Viscoplastic law of behavior LETK

Summary:

Law L&K describes a behavior élasto-visco-plastic of the rocks. Elastoplasticity is characterized by a positive work hardening in pre-peak and a lenitive behavior beyond resistance. Viscoplasticity translates the effect of time on the behavior. It is described by a law in power of Perzyna.

The initiation of the phenomena elastoplastic or viscoplastic starts as of the crossing of the corresponding threshold. The behavior related to each phase is described by the evolution of these various thresholds. This evolution is governed by functions of plastic or viscoplastic work hardening.

For the elastoplastic mechanism, a surface of load evolves through various thresholds:

- A threshold of damage confused with the threshold of initial viscosity,
- A macroscopic of peak, definite threshold starting from the laboratory tests,
- An intermediate threshold, qualified of limit of cleavage, determined analytically,
- A definite threshold characteristic as the envelope of the threshold of damage and limit of cleavage, called also limiting of contractance/dilatancy (this limit is confused with the threshold of maximum viscoplasticity),
- A threshold of residual resistance.

For the viscous mechanism, a viscoplastic surface evolves through:

- An initial threshold confused with the threshold of damage
- A maximum threshold of viscosity considered confused with the limit of contractane/dilatancy

Contents

1 Notations

1.1 General information

σ indicate the tensor of the effective constraints in small disturbances, noted in the shape of the following vector:

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sqrt{2}\,\sigma_{12} \\ \sqrt{2}\,\sigma_{13} \\ \sqrt{2}\,\sigma_{23} \end{pmatrix}$$

One notes:

$I_1 = tr(\boldsymbol{\sigma})$	first invariant of the constraints
$\boldsymbol{s} = \boldsymbol{\sigma} - \frac{I_1}{3}\boldsymbol{I}$	tensor of the constraints déviatoires
$s_{II} = \sqrt{\boldsymbol{s}\,.\,\boldsymbol{s}}$	second invariant of the tensor of the constraints déviatoires
$\sigma_{\max}$	major principal constraint
$\sigma_{\min}$	minor principal constraint
$\tilde{\boldsymbol{\varepsilon}} = \boldsymbol{\varepsilon} - \frac{Tr(\boldsymbol{\varepsilon})}{3}\boldsymbol{I}$	diverter of the deformations
$\varepsilon_V = tr(\boldsymbol{\varepsilon})$	voluminal deformation
$\cos(3\theta) = 2^{1/2}\ 3^{3/2}\ \frac{\det(\boldsymbol{s})}{s_{II}^3}$	θ being the angle of Lode
$\dot{\gamma}_p = \sqrt{\frac{2}{3}\dot{\tilde{\varepsilon}}_{ij}^p \dot{\tilde{\varepsilon}}_{ij}^p}$	cumulated plastic deviatoric deformations
$\dot{\gamma}_{vp} = \sqrt{\frac{2}{3}\dot{\tilde{\varepsilon}}_{ij}^{vp} \dot{\tilde{\varepsilon}}_{ij}^{vp}}$	cumulated viscoplastic deviatoric deformations
ξ_p	plastic parameter of work hardening
ξ_{vp}	viscoplastic parameter of work hardening
$\boldsymbol{G}^{visc}$	function controlling the evolution of the viscous deformations and describing the direction of flow
$\tilde{\boldsymbol{G}} = \boldsymbol{G} - \frac{Tr(\boldsymbol{G})}{3}\boldsymbol{I}$	diverter of $\boldsymbol{G}$
$G = Tr(\boldsymbol{G})$	trace of $\boldsymbol{G}$
$\tilde{G}_{II} = \sqrt{\tilde{\boldsymbol{G}}\cdot\tilde{\boldsymbol{G}}}$	normalizes $\tilde{\boldsymbol{G}}$
ψ	angle of dilatancy
f^d	elastoplastic surface of load
f^{vp}	viscoplastic surface of load

1.2 Convention of signs

- In Code_Aster, the convention of signs is that of the mechanics of the continuous mediums:

  In compression: $\sigma<0$; $\varepsilon=\frac{\partial u}{\partial x}<0$

  In traction : $\sigma>0$; $\varepsilon=\frac{\partial u}{\partial x}>0$

- In the model LETK , the convention of sign is that of the soil mechanics:

  In compression : $\sigma>0$

  Contractance : $\varepsilon_v>0$

  In traction : $\sigma<0$

  Dilatancy : $\varepsilon_v<0$

Note:

To integrate this law in *Code_Aster* such as it is presented, it is necessary to change the sign of all the fields at the entrance of the routine corresponding to the law of behavior and its exit.

At the entrance of the routine:

$\sigma^-_{\mathrm{L\&K}}=-\sigma^-$

$\varepsilon^-_{\mathrm{L\&K}}=-\varepsilon^-$

$\Delta\varepsilon_{\mathrm{L\&K}}=-\Delta\varepsilon$

At the exit of the routine:

$\sigma=-\sigma_{\mathrm{L\&K}}$

$\varepsilon=-\varepsilon_{\mathrm{L\&K}}$

$\Delta\varepsilon=-\Delta\varepsilon_{\mathrm{L\&K}}$

1.3 Parameters of the model

Notation	Description
P_a	atmospheric pressure
σ_c	resistance in simple compression, intervening in the expression of the criteria
H_0^{ext}	parameter controlling resistance in extension, intervening in the expression of the criteria
σ_{point1}	$\sigma_{\min}$ intersection enters the thresholds of peak and intermediary
x_{ams}	parameter not no one intervening in the laws of work hardening pre-peak
η	parameter not no one intervening in the laws of work hardening post-peak
a_0	value of a on the threshold of damage
m_0	value of m on the threshold of damage
s_0	value of S on the threshold of damage
a_{pic}	value of a on the threshold of peak
m_{pic}	value of m on the threshold of peak
ξ_{pic}	level of work hardening necessary to ξ_p to reach the threshold of peak
a_e	value of a on the threshold of cleavage
m_e	value of m on the threshold of cleavage
ξ_e	level of work hardening necessary to ξ_p to reach the threshold of cleavage
m_{ult}	value of m on the residual threshold
ξ_{ult}	level of work hardening necessary to ξ_p to reach the residual threshold
$m_{v-\max}$	value of m on the maximum viscoplastic threshold
$\xi_{v-\max}$	value of ξ_v for which the maximum viscoplastic criterion is reached
A_v	parameter characterizing the amplitude the speed of creep
n_v	exhibitor intervening in the formula controlling the kinetics of creep

$\mu_{0,v}$ parameter relating to dilatancy in pre-peak

$\xi_{0,v}$ parameter relating to dilatancy in pre-peak

μ_1 parameter relating to dilatancy in post peak

ξ_1 parameter relating to dilatancy in post-peak

2 Introduction

This document presents rheological model L&K developed with the CIH by F. Laigle and A. Kleine. It is a model élasto-visco-plastic dedicated to the rocks. The specificity of elastoplasticity resides in the modeling of a non-linear behavior in phase pre-peak and of a behavior post peak softening. Viscosity characterizes the effect of time on the behavior of the rock. The initiation of each one of these phenomena starts as of the crossing of a threshold. The behavior related to each phase is described by the evolution of these various thresholds governed by functions of plastic or viscoplastic work hardening.

3 Equations of model L&K

3.1 Simplification of the model

In order to describe as well as possible and in a concise way this version of the model, it is necessary to give an outline on the original model. The difference between the two versions will be perceived better.

