Algorithms of temporal integration of the operator DYNA_TRAN_MODAL

Summary

This document describes the diagrams of temporal integration which are used to solve within the space of modes of the problems of transitory dynamics in linear mechanics, with, for certain diagrams, the taking into possible account of nonlocalised linearities of shocks type, frictions, relations effort-displacement and effort-speed, and the possible use of the under-structuring. Diagrams of NEWMARK (implicit, limited here to linear), EULER, DEVOGELAERE, and four explicit diagrams with step of adaptive time, ADAPT_ORDRE1 and ADAPT_ORDRE2 RUNGE_KUTTA_32 and RUNGE_KUTTA_54 are presented.

Contents

1 Introduction

The goal of the transitory dynamic analysis is to determine according to time the answer of a structure, being given a loading external or boundary conditions the functions of time, in cases where the effects of inertia cannot be neglected.

In a certain number of physical configurations, one cannot be satisfied with an analysis modal or harmonic and one must carry out a transitory analysis. It is in particular the case if:

- the history of the phenomenon has an importance in the study,
- if the external loading is complex (earthquake, excitations multi-components, etc…),
- if the system is nonlinear (plasticity, shocks, frictions, etc…).

The methods of analysis transient which can be then used divide into two main categories:

- methods known as of direct integration,
- the methods of Ritz, which understand inter alia the recombination of modal projections.

The methods of direct integration are thus called because no transformation is carried out on the dynamic system after the discretization by finite elements. They are presented in the document [R5.05.02], algorithms of direct integration of the operator `DYNA_LINE_TRAN`.
The methods of Ritz, on the other hand, proceed to a transformation of the initial dynamic system, by a projection on a subspace of the space of discretization departure. The resolution is done then on a modified system, which, if it is reduced, gives access only one approximation of the answer of the real system.

Algorithms of temporal integration on a frame of reference generalized are used to solve the dynamic problems in mechanics for linear structures, with taking into account possible of nonthe localised linearities the such shocks, frictions, and the relations effort-displacement and effort-speed. Certain algorithms allow moreover the under-structuring.
These algorithms are programmed in the operator `DYNA_TRAN_MODAL` of *Code_Aster* [U4.53.21].

2 Methods of temporal integration of a dynamic problem

2.1 Introduction

It is supposed that the equations governing the dynamic balance of the solids were discretized by finite elements. One obtains a discrete system of equations which it is a question of integrating in time. For that one chooses a discretization $\{t_i, i \in \mathbb{N}\}$ time interval of the study $[0, T]$ and one writes the balance of the structure at the moments t_i .

In a general way these equations take the following shape:

$$\mathbf{M}.\ddot{X}_t+\mathbf{C}.\dot{X}_t+\mathbf{K}.X_t=\mathbf{R}_{ext}(t)+\mathbf{R}_{nl}(X_t, \dot{X}_t, \ddot{X}_t)$$ **éq 2.1-1**

where

- $\mathbf{M}$ is the matrix of mass of the system,
- $\mathbf{K}$ is the matrix of rigidity of the system,
- $\mathbf{C}$ is the matrix of damping of the system,
- $\mathbf{R}_{ext}(t)$ is the vector of the external forces,
- $\mathbf{R}_{nl}(X_t, \dot{X}_t, \ddot{X}_t)$ is the vector of the nonlinear forces.

The matrix of damping $\mathbf{C}$ is in general difficult to evaluate because damping is often function of the frequency. It is however frequent to simplify the catch in depreciation account by employing the model of damping proportional, or model of Rayleigh.

The methods of reduction of Rayleigh-Ritz are presented in the document [R5.06.01], Methods of Ritz in linear and nonlinear dynamics.

If the term $\mathbf{R}_{nl}\left(X_t, \dot{X}_t, \ddot{X}_t\right)$ is not null, the technique of the pseudo-forces consists in projecting on the basis of linear system and maintaining the forces nonlinear with the second member. The technique of the pseudo-forces is always associated with a diagram of explicit integration. This fact the taking into account of nonthe linearities is available only for explicit diagrams. The addition of nonthe linearities does not modify the form of the equations.

In the method of Ritz, the field of displacement X_t is replaced by its projection on the modal basis such as $X_t = \Phi \eta_t$ where η_t is the vector of the generalized coordinates and Φ is the modal base, generally reduced.

The dynamic system project takes the following shape, with $\eta_t \in \mathbb{R}^p$:

$$\Phi^t.\mathbf{M}.\Phi.\ddot{\eta}_t + \Phi^t.\mathbf{C}.\Phi.\dot{\eta}_t + \Phi^t.\mathbf{K}.\Phi.\eta_t = \Phi^t.\mathbf{R}_{ext}(t) + \Phi^t.\mathbf{R}_{nl}\left(\Phi.\eta_t, \Phi.\dot{\eta}_t, \Phi.\ddot{\eta}_t\right) \quad \textbf{éq 2.1-2}$$

When the assumption of Basile does not apply (damping nonproportional), the matrix of damping projected is not diagonal. The integration of the coupled system is done then obligatorily with one of the three following diagrams: the implicit scheme `NEWMARK`, the explicit diagram `EULER` or explicit diagrams `ADAPT_ORDRE1` and `ADAPT_ORDRE2`.

The equation obtained in η_t is same form as the equation in X_t . So in the continuation of the document, one will use the notation X_t as well for displacement in coordinates generalized as for displacement in physical space. In the case of the operator `DYNA_TRAN_MODAL`, they are generalized coordinates.

Two classes of method can be distinguished in integration step by step from the equilibrium equations, the methods of explicit integration and the methods of implicit integration.

That is to say the linear dynamic system according to integrating in time:

$$\mathbf{M}.\ddot{X}_t + \mathbf{C}.\dot{X}_t + \mathbf{K}.X_t = \mathbf{R}_{ext}(t) \quad \textbf{éq 2.1-3}$$

This differential connection of the second order can be brought back to a first order system:

$$\mathbf{N}.\dot{U}_t = \mathbf{H}.U_t + \mathbf{F}_t \quad \textbf{éq 2.1-4}$$

where $U_t = \begin{pmatrix} X_t \\ \dot{X}_t \end{pmatrix}$, $\mathbf{N} = \begin{pmatrix} \mathbf{Id} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{pmatrix}$, $\mathbf{H} = \begin{pmatrix} \mathbf{0} & \mathbf{Id} \\ -\mathbf{K} & -\mathbf{C} \end{pmatrix}$, $\mathbf{F}_t = \begin{pmatrix} \mathbf{0} \\ \mathbf{R}_t \end{pmatrix}$

To integrate this system of differential equations, a discretization is used $\left\{t_i, i \in \mathbb{N}\right\}$, as well as a formula of differences finished to express the derivative $\dot{U}_t$.

