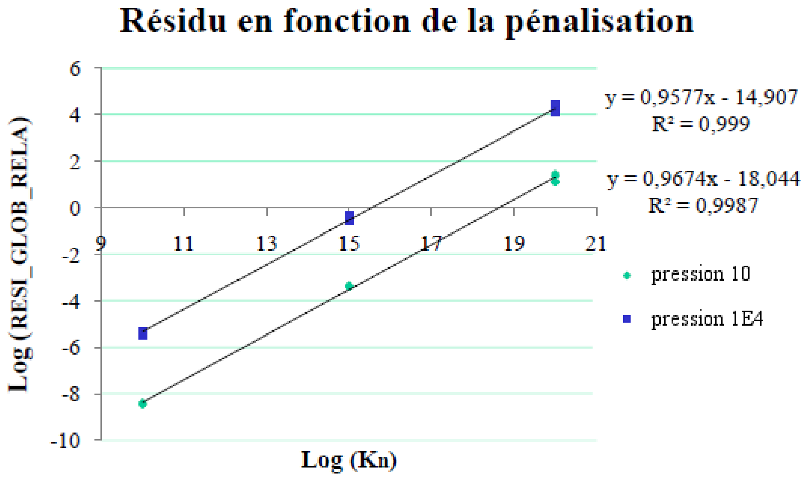
$\epsilon_{\mathrm{machine}} h<\eta\frac{\sigma}{\kappa_n}$ that is to say $\kappa_n<\frac{\eta\sigma}{\epsilon_{\mathrm{machine}} h}$

The validity of this upper limit can be illustrated by digital tests. For that, we consider a uniform test of compression SSNV182G adapted in great slips penalized, for which we inform in the method of Newton a tolerance η' excessively small, for example $\eta'=1.10^{-20}$. We note whereas the residue `RESI_GLOB_RELA` "reach a maximum", i.e. reached a minimal value in on this side which it cannot go down because of precision machine. We trace this minimal value then atteignable relative residue, which we call $\eta_{\min}$ according to the coefficient of penalization κ_n , for two values different from the contact pressure. Taking into account the study carried out in the preceding paragraph we expect to find:

$$\kappa_n=\frac{\eta_{\min}\sigma}{\epsilon_{\mathrm{machine}} h}$$

The fact that the coefficients associated directors with the representation of $\log(\eta_{\min})$ according to $\log(\kappa_n)$ are very close to 1 confirms the proportionality enters η and κ_n . The precision machine of the machine used being 10^{-14} and $h\sim 1$, one expects according to the analytical formula to find ordinates in the beginning:

$\log\left(\frac{\epsilon_{\mathrm{machine}} h}{\sigma}\right)=\left|\begin{array}{l}-15 \text{ si } \sigma=10\\ -18 \text{ si } \sigma=10^4\end{array}\right.$, which is close to the actual values numerically (respectively $-14{,}90$ and $-18{,}04$) and confirms the analytical reasoning.



4.2 Linearization of the mixed variational formulation

Let us consider an iteration of Newton for which fields u , λ and Λ are initially given. One notes δu , $\delta\lambda$ and $\delta\Lambda$ their variations so that new values at the end of the iteration, u^f , λ^f and Λ^f are determined by:

$$\begin{aligned} u^f &= u+\delta u \ , \\ \lambda^f &= \lambda+\delta\lambda \ , \\ \Lambda^f &= \Lambda+\delta\Lambda \ . \end{aligned} \quad (22)$$

Because of dependence of the threshold of slip to the contact pressure, the law of Coulomb is a nonassociated law. This point is largely developed in [R5,03,50], §2,3 to which one will be able to refer for more precise details. We can rewrite the condition of admissibility of the constraints of the law of Coulomb (7) in the form:

$$f(\boldsymbol{r},\mu) \stackrel{\text{dèf}}{=} \|\boldsymbol{r}_\tau\| - \mu|\lambda| \leq 0$$

The law of flow is written when with it:

$$[\![\dot{\boldsymbol{u}}]\!] = -\alpha\,\boldsymbol{r}_\tau \text{ with } \alpha\, f(\boldsymbol{r},\mu) = 0$$

We see whereas direction of flow, in other words the direction relative speed, is not colinéaire with the normal on the surface threshold. This nonassociativeness is source of non-linearity (known as not linearity of threshold). One can deal with the problem with a fixed threshold of friction, which one reactualizes in an external loop. One then finds a pseudopotential (known as criterion of Tresca) which is associated.
This strategy was put in work in contact X-FEM in small slips (see [R7,02,12]).

We benefit here from recent developments in the method of contact continues (see [R5,03,52]) to adopt a direct linearization. With this intention, we linearize by considering the threshold λ variable, put except for when one is on the surface of the cone of Coulomb, because that would thus return as we saw to recommend a nontangent relative speed, and incorrect. In short, the threshold is linearized for the adherent points, not for the slipping points. This method allows in particular to bring back in the cone of adherence of the points considered slipping wrongly. It was already implemented in continuous formulation, with conclusive results.