3.1.1 Short outline on the thresholds of the original model

In the original version of model L&K such as it is developed under the software Splash with the CIH (cf. R 1) or the thesis of A. Kleine (cf. R 4), there exist three distinct mechanisms:

- An elastoplastic mechanism pre-peak, governed by a positive work hardening,
- A viscoplastic mechanism also governed by a positive work hardening,
- An elastoplastic mechanism post peak governed by a negative work hardening describing the fracturing.

The characteristic of this original model lies in the fact that the coupling of the two mechanisms pre peak thus starts the fracturing the mechanism post-peak. Indeed, the cracks of extension induce a degradation of the mechanical properties of materials with the increase in dilatancy.

For the elastoplastic mechanism, a surface of load evolves through various thresholds. For the viscous mechanism, a viscoplastic surface evolves of an initial threshold to a final threshold.

The various thresholds delimit fields associated with particular physical mechanisms:

- A threshold of damage confused with the threshold of initial viscosity,
- An intermediate threshold, qualified of limit of cleavage,
- A definite threshold characteristic as the envelope of the threshold of damage and limit of cleavage, called also limiting of contractance/dilatancy (this limit is confused with the threshold of maximum viscoplasticity),
- A macroscopic of peak, definite threshold starting from the laboratory tests,
- A definite purely conceptual intrinsic threshold like extrapolation of the threshold of peak, (this threshold is eliminated in the version simplified from the model),
- A threshold of residual resistance.

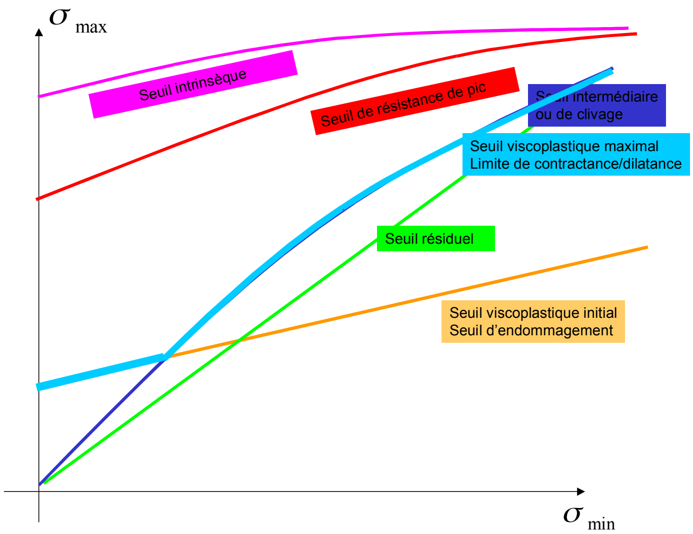


Figure 3.1.1-a. Thresholds of the original model presented in the plan $(\sigma_{min}, \sigma_{max})$

3.1.2 Characteristics of the simplified model

The simplified version proposed by the CIH (cf.R 2) rest only on two mechanisms: an elastoplastic mechanism and a viscoplastic mechanism.

- The intrinsic threshold is eliminated in this version.
- The threshold characteristic delimiting the fields of contractance and dilatancy in phase pre-peak, is linearized to avoid any digital problem. It is supposed to be confused with the threshold of maximum viscosity.

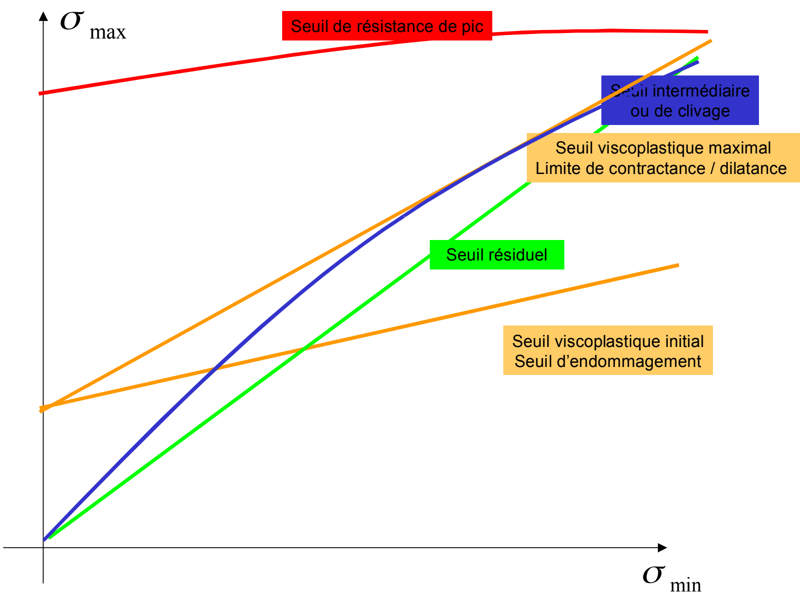


Figure 3.1.2-a. Thresholds of the model simplified in the plan $(\sigma_{min}, \sigma_{max})$

3.2 Description of the mechanisms

3.2.1 The viscoplastic mechanism

This mechanism is activated as soon as the point of load exceeds the initial viscoelastic threshold (compare to the initial elastic limit). Unrecoverable deformations are generated. These speeds of unrecoverable deformations are proportional to the distance from the point of load compared to the threshold of viscosity. Surface associated with the viscoplastic mechanism evolves of the initial elastic limit to the maximum viscoplastic threshold according to the generated unrecoverable deformations.

3.2.2 The elastoplastic mechanism

3.2.2.1 pre peak

The plastic mechanism élasto is activated at the same time as the viscoplastic mechanism. As soon as the point of load exceeds the initial elastic limit, the surface of load starts with écrouire positively.

3.2.2.2 post peak

In the version simplified model, this mechanism is governed by:
- a negative work hardening of the threshold of peak towards the intermediate threshold,
- a negative work hardening of the intermediate threshold towards the residual threshold.

3.2.3 The voluminal behavior

The voluminal behavior, during the phase pre-peak, can be contracting or dilating.
Below the initial elastic limit, the behavior is contracting.
Below the limit contractance/dilatancy, the voluminal behavior is contracting plastic.
Beyond this limit, the voluminal behavior is dilating.

N.B: In the simplified version of the model, the limit contractance/dilatancy is confused with the maximum viscoplastic threshold.

3.3 Decomposition of the tensor of deformation

The decomposition of the increment of total deflection is written:
$\underline{\dot{\varepsilon}}=\underline{\dot{\varepsilon}}^{e}+\underline{\dot{\varepsilon}}^{p}+\underline{\dot{\varepsilon}}^{vp}$
where $\underline{\dot{\varepsilon}}^{e}$, $\underline{\dot{\varepsilon}}^{p}$ and $\underline{\dot{\varepsilon}}^{vp}$ are the increments of the tensors elastic, irreversible instantaneous (plastic) and irreversible differed (viscoplastic).

3.3.1 Hypo-elasticity

The selected elastic law is a law hypo-rubber band:
$\dot{\varepsilon}_{ij}^{e}=\frac{1+\nu}{E}\dot{\sigma}_{ij}-\frac{3\nu}{E}\dot{p}\delta_{ij}$ or $\dot{\varepsilon}_{ij}^{e}=\frac{1}{2G}\dot{s}_{ij}+\frac{1}{3K}\dot{p}\delta_{ij}$
that one also notes: $\dot{\sigma}_{ij}=D^{e}\dot{\varepsilon}_{ij}^{e}$.