One will call methods of integration clarifies the methods where only the derivative $\dot{U}_t$ fact of intervening unknown factors at time t_{i+1} . In this way determination of the sizes sought at the moment t_{i+1} do not result from an inversion of system utilizing the operator $\mathbf{H}$. So moreover, one carries out a "farmhouse-lumping" in order to return the matrix $\mathbf{M}$ diagonal, determination of U_t is particularly simple. They are there the main features of the methods of explicit integration.

The implicit or semi-implicit methods utilize the discretization of U_t at one posterior moment with t_i, generally t_{i+1}. The determination of the variables thus passes by the resolution of a system utilizing the operator $\mathbf{H}$.

Two concepts are important: consistency, or the order of the diagram of integration, and stability.

The approximations used to obtain the differential operators define consistency, or the order of the diagram of integration. One can indeed consider that the approximation with which one obtains displacement with each step of time is related to the order of approximation of the derivative first and seconds compared to time.

The study of stability of a diagram consists in analyzing the propagation of the digital disturbances in the course of time. A stable diagram preserves a finished solution, in spite of the disturbances, whereas an unstable diagram led to a digital explosion or divergence of the solution.

To carry out a study of stability of a diagram of integration, one puts this last in the form of a linear recursive system and one determines the particular characteristics of this system. If all the eigenvalues of the operator of recursivity are smaller than 1 modulates some, the diagram is stable. If not it is unstable.
The diagrams of integration clarifies are generally conditionally stable, which means that the step of time must be sufficiently small to ensure the stability of temporal integration.
Certain implicit algorithms have the property to be unconditionally stable, which makes their interest and makes it possible to use a step of arbitrarily large time.

Diagrams retained for the operator `DYNA_TRAN_MODAL` are an implicit scheme, `NEWMARK`, and four explicit diagrams, `EULER`, `DEVOGELAERE`, `ADAPT_ORDRE1` and `ADAPT_ORDRE2` (with step of adaptive time). The choice is done by the keyword `METHOD`: `'EULER'`, `'DEVOGE'`, `'NEWMARK'`, `'ADAPT_ORDRE1'` or `'ADAPT_ORDRE2'`.

2.2 Method of integration implicit

2.2.1 Introduction

The implicit methods utilize the resolution of a matric system with the operator previously definite. If the solids are supposed to be elastic linear, that results in the resolution of a linear system to each step of time.

The advantage of these methods is their unconditional stability, which enables them to integrate the equations of dynamics with a step of relatively important time while correctly representing the behaviour of the modes low in frequency of the structure.
An implicit version of the method of `NEWMARK`, which was programmed in `DYNA_TRAN_MODAL` for the linear problems.

2.2.2 Method of NEWMARK [bib1]

2.2.2.1 Presentation of the diagram

NEWMARK [bib1] introduced two parameters γ and β for the calculation positions and speeds to the step of time $t+\Delta t$:

$$\dot{\mathbf{X}}_{t+\Delta t}=\dot{\mathbf{X}}_t+\Delta t\left[(1-\gamma)\ddot{\mathbf{X}}_{\mathbf{t}}+\gamma\ddot{\mathbf{X}}_{t+\Delta t}\right]$$

$$\mathbf{X}_{t+\Delta t}=\mathbf{X}_t+\Delta t\,\dot{\mathbf{X}}_{\mathbf{t}}+\Delta t^2\left[(\frac{1}{2}-\beta)\ddot{\mathbf{X}}_{\mathbf{t}}+\beta\ddot{\mathbf{X}}_{t+\Delta t}\right]$$

Let us consider the equilibrium equations at time $t+\Delta t$:

$$\mathbf{M}\cdot\ddot{\mathbf{X}}_{t+\Delta t}+\mathbf{C}\dot{\mathbf{X}}_{t+\Delta t}+\mathbf{K}\mathbf{X}_{t+\Delta t}=\mathbf{R}_{\mathbf{t+\Delta t}}$$

Let us defer the preceding relations while eliminating $\dot{\mathbf{X}}_{t+\Delta t}$ and $\ddot{\mathbf{X}}_{t+\Delta t}$, it comes:

$$\tilde{\mathbf{K}}\cdot\mathbf{X}_{t+\Delta t}=\tilde{\mathbf{R}}_{t+\Delta t} \text{ where } \tilde{\mathbf{K}}=\mathbf{K}+a_0\mathbf{M}+a_1\mathbf{C}$$

$$\tilde{\mathbf{R}}=\mathbf{R}_{t+\Delta t}+\mathbf{C}\cdot\left(a_1\mathbf{X}_t+a_4\dot{\mathbf{X}}_t+a_5\ddot{\mathbf{X}}_t\right)+\mathbf{M}\cdot\left(a_0\mathbf{X}_t+a_2\dot{\mathbf{X}}_t+a_3\ddot{\mathbf{X}}_t\right)$$

with:

$$a_0=\frac{1}{\left(\beta.\Delta t^2\right)} \quad a_1=\frac{\gamma}{\left(\beta.\Delta t\right)} \quad a_2=\frac{1}{\left(\beta.\Delta t\right)} \quad a_3=\frac{1}{2\beta}-1$$

$$a_4=\frac{\gamma}{\beta}-1 \quad a_5=\frac{\Delta t}{2}\left(\frac{\gamma}{\beta}-2\right) \quad a_6=\Delta t.\left(1-\gamma\right) \quad a_7=\gamma.\Delta t$$

2.2.2.2 Complete algorithm of the method of NEWMARK

a) initialization

1) initial conditions $\mathbf{X_0}$, $\dot{\mathbf{X}}_\mathbf{0}$ and $\ddot{\mathbf{X}}_\mathbf{0}$
2) choice of Δt γ and β and calculation of the coefficients $a_{1,}\dots,a_8$ (cf above)
3) to assemble the matrices of stiffness $\mathbf{K}$, of mass $\mathbf{M}$ and of damping $\mathbf{C}$
4) to form the matrix of effective rigidity $\tilde{\mathbf{K}}=\mathbf{K}+a_0\mathbf{M}+a_1\mathbf{C}$
5) to factorize $\tilde{\mathbf{K}}$

b) with each step of time

1) to calculate the effective loading $\tilde{\mathbf{R}}$:
   $$\tilde{\mathbf{R}}=\mathbf{R}_{t+\Delta t}+\mathbf{C}\cdot\left(a_1\mathbf{X}_t+a_4\dot{\mathbf{X}}_t+a_5\ddot{\mathbf{X}}_t\right)+\mathbf{M}\cdot\left(a_0\mathbf{X}_t+a_2\dot{\mathbf{X}}_t+a_3\ddot{\mathbf{X}}_t\right)$$
2) to solve $\tilde{\mathbf{K}}\cdot\mathbf{X}_{t+\Delta t}=\tilde{\mathbf{R}}_{t+\Delta t}$
3) to calculate speeds and accelerations at time $t+\Delta t$
   $$\ddot{\mathbf{X}}_{t+\Delta t}=a_0\left(\mathbf{X}_{t+\Delta t}-\mathbf{X}_t\right)-a_2\dot{\mathbf{X}}_t-a_3\ddot{\mathbf{X}}_t$$
   $$\dot{\mathbf{X}}_{t+\Delta t}=\dot{\mathbf{X}}_t+a_6\ddot{\mathbf{X}}_t+a_7\ddot{\mathbf{X}}_{t+\Delta t}$$
4) calculation of the step of next time: return out of B) 1)

2.2.2.3 Stability conditions of the diagram of NEWMARK

Method of NEWMARK used in a rather widespread way in the field of mechanics, because it makes it possible to choose the order of integration, to introduce or not digital damping, and has a very good precision.