Linearization of the formulation (12-14), while developing δg_n and δg_τ , gives:

$$\begin{aligned}&\int_\Omega tr(\Pi(\nabla_p(\delta u)):\nabla_p(u^*))\,d\Omega-\int_{\Gamma_c}\chi\delta\lambda[u^*]\cdot n\,d\Gamma+\int_{\Gamma_c}\chi\rho_n[\delta u]\cdot n[u^*]\cdot n\,d\Gamma\\&-\int_{\Gamma_c}\chi S_f\mu\delta\lambda\, g_\tau[u^*]_\tau d\Gamma\\&-\int_{\Gamma_c}\chi\mu\lambda K(g_\tau)\delta\Lambda[u^*]_\tau d\Gamma-\int_{\Gamma_c}\chi\mu\lambda\rho_\tau K(g_\tau)[\delta u]_\tau[u^*]_\tau d\Gamma\\&=-\int_\Omega tr(\Pi(\nabla_p(u)):\nabla_p(u^*))\,d\Omega+L_{meca}(u^*)\\&+\int_{\Gamma_c}\chi g_n[u^*]\cdot n\,d\Gamma+\int_{\Gamma_c}\chi\mu\lambda P_B(g_\tau)[u^*]_\tau d\Gamma\end{aligned}\tag{23}$$

$$-\int_{\Gamma^c}\frac{1-\chi}{\rho_n}\delta\lambda\,\lambda^*\,d\Gamma-\int_{\Gamma_c}\chi[\delta u]\cdot n\,\lambda^*\,d\Gamma=+\int_{\Gamma_c}\left(\frac{1-\chi}{\rho_n}\lambda\,\lambda^*+\chi[u_e-u_m]\cdot n\,\lambda^*\right)d\Gamma\tag{24}$$

$$\begin{aligned}&-\int_{\Gamma_c}\chi\mu\lambda K(g_\tau)[\delta u]_\tau\Lambda^*\,d\Gamma+\int_{\Gamma_c}\frac{\chi\mu\lambda}{\rho_\tau}(I_d-K(g_\tau))\delta\Lambda\,\Lambda^*\,d\Gamma+\int_{\Gamma_c}(1-\chi)\delta\Lambda\,\Lambda^*\,d\Gamma\\&-\int_{\Gamma_c}\chi S_f\mu\,\delta\lambda\,\nu_\tau[\delta u]_\tau\Lambda^*\,d\Gamma\\&=-\int_{\Gamma_c}\frac{\mu\lambda\chi}{\rho_\tau}(\Lambda-P_B(g_\tau))\Lambda^*\,d\Gamma-\int_{\Gamma_c}(1-\chi)\Lambda\,\Lambda^*\,d\Gamma\end{aligned}\tag{25}$$

With:

$K(g_\tau)=Id$ if adherence

$K(g_\tau)=\frac{1}{\|g_\tau\|}\left(Id-\frac{g_\tau\cdot g_\tau^T}{\|g_\tau\|^2}\right)$ if slip

Let us note that one solves to it not geometrical linearity by a fixed problem of point, by supposing that n is fixed during the variation of displacement.

With regard to the penalized formulation, an iteration of the method of Newton is written:

$$\begin{aligned}&\int_M tr(\Pi(\nabla_p(\delta u)):\nabla_p(u^*))\,dM-\int_{\Gamma_c}\chi\delta\lambda[u^*]\cdot n\,d\Gamma\\&-\int_{\Gamma_c}\chi S_f\mu\delta\lambda\kappa_\tau v_\tau[u^*]_\tau d\Gamma-\int_{\Gamma_c}\chi\mu\lambda\kappa_\tau K(\kappa_\tau v_\tau)[\delta u]_\tau[u^*]_\tau d\Gamma\\&=-\int_M tr(\Pi(\nabla_p(u)):\nabla_p(u^*))\,dM+L_{meca}(u^*)\\&+\int_{\Gamma_c}\chi\lambda[u^*]\cdot n\,d\Gamma+\int_{\Gamma_c}\chi\mu\lambda P_B(\kappa_\tau v_\tau)[u^*]_\tau d\Gamma\end{aligned}\tag{26}$$

$$-\int_{\Gamma^c}\frac{1}{\kappa_n}\delta\lambda\,\lambda^*\,d\Gamma-\int_{\Gamma_c}\chi[\delta u]\cdot n\,\lambda^*\,d\Gamma=+\int_{\Gamma_c}\left(\frac{1}{\kappa_n}\lambda\,\lambda^*+\chi[u]\cdot n\,\lambda^*\right)d\Gamma\tag{27}$$

$$\begin{aligned}&\int_{\Gamma_c}\delta\Lambda\,\Lambda^*\,d\Gamma\\&=-\int_{\Gamma_c}\chi(\Lambda-P_B(\kappa_\tau v_\tau))\Lambda^*\,d\Gamma-\int_{\Gamma_c}(1-\chi)\Lambda\,\Lambda^*\,d\Gamma\end{aligned}\tag{28}$$