Moduli of rigidity G and of compressibility K depend on the state of stresses: $K=K_0^e\left[\frac{I_1^-}{3P_a}\right]^{n_{elas}}$
and $G=G_0^e\left[\frac{I_1^-}{3P_a}\right]^{n_{elas}}$ with $I_1^-=tr(\boldsymbol{\sigma}^-)$.

$I_1^-=tr(\boldsymbol{\sigma}^-)$ being the trace of the constraints at the moment $-$.

3.3.2 Plasticity

As in the simplified version of the model, only the mechanism deviatoric is taken into account, the instantaneous unrecoverable deformation is written:

$$\dot{\varepsilon}_{ij}^{p}=\dot{\lambda}\, G_{ij}$$

λ being the plastic multiplier and G is the function of flow.

That is to say f^{d} the criterion of plasticity:

If $f^{d}\le 0$ then $\lambda=0$
If $f^{d}=0$ then $\lambda>0$

The expression of G rest on several work quoted in the note R 2 and is form:

$$G_{ij}=\frac{\partial f^{d}}{\partial\sigma_{ij}}-\left(\frac{\partial f^{d}}{\partial\sigma_{kl}}n_{kl}\right)n_{ij}\ ,\quad n_{ij}=\frac{\beta'\frac{s_{ij}}{s_{II}}-\delta_{ij}}{\sqrt{\beta'^{2}+3}}\ ,\quad \beta'=-\frac{2\sqrt{6}\sin(\Psi)}{3-\sin(\Psi)}$$

Expressions of $\sin(\Psi)$ are detailed in the paragraph 3.6.1 and 3.6.2.
The calculation of $\dot{\lambda}$ fact the object of the paragraph 4.2.1.2.
The calculation of $\frac{\partial f}{\partial\sigma_{ij}}$ is detailed in the paragraph 3.7.1
The evolution of elastoplasticity induces a plastic deformation: ε_{p} connected through its deviatoric component $\tilde{\varepsilon}_{p}$ with the parameter of work hardening γ_{p} such as: $\gamma_{p}=\int\sqrt{\frac{2}{3}\dot{\tilde{\varepsilon}}_{p}\dot{\tilde{\varepsilon}}_{p}}dt$

from where the relation $\dot{\gamma}_{p}=\dot{\lambda}\sqrt{\frac{2}{3}\tilde{G}_{ij}\tilde{G}_{ij}}=\dot{\lambda}\sqrt{\frac{2}{3}}G_{II}$

3.3.3 Viscoplasticity

The calculation of the differed unrecoverable deformations $\dot{\underline{\varepsilon}}^{vp}$ be based on the theory of Perzyna.
$\dot{\underline{\underline{\varepsilon}}}^{vp}=\langle\Phi(f^{vp})\rangle G_{ij}^{visc}$ where $\Phi(f^{vp})$ and G^{visc} the amplitude and the direction the speed of the unrecoverable deformations characterize:

$$\Phi(f^{vp})=A_{v}\left(\frac{f^{vp}}{Pa}\right)^{n_{v}}\quad\text{and}\quad G_{ij}^{visc}=\frac{\partial f^{vp}}{\partial\sigma_{ij}}-\left(\frac{\partial f^{vp}}{\partial\sigma_{kl}}n_{kl}\right)n_{ij}$$

f^{vp} being the criterion of viscoplasticity, A_{v} and n_{v} are parameters of the model. P_{a} is the atmospheric pressure.

The evolution of viscoplasticity induces a viscous deformation: connected through its deviatoric component $\tilde{\varepsilon}_{vp}$ with the parameter of work hardening γ_{vp} such as: $\gamma_{vp}=\int\sqrt{\frac{2}{3}\dot{\tilde{\varepsilon}}_{vp}\dot{\tilde{\varepsilon}}_{vp}}dt$.

Note:
That is to say S^{vp} the surface defined within the space of constraints by: $S^{vp}=\{\sigma, f^{vp}(\sigma,\xi^{vp})=0\}$
The speed of creep for a state of stresses σ is proportional to the distance from σ with S^{vp} . That is to say P_{σ}^{vp} the projection of σ on S^{vp} and $d=\|\sigma-P_{\sigma}^{vp}\|$.

One can also write $d=\|\frac{\partial f^{vp}}{\partial\sigma}\left(P_\sigma^{vp}\right)\|C$. C being a constant which depends on the viscous parameters. At first approximation, one writes: $d=\|\frac{\partial f^{vp}}{\partial\sigma}(\sigma)\|C$. But this approximation poses a problem insofar as the function f^{vp} can not be defined for the value σ whereas it is for P_σ^{vp} . For the moment, in *Code_Aster* this distance is not calculated. If the situation arises, a message of alarm warns the user.

3.4 Expressions of the criteria

Expressions of the two criteria viscoplastic f^{vp} and elastoplastic f^d depend on the constraints and functions on work hardening. In these expressions, one finds I_1 the first invariant of the constraints and s_{II} the second invariant of the tensor of the constraints déviatoires. In the two criteria, the same definitions are adopted for:

$$H(\theta)=\frac{H_0^c+H_0^e}{2}+\left(\frac{H_0^c-H_0^e}{2}\right)\left(\frac{2\mathrm{h}(\theta)-\left(h_0^c+h_0^e\right)}{h_0^c-h_0^e}\right)$$

$$h(\theta)=(1-\gamma\cos 3\theta)^{\frac{1}{6}},\quad h_0^c=H_0^c=h(0^\circ)=(1-\gamma)^{\frac{1}{6}},\quad h_0^e=h(60^\circ)=(1+\gamma)^{\frac{1}{6}},$$

H_0^e is a parameter of the model. θ is the angle of Lode.

3.4.1 The viscoplastic criterion f^{vp}

$$f^{vp}(\sigma)=s_{II}H(\theta)-\sigma_c H_0^c\left[A^{vp}(\xi_{vp})s_{II}H(\theta)+B^{vp}(\xi_{vp})I_1+D^{vp}(\xi_{vp})\right]^{a^{vp}(\xi_{vp})}$$

with $A^{vp}(\xi_{vp})=-\frac{m^{vp}(\xi_{vp})k^{vp}(\xi_{vp})}{\sqrt{6}\sigma_c h_c^0}$, $B^{vp}(\xi_{vp})=\frac{m^{vp}(\xi_{vp})k^{vp}(\xi_{vp})}{3\sigma_c}$, $D^{vp}(\xi_{vp})=s^{vp}(\xi_{vp})k^{vp}(\xi_{vp})$,

$$k^{vp}(\xi_{vp})=\left(\frac{2}{3}\right)^{\frac{1}{2a^{vp}(\xi_{vp})}}$$

Functions of work hardening $A^{vp}(\xi_{vp})$, $B^{vp}(\xi_{vp})$ and $D^{vp}(\xi_{vp})$ depend on the parameters of work hardening $a^{vp}(\xi_{vp})$, $m^{vp}(\xi_{vp})$ and $s^{vp}(\xi_{vp})$ whose expressions evolve with the variables of work hardening ξ_{vp} (see § 3.5). When ξ_{vp} reached certain particular values, surface f^{vp} reached the corresponding thresholds.