It is unconditionally stable if: $\gamma>0.5$ and $\beta>\frac{(2\gamma+1)^2}{4}$

One introduces a positive digital damping if $\gamma>\frac{1}{2}$ and negative if $\gamma<\frac{1}{2}$.

When $\gamma=\frac{1}{2}$ and $\beta=0$, the formula of NEWMARK is reduced to the diagram centered differences. It is thus then an explicit diagram.

A combination very often employed is $\gamma=\frac{1}{2}$ and $\beta=\frac{1}{4}$, because it leads to a diagram of a nature 2, unconditionally stable without digital damping. In fact the choice was made in the operator DYNA_TRAN_MODAL. The diagram of Newmark of this operator is thus implicit.

2.2.2.4 Employment

In DYNA_TRAN_MODAL, this diagram allows integration only linear problems. Within the framework of the dynamic under-structuring, it makes it possible to employ a modal base calculated by under-structuring but it does not support direct calculation on the basis of modal substructure.

2.2.2.5 Digital damping of the implicit schemes

The digital advantage of the direct diagrams of implicit integration lies in the fact that the step of time can be substantially large compared to the smallest clean period of the system without being likely to cause an instability of the results.

For modes of period clean about the step of time or lower than the step of time, the algorithms of integration introduce a strong damping which contributes to erase the contribution of high modes (cf [R5.05.02]).

There is no digital damping in the typical case of the algorithm of NEWMARK with $\beta=\frac{1}{4}$ and $\gamma=\frac{1}{2}$.

On the other hand, implicit algorithms one a significant effect of lengthening of the periods of the answer of the structure. It is noted that to guarantee a good precision on the amplitude and the phase of calculated displacements, it is necessary to respect a criterion close to:

$$\Delta t<\frac{1}{(10\times F_{max})} \text{ à } \frac{1}{(100\times F_{max})}$$

where $F_{\max}$ is the high frequency of the movement which one wishes to capture.

2.3 Explicit methods of integration

2.3.1 Introduction

Several methods of integration clarifies are presented: a diagram of modified Euler of order 1, a diagram of Devogelaere-Fu of order 4 and diagrams with step of adaptive time ADAPT and Runge-Kutta. These three methods are available in the operator DYNA_TRAN_MODAL. Schémas is presented by considering only linear forces. However the taking into account of the nonlinear forces from of easily deduced with the technique from the pseudo-forces.

2.3.2 Diagram clarifies of modified Euler of order 1

2.3.2.1 Presentation

This diagram is commonly called "modified Euler" because it is about a very simple but conditionally stable alternative of the diagram of Euler of order 1, which is, him, unstable. It is thus a diagram often employed into explicit for mechanics. In *Code_Aster*, it is quite simply called EULER.

This diagram was used in the module STIFF with LICE [bib3], code finite elements of beam, and in code CADYRO [bib4] for the calculation of the lines of trees in rotation.

The diagram uses formulates it of Euler of order 1 to estimate the derivative in time, with a formula of front Euler for the speed and a formula of back Euler for displacement, as follows:

$$\dot{\mathbf{X}}_{n+1}=\dot{\mathbf{X}}_n+\Delta t\,\mathbf{M}^{-1}\left(\mathbf{R}-\mathbf{K}\,\mathbf{X}_n-\mathbf{C}\,\dot{\mathbf{X}}_n\right)+o(\Delta t)$$
$$\boldsymbol{X}_{n+1}=\boldsymbol{X}_n+\Delta t\,\dot{\boldsymbol{X}}_{n+1}+o(\Delta t)$$

The algorithm is thus the following:

a) initialization: $X_0, \dot{X}_0$ given

b) with each step of time:

$$\dot{\boldsymbol{X}}_{n+1}=\dot{\boldsymbol{X}}_n+\Delta t\,.\,\mathbf{M}^{-1}.\left(\mathbf{R}-\mathbf{K}.\,\boldsymbol{X}_n-\mathbf{C}.\,\dot{\boldsymbol{X}}_n\right)$$
$$\boldsymbol{X}_{n+1}=\boldsymbol{X}_n+\Delta t\,.\,\dot{\boldsymbol{X}}_{n+1}$$

2.3.2.2 Order and stability of the diagram

the approximations used in obtaining this diagram are of order 1. One can thus consider that the approximation with which one obtains displacement with each step of time is of order 1. It is the consistency of the diagram.

If one puts the diagram of integration in recursive form by eliminating the terms speed, one obtains the relation of following recurrence (without external force, nor damping):

$$\boldsymbol{X}_{n+1}+\left(\mathbf{M}^{-1}.\mathbf{K}.\Delta t^2-2\right).\boldsymbol{X}_n+\boldsymbol{X}_{n-1}=\mathbf{0}$$

The eigenvalue of this diagram are for a system with a degree of freedom:

$$\lambda=\frac{2-\mathbf{M}^{-1}\boldsymbol{K}\,\Delta t^2\pm i\sqrt{\mathbf{M}^{-1}.\mathbf{K}.\Delta t^2\left(4-\mathbf{M}^{-1}.\mathbf{K}.\Delta t^2\right)}}{2}\quad \text{if}\quad \Delta t<\frac{2}{\mathbf{M}^{-1}.\mathbf{K}}$$

The module of the eigenvalues λ 1 is worth. One realizes that one is in a limiting but favorable situation. There will not be uncontrolled increase in the error. Without damping, one is right on the terminal of stability. It can be an asset for the diagram: it does not introduce digital dissipation.

If $\Delta t<\frac{2}{\mathbf{M}^{-1}.\mathbf{K}}$ one can show that one of the two eigenvalues has a module larger than the unit and thus that the diagram is unstable.

The criterion of stability of the diagram `EULER` is thus: $\Delta t<\frac{2}{\mathbf{M}^{-1}.\mathbf{K}}$.

This study can be wide with a system with finished number of degrees of freedom. In this case, the criterion of stability becomes:

$$\Delta t<\frac{2}{\omega_{\max}}$$

The analysis can be refined by considering a system with damping [bib13].