While introducing (20-21) into the linearized formulation of the problem (23-25), one obtains a system which can be put in the following matric form:

$$\begin{bmatrix} K_{meca}+A^u+B^u & A^T+D^T & B_r^T \\ A & C & 0 \\ B_r & E & F \end{bmatrix} \begin{pmatrix} \delta u \\ \delta\lambda \\ \delta\Lambda \end{pmatrix} = \begin{pmatrix} L_{meca}+L^1_{cont}+L^1_{frott} \\ L^2_{cont} \\ L^3_{frott} \end{pmatrix} \quad (29)$$

where:

- K_{meca} is the mechanical matrix of rigidity,
- A^u is the matrix of increased rigidity due to the contact,
- B^u is the matrix of increased rigidity due to friction,
- A is the matrix binding the terms of displacement to those of contact (matrix of the law of contact),
- B_r is the matrix binding the terms of displacement to those of friction (matrix of the law of friction),
- C is the matrix allowing to determine contact pressures in the lack of contact case,
- D is a matrix binding the terms of displacement and the contact pressure. It comes from the linearization of the threshold of friction for the adherent points.
- E is a matrix binding and multiplier contact pressure of friction. It comes from the linearization of the threshold of friction for the adherent points.
- F is the matrix allowing to determine the multipliers of friction in the cases not contacting, not-rubbing, or contacting rubbing slipping,
- L_{meca} is the second member representing the internal forces and the increments of loadings,
- L^1_{cont} and L^2_{cont} are the second members due to the contact,
- L^1_{frott} and L^3_{frott} are the second members related to friction.

It is noted that the choice of a direct linearization of the threshold of friction led to a nonsymmetrical matrix.

One discretizes by keeping only the contributions of contact and one obtains:

$$\begin{bmatrix} \begin{bmatrix} A+B_{aa} & A+B_{ab} & A+B_{ac} \\ A+B_{ba} & A+B_{bb} & A+B_{bc} \\ A+B_{ca} & A+B_{cb} & A+B_{cc} \end{bmatrix}^{u}_{es:es} & \begin{bmatrix} A_a+D_a \\ A_b+D_b \\ A_c+D_c \end{bmatrix}_{es:cont} & \begin{bmatrix} B_a \\ B_b \\ B_c \end{bmatrix}_{es:cont} & \begin{bmatrix} A+B_{aa} & A+B_{ab} & A+B_{ac} \\ A+B_{ba} & A+B_{bb} & A+B_{bc} \\ A+B_{ca} & A+B_{cb} & A+B_{cc} \end{bmatrix}^{u}_{es:ma} \\ [A_a \quad A_b \quad A_c]_{cont:es} & C & 0 & [A_a \quad A_b \quad A_c]_{cont:ma} \\ [B_a \quad B_b \quad B_c]_{cont:es} & E & F & [B_a \quad B_b \quad B_c]_{cont:ma} \\ \begin{bmatrix} A+B_{aa} & A+B_{ab} & A+B_{ac} \\ A+B_{ba} & A+B_{bb} & A+B_{bc} \\ A+B_{ca} & A+B_{cb} & A+B_{cc} \end{bmatrix}^{u}_{ma:es} & \begin{bmatrix} A_a+D_a \\ A_b+D_b \\ A_c+D_c \end{bmatrix}_{ma:cont} & \begin{bmatrix} B_a \\ B_b \\ B_c \end{bmatrix}_{ma:cont} & \begin{bmatrix} A+B_{aa} & A+B_{ab} & A+B_{ac} \\ A+B_{ba} & A+B_{bb} & A+B_{bc} \\ A+B_{ca} & A+B_{cb} & A+B_{cc} \end{bmatrix}^{u}_{ma:ma} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \delta a \\ \delta b \\ \delta c \end{bmatrix}_{es} \\ \delta\lambda \\ \delta\Lambda \\ \begin{bmatrix} \delta a \\ \delta b \\ \delta c \end{bmatrix}_{ma} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} L_a \\ L_b \\ L_c \end{bmatrix}^{1\,cont\,frot}_{es} \\ L^2 \\ L^3 \\ \begin{bmatrix} L_a \\ L_b \\ L_c \end{bmatrix}^{1\,cont\,frot}_{ma} \end{bmatrix}$$

(30)

Indices es and ma the contributions slave and Master indicate, respectively, while a , b and c the parts classic, enriched Heaviside indicate and enriched Tip Ace, respectively.