Since the viscoplastic threshold is hammer-hardened only by viscosity, one always has $\dot{\xi}_{vp}=Min\left[\dot{\gamma}_{vp},\xi_{v-\max}-\xi_{vp}\right]$. $\xi_{vp-\max}$ corresponds to the maximum viscoplastic criterion and is a parameter of the model

3.4.2 The elastoplastic criterion f^d

$$f^d(\sigma)=s_{II}H(\theta)-\sigma_c H_0^c\left[A^d(\xi_p)s_{II}H(\theta)+B^d(\xi_p)I_1+D^d(\xi_p)\right]^{a^d(\xi_p)}$$

with $A^d(\xi_p)=-\frac{m^d(\xi_p)k^d(\xi_p)}{\sqrt{6}\sigma_c h_c^0}$, $B^d(\xi_p)=\frac{m^d(\xi_p)k^d(\xi_p)}{3\sigma_c}$, $D^d(\xi_p)=s^d(\xi_p)k^d(\xi_p)$,

$$k^d(\xi_p)=\left(\frac{2}{3}\right)^{\frac{1}{2a^d(\xi_p)}}$$

Functions of work hardening $A^d(\xi_p)$, $B^d(\xi_p)$ and $D^d(\xi_p)$ depend on the parameters of work hardening $a^d(\xi_p)$, $m^d(\xi_p)$ and $s^d(\xi_p)$ whose expressions evolve with the variables of work hardening ξ_p (see § 3.5). When ξ_p reached certain particular values, surface f^d reached the corresponding thresholds.

Elastoplastic work hardening depends on the position of the point of load compared to the limit contractance/dilatancy:

- if the point of load is below this limit, $\dot{\xi}_p=\dot{\gamma}_p$,
- if the point of load is with the top of this limit, $\dot{\xi}_p=\dot{\gamma}_p+\dot{\gamma}_{vp}$.

3.5 Functions of work hardening

3.5.1 Functions of work hardening of the viscous criterion

The viscoplastic criterion is governed by the following functions of work hardening:

$a(\xi_{vp})=a_0+(a_{v-\max}-a_0)\frac{\xi_{vp}}{\xi_{v-\max}}$ with $a_{v-\max}=1$.

$$m(\xi_{vp})=m_0+(m_{v-\max}-m_0)\frac{\xi_{vp}}{\xi_{v-\max}}$$

$s(\xi_{vp})=s_0+(s_{v-\max}-s_0)\frac{\xi_{vp}}{\xi_{v-\max}}$ with $s_{v-\max}=s_0$

3.5.2 Functions of work hardening of the elastoplastic criterion and their derivative

The expressions of the functions of work hardening which govern the elastoplastic criterion vary according to the value of the parameters ξ_p :

Evolution enters the threshold of damage and the threshold of peak : If $0\le\xi_p<\xi_{pic}$

$$a(\xi_p)=a_0+\ln\left(1+\frac{\xi_p}{x_{ams}\xi_{pic}}\right)\left(\frac{a_{pic}-a_0}{\ln(1+1/x_{ams})}\right) \qquad \frac{\partial a}{\partial\xi_p}=\left(\frac{a_{pic}-a_0}{\ln(1+1/x_{ams})}\right)\left(\frac{1}{\xi_p+x_{ams}\xi_{pic}}\right)$$

$$m(\xi_p)=m_0+\ln\left(1+\frac{\xi_p}{x_{ams}\xi_{pic}}\right)\left(\frac{m_{pic}-m_0}{\ln(1+1/x_{ams})}\right) \qquad \frac{\partial m}{\partial\xi_p}=\left(\frac{m_{pic}-m_0}{\ln(1+1/x_{ams})}\right)\left(\frac{1}{\xi_p+x_{ams}\xi_{pic}}\right)$$

$$s(\xi_p)=s_0+\ln\left(1+\frac{\xi_p}{x_{ams}\xi_{pic}}\right)\left(\frac{s_{pic}-s_0}{\ln(1+1/x_{ams})}\right)$$

$$\frac{\partial s}{\partial\xi_p}=\left(\frac{s_{pic}-s_0}{\ln(1+1/x_{ams})}\right)\left(\frac{1}{\xi_p+x_{ams}\xi_{pic}}\right)$$

with $s_{pic}=1$.

Evolution enters the threshold of peak and the intermediate threshold or limit of cleavage: If $\xi_{pic} \le \xi_p < \xi_e$

$$a(\xi_p) = a_{pic} + (a_e - a_{pic}) \left(\frac{\xi_p - \xi_{pic}}{\xi_e - \xi_{pic}} \right) \qquad \frac{\partial a}{\partial \xi_p} = \frac{a_e - a_{pic}}{\xi_e - \xi_{pic}}$$

$$s(\xi_p) = 1 - \left(\frac{\xi_p - \xi_{pic}}{\xi_e - \xi_{pic}} \right) \qquad \frac{\partial s}{\partial \xi_p} = \frac{-1}{\xi_e - \xi_{pic}}$$

$$\frac{\partial m}{\partial \xi_p} = \frac{\partial m}{\partial a}\frac{\partial a}{\partial \xi_p} + \frac{\partial m}{\partial s}\frac{\partial s}{\partial \xi_p}$$

$$\frac{\partial m}{\partial \xi_p} = \frac{\sigma_c}{\sigma_{po\text{int}1}} \left[\left(-\frac{a_{pic}}{a(\xi_p)^2} \right) \left(m_{pic} \frac{\sigma_{po\text{int}1}}{\sigma_c} + s_{pic} \right)^{\frac{a_{pic}}{a(\xi_p)}} \ln \left(m_{pic} \frac{\sigma_{po\text{int}1}}{\sigma_c} + s_{pic} \right) \frac{\partial a}{\partial \xi_p} - \frac{\partial s}{\partial \xi_p} \right]$$

Evolution enters the intermediate threshold and the residual threshold : If $\xi_e \le \xi_p < \xi_{ult}$

$$a(\xi_p) = a_e + \ln \left(1 + \frac{1}{\eta} \frac{\xi_p - \xi_e}{\xi_{ult} - \xi_e} \right) \left(\frac{a_{ult} - a_e}{\ln(1 + 1/\eta)} \right) \qquad \frac{\partial a}{\partial \xi_p} = \left(\frac{a_{ult} - a_e}{\ln(1 + 1/\eta)} \right) \left(\frac{1}{\xi_p + \eta \xi_{ult} - (1+\eta) \xi_e} \right)$$

$$s(\xi_p) = 0 \qquad \frac{\partial s}{\partial \xi^p} = 0$$

$$m(\xi_p) = \frac{\sigma_c}{\sigma_{po\text{int}2}} \left(m_e \frac{\sigma_{po\text{int}2}}{\sigma_c} \right)^{\frac{a_e}{a(\xi_p)}}$$

$$\frac{\partial m}{\partial \xi_p} = \frac{\sigma_c}{\sigma_{po\text{int}2}} \left[\left(-\frac{a_e}{a(\xi_p)^2} \right) \ln \left(m_e \frac{\sigma_{po\text{int}2}}{\sigma_c} \right) \left(m_e \frac{\sigma_{po\text{int}2}}{\sigma_c} \right)^{\frac{a_e}{a(\xi_p)}} \right] \frac{\partial a}{\partial \xi_p}$$

On the residual criterion : If $\xi_p \ge \xi_{ult}$

$$a(\xi_p) = a_{ult} = 1. \qquad \frac{\partial a}{\partial \xi_p} = 0$$

$$s(\xi_p) = 0 \qquad \frac{\partial s}{\partial \xi_p} = 0$$

$$m(\xi_p) = m_{ult} \qquad \frac{\partial m}{\partial \xi_p} = 0$$

3.6 Laws of dilatancy

The elastoplastic and viscoplastic mechanisms are non-aligned. Laws of evolution of $\dot{\varepsilon}_{ij}^{p}$ and of $\dot{\varepsilon}_{ij}^{vp}$ are controls respectively by a function G and a function G^{visc} , such as:

$G_{ij} = \frac{\partial f^d}{\partial \sigma_{ij}} - \left(\frac{\partial f^d}{\partial \sigma_{kl}} n_{kl} \right) n_{ij}$ and $G_{ij}^{visc} = \frac{\partial f^{vp}}{\partial \sigma_{ij}} - \left(\frac{\partial f^{vp}}{\partial \sigma_{kl}} n_{kl} \right) n_{ij}$ with

$$n_{ij}=\frac{\beta'\frac{s_{ij}}{s_{II}}-\delta_{ij}}{\sqrt{\beta'^2+3}} \quad \text{and} \quad \beta'=-\frac{2\sqrt{6}\sin(\Psi)}{3-\sin(\Psi)}$$

The calculation of the angle of dilatancy Ψ differ according to the viscous mechanisms or elastoplastic pre-peak and elastoplastic post-peak.

