Harmonic answer

Summary

This document presents the theoretical bases of the calculation of the permanent mode of the answer of a mechanical system complexes, with linear behavior, subjected to a harmonic dynamic stress. Calculation relates indifferently directly to the system modelled in finite elements, or represented by a modal base; in this last case if the modal base is the product of the technique of under-structuring see document [R4.06.03].

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1 Introduction

In the harmonic problems, the studied system is subjected to an excitation varying like the product of an unspecified function of space by a sinusoidal function of time.
To search the answer consists in calculating the field of the sizes represented by the ddls of modeling in finite elements of the system. When the system has a linear behavior the answer of the field of the sizes observed tends quickly (because of extinction of its transitory component by dissipation interns) towards a permanent mode: the resulting field varies finally harmonically like the excitation. It is this permanent mode of the answer that one proposes to calculate.

General notations:

t	:	time
P	:	Not running of the model
ω	:	Pulsation ($rad.s^{-1}$)
j	:	Imaginary pure unit $\left(j^2=-1\right)$
$\mathbf{M}$	:	Matrix of mass resulting from modeling finite elements
$\mathbf{K}$	:	Matrix of rigidity resulting from modeling finite elements
$\mathbf{C}$	:	Matrix of damping resulting from modeling finite elements
$\mathbf{q}$	:	Vector of the degrees of freedom resulting from modeling finite elements
$\mathbf{f}_{\mathbf{ext}}$	:	Vector of the forces external with the system
$\mathbf{\Phi}$	:	Matrix of the vectors of the base of the substructures
$\mathbf{\eta}$	:	Vector of the generalized degrees of freedom

2 Harmonic equation

We establish the dynamic equation in the case of a harmonic request for three kinds of mechanical systems:

- pure structures (without fluid),
- pure fluids (without structure) with linear 'acoustic' behavior,
- analog and digital systems structures and fluids in interaction fluid-structure.

2.1 Harmonic equation of the structures

The vibratory behavior of a pure structure results from the external forces which are applied to him.
The size to be calculated is displacement in any point P model.

2.1.1 Direct calculation

In the case of direct calculation on the model in finite elements we can write:

$$\mathbf{M}\,\ddot{\mathbf{u}}+\mathbf{C}\,\dot{\mathbf{u}}+\mathbf{K}\;\mathbf{u}=\mathbf{f}_{ext}(P,t) \qquad \text{éq 2.1.1-1}$$

where:

$\mathbf{M}$	is the matrix (real) of mass resulting from modeling finite elements from S ,
$\mathbf{C}$	is the matrix (real) of damping exit of modeling finite elements of S ,
$\mathbf{K}$	is the matrix (real) of rigidity resulting from modeling finite elements from S ,
$\mathbf{f}_{ext}(P,t)$	is the vector (complex) of field of the external forces applied to S ,
$\mathbf{u}$, $\dot{\mathbf{u}}$ and $\ddot{\mathbf{u}}$	are the vectors (complex) displacement, speed and acceleration, functions of P and t , resulting from modeling finite elements.

In a harmonic problem, one imposes a loading dynamic, spatially unspecified, but sinusoidal in time. One is interested then has the stabilized answer of the system, without taking account of the transitory part.

The field of the external forces is written :

$$\mathbf{f}_{ext}(P,t)=\{\mathbf{f}_{ext}(P)\}e^{j\omega t}$$

The field of displacements is written:

$$\mathbf{u}(P,t)=\{\mathbf{u}(P)\}e^{j\omega t}$$

The fields speed and acceleration are written:

$$\dot{\mathbf{u}}(P,t)=j\omega\{\mathbf{u}(P)\}e^{j\omega t}$$
$$\ddot{\mathbf{u}}(P,t)=-\omega^2\{\mathbf{u}(P)\}e^{j\omega t}$$

Finally the structure S check the following equation:

$$(\mathbf{K}+j\omega\mathbf{C}-\omega^2\mathbf{M})\{u\}=\{\mathbf{f}_{ext}(P)\} \quad \text{éq 2.1.1-2}$$

Typical case : if damping is of type **hysteretic** “total” the equation [éq 2.1.1-1] becomes:

$$\mathbf{M}\ddot{\mathbf{u}}+\mathbf{K}(1+j\eta)\mathbf{u}=\mathbf{f}_{ext}(P,t) \quad \text{éq 2.1.1-3}$$

where η is a total loss ratio (cf [R5.05.04]).
Then the equation [éq 2.1.1-2] is replaced by:

$$(\mathbf{K_c}-\omega^2\mathbf{M})\{\mathbf{u}\}=\{\mathbf{f}_{ext}(P)\} \quad \text{éq 2.1.1-4}$$

where:

$\mathbf{M}$	is the matrix (real) of mass resulting from modeling finite elements from S ,
$\mathbf{K_c}=\mathbf{K}+j\eta\mathbf{K}$	is a matrix of rigidity *complex* .