If one treats an element of the type exclusively Tip Ace, one regains the shape HP and the matrix becomes:

$$\begin{bmatrix} 4[A_{c:c}+B_{c:c}]^u & 2[A_{c:\lambda}+D_{c:\lambda}] & 2[B_{c:\Lambda}] \\ 2[A_{\lambda:c}] & C & 0 \\ 2[B_{\Lambda:c}] & E & F \end{bmatrix} \begin{bmatrix} \delta c \\ \delta\lambda \\ \delta\Lambda \end{bmatrix} = \begin{bmatrix} 2\mathrm{L}_c^{1\ cont}+2\mathrm{L}_c^{1\ frot} \\ L^2 \\ L^3 \end{bmatrix}$$

The expressions of the terms present in the system are:

- A^u and B^u , block slave – slave:

$$[A^u_{es:es}]_{ij} = \int_{\Gamma_c} \rho_n \chi \, \phi_i^{es} \phi_j^{es} \, n \otimes n [MM] \, d\Gamma$$

$$[B^u_{es:es}]_{ij}=\int_{\Gamma_c}-\rho_\tau\chi\mu\lambda\phi_i^{es}\phi_j^{es}[P_\tau]^T\cdot[K(g_\tau)]\cdot[P_\tau][MM]d\Gamma$$

where:

$$[MM]=\begin{bmatrix}1 & He_j^{1,e} & -\sqrt{r_e}\\ & He_i^{1,e}He_j^{1,e} & -He_i^{1,e}\sqrt{r_e}\\ sym & & r_e\end{bmatrix}\text{ if ace-tip}$$

$[MM]_{1,1}=1$, $[MM]_{1,1+l}=He_j^{l,e}$, $[MM]_{1+k,1}=He_i^{k,e}$ and $[MM]_{1+k,1+l}=He_i^{k,e}He_j^{l,e}$ if multi-Heaviside

The cases ace-tip are distinguished and multi-Heaviside bus currently, an element cannot be at the same time ace-tip and multi-Heaviside. Let us note that if one would have such an element, the forms of the elementary matrices would be obtained easily by combination of the preceding terms. In other words there would not be great a deal to make in the TE of assembly of these terms, as long as there is only one enrichment ace-tip associated with the one with the cracks with the element.

- A^u and B^u , block slave – Master and master-slave:

$$[A^u_{es:ma}]_{ij}=\int_{\Gamma_c}\rho_n\chi\phi_i^{es}\phi_j^{ma}n\otimes n[MM]d\Gamma$$

$$[A^u_{ma:es}]_{ij}=\int_{\Gamma_c}\rho_n\chi\phi_i^{ma}\phi_j^{es}n\otimes n[MM]^Td\Gamma=[A^u_{es:ma}]^T_{ji}$$

$$[B^u_{es:ma}]_{ij}=\int_{\Gamma_c}-\rho_\tau\chi\mu\lambda\phi_i^{es}\phi_j^{ma}[P_\tau]^T\cdot[K(g_\tau)]\cdot[P_\tau][MM]d\Gamma$$

$$[B^u_{ma:es}]_{ij}=\int_{\Gamma_c}-\rho_\tau\chi\mu\lambda\phi_i^{ma}\phi_j^{es}[P_\tau]^T\cdot[K(g_\tau)]\cdot[P_\tau][MM]^Td\Gamma$$

where:

$$[MM]=\begin{bmatrix}-1 & -1 & -\sqrt{r_m}\\ 1 & 1 & \sqrt{r_m}\\ \sqrt{r_e} & \sqrt{r_e} & \sqrt{r_e}\cdot\sqrt{r_m}\end{bmatrix}\text{ if ace-tip}$$

$[MM]_{1,1}=-1$, $[MM]_{1,1+l}=-He_j^{l,m}$, $[MM]_{1+k,1}=-He_i^{k,e}$ and $[MM]_{1+k,1+l}=-He_i^{k,e}He_j^{l,m}$ if multi-Heaviside.

- A^u and B^u , main block – Master

$$[A^u_{ma:ma}]_{ij}=\int_{\Gamma_c}\rho_n\chi\phi_i^{ma}\phi_j^{ma}n\otimes n[MM]d\Gamma$$

$$[B^u_{ma:ma}]_{ij}=\int_{\Gamma_c}-\rho_\tau\chi\mu\lambda\phi_i^{ma}\phi_j^{ma}[P_\tau]^T\cdot[K(g_\tau)]\cdot[P_\tau][MM]d\Gamma$$

where:

$$[MM]=\begin{bmatrix}1 & He_j^{1,m} & \sqrt{r_m}\\ & He_i^{1,m}He_j^{1,m} & He_i^{1,m}\sqrt{r_m}\\ sym & & r_m\end{bmatrix}\text{ if ace-tip}$$

$[MM]_{1,1}=1$, $[MM]_{1,1+l}=He_j^{l,m}$, $[MM]_{1+k,1}=He_i^{k,m}$ and $[MM]_{1+k,1+l}=He_i^{k,m}He_j^{l,m}$ if multi-Heaviside.

Note

The term $n\otimes n$ being symmetrical, of this fact $[A^u]$ is symmetrical. The term $[P_\tau]^T\cdot[K(g_\tau)]\cdot[P_\tau]$ is also symmetrical, of this fact $[B^u]$ is symmetrical.