3.6.1 Elastoplastic pre-peak and viscoplastic angle of dilatancy of the mechanisms

$\sin(\Psi)=\mu_{0,v}\left(\frac{\sigma_{\max}-\sigma_{\lim}}{\xi_{0,v}\sigma_{\max}+\sigma_{\lim}}\right)$ with $\mu_{0,v}$ and $\xi_{0,v}$ parameters of the model.

where

$\sigma_{\lim}=\sigma_{\min}+\sigma_c\left(m_{v-\max}\frac{\sigma_{\min}}{\sigma_c}+s_{v-\max}\right)^{a_{v-\max}}$ with $s_{v-\max}=s_0$ and $a_{v-\max}=1$. σ_c and $m_{v-\max}$ are parameters of the model.

There exist conditions on the parameters $\mu_{0,v}$ and $\xi_{0,v}$ who are:

- $\mu_{0,v}<\xi_{0,v}$ or
- $\begin{cases}\mu_{0,v}>\xi_{0,v}\\ \frac{(s_{\text{pic}})^{a_{\text{pic}}}}{(s_0)^{a_0}}\leq\frac{1+\mu_{0,v}}{\mu_{0,v}-\xi_{0,v}}\end{cases}$

3.6.2 Angle of dilatancy of the elastoplastic mechanism post-peak

$\sin(\Psi)=\mu_1\left(\frac{\alpha-\alpha_{res}}{\xi_1\alpha+\alpha_{res}}\right)$ with μ_1 and ξ_1 parameters of the model

where

$$\alpha=\frac{\sigma_{\max}+\tilde{\sigma}}{\sigma_{\min}+\tilde{\sigma}} \quad \text{and} \quad \alpha_{res}=\frac{\sigma_{\max}}{\sigma_{\min}}=1+m_{ult}$$

$$\tilde{\sigma}=\begin{cases}\frac{\tilde{c}(\xi_p)}{\tan(\tilde{\varphi}(\xi_p))} & si \quad \xi_p\leq\xi_e\\ 0 & si \quad \xi_p>\xi_e\end{cases}$$

with $\tilde{c}(\xi_p)=\frac{\sigma_c\,.\,s(\xi_p)^{a(\xi_p)}}{2\sqrt{1+a(\xi_p)m(\xi_p)s(\xi_p)^{a(\xi_p)-1}}}$ and

$$\tilde{\varphi}(\xi_p)=2.\,arctg\left(\sqrt{1+a(\xi_p)m(\xi_p)s(\xi_p)^{a(\xi_p)-1}}\right)-\frac{\pi}{2}$$

$\sigma_{\min}$ and $\sigma_{\max}$ are calculated starting from the invariants of the constraints:

$$\sigma_{\min}=\frac{1}{3}\left(I_1-\left(\frac{3}{2}-\frac{2\mathrm{H}(\theta)-\left(H_0^c+H_0^e\right)}{2\left(H_0^c-H_0^e\right)}\right)\sqrt{\frac{3}{2}}s_{II}\right)$$

$$\sigma_{\max}=\frac{1}{3}\left(I_1+\left(\frac{3}{2}+\frac{2\mathrm{H}(\theta)-\left(H_0^c+H_0^e\right)}{2\left(H_0^c-H_0^e\right)}\right)\sqrt{\frac{3}{2}}s_{II}\right)$$

3.7 Derived from the criterion

3.7.1 Calculation of $\frac{\partial f}{\partial\sigma_{ij}}$

$$\frac{\partial I_1}{\partial\sigma_{ij}}=\frac{\partial tr\left(\sigma_{ij}\right)}{\partial\sigma_{ij}}=\delta_{ij}$$

$$\frac{\partial\left(s_{II}H(\theta)\right)}{\partial\sigma_{ij}}=\frac{\partial\left(s_{II}H(\theta)\right)}{\partial s_{kl}}\frac{\partial s_{kl}}{\partial\sigma_{ij}}=\left(\frac{\partial H(\theta)}{\partial s_{kl}}s_{II}+H(\theta)\frac{\partial s_{II}}{\partial s_{kl}}\right)\frac{\partial s_{kl}}{\partial\sigma_{ij}}$$

$$\frac{\partial s_{II}}{\partial s_{kl}}=\frac{s_{kl}}{s_{II}}\ ;\ s_{II}=\sqrt{s_{kl}.s_{kl}}$$

$$\frac{\partial s_{kl}}{\partial\sigma_{ij}}=\frac{\partial\left(\sigma_{kl}-\frac{1}{3}tr(\sigma)\delta_{kl}\right)}{\partial\sigma_{ij}}=\delta_{ik}.\delta_{jl}-\frac{1}{3}\delta_{ij}.\delta_{kl}$$

Note: $s_{kl}.\left(\delta_{ik}.\delta_{jl}-\frac{1}{3}\delta_{ij}.\delta_{kl}\right)=s_{ij}$

$$\frac{\partial H(\theta)}{\partial s_{kl}}=\left(\frac{H_0^c-H_0^e}{h_0^c-h_0^e}\right)\frac{\partial h(\theta)}{\partial s_{kl}}$$

from where $\frac{\partial\left(s_{II}H(\theta)\right)}{\partial\sigma_{ij}}=\left(\left(\frac{H_0^c-H_0^e}{h_0^c-h_0^e}\right)\frac{\partial h(\theta)}{\partial s_{kl}}s_{II}+H(\theta)\frac{s_{kl}}{s_{II}}\right).\left(\delta_{ik}.\delta_{jl}-\frac{1}{3}\delta_{ij}.\delta_{kl}\right)$