2.3.3 Method of Devogelaere-Fu

2.3.3.1 Presentation

To present the algorithm of Devogelaere-Fu, shortened in `DEVOGE` in *Code_Aster*, one puts the dynamic problem in the form:

$\mathbf{M}.\ddot{X}_t+\mathbf{C}.\dot{X}_t=\mathbf{G}(t,X_t)$ where the matrix $\mathbf{C}$ is supposed to be diagonal.

Displacements and speeds are calculated as follows:

a) initialization

$$X_{-\frac{1}{2}}=X_0-\frac{\Delta t}{2}\dot{X}_0+\frac{\Delta t^2}{8}\left(\mathbf{M}^{-1}.\mathbf{G}(t_0,X_0)-\mathbf{M}^{-1}.\mathbf{C}.\dot{X}_0\right)$$
$$\dot{X}_{-\frac{1}{2}}=\left(4I-\Delta t.\mathbf{M}^{-1}.\mathbf{C}\right)^{-1}\left(\left(4I+\Delta t.\mathbf{M}^{-1}.\mathbf{C}\right).\dot{X}_0-\Delta t\left(\mathbf{G}\left(t_{-\frac{1}{2}},X_{-\frac{1}{2}}\right)+\mathbf{G}(t_0,X_0)\right)\right)$$

b) with each step of time

$$X_{n+\frac{1}{2}}=X_n+\frac{\Delta t}{2}\dot{X}_n+\frac{\Delta t^2}{24}\left(4\mathbf{M}^{-1}.\mathbf{G}(t_n,X_n)-\mathbf{M}^{-1}.\mathbf{G}\left(t_{n-\frac{1}{2}},X_{n-\frac{1}{2}}\right)-\mathbf{M}^{-1}.\mathbf{C}\left(4\dot{X}_n-\dot{X}_{n-\frac{1}{2}}\right)\right)$$
$$\dot{X}_{n+\frac{1}{2}}=4\left(4I+\Delta t.\mathbf{M}^{-1}.\mathbf{C}\right)^{-1}\left(\dot{X}_n+\frac{\Delta t}{4}\left(\mathbf{G}(t_n,X_n)+\mathbf{G}\left(t_{n+\frac{1}{2}},X_{n+\frac{1}{2}}\right)-\mathbf{M}^{-1}.\mathbf{C}.\dot{X}_n\right)\right)$$
$$X_{n+1}=X_n+\frac{\Delta t}{2}\dot{X}_n+\frac{\Delta t_2}{6}\left(4\mathbf{M}^{-1}.\mathbf{G}(t_n,X_n)+2\mathbf{M}^{-1}.\mathbf{G}\left(t_{n-\frac{1}{2}},X_{n-\frac{1}{2}}\right)-\mathbf{M}^{-1}.\mathbf{C}\left(\dot{X}_n+2\dot{X}_{n+\frac{1}{2}}\right)\right)$$
$$\dot{X}_{n+1}=6\left(6I+\Delta t.\mathbf{M}^{-1}.\mathbf{C}\right)^{-1}.\left(\dot{X}_n+\frac{\Delta t}{6}\left(\mathbf{G}(t_{n+1},X_{n+1})+4\mathbf{G}\left(t_{n+\frac{1}{2}},X_{n+\frac{1}{2}}\right)+\mathbf{G}(t_n,X_n)-\mathbf{M}^{-1}.\mathbf{C}.\left(4\dot{X}_{n+\frac{1}{2}}+\dot{X}_n\right)\right)\right)$$

2.3.3.2 Order and stability of the diagram

The diagram is of order 4, the approximations in the writing of the derivative temporal being in. $o(\Delta t^4)$ It thus has an excellent aptitude for the integration of regular solutions. Its interest is on the other hand less expresses if the functions to be integrated have discontinuities (shocks, friction, etc)

One can show that for a linear system not deadened the step of time guaranteeing stability is worth: $\Delta t<\frac{2\sqrt{2}}{\omega_{max}}$.

2.3.3.3 Employment

This method is expensive in computing times because it twice requires the evaluation of the vector of the internal forces $\mathbf{G}$, particularly heavy operation. Consequently it is used little in mechanics for direct integration. On the other hand it is employed by the ECA [bib2] in the case of the systems projected on modal basis.

This diagram allows the taking into account of nonlocalised linearities of shocks type and frictions.
Within the framework of the dynamic under-structuring, it makes it possible to employ a modal base calculated by under-structuring but it does not support direct calculation on the basis of modal substructure.

2.3.4 Diagrams of integration to step of adaptive time

2.3.4.1 Introduction: interest of a step of adaptive time

To carry out the temporal integration of the transient of a structure in a nonlinear phase always poses problems as for the choice of the step time. The estimate of the error is seldom accessible during integration.

The diagrams of explicit integration oblige to respect a step of maximum time not to diverge. In the case of nonlinear behavior, this step cannot be given *a priori* and can change with each iteration. When rigidity very strongly varies, a step of constant and very fine time to preserve the stability of the diagram led to a very large iteration count and to a considerable computing time.

Several algorithms of integration to step of adaptive time were thus developed for `DYNA_TRAN_MODAL: ADAPT_ORDRE1, ADAPT_ORDRE2, RUNGE_KUTTA_32 and RUNGE_KUTTA54`. The two first are based on the diagram of centered differences, order 2 and on the diagram of Euler, of order 1. The two last, they are diagrams of the family of Runge-Kutta with control of L 'error. In the continuation of this chapter one will not represent L' algorithm of the adaptive diagram of order 1 because it is copied on the diagram of Euler. The management of the adaptation of the step of time is, as for it, the same one as for the adaptive diagram of order 2.

One can notice that this kind of diagram was also programmed in `DYNA_LINE_TRAN` (cf [R5.05.02]).

2.3.4.2 Diagrams with adaptive steps ADAPT_ORDRE1 and ADAPT_ORDRE2

2.3.4.2.1 Diagram of the centered differences with constant step

One presents initially the diagram of the differences centered to constant step on which the diagram `ADAPT_ORDRE2` bases itself. He is written as follows:

$$\dot{X}_{n+\frac{1}{2}}=\dot{X}_{n-\frac{1}{2}}+\Delta t.\ddot{X}_n\left(t_n,X_n,\dot{X}_n\right)+o\left(\Delta t^2\right)$$
$$X_{n+1}=X_n+\Delta t\,\dot{X}_{n+\frac{1}{2}}+o\left(\Delta t^2\right)$$

with the following notations:

It is noted that speed is expressed with indices half entireties of the discretization in time whereas displacements and accelerations are expressed with the whole indices. Written this way the diagram is of order 2. However acceleration is not immediately calculable because speed is known only with the half not preceding. To circumvent this difficulty, one can use several approximations speed to the step of whole time.