- A , B and D , blocks slave – contact and contact – slave:

$$[(A+D)_{es:cont}]_{ij}=\int_{\Gamma_c}\chi\phi_i^{es}\phi_j^{es}(n+S_f\mu g_\tau\cdot[P_\tau])[V]d\Gamma$$

$$[A_{cont:es}]_{ij}=\int_{\Gamma_c}\chi\phi_i^{es}\phi_j^{es}n^T[V]^Td\Gamma$$

$$[B_{es:cont}]_{ij}=\int_{\Gamma_c}\chi\,\mu\lambda\,\phi_i^{es}\phi_j^{es}[P_\tau]^T\cdot[K(g_\tau)]\cdot[\tau_1,\tau_2][V]\,d\Gamma$$
$$[B_{cont:es}]_{ij}=\int_{\Gamma_c}\chi\,\mu\,\lambda\,\phi_i^{es}\phi_j^{es}[\tau_1,\tau_2]^T[K(g_\tau)]\cdot[P_\tau][V]^T\,d\Gamma$$

where:

$$[V]=\begin{bmatrix}-1\\-He_i^{1,e}\\\sqrt{r_e}\end{bmatrix}\text{ if ace-tip}$$
$$[V]_1=-1\;,\;[V]_{1+k}=-He_i^{k,e}\text{ and if multi-Heaviside}$$

- A and B, blocks Master – contact and contact – Master:

$$[(A+D)_{ma:cont}]_{ij}=\int_{\Gamma_c}\chi\,\phi_i^{ma}\phi_j^{es}(n+S_f\,\mu\,g_\tau\cdot[P_\tau])[V]\,d\Gamma$$
$$[A_{cont:ma}]_{ij}=\int_{\Gamma_c}\chi\,\phi_i^{ma}\phi_j^{es}n^T[V]^T\,d\Gamma$$
$$[B_{ma:cont}]_{ij}=\int_{\Gamma_c}\chi\,\mu\lambda\,\phi_i^{ma}\phi_j^{es}[P_\tau]^T\cdot[K(g_\tau)]\cdot[\tau_1,\tau_2][V]\,d\Gamma$$
$$[B_{cont:ma}]_{ij}=\int_{\Gamma_c}\chi\,\mu\,\lambda\,\phi_i^{ma}\phi_j^{es}[\tau_1,\tau_2]^T[K(g_\tau)]\cdot[P_\tau][V]^T\,d\Gamma$$

where:

$$[V]=\begin{bmatrix}1\\He_i^{1,m}\\\sqrt{r_m}\end{bmatrix}\text{ if ace-tip}$$
$$[V]_1=1\;,\;[V]_{1+k}=He_i^{k,m}\text{ if multi-Heaviside}$$

Note

If the contact were applied, A is symmetrical only if there is no friction and B is symmetrical if and only if I $\mu=1$.

- C:

$$[C]_{ij}=-\int_{\Gamma_c}\frac{1-\chi}{\rho_n}\phi_i^{es}\phi_j^{es}\,d\Gamma$$

- E:

$$[E]_{ij}=-\int_{\Gamma_c}\chi\,S_f\,\mu\,\phi_i^{es}\phi_j^{es}(\nu_\tau\cdot[\tau_1,\tau_2]^T)[\tau_1,\tau_2]\,d\Gamma$$

- F:

$$[F]_{ij}=\int_{\Gamma_c}(1-\chi)\phi_i^{es}\phi_j^{es}[\tau_1,\tau_2]^T\cdot[\tau_1,\tau_2]\,d\Gamma+\int_{\Gamma_c}\frac{\chi\,\mu\,\lambda}{\rho_\tau}\phi_i^{es}\phi_j^{es}[\tau_1,\tau_2]^T\cdot[I_d-K(g_\tau)]\cdot[\tau_1,\tau_2]\,d\Gamma$$

Note

In the absence of friction or when $\mu=0$, the matrix is symmetrical because the terms D, and E are not assembled. With friction, it is not it any more.

- L^1, block slave:

$$[L_{es}^{1,cont}]_i=-\int_{\Gamma_c}\chi(\lambda-\rho_n d_n)\phi_i^{es}n[V]\,d\Gamma$$
$$[L_{es}^{1,frot}]_i=-\int_{\Gamma_c}\chi\mu\lambda[P_\tau]^T\cdot P_B(\Lambda+\rho_\tau[u_e-u_m]_\tau)\phi_i^{es}[V]\,d\Gamma$$

where:

$$[V]=\begin{bmatrix}-1\\-He_i^{1,e}\\\sqrt{r_e}\end{bmatrix}\text{ if ace-tip}$$

$$[V]_1=-1\ ,\ [V]_{1+k}=-He_i^{k,e}\text{ if multi-Heaviside}$$

- L^1 , main block:

$$[L_{ma}^{1,cont}]_i=-\int_{\Gamma_c}\chi(\lambda-\rho_n d_n)\phi_i^{ma}n[V]d\Gamma$$

$$[L_{es}^{1,frot}]_i=-\int_{\Gamma_c}\chi\mu\lambda[P_\tau]^T\cdot P_B(\Lambda+\rho_\tau[u_e-u_m]_\tau)\phi_i^{ma}[V]d\Gamma$$

where:

$$[V]=\begin{bmatrix}1\\He_i^{1,m}\\\sqrt{r_m}\end{bmatrix}\text{ if ace-tip}$$

$$[V]_1=1\ ,\ [V]_{1+k}=He_i^{k,m}\text{ if multi-Heaviside}$$

- L^2 :

$$[L^2]_i=\int_{\Gamma_c}(\frac{1-\chi}{\rho_n}\lambda+\chi[u_e-u_m]\cdot n)\phi_i^{es}d\Gamma$$