There is the relation: $\cos(3\theta)=\sqrt{54}\frac{\det(\boldsymbol{s})}{s_{II}^3}$ (see Documentation R7.01.13-A: Law CJS in mechanics)

$$\frac{\partial h(\theta)}{\partial s_{kl}}=\frac{1}{6}\left(1-\gamma\cos(3\theta)\right)^{-\frac{5}{6}}\frac{\partial\left(1-\gamma\cos(3\theta)\right)}{\partial s_{kl}}=\frac{1}{6\mathrm{h}(\theta)^5}\frac{\partial}{\partial s_{kl}}\left(\frac{s_{II}^3-\gamma\sqrt{54}\det(s)}{s_{II}^3}\right)$$

$$\frac{\partial h(\theta)}{\partial s_{kl}}=\frac{1}{6\mathrm{h}(\theta)^5}\left\{\left[\frac{\partial s_{II}^3}{\partial s_{kl}}-\gamma\sqrt{54}\left(\frac{\partial \det(\underline{\underline{s}})}{\partial s_{kl}}\right)\right]\frac{s_{II}^3}{s_{II}^6}-\left(s_{II}^3-\gamma\sqrt{54}\det(\underline{\underline{s}})\right)\frac{3\mathrm{s}_{kl}s_{II}}{s_{II}^6}\right\}$$

$$=\frac{1}{6\mathrm{h}(\theta)^5}\left\{\frac{3\mathrm{s}_{kl}}{s_{II}^2}-\gamma\sqrt{54}\left(\frac{\partial \det(\underline{\underline{s}})}{\partial s_{kl}}\right)\frac{1}{s_{II}^3}-(1-\gamma\cos(3\theta))\frac{3\mathrm{s}_{kl}}{s_{II}^2}\right\}$$

$$=\frac{\gamma\cos(3\theta)}{6\mathrm{h}(\theta)^5}\frac{3\mathrm{s}_{kl}}{s_{II}^2}-\frac{\gamma\sqrt{54}}{6\mathrm{h}(\theta)^5 s_{II}^3}\left(\frac{\partial \det(\underline{\underline{s}})}{\partial s_{kl}}\right)$$

One thus finds:

$$\frac{\partial\left(s_{II}H(\theta)\right)}{\partial\sigma_{ij}}=$$

$$\left(\left(\frac{H_0^c-H_0^e}{h_0^c-h_0^e}\right)\left(\frac{\gamma\cos(3\theta)}{6\mathrm{h}(\theta)^5}\frac{3\mathrm{s}_{kl}}{s_{II}^2}-\frac{\gamma\sqrt{54}}{6\mathrm{h}(\theta)^5 s_{II}^3}\left(\frac{\partial \det(\underline{\underline{s}})}{\partial s_{kl}}\right)\right)s_{II}+H(\theta)\frac{s_{kl}}{s_{II}}\right).\left(\delta_{ik}.\delta_{jl}-\frac{1}{3}\delta_{ij}.\delta_{kl}\right)$$

Finally:

For the elastoplastic criterion :

$$\frac{\partial f^d}{\partial\sigma_{ij}}=$$

$$\frac{\partial\left(s_{II}H(\theta)\right)}{\partial\sigma_{ij}}-a^d(\xi_p)\sigma_c H_0^c\left[A^d(\xi_p)s_{II}H(\theta)+B^d(\xi_p)I_1+D^d(\xi_p)\right]^{a^d(\xi_p)-1}$$

$$\left(A^d(\xi_p)\frac{\partial\left(s_{II}H(\theta)\right)}{\partial\sigma_{ij}}+B^d(\xi_p)I_d\right)$$

and for the viscous criterion:

$$\frac{\partial f^{vp}}{\partial\sigma_{ij}}=$$

$$\frac{\partial\left(s_{II}H(\theta)\right)}{\partial\sigma_{ij}}-a^{vp}(\xi_{vp})\sigma_c H_0^c\left[A^{vp}(\xi_{vp})s_{II}H(\theta)+B^{vp}(\xi_{vp})I_1+D^{vp}(\xi_{vp})\right]^{a^{vp}(\xi_{pp})-1}$$

$$\left(A^{vp}(\xi_{vp})\frac{\partial\left(s_{II}H(\theta)\right)}{\partial\sigma_{ij}}+B^{vp}(\xi_{vp})I_d\right)$$

with $\frac{\partial\left(s_{II}H(\theta)\right)}{\partial\sigma_{ij}}=$

$$\left(\left(\frac{H_0^c-H_0^e}{h_0^c-h_0^e}\right)\left(\frac{\gamma\cos(3\theta)}{6\mathrm{h}(\theta)^5}\frac{3\mathrm{s}_{kl}}{s_{II}^2}-\frac{\gamma\sqrt{54}}{6\mathrm{h}(\theta)^5 s_{II}^3}\left(\frac{\partial \det(\underline{\underline{s}})}{\partial s_{kl}}\right)\right)s_{II}+H(\theta)\frac{s_{kl}}{s_{II}}\right).\left(\delta_{ik}.\delta_{jl}-\frac{1}{3}\delta_{ij}.\delta_{kl}\right)$$

3.7.2 Calculation of $\frac{\partial f^d}{\partial \xi_p}$

Expression of the threshold in constraints:

$$f^d(\sigma)=s_{II}H(\theta)-\sigma_c H_0^c\left[A^d(\xi_p)s_{II}H(\theta)+B^d(\xi_p)I_1+D^d(\xi_p)\right]^{a^d(\xi_p)}$$

with $A^d(\xi^p)=-\frac{m^d(\xi_p)k^d(\xi_p)}{\sqrt{6}\sigma_c h_c^0}$, $B^d(\xi_p)=\frac{m^d(\xi_p)k^d(\xi_p)}{3\sigma_c}$, $D^d(\xi_p)=s^d(\xi_p)k(\xi_p)$,

$$k^d(\xi_p)=\left(\frac{2}{3}\right)^{\frac{1}{2a^d(\xi_p)}}$$

$$\frac{\partial f^d}{\partial \xi_p}=\frac{\partial f^d}{\partial a^d}.\dot{a}^d(\xi_p)+\frac{\partial f}{\partial m^d}.\dot{m}^d(\xi_p)+\frac{\partial f}{\partial s^d}.\dot{s}^d(\xi_p)$$

$$\frac{\partial f^d}{\partial s^d}=-a^d k^d \sigma_c H_0^c\left[A^d s_{II}H(\theta)+B^d I_1+D^d\right]^{a^d-1}$$

$$\frac{\partial f^d}{\partial m^d}=-a^d\sigma_c H_0^c\left[\frac{A^d}{m^d}s_{II}H(\theta)+\frac{B^d}{m^d}I_1\right]\left[A^d s_{II}H(\theta)+B^d I_1+D^d\right]^{a^d-1}$$

$$\frac{\partial f^d}{\partial a^d}=$$

$$\sigma_c H_0^c\dot{a}^d\left[A^d s_{II}H(\theta)+B^d I_1+D^d\right]^{a^d}.\left[\ln\left[A^d s_{II}H(\theta)+B^d I_1+D^d\right]-\frac{\frac{s^d}{2a^d}\ln\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)^{\left(\frac{1}{2a}\right)}}{\left[A^d s_{II}H(\theta)+B^d I_1+D^d\right]}\right]$$

4 Integration in Code_Aster

The integration of the model `LETK` can be realized according to two distinct diagrams of integration. The first diagram of integration ("historical") is described like `SPECIFIC` and corresponds to a diagram of explicit integration. The second diagram is built on the basis of diagram of implicit integration. It is accessible under the keyword `NEWTON_PERT` .