- method 1: to suppose that $\ddot{X}_n\left(X_n,\dot{X}_n,t_n\right)\equiv\ddot{X}_x\left(X_n,\dot{X}_{n-\frac{1}{2}},t_n\right)$ what constitutes a valid approximation if damping is sufficiently weak $\dot{X}_n=\dot{X}_{n-\frac{1}{2}}+o(1)$. If damping is important, the diagram loses then its precision of order 2.
- method 2: to use an approximation of order 1 for speed: what allows $\dot{X}_n=\dot{X}_{n-\frac{1}{2}}+\frac{\Delta t}{2}\ddot{X}_{n-1}+o(\Delta t)$ to preserve order 2 of the diagram.
- method 3: to use a diagram of type correct predictor/:

$$\text{predictor:}\quad \begin{cases} \dot{X}_{n^p} = \dot{X}_{n-\frac{1}{2}} + \gamma \Delta t \, \ddot{X}_{n-1} \\ \ddot{X}_{n^p} = \ddot{X}\left(t_n, X_n, \dot{X}_{n^p}\right) \end{cases}$$

$$\text{corrector:}\quad \begin{cases} \dot{X}_n = \dot{X}_{n-\frac{1}{2}} + \frac{\Delta t}{2}\left(\beta \ddot{X}_{n^p} + (1-\beta)\ddot{X}_{n-1}\right) \\ \ddot{X}_n = \ddot{X}\left(t_n, X_n, \dot{X}_n\right) \end{cases}$$

where α and β are two parameter to be chosen. Park and Underwood [bib6] report that to carry out additional iterations does not improve in a significant way the stability of the diagram.

2.3.4.2.2 Adaptation of the diagram to the variable step of time

When the step of time varies, the expressions of the preceding paragraph are not valid any more, acceleration $\ddot{X}_n$ being more necessarily expressed in the center of the interval $\left[\dot{X}_{n-\frac{1}{2}}, \dot{X}_{n+\frac{1}{2}}\right]$, as one sees it on the diagram below.
To take account of this, speed is calculated as follows:

$$\dot{X}_{n+\frac{1}{2}} = \dot{X}_{n-\frac{1}{2}} + \frac{\Delta t_{n-1} + \Delta t_n}{2} \ddot{X}_n$$

The diagram `ADAPT_ORDRE2` complete is written then as follows:

1. estimate of $\dot{X}_x$ according to methods 1.2 or 3
2. $\dot{X}_{n+\frac{1}{2}} = \dot{X}_{n-\frac{1}{2}} + \frac{\Delta t_{n-1} + \Delta t_n}{2} \ddot{X}_n\left(t_n, X_n, \dot{X}_n\right) + o\left(\Delta t^2\right)$
3. $X_{n+1} = X_n + \Delta t \, \dot{X}_{n+\frac{1}{2}} + o\left(\Delta t^2\right)$

The order of the diagram is not rigorously any more equal to 2, the diagram having lost its centered character.
More Δt_n and Δt_{n+1} are different, more the order of the diagram tends towards 1. Strong variations of the step of time thus lead to a loss of precision.

It is possible to find expressions more complex, which use speed or acceleration with the preceding iteration [bib7]. However the formula presented here gives satisfactory results when the step of time decreases but it cause a drop in the limit of stability when the step of time increases. The remedy is to control the step so that it increases only slowly.

2.3.4.2.3 Stability and precision of the diagram

To study the diagram, one was satisfied with the analysis of a system to only one degree of freedom, free and linear, of own pulsation ω and of reduced damping ξ :

$$\ddot{\xi} + 2\omega\xi\dot{\xi} + \omega^2\xi = 0$$

The approached solution, by using the constant diagram with pas de, is obtained by the relation of following recurrence:

$$\mathbf{A}.Y_n + \mathbf{B}.Y_{n-1} = \mathbf{0}$$

with $\boldsymbol{Y}_n = \begin{bmatrix} \ddot{x}_n \\ \dot{x}_{n+\frac{1}{2}} \\ x_{n+1} \end{bmatrix}$

$\mathbf{A}$ and $\mathbf{B}$ are two matrices which depend on the selected method to calculate the contribution of the term of damping. A solution of the form is sought: $\boldsymbol{Y}_n = \lambda \boldsymbol{Y}_{n-1}$.

λ is an eigenvalue of $\mathbf{A}^{-1}.\mathbf{B}$ and can be written in the following form: $\lambda_c = \exp\left(\omega_c \Delta t \left(-\xi_c \pm i\sqrt{1-\xi_{c^2}}\right)\right)$ where ω_c and ξ_c are the pulsation and reduced damping calculated by the algorithm.

One can compare them with the exact solution, $\lambda_e = \exp\left(\omega \Delta t \left(-\xi \pm i\sqrt{1-\xi^2}\right)\right)$, which makes it possible to evaluate the error on the pulsation and the error on damping: $\dfrac{|\omega_c - \omega|}{\omega}$ and. $\dfrac{|\xi_c - \xi|}{\xi}$.

One studied [feeding-bottle 8] and [bib10] the properties of the diagram according to the method employed to estimate speed with the whole steps. It was empirically found that method 3 is at the same time more precise and more stable than methods 1 and 2. The method, without overcost of calculation, makes it possible to increase the order of the diagram and gives in most case a better precision, except in the event of weak damping. It is however less stable than method 1. It is the method 2 which was finally adopted in the diagram `ADAPT`. These studies made it possible moreover to estimate the number of points per period necessary to guarantee a stable integration. 20 is a value which gives a good safety margin. It is the selected value by default.

2.3.4.2.4 Criteria of adaptation of the step of time

The preceding developments make it possible to quantify the errors introduced during the calculation of a free and linear system. These criteria do not make it possible however to adapt the step of time. They are indeed delicate to implement in the nonlinear cases and do not take account of the variations of the excitation.

Another approach consists in studying the site error introduced by the diagram using limited developments.

The exact solution of a system to a degree of freedom checks:

$$\begin{cases} X\left(t+\dfrac{\Delta t}{2}\right) = X(t) + \dfrac{\Delta t}{2}\dot{X}(t) + \dfrac{\Delta t^2}{8}\ddot{X}(t) + \dfrac{\Delta t^3}{48}\dddot{X}(t) + o(\Delta t^3) \\ X\left(t-\dfrac{\Delta t}{2}\right) = X(t) - \dfrac{\Delta t}{2}\dot{X}(t) + \dfrac{\Delta t^2}{8}\ddot{X}(t) - \dfrac{\Delta t^3}{48}\dddot{X}(t) + o(\Delta t^3) \end{cases}$$

$$\Rightarrow X\left(t+\frac{\Delta t}{2}\right) = X\left(t-\frac{\Delta t}{2}\right) + \Delta t\,\dot{X}(t) + \frac{\Delta t^3}{24}\dddot{X}(t) + o(\Delta t^3)$$

The formula of integration of the differences thus leads to a truncation error being worth:

$$E_n = \frac{\Delta t^3}{24}\dddot{X}(t_n) \equiv \frac{\Delta t^2}{12}\left(\ddot{X}_n - \ddot{X}_{n-1}\right)$$

One can normalize this error to obtain a relative error:

$$e_n = \frac{\Delta t^2}{12} \frac{|\ddot{X}_n - \ddot{X}_{n-1}|}{X_n} \quad , \quad X_n \neq 0$$

Park and Underwood [bib7] interpreted this error by defining a "apparent pulsation":

$$\omega_{A_n^2} = \frac{\ddot{X}_n}{X_n}$$

Applied to the diagram with centered difference, this definition makes it possible to interpret the relative error e_n like a variation of the apparent pulsation:

$$e_n \equiv \frac{\Delta t^2}{12} |\omega_{A_n^2} - \omega_{A_{n-1}^2}|$$

Many algorithms use a criterion of adaptation of the step of time based on the truncation error ([bib9], [bib11]). However in the case of a conditionally stable diagram, this method neither to make sure of the stability of integration, nor to guarantee a precision for the calculation of the transients.