- L^3 :

$$[L^3]_i=-\int_{\Gamma_c}(1-\chi)\phi_i^{es}[\tau_1,\tau_2]^T\cdot[\tau_1,\tau_2]\Lambda d\Gamma-\int_{\Gamma_c}\frac{\chi\mu\lambda}{\rho_\tau}\phi_i^{es}[\tau_1,\tau_2]^T\cdot[\tau_1,\tau_2](\Lambda-P_B(\Lambda+\rho_\tau[u_e-u_m]_\tau))d\Gamma$$

Note

Attention, in routines FORTRAN calculating the second members, all the terms L expressed above are multiplied by -1 because in Code_Aster one considers, historically, that the second member is located in the left part of the system.

In the Lagrangian cases increased as in penalization, the matrix is singular if $\chi=1$ and $\lambda=0$. It is thus considered, in the equation of friction only, that $\chi=0$ if $\lambda=0$. The initial linear system takes the shape then

$$\begin{bmatrix}K_{meca}+A^u+B^u & A^T & 0\\A & C & 0\\0 & 0 & F\end{bmatrix}\begin{pmatrix}\delta u\\\delta\lambda\\\delta\Lambda\end{pmatrix}=\begin{pmatrix}L_{meca}\\0\\0\end{pmatrix}$$

5 Improvement of integration for the contact

5.1 Conflict enters the relations imposed by condition LBB and the changes of status of contact

5.1.1 Linear relation on the way contacting/not contacting

We had noted that the linear relations introduced on the degrees of freedom of contact-friction, in order to satisfy condition LBB in X-FEM, posed a problem in great slips. Indeed at the time of the contacting passage/not contacting, they introduced an useless relation which was at the origin of oscillations on the profile of pressure [bib10]. This generated difficulties to converge on the statutes of contact. To illustrate this phenomenon, let us defer on figure 7.

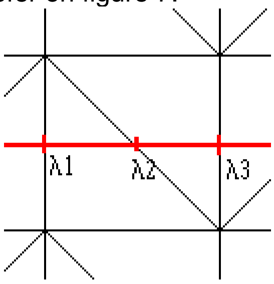


Figure 7 - the 3 nodes are seen imposing the linear relation $\lambda_1 - 2\lambda_2 + \lambda_3 = 0$ by the algorithm of stabilization of the LBB.

If items 1 and 2 are not contacting and item 3 is contacting on this figure, there is the linear relation $\lambda_1 - 2\lambda_2 + \lambda_3 = 0$ besides $\begin{cases}\lambda_1 = 0\\ \lambda_2 = 0\end{cases}$, which imposes $\lambda_3 = 0$.

However the contribution to node 3 is normally not worthless. It is determined by the weak formulation of the Principle of Virtual Work.

The lost pressure is then compensated by a peak on Lagrange of contact of the node contacting according to. This peak is followed of oscillations on the profile of pressure. These oscillations cause difficulties of convergence on the statutes of contact.

To solve this problem, one should not take account of the linear relation when one finds oneself in such a case, which is difficult given that it was introduced previously via another operator.

The made choice is thus not to assemble in the equation of contact the contribution of a point which 'is taken' in a linear relation and which is not contacting.

On the example of figure 7, that amounts not imposing more $\lambda_2 = 0$ because item 2 'is then taken' in the linear relation. One thus has only $\begin{cases}\lambda_1 - 2\lambda_2 + \lambda_3 = 0\\ \lambda_1 = 0\end{cases}$, which does not impose any more $\lambda_3 = 0$.

With the new formulation [bib9] which consists in storing the degrees of freedom of contact only to nodes already having degrees of freedom of displacement, that amounts not assembling on the level of the law of contact a point of integration not contacting which is not on a vital edge: one then assembles neither the contribution in the elementary matrix, nor the contribution to the second member of this point of integration (but the assembly is made for balance).

For more details on the definition of a vital edge in Code_Aster, one can refer to part LBB condition of chapter 4 of [bib1].

5.1.2 Relation of equality on the way contacting/not contacting

Just as for the linear relations, and the same reasons, we had noted that the relations of equality introduced on the degrees of freedom of contact-friction in order to satisfy condition LBB in X-FEM posed also a problem in great slips. They introduced also an useless relation at the time of the contacting passage/not contacting, the final consequence being the difficulty in converging on the statutes of contact. To illustrate this phenomenon, let us defer on Figure 8.