2.1.2 Calculation on modal basis

The calculation of the harmonic answer by the method of modal synthesis consists has to search the field of unknown displacement, resulting from modeling finite elements, on an adapted space, of reduced size (transformation of Ritz).
One will refer to the documents [R4.06.02] and [R4.06.03].
If this method rather is used the equation [éq 2.1.1-2] is projected on the basis of modal S and one leads to the following harmonic equation :

$$(\overline{\mathbf{K}}+j\omega\overline{\mathbf{C}}-\omega^2\overline{\mathbf{M}})\{\eta\}=\{\overline{\mathbf{f}}_{ext}\} \quad \text{éq 2.1.2-1}$$

where:

$\bar{\mathbf{M}}=\mathbf{\Phi}^{\mathbf{T}}\mathbf{M}\mathbf{\Phi}$	is the matrix (real) of mass generalized of S ,
$\bar{\mathbf{C}}=\mathbf{\Phi}^{\mathbf{T}}\mathbf{C}\mathbf{\Phi}$	is the matrix (real) of damping generalized of S ,
$\bar{\mathbf{k}}=\mathbf{\Phi}^{\mathbf{T}}\mathbf{k}\,\mathbf{\Phi}$	is the matrix (real) of rigidity generalized of S ,
$\{\bar{\mathbf{f}}_{ext}\}=\mathbf{\Phi}^{\mathbf{T}}\{\mathbf{f}_{ext}\}$	is the vector (complex) generalized harmonic external forces applied to S ,
$\mathbf{\Phi}$	is the matrix (real) modal vectors of the base of Ritz of S ,
$\{\boldsymbol{\eta}(P)\}$	is the vector (complex) generalized harmonic displacements.

Once $\{\boldsymbol{\eta}(P)\}$ determined by éq 2.1.2-1 one makes a restitution on physical basis (cf [R4.06.02]).

2.2 Harmonic equation of the acoustic fluids

The document [R4.02.01] described modeling by finite elements of a fluid system (without transport) having a linear acoustic behavior.
The fluid system F a harmonic request acoustic speed undergoes on part of its border. The harmonic answer is described by the following equation [éq 2.2-1], where the size to be calculated is the acoustic pressure in any point P model.

$$(\mathbf{K}+j\omega\,\mathbf{C}-\omega^2\mathbf{M})\{\mathbf{p}(P)\}=-j\omega\,\{\mathbf{v_n}(P)\} \qquad \text{éq 2.2-1}$$

where:

$\mathbf{M}$	is the matrix (complex) of "mass" acoustic exit of modeling finite elements of F ,
$\mathbf{C}$	is the matrix (complex) of "damping" acoustic exit of modeling finite elements of F , and in the species of the edge $\partial_z F$ where one applies an acoustic impedance,
$\mathbf{K}$	is the matrix (complex) of "rigidity" acoustic exit of modeling finite elements of F ,
$\mathbf{v_n}(P,t)=\{\mathbf{v_n}(P)\}e^{j\omega t}$	Where $\{\mathbf{V_n}(P)\}$ is the vector (complex) of field the normal acoustic speeds applied to the border $\partial_v F$ of F where one applies acoustic speeds,
$\mathbf{p}(P,t)=\{\mathbf{p}(P)\}e^{j\omega t}$	Where $\{\mathbf{p}(P)\}$ is the vector (complex) acoustic pressures resulting from modeling finite elements from F .

2.3 Harmonic equation of the systems fluid-structures

The document [R4.02.02] described modeling by finite elements of a system $F+S$ constituted of a fluid part (without transport) F in interaction with a structure part S (interaction in $F\cup S$). Fluid and structure have a linear behavior.
The fluid system F a harmonic request normal acoustic speed undergoes on part of its border. The harmonic answer is described by the following equation [éq 2.3-1], where the sizes to be calculated are:

- acoustic pressure in any point P fluid F ,
- displacement in any point P structure S ,
- as an auxiliary potential ϕ of displacement in any point P fluid F ,

$$(\mathbf{K}-\omega^2\mathbf{M}-j\omega^3\mathbf{I})\begin{Bmatrix}\mathbf{u}(P)\\\mathbf{p}(P)\\\phi(P)\end{Bmatrix}=+j\omega\{\mathbf{v_n}(P)\}$$ éq 2.3-1

where:

$\mathbf{M}$	is the matrix (real) of “mass” fluid-structure resulting from modeling finite elements of the fields F and S
$\mathbf{I}$	is the matrix (real) of “impedance” fluid exit of modeling finite elements of the edge $\partial_z F$ field F where an impedance is applied
$\mathbf{K}$	is the matrix (real) of “rigidity” fluid-structure resulting from modeling finite elements of the fields F and S
$\mathbf{v_n}(P,t)=\{\mathbf{v_n}(P)\}e^{j\omega t}$	Where $\mathbf{v_n}(P)$ is the vector (real) field the normal acoustic speeds applied to the border $\partial_v F$ of F
$\mathbf{u}(P,t)=\{\mathbf{u}(P)\}e^{j\omega t}$	is the vector (complex) field of displacement in the structure S
$\mathbf{p}(P,t)=\{\mathbf{p}(P)\}e^{j\omega t}$	is the vector (complex) acoustic field of pressure in the fluid F
$\phi(P,t)=\{\phi(P)\}e^{j\omega t}$	is the vector (complex) field of potential of displacement in the fluid F

2.4 General harmonic equation

With an aim of taking into account all the harmonic cases of equations the operator DYNA_LINE_HARM of *Code_Aster* solves the following general harmonic equation (cf [U4.53.11]):

$$(-j\omega^3\mathbf{I}-\omega^2\mathbf{M}+j\omega\mathbf{C}+\mathbf{K})\{\mathbf{q}\}=\left\{\sum_{i=1}^{k}h_i(f)\cdot\omega^{n_i}\cdot e^{j\pi\frac{\phi_i}{180}}\cdot\mathbf{g_i}(P)\right\}$$ éq 2.4-1

where:

$\mathbf{I}$	Matrix of fluid “impedance” possible exit of modeling finite elements,
$\mathbf{M}$	Matrix of “mass” resulting from modeling finite elements,
$\mathbf{C}$	Matrix of “damping” exit of modeling finite elements,
$\mathbf{K}$	Matrix of “rigidity” resulting from modeling finite elements,
$\{\mathbf{q}(P)\}$	Vector of the degrees of freedom resulting from modeling finite elements,
$\{\mathbf{g_i}(P)\}$	Vector field with the nodes corresponding to one or more loads of force or acoustic or potential speed or imposed movement,
$h_i(f)$	Real or complex function of the frequency f ,
$\omega=2\mathrm{pf}$	Pulsation
n_i	Power of the pulsation when the loading is function of the pulsation,
ϕ_i	Phase in degrees of each component of the excitation compared to a reference of phase.

As example if one takes the case of a system of fluid modelled in acoustics, without degrees of freedom imposed, simply solicited on part of his border by a field normal speed $\mathbf{v_n}(P,t)=\{\mathbf{v_n}(P)\}e^{j\omega t}$, terms of the equation [**éq 2.4-1**] become:

$\mathbf{I}$	*non-existent* ,
$\mathbf{M}$	Matrix of mass resulting from acoustic modeling finite elements,
$\mathbf{C}$	*Possibly* matrix of damping resulting from acoustic modeling finite elements if impedance on border,
$\mathbf{K}$	Matrix of rigidity resulting from acoustic modeling finite elements,
$\{\mathbf{q}(P)\}$	$=\{\mathbf{p}(P)\}$, vector of the pressures to the nodes,
$\{\mathbf{g_i}(P)\}$	$=\{\mathbf{v_n}(P)\}$, vector field normal speed to the faces (finite elements)
$h_i(f)$	$=-1.$ (constant),
$\omega=2\mathrm{pf}$	Pulsation,
n_i	$=1$
ϕ_i	$=0$

Note:

*Outre the solution of the harmonic equation [**éq 2.4-1**], Code_Aster allows to calculate the derivativeS of this solution by report with the loading $\{\mathbf{g_i}(P)\}$ OU with parameters of mass, stiffness or damping ($\mathbf{M}$, $\mathbf{K}$, $\mathbf{C}$). The equations whose these derivative are solutions and the theoretical developments relative are in [R4.03.04].*

3 Bibliography

1) R. DAUTRAY, J-L. LIONS, “Analyzes mathematical and digital calculation for sciences and technology”, Volume 2, Masson, 1985.

2) G. DHATT, G. TOUZOT, “a presentation of the finite element method”, Maloine S.A., Paris, 1984.

4 Description of the versions of the document

Version Aster	Author (S) Organization (S)	Description of the modifications
6	F.STIFKENS EDF-R&D/AMA	Initial text