Other methods use an approximation of the instantaneous own pulsation of the system [bib12], using the matrices of mass and stiffness. They have the defect not to adapt to the external forces and their fluctuations in frequency.

It is thus useful to find a criterion which takes account of the two approaches. This is why Park and Underwood introduced the concept of "frequency connect disturbed":

$$f_{AP_n} = \frac{1}{2p} \sqrt{|\frac{\ddot{X}_n - \ddot{X}_{n-1}}{X_n - X_{n-1}}|}$$

This size is interpreted like the "instantaneous" frequency of the system.

In the case of a system with several degrees of freedom, it is necessary to calculate an apparent frequency for each degree of freedom and to take the maximum. The step of time can be then selected to respect a minimum of points per apparent period.

If the denominator of the expression of the apparent frequency tends towards zero, this one can become very large and not to have meaning more. This leads to an unjustified refinement when speed is cancelled. To cure it a criterion of the type is added:

$$\frac{|X_n - X_{n-1}|}{Dt} < \dot{X}_{\min} \Rightarrow f_{AP_n} = \frac{1}{2p} \sqrt{|\frac{\ddot{X}_n - \ddot{X}_{n-1}}{\dot{X}_{\min} Dt}|}$$

It is an intermediary between the disturbed apparent frequency and the truncation error. The adequate value of $f_{AP_n} = \frac{1}{2\pi} \sqrt{|\frac{\ddot{X}_n - \ddot{X}_{n-1}}{X_n - X_{n-1}}|}$ is difficult to choose *a priori* and an unsuited value involves an artificial reduction in the apparent frequency. In the case of a system with several degrees of freedom one circumvents this difficulty by employing the "close" degrees of freedom:

$$f_{AP_n} = \max_{1 \le i \le \mathrm{nb}\, dll} \left(\frac{1}{2\pi} \sqrt{|\frac{\ddot{X}_n^i - \ddot{X}_{n-1}^i}{b_n^i}|} \right)$$

$$b_n^i = \max \left(\max_{|i-j| \le lb} |X_n^j - X_{\mathrm{n-1}}^j|, \dot{X}_{\min} \Delta t \right)$$

where lb indicate for example the bandwidth of the matrix of stiffness and where can be $\dot{X}_{\min}$ chosen very small.

This method appears very effective if them X_n^i indicate physical components (displacements). In the case of a projection on modal basis it is not relevant to employ the close components to calculate the apparent frequency. In this case it is to better return to the first criterion and to use one of the two following methods, specified by the key word `VITE_MIN` :

- if `VITE_MIN: 'NORM'` then it is a variable parameter equal to $\frac{\|\dot{X}_n\|}{100}$ $(\|\dot{X}_n\|=\sqrt{\sum_{1\le i\le \mathrm{nb}\,ddl}(\dot{X}_n^i)^2})$. This method gives good performances when the number of degrees of freedom is large and is inapplicable with the case with only one degree of freedom. It is not indicated any more if the order of magnitude speed is very different from a degree of freedom to another.
- if `VITE_MIN: 'MAXIMUM'` then it is a variable and different parameter for each degree of freedom, $\dot{X}_{\min}^j=\max_{1\le m\le n}\frac{|\dot{X}_m^j|}{1001}$. This method has the advantage of functioning whatever the number of degree of freedom of the system but it cannot be used if the order of magnitude speed varies too much during calculation because, in this case, one would obtain systematically: $\frac{|X_n^j-X_{n-1}^j|}{\Delta t}\le\dot{X}_{\min}^j$.

2.3.4.2.5 Algorithm of the diagram of the centered differences with adaptive step

Rules mentioned above allow to fix a number of steps of time desired per period of the answer according to the wanted precision, N . It is adjustable by the key word `NB_POINT_PERIODE`. The step of time Δt_n must then be lower than $\frac{1}{Nf_{AP_n}}$. The key word `NOT` give the step of initial time, Δt_{ini} , and the key word `PAS_MAXI` the step of maximum time not to exceed, $\Delta t_{\max}$. In the old versions of the code (before version 10.1.20) the key word `NOT` defined at the same time the maximum step, the key word `PAS_MAXI` not existing.

The algorithm is described schematically below:

1) initialization: X_0 and $\dot{X}_0$ given

$\Delta t_{-1}=0$, $\Delta t_0=\Delta t_{ini}$ and $\ddot{X}_0=\ddot{X}(t_0, X_0, \dot{X}_0)$

2) initialization of $\dot{X}_{\min}$

3) With each step of time

1) initialization of the research of the step of time: $N_{iter}=0$

2) calculation of $\dot{X}_{n+\frac{1}{2}}$ then of X_{n+1} :

estimate speed (method 2): $\dot{X}_n=\dot{X}_{n-\frac{1}{2}}+\frac{\Delta t}{2}\ddot{X}_{n-1}$

speed with the semi step: $\dot{X}_{n+\frac{1}{2}}=\dot{X}_{n-\frac{1}{2}}+\frac{\Delta t_{n-1}+\Delta t_n}{2}\ddot{X}_n(t_n, X_n, \dot{X}_n)$

displacement: $X_{n+1}=X_n+\Delta t\,\dot{X}_{n+\frac{1}{2}}$

- calculation of acceleration $\ddot{X}_{n+1}$
- calculation of the apparent frequency:

$$\frac{|X_n-X_{n-1}|}{\Delta t}\geq \dot{X}_{\min}\Rightarrow f_{AP_n}=\frac{1}{2\pi}\sqrt{\left|\frac{\ddot{X}_n-\ddot{X}_{n-1}}{X_n-X_{n-1}}\right|}$$

$$\frac{|X_n-X_{n-1}|}{\Delta t}<\dot{X}_{\min}\Rightarrow f_{AP_n}=\frac{1}{2\pi}\sqrt{\left|\frac{\ddot{X}_n-\ddot{X}_{n-1}}{\dot{X}_{\min}\Delta t}\right|}$$

3) checking of the adequacy enters the step of time and the apparent frequency:

calculation of the indicator $err=\Delta t_n N f_{AP_n}$

- if $err\geq 1$ and $N_{itez}<N_{iter\ \max}$ then reduction of the step of time and new iteration of research of the step: $\Delta t_n\leftarrow 0,75\Delta t_n$, $N_{iter}\leftarrow N_{iter}+1$, return in 2)
- if $err\leq 0,75$ since more 5 pas de consecutive times, then increase in the step of time $N_{iter}\leftarrow N_{iter}+1$
- filing of the solution, possible calculation of $\dot{X}_{\min}$ and return in 1) for the following iteration

2.3.4.2.6 Comments on the parameters of the algorithm

The fact of fixing an upper limit $N_{iter\ \max}$ by the key word NMAX_ITER_PAS the number of reductions of the step of time allows to make sure of the convergence of the algorithm in the difficult cases (for example in the event of discontinuity in the external forces).