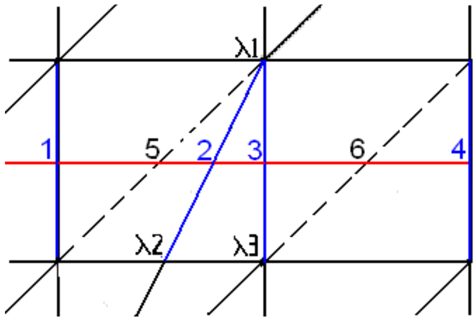


Figure 8 - edges 5 and 6 (in dotted lines) are nonvital, edges 1,2,3 and 4 (in blue) are vital. On edges 2 and 3, there are the relations λ1 = λ2 and λ1 = λ3

If the point of contact of edge 2 is not contacting, and the point of edge 3 contacting on this figure, there are the relations $\begin{cases}\lambda_1=\lambda_2=0\\ \lambda_1=\lambda_3\end{cases}$, which imposes $\lambda_3=0$ on the level of the law of contact.

However the contribution on edge 3 is normally not worthless. It is determined by the weak formulation of the Principle of Virtual Work.
It acts by way of same problem as into 5.1.1.
The choice made to solve this problem is the same one as for the problem with the linear relations, i.e not to assemble in the equation of contact the contribution of a point which 'is taken' in a relation of equality and which is not contacting, if another point 'taken' in this relation of equality is contacting.
In the example of Figure 8, that amounts not imposing more $\lambda_2=0$ because the point of integration not contacting of the vital edge 2 'is taken' in the relation between equality and the contacting point of the vital edge 3. One thus has only $\lambda_1=\lambda_2=\lambda_3$, which does not impose any more $\lambda_3=0$.
More generally, one should not assemble in the equation of contact the contributions of contact-friction of the points of integrations not contacting when a relation of equality connects them to a contacting point.
To do that, in formulation with the nodes tops [bib9], one defines as connecting a node which connects several vital edges, i.e the score of this node at the end of the algorithm to satisfy the condition LBB is strictly higher than 1 (on Figure 8, node 1 which connects edges 2 and 3 is regarded as connecting).
Then, each connecting node defines a group of edges connected between them. One numbers the groups of edges of 1 with n_{group} and in each group one numbers the edges of 1 with $n_{arête}$.
One then defines in each group a determining edge. It will be only group which one will assemble the contributions in the equation of contact if no point of integration of the group is contacting. In addition so for the group there exist then contacting points of integration one assembles in the equation of contact all the contacting contributions and one does not assemble the contributions not contacting.
One describes the choice of the determining edge now. For the first iteration of contact of the first step of time, all the statutes are identical, therefore one can choose the determining edge arbitrarily. Then, each reactualization of the statutes of contact, one carries out in the wake the reactualization of the determining edges.
One presents below the principal stages of the algorithm which makes this reactualization and which is launched just after having reactualized the statutes of contact:

- Boucle on the points of integration
  - If the point of integration in progress belongs to a group of connected vital edges, that it becomes contacting and that it is not on the determining edge:
    - Loop on the points of integration of the group
      The sta is searchedconcealed points on the determining edge
      - If the statute of the points of the determining edge is not contacting:
        the statute of the edge in progress becomes determining instead of the other
      - Fine if
    - Fine of loop on the points of integration of the group
  - Fine if
- Fin loop on the points of integration

5.1.3 Summary

The algorithm of selection detailed in the two preceding paragraphs consists in integrating the equation of contact on all the contacting edges, and integrating the contribution for the edges not contacting only if the group of vital edges does not comprise any contacting edge. Only one edge of the group, known as determining, is enough then.

This is explained by the fact why the force of contact in the equilibrium equation is implicit in the case contacting and explicit in the lack of contact case. In this last case, the multipliers of contact do not intervene nowhere in balance: the force is directly put at 0 via the statutes of contact. Since in the equation of contact one cannot assemble terms of different statutes within the same group, one preferentially assembles the contributions due to the contact where they describe a force of reaction intervening in balance, i.e. on the contacting edges.

5.2 Conflict enters the relations of equalities imposed by condition LBB and the changes of status of adherence

If one takes again the example of figure 7, in the case where, this time, the 3 points are contacting, but where items 1 and 2 are slipping and item 3 is adherent. One has, in 2D for example, the linear relation: $\Lambda_1-2\Lambda_2+\Lambda_3=0$ besides $\Lambda_1=1;\Lambda_2=1$ what imposes $\Lambda_3=1$.
However the contribution to node 3 is normally strictly lower than 1 since the point is adherent (inside the cone of friction), it is determined by the weak formulation of the Principle of Virtual Work. One finds a risk of not-convergence similar to the contact. It is necessary to remove in the equation of friction at least the assembly of the contribution of a point which 'is taken' in a linear relation and which is slipping into the law of friction. On the example of figure 7, that amounts not imposing more $\Lambda_2=1$ because item 2 'is then taken' in the linear relation.
One thus has only $\begin{cases}\Lambda_1-2\Lambda_2+\Lambda_3=0\\ \Lambda_1=1\end{cases}$, which does not impose any more $\Lambda_3=1$.