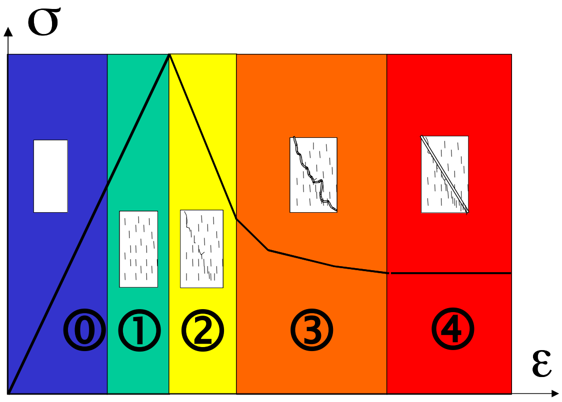
4.1 Internal variables

V_1 : elastoplastic variable of work hardening ξ_p
V_2 : plastic deviatoric deformation γ_p
V_3 : viscoplastic variable of work hardening ξ_{vp}
V_4 : viscoplastic deviatoric deformation γ_{vp}
V_5 : 0 if contractance, 1 if dilatancy
V_6 : indicator of viscoplasticity
V_7 : indicator of plasticity
V_8 : Fields of behavior of the rock in plasticity

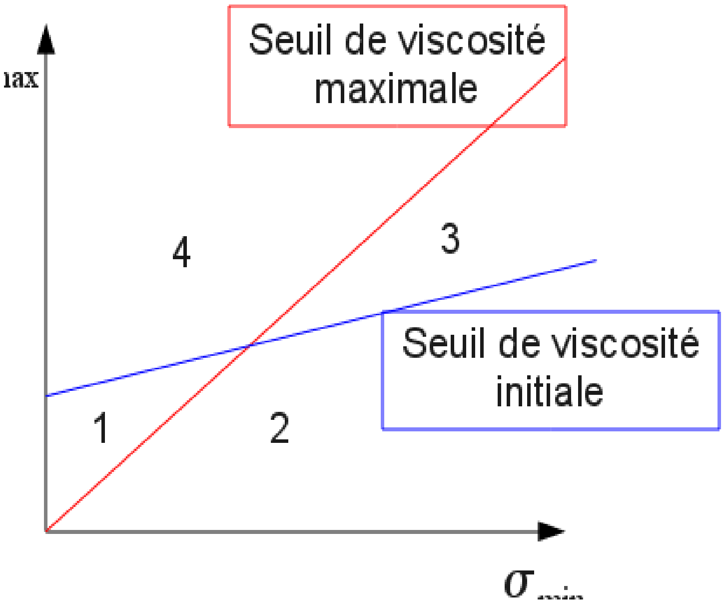
Five fields of behavior, numbered from 0 to 4 (cf appears), are identified to make it possible to have a relatively simple representation of the state of damage of the rock, since the intact rock to the rock in a residual state. These fields are function of the cumulated plastic deformation déviatoire γ^p and of the state of stress. Each increment of number of field defines the passage in a field of higher damage.

- If the diverter is lower than 70% of the diverter of peak, then the material is in field 0;
- If not:
  - If $\gamma^p=0$ then the material is in field 1;
    1) If $0<\gamma^p<\gamma^e$ then the material is in field 2;
  - If $\gamma_e<\gamma^p<\gamma_{\mathrm{ult}}$ then the material is in field 3;
  - If $\gamma^p>\gamma_{\mathrm{ult}}$ then the material is in field 4.

Domaine	Etat de la roche
0	Intacte
1	Endommagement pré-pic
2	Endommagement post-pic
3	Fissurée
4	Fracturée



V_9 : position of the state of stresses compared to the thresholds of viscosity.



Four fields of behavior, numbered from 1 to 4 (cf appears), are identified to make it possible to have a simple representation of the viscous behavior of material, since the intact rock to the rock in a residual state. These fields are function of the cumulated viscoplastic deformation déviatoire γ^{vp} and of the state of stress.

4.2 Diagram of integration clarifies (`SPECIFIC`)

4.2.1 Update of the constraints

One expresses the constraints brought up to date at the moment + compared to those calculated at the moment -:

$$\sigma=\sigma^{-}+D^{e}\Delta\varepsilon^{e} \;\; ; \;\; s=s^{-}+2G\Delta\tilde{\varepsilon}^{e} \;\; ; \;\; I_1=I_1^{-}+3K\Delta\varepsilon_v^{e}$$

$$\sigma_{ij}=s_{ij}+\frac{I_1}{3}\delta_{ij} \;\; ; \;\; \Delta\varepsilon_{ij}=\Delta\tilde{\varepsilon}_{ij}+\frac{tr(\Delta\varepsilon)}{3}\delta_{ij}=\Delta\tilde{\varepsilon}_{ij}+\frac{\Delta\varepsilon_v}{3}\delta_{ij} \;\; ; \;\; I_1=tr(\sigma) \;\; ; \;\; \varepsilon_v=tr(\Delta\varepsilon)$$

Elastic prediction:

$$\sigma^{e}=\sigma^{-}+D^{e}\Delta\varepsilon \;\; ; \;\; s^{e}=s^{-}+2G\Delta\tilde{\varepsilon} \;\; ; \;\; I_1^{e}=I_1^{-}+3K\Delta\varepsilon_v$$

$$K=K_0^{e}\left[\frac{I_1^{-}}{3P_a}\right]^{n_{elas}} \text{ and } G=G_0^{e}\left[\frac{I_1^{-}}{3P_a}\right]^{n_{elas}}$$

Note: The coefficient of compressibility K and it modulus of rigidity G are considered at the moment -.

4.2.1.1 Hypoelasticity

$$\Delta\sigma_{ij}=\Delta s_{ij}+\frac{\Delta I_1}{3}\delta_{ij} \qquad \Delta\varepsilon_{ij}=\Delta\tilde{\varepsilon}_{ij}+\frac{\Delta\varepsilon_v}{3}\delta_{ij} \qquad \Delta\sigma_{ij}=2G\Delta\varepsilon_{ij}+\left(K-\frac{2G}{3}\right)tr(\Delta\varepsilon)\delta_{ij}$$

$$\begin{Bmatrix} \Delta\sigma_{11} \\ \Delta\sigma_{22} \\ \Delta\sigma_{33} \\ \sqrt{2}\Delta\sigma_{12} \\ \sqrt{2}\Delta\sigma_{13} \\ \sqrt{2}\Delta\sigma_{23} \end{Bmatrix} = \underbrace{\begin{bmatrix} \frac{4G}{3}+K & K-\frac{2G}{3} & K-\frac{2G}{3} & 0 & 0 & 0 \\ K-\frac{2G}{3} & \frac{4G}{3}+K & K-\frac{2G}{3} & 0 & 0 & 0 \\ K-\frac{2G}{3} & K-\frac{2G}{3} & \frac{4G}{3}+K & 0 & 0 & 0 \\ 0 & 0 & 0 & 2G & 0 & 0 \\ 0 & 0 & 0 & 0 & 2G & 0 \\ 0 & 0 & 0 & 0 & 0 & 2G \end{bmatrix}}_{D^e} . \begin{Bmatrix} \Delta\varepsilon_{11} \\ \Delta\varepsilon_{22} \\ \Delta\varepsilon_{33} \\ \sqrt{2}\Delta\varepsilon_{12} \\ \sqrt{2}\Delta\varepsilon_{13} \\ \sqrt{2}\Delta\varepsilon_{23} \end{Bmatrix}$$

4.2.1.2 Plasticity and viscoplasticity

One expresses the stress field at the moment $+$:

$$\sigma_{ij}=\sigma_{ij}^{-}+D_{ijkl}^{e}\Delta\varepsilon_{kl}^{e}=\sigma_{ij}^{-}+D_{ijkl}^{e}\left(\Delta\varepsilon_{kl}-\Delta\varepsilon_{kl}^{p}-\Delta\varepsilon_{kl}^{vp}\right)$$

who is written by replacing the increase in the plastic deformations and viscous by their expressions in the form:

$$\sigma_{ij}=\sigma_{ij}^{-}+D_{ijkl}^{e}\left(\Delta\varepsilon_{kl}-\Delta\lambda G_{kl}(\sigma^{-},\xi_{p}^{-})-\langle\varphi\rangle G_{kl}^{visc}\left(\sigma^{-},\xi_{v}^{-}\right)\Delta t\right)$$

The principal unknown factor is the plastic multiplier $\Delta\lambda$.