When the indicator err is higher than 1, the step of time is multiplied by a factor fixes (0.75 by defaults but it can be modified by the user thanks to the operand COEF_DIVI_PAS). It would have been possible to write directly: $\Delta t_n\rightarrow\frac{1}{err}\Delta t_n=Nf_{AP_n}$, which more intuitive. But this strategy leads to an excessive refinement, the calculated apparent frequency being often largely higher than the real frequency, when the error is large. However, in only one step of time, Δt_n can be considerably reduced (factor $0{,}75^{N_{iter}}$).

On the other hand the increase in the step of time is always much slower (ratio of intensification by defaults of 1.1 definable by `COEF_MULT_PAS`) and has place only if the indicator is lower than 1 during five steps of consecutive time. These restrictions are justified by the risks of loss of stability or precision of the diagram when the step of time varies too quickly. A coefficient of 1.2 or 1.3 can allow a faster calculation but exposes sometimes at the risks of error.

In short the values by default were validated by many tests and in general give satisfaction in terms of precision and stability [bib8].

2.3.4.2.7 Performance of the algorithm

With equal precision, the iteration count carried out by a diagram adaptive is at least five times weaker than with a constant step in the phenomena which justify the use of a variable step by the irregular aspect of their evolution (shocks, discontinuous excitations, etc).

The empirical studies showed that the step of adaptive time makes it possible in the successful outcomes to gain a factor two or three in computing times. This diagram makes it possible moreover to control the precision of integration by the method of the control amongst points per period of the answer.
In the case of the very deadened systems, the profits can be even more important (calculations five to ten times faster).

On the other hand when the step of "ideal" time is about constant, the use of the diagram adaptive appears useless.

It of course allows the taking into account of nonthe localised linearities of shocks type or frictions.

In dynamic under-structuring, it is compatible as well with the transitory analysis on the modal basis restored on the whole system or transitory calculation on the bases distinct from the substructures.

2.3.4.3 Diagrams with step of adaptive times of the family of Runge-Kutta

The explicit methods of integration to a step of the Runge-Kutta type seek to determine an approximate solution of the problem of following Cauchy:

$$\begin{cases} \dot{y}(t) = f(t, y(t)) & t \in [t_{0,} T] \\ y(t_0) = y_0 & y_0 \in \mathbb{R} \end{cases}$$

For that, the interval is subdivided $[t_0 : T] : t_0 < t_1 < t_2 < \cdots < t_N$ and one poses $\Delta t = t_{n+1} - t_n$. At every moment of the interval the solution of problem of Cauchy is given by the integral expression:

$$y(t_{n+1}) = y(t_n) + \int_{t_n}^{t_n+\Delta t} f(t, y(t))$$

The approximate solution $y_n \approx y(t_n)$ proposed by the recursive diagrams of Runge-Kutta take the shape:

$$y_{n+1} = y_n + \Delta t \Psi(t_n, y_n, \Delta t)$$

with:

$$\Psi : [t_{0,} T] \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^d$$

where the term easily is identified $\Delta t \Psi(t_n, y_n, \Delta t)$ like the integral approximation of the term $\int_{t_n}^{t_n+\Delta t} f(t, y(t))$.

Thus, a method of Runge-Kutta with s stages allowing to approach the solution of the problem of Cauchy is given by the method of following squaring:

$$\begin{cases} k_1 = f(t_n, y_n) \\ k_2 = f(t_n + c_2 \cdot \Delta t, y_n + a_{21} \cdot k_1) \\ \vdots \\ k_s = f(t_n + c_s \cdot \Delta t, y_n + a_{s1} \cdot k_1 + a_{s2} \cdot k_2 + \cdots + a_{s\ s-1} \cdot k_{s-1}) \\ y_{n+1} = y_n + \Delta t \cdot (b_1 \cdot k_1 + b_2 \cdot k_2 + \cdots + b_s \cdot k_s) \end{cases}$$

A diagram of integration of Runge-Kutta is thus entirely defined by the coefficients b_i , c_i and a_{ij} with, in addition, $c_i = \sum_j a_{ij}$. In the case of an explicit diagram of Runge-Kutta, the coefficients must also check the condition $a_{ij} = 0 \; \forall \; j \geq i$.

In practice, the whole of the coefficients of a diagram of Runge-Kutta is presented in the shape of a table of Butcher like illustrates it below the figure:

$$\begin{array}{c|cccc} c_1 & & & & \\ c_2 & a_{21} & & & \\ c_3 & a_{31} & a_{32} & & \\ \vdots & \cdots & \cdots & \ddots & \\ c_s & a_{s1} & a_{s2} & \cdots & a_{s\ s-1} \\ \hline y_{n+1} & b_1 & b_2 & \cdots & b_s \end{array}$$

Moreover, it will be said that a method is of order p if $\forall n, 0 \leq n \leq N, e_n = O(\Delta t^{p+1}), \Delta t \to 0$

2.3.4.3.1 Diagrams of Runge-Kutta encased for the control of the step of adaptive time

At the time of the digital resolution of the problem of Cauchy describes higher, the choice of the step of time Δt is determining on the order of magnitude of the total error. Thus, the control of the made mistake e_n during the calculation of the solution $y_n \approx y(t_n)$ allows to determine a choice of step of time known as "optimal" so as to guarantee an increase of the error by a tolerance provided by the user.

With this intention, a classical procedure consists in employing two methods of Runge-Kutta known as encased. First method of order p with s stages is used to calculate the approximate solution y_{n+1} , whereas second method of order $\hat{p} < p$ is used to estimate the error $e_n = \| y_{n+1} - \hat{y}_{n+1} \|$ for the control of the step of time. In general, one has $\hat{p} = p - 1$ and the method is noted $RK \;\; p(\hat{p})$.
The advantage of this approach is that the weakest approximation of the order $\hat{p}$ use the same evaluations of f and thus same coefficients a_{ij} .