With the new formulation [bib9] which consists in storing the degrees of freedom of contact only into cubes nodes already having degrees of freedom of displacement, that amounts not assembling on the level of the law of friction a slipping point of integration which is not on a vital edge: one then assembles neither the contribution in the elementary matrix, nor the contribution to the second member of this point of integration (but the assembly is made for balance). The major difference with the case of the statute in contact, is that while thus making for friction, the matrix becomes asymmetrical. It is not a problem because that joined the choice already made in the chapter 4 precedent. Another difference with the contact comes owing to the fact that item 2 not contacting does not intervene in balance because χ 0 in this case are worth: if one does not assemble his contribution in the law of contact, and that its normally worthless value is not put at zero by the following iteration of contact then it will be worth $\lambda_2=\lambda_3/2$ because $\lambda_1=0$ by the linear relation $\lambda_1-2\lambda_2+\lambda_3=0$ but without that not modifying balance.

For friction, slipping item 2 intervenes in balance: if one does not assemble his contribution in the law of friction, one does not impose only the value of Λ that is to say equal to 1 with the following iteration of Newton (it will be worth $\Lambda_2=(\Lambda_1+\Lambda_3)/2=(1+\Lambda_3)/2<1$ because $\Lambda_3<1$). One is then likely to modify balance contrary to the case of the contact. Knowing that the law of friction does not impose either $v_{\tau 2}=0$, two cases are then possible:

- if ρ_τ is rather large so that $g_{\tau 2}=\Lambda_2+\rho_\tau v_{\tau 2}>1$. The statute of item 2 remains slipping and balances it is not modified since one uses $P_B(g_{\tau 2})$ in the dissipative term associated with friction;

- if ρ_τ is not large enough and only $g_{\tau 2}<1$ the statute of item 2 changes and becomes adherent. One assembles with the following iteration the law of friction which imposes $v_{\tau 2}=0$ and which consolidates the adherent statute.

In the same way and to take again the example of figure 8, in the case where, this time, the 2 points are contacting, but where the point of contact of edge 2 is slipping and the point of contact of edge 3 is adherent, one has the relations $\Lambda_1=\Lambda_2=1;\Lambda_1=\Lambda_3$, which imposes $\Lambda_3=1$ on the level of the law of friction. However the contribution on edge 3 is normally strictly lower than 1 since the point is adherent (inside the cone of friction). The same risk of not-convergence that previously is present.

As for the contact pressure, the semi-multiplier of friction is implicit in the adherent and explicit case in the slipping case where we can express it in the form:

$$\Lambda=-\frac{v_\tau}{\|v_\tau\|} \quad (31)$$

Like, differential of Λ compared to v_τ data by:

$$\frac{\partial\Lambda}{\partial v_\tau}=\frac{1}{\|v_\tau\|}\left(Id-\frac{v_\tau\otimes v_\tau}{\|v_\tau\|^2}\right)\underset{v_\tau\to 0}{\to}\infty$$

associated with the choice of (31) is not limited when the speed of slip becomes small, the purely explicit formulation of the slip is not retained, and one retains the following shape of the semi-multiplier to correct this defect:

$$\Lambda=P_{B(0,1)}(\Lambda+\rho_\tau v_\tau)$$

Consequently and contrary to the contact pressure which intervenes only in the contacting case, the semi-multiplier of friction intervenes at the same time in adherence or slip. If one assembles selectively the adherent edges in the equation of friction – in the same way that what was made for the edges contacting in the equation of contact – one takes the chance to observe oscillations of the statute of adherence preventing convergence because the semi-multiplier determined by balance contains the contributions of the slipping terms which are not taken into account in the equation of friction (comparatively, in the case of the contact, all the contributions not contacting do not intervene in the equilibrium equation). One could observe such oscillations on the test ssnv209j by implementing this technique.

This is why it is necessary to fix a priori a determining edge by group of vital edge. Finally and whatever its statute, the equation of friction is assembled on this only edge only (what means that the determining edge gives to the group its slipping or adherent characteristic and that within the same group one cannot have the presence of two distinct statutes, the value resulting from semi-mulitplicateur being realised).

Finally one recapitulates here the call to the algorithms presented higher:

- Not fixes on the statutes of contact
  - Iterations of Newton
    - Calculation of the contributions of contact-friction
  - Fine of the iteration of Newton

  Calculation of statutes of contact

  Algorithm of reactualization of the determining edge according to the statutes of contact
- Fine point fixes on the statutes of contact

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11 SSNP504 – Contact in great slips with X-FEM for oblique cracks, Documentation of Validation of Code_Aster n° [V6.03.504]

7 Description of the versions of the document

Index Doc.	**Version Aster**	**Author (S) or contributor (S), organization**	**Description of the modifications**
With	8.4	I.Nistor EDF/R & D /AMA	Initial text
B	9.7.4	I.Nistor, P.Massin EDF/R & D /AMA MR. SIAVELIS, MR. GUITON IFP	Card 12608: elements triangles, diagram of Simpson and Newton-Dimensions, calculation of the game.
C	10.1.1	P.Massin EDF/R & D /AMA MR. SIAVELIS, MR. GUITON IFP	Card 14123: introduction of the funds of crack.