Within the framework of the operator DYNA_TRAN_MODAL , two explicit diagrams of integration of the Runge-Kutta family are available:

1. The diagram 'RUNGE_KUTTA_32' : It is the diagram of Bogacki-Shampine here 3(2). This diagram is of order 3 with 4 stages. It also integrates an approximation of a nature 2 allowing the control of the error. Although there are 4 stages, it uses really 3 stages because he has property FSAL (First Same Ace Last). The table of associated Butcher is presented below:

$$\begin{array}{c|ccc} 0 & & & \\ 1/2 & 1/2 & & \\ 3/4 & 0 & 3/4 & \\ 1 & 2/9 & 1/3 & 4/9 \\ \hline \end{array}$$

y_{n+1}	2/9 1/3 4/9 0
$\hat{y}_{n+1}$	7/24 1/4 1/3 1/8

2. The diagram `'RUNGE_KUTTA_54'` : It is here the diagram of Dormand-Prince also known under the name of DOPRI 5(4). This diagram is of order 5 with 7 stages. With an approximation of a nature 4 allowing the control of the error. As the preceding diagram it has property FSAL and it uses really only 6 stages. The table of associated Butcher is presented below:

c0	0							
c1	1/5	1/5						
c2	3/10	3/40	9/40					
c3	4/5	44/45	-56/15	32/9				
c4	8/9	19372/6561	−25360/2187	64448/6561	−212/729			
c5	1	9017/3168	−355/33	46732/5247	49/176	−5103/18656		
c6	1	35/384	0	500/1113	125/192	−2187/6784	11/84	
B	$\hat{y}_{n+1}$	5179/57600	0	7571/16695	393/640	−92097/339200	187/2100	1/40
D	y_{n+1}	35/384	0	500/1113	125/192	−2187/6784	11/84	0

These two diagrams programmed in the operator `DYNA_TRAN_MODAL` consider a vector of state y_n as being concatenation of the vector of displacements and speeds, namely:

$$y_n=\begin{Bmatrix} X_n \\ \dot{X}_n \end{Bmatrix}$$

In addition, the standard used for the control of the relative error is given by [bib14]:

$$err=\frac{1}{d}\sum_{k=1}^{d}\sqrt{\left(\frac{y_{n+1}^k-\hat{y}_{n+1}^k}{sc^k}\right)^2}$$

where d is the dimension of the vector of state y , y_{n+1}^k and $\hat{y}_{n+1}^k$ k - ièmes component of the vectors y_{n+1} and $\hat{y}_{n+1}$ respectively. Lastly, sc^k is given by:

$$sc^k=MAX\left(|y_n^k|,|y_{n+1}^k|\right)+\alpha$$

where α is a parameter regularization (value by default 0.001). Thus, the algorithm controls the increase of the relative error by the expression $err\leq tol$ where tol is a relative tolerance given by the user.

Lastly, the expression of the step of optimal time function of the made mistake is:

$$\Delta t_{opt}=0{,}9\cdot\Delta t_n\cdot\left(\frac{tol}{err}\right)^{\frac{1}{p+1}}$$

with $p=5$ for the diagram `'RUNGE_KUTTA_54'` and $p=3$ for the diagram `'RUNGE_KUTTA_32'`. For a better comprehension, the two algorithms are presented in the table below.

1) Initialization: X_0 **,** $\dot{X}_0$ **,** t_0 **and** Δt_0 **given:**
$n=0$, $\Delta t=\Delta t_0$ and $\ddot{X}_0=\ddot{X}(t_0, X_0, \dot{X}_0)$

2) As long as the condition $t_n<T$ **is satisfied**

3) Calculation of the following state y_{n+1} and of the relative error err by a method of Runge_Kutta

4) If the condition $err\leq tol$ is satisfied (the step of time is accepted):

- Filing of X_0 , $\dot{X}_0$ and $\ddot{X}_0$
- Passage à l' following state:
  - $y_n:=y_{n+1}$
  - $t_n:=t_n+\Delta t$
  - $n:=n+1$

5) Calculation of Δt_{opt} under the constraint $0,2\cdot\Delta t_n\leq\Delta t_{opt}\leq 5\cdot\Delta t_n$ (in order to avoid brutal changes)

6) Selection of the step of time $\Delta t:=min(\Delta t_{opt}, t_n-T)$

One can notice several things, particular with the method `'RUNGE_KUTTA_54'`:

- stages 5 and 6, calculate with t+dt.
- that the values of "D" correspond to the coefficients of the 6th stage of the method.
- the error is calculated with assistance (comic) *ki, and the step is adapted to the need. One can thus take either "B" or "D" to calculate the solution with t+dt.
- "D" makes it possible to calculate the solution with order 4 with a control of the error. One thus keeps the evaluation carried out on the 6th floor of the method for reasons of performance CPU. There needs more to use neither "B" nor "D" to evaluate the solution with t+dt.

For reasons of performance CPU and digital error, one calculates in rational form "comic" in the programming, for the evaluation of the error. The coefficients "B" and "D" are not thus useful any more at the time the programming of this method. (To trace the operations carried out they are put in comments in the code).

3 Conclusion

As a conclusion, summarized here various possibilities of temporal integration which offer the operator:

- Euler (`'EULER'`) clarify modified to ensure a conditional stability,
- Diagram of Newmark (`'NEWMARK'`) parameterized in order to be implicit,
- Diagram of Devogelaere-Fu (`'DEVOGE'`) of order 4,
- Four explicit adaptive diagrams:
  - `'ADAPT_ORDRE1'` being based on the diagram of Euleur,
  - `'ADAPT_ORDRE2'` based on the diagram of centered differences,
  - two diagrams of the Runge-Kutta family: 'RUNGE_KUTTA_32`'and RUNGE_KUTTA_54'.`

The diagram by default is `EULER` but it is not systematically adapted more. The diagram of `NEWMARK` available in *Code_Aster* is implicit and guarantees an unconditional stability but is usable only for purely linear problems.

The diagram `DEVOGE` is of order 4 and thus is more precise but is it is expensive in computing times.

Diagrams adaptive are more particularly indicated for the problems with located non-linearities, where the step of "ideal" time is not constant during the transient. It is thus the experiment of the modeling which makes it possible to choose the diagram best adapted to the problem according to the report (computing time) /précision.

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5 Description of the versions of the document

Version Aster	Author (S) Organization (S)	Description of the modifications
5	*G. JACQUART EDF-R&D/AMV*	Initial text
6.4	*E. BOYERE,* R & D /AMA *A.C. LIGHT,* EDF-R&D/TESE,	

	G. JACQUART DER/AMV	
10.4	N.GREFFET, F.VOLDOIRE R & D /AMA	Addition of syntax PAS_MAXI for the diagram ADAPT (card 14906). Addition ofa diagram ADAPT based on the diagram of Euler (card 16222).