One seeks $\Delta\lambda / f^{d}\left(\sigma,\xi_{p}\right)=0$

$$f^{d}\left(\sigma_{ij}^{-}+D_{ijkl}^{e}\left(\Delta\varepsilon_{kl}-\Delta\lambda G_{kl}(\sigma^{-},\xi_{p}^{-})-\langle\varphi\rangle G_{kl}^{visc}\left(\sigma^{-},\xi_{v}^{-}\right)\Delta t\right),\xi_{p}^{-}+\Delta\xi_{p}\right)=0$$

with $\Delta\gamma_{p}=\Delta\lambda\sqrt{\frac{2}{3}\tilde{G}_{ij}\tilde{G}_{ij}}=\Delta\lambda\sqrt{\frac{2}{3}}G_{II}$

One chooses to make an explicit resolution with a development of Euler:

$$f^{d}\left(\sigma_{ij}^{-}+D_{ijkl}^{e}\Delta\varepsilon_{kl}-D_{ijkl}^{e}\langle\Phi\rangle G_{kl}^{visc}\left(\sigma^{-},\xi_{v}^{-}\right)\Delta t,\xi_{p}^{-}\right)-\frac{\partial f^{d}}{\partial\sigma_{ij}}D_{ijkl}^{e}G_{kl}\left(\sigma^{-},\xi_{p}^{-}\right)\Delta\lambda+\frac{\partial f^{d}}{\partial\xi_{p}}\Delta\xi_{p}=0$$

The two cases are distinguished:

- $\Delta\xi_{p}=\Delta\gamma_{p}+\Delta\gamma_{vp}$ (dilating case: the state of stresses exceeds the limit contractance/dilatancy)

$$f^{d}\left(\sigma_{ij}^{-}+D_{ijkl}^{e}\Delta\varepsilon_{kl}-D_{ijkl}^{e}\langle\Phi\rangle G_{kl}^{visc}\left(\sigma^{-},\xi_{v}^{-}\right)\Delta t,\xi_{p}^{-}\right) -\frac{\partial f^{d}}{\partial\sigma_{ij}}D_{ijkl}^{e}G_{kl}\left(\sigma^{-},\xi_{p}^{-}\right)\Delta\lambda+\frac{\partial f^{d}}{\partial\xi_{p}}\left(\Delta\gamma_{p}+\Delta\gamma_{vp}\right)=0$$

$$f^{d}\left(\sigma_{ij}^{-}+D_{ijkl}^{e}\Delta\epsilon_{kl}-D_{ijkl}^{e}<\Phi>G_{kl}^{visc}\left(\sigma^{-},\xi_{v}^{-}\right)\Delta t,\xi_{p}^{-}\right)=$$
$$\left(\frac{\partial f^{d}}{\partial\sigma_{ij}}D_{ijkl}^{e}G_{kl}(\sigma^{-},\xi_{p}^{-})-\frac{\partial f^{d}}{\partial\xi_{p}}\sqrt{\frac{2}{3}}G_{II}\left(\sigma^{-},\xi_{p}^{-}\right)\Delta\lambda-\frac{\partial f^{d}}{\partial\xi_{p}}\Delta\gamma_{vp}\right)$$

$$\Delta\lambda=\frac{f^d\left(\sigma_{ij}^-,\xi_p^-\right)+\frac{\partial f^d}{\partial\xi_p}\Delta\gamma_{vp}+\frac{\partial f^d}{\partial\sigma_{ij}}\left[D_{ijkl}^e\Delta\varepsilon_{kl}-D_{ijkl}^e\langle\Phi\rangle G_{kl}^{visc}\left(\sigma^-,\xi_{vp}^-\right)\Delta t\right]}{\left(\frac{\partial f^d}{\partial\sigma_{ij}}D_{ijkl}^e G_{kl}\left(\sigma^-,\xi_p^-\right)-\frac{\partial f^d}{\partial\xi_p}\sqrt{\frac{2}{3}}\tilde{G}_{II}\left(\sigma^-,\xi_p^-\right)\right)}$$

- $\Delta\xi_p=\Delta\gamma_p$ (case contracting: the state of stresses is below the limit contractance/dilatancy)

$$\Delta\lambda=\frac{f^d\left(\sigma_{ij}^-,\xi_p^-\right)+\frac{\partial f^d}{\partial\sigma_{ij}}\left[D_{ijkl}^e\Delta\varepsilon_{kl}-D_{ijkl}^e\langle\Phi\rangle G_{kl}^{visc}\left(\sigma^-,\xi_{vp}^-\right)\Delta t\right]}{\left(\frac{\partial f^d}{\partial\sigma_{ij}}D_{ijkl}^e G_{kl}\left(\sigma^-,\xi_p^-\right)-\frac{\partial f^d}{\partial\xi_p}\sqrt{\frac{2}{3}}\tilde{G}_{II}\left(\sigma^-,\xi_p^-\right)\right)}$$

with $\Phi=A_v\left(\frac{f^{vp}\left(\sigma^e,\xi_{vp}^-\right)}{Pa}\right)^{n_v}$, A_v and n_v are parameters of the model.

4.2.2 Tangent operator

The constraint with the state $+$:

$$\sigma_{ij}=\sigma_{ij}^-+D_{ijkl}^e\Delta\epsilon_{kl}^e=\sigma_{ij}^-+D_{ijkl}^e\left(\Delta\epsilon_{kl}-\Delta\epsilon_{kl}^p-\Delta\epsilon_{kl}^{vp}\right)=$$
$$\sigma_{ij}^-+D_{ijkl}^e\left(\Delta\epsilon_{kl}-\Delta\lambda\, G_{kl}(\sigma^-,\xi_p^-)-\langle\Phi\rangle G_{kl}^{visc}(\sigma^-,\xi_{vp}^-)\Delta t\right)$$

$$\frac{\partial\sigma_{ij}}{\partial\Delta\varepsilon_{kl}}=D_{ijkl}^e-D_{ijmn}^e G_{mn}^-\frac{\partial\Delta\lambda}{\partial\Delta\varepsilon_{kl}}-D_{ijmn}^e G_{imn}^{visc-}\frac{\partial\langle\varphi\rangle}{\partial\Delta\varepsilon_{kl}}\Delta t$$

The two cases are distinguished:

- $\Delta\xi_p=\Delta\gamma_p$ (case contracting: the state of stresses is below characteristic threshold)
-

$$\Delta\lambda=\frac{f^d\left(\sigma_{ij}^-,\xi_p^-\right)+\frac{\partial f^d}{\partial\sigma_{ij}}\left[D_{ijkl}^e\Delta\varepsilon_{kl}-D_{ijkl}^e\langle\Phi\rangle G_{kl}^{visc}\left(\sigma^-,\xi_v^-\right)\Delta t\right]}{\left(\frac{\partial f^d}{\partial\sigma_{ij}}D_{ijkl}^e G_{kl}\left(\sigma^-,\xi_p^-\right)-\frac{\partial f^d}{\partial\xi_p}\sqrt{\frac{2}{3}}\tilde{G}_{II}\left(\sigma^-,\xi_p^-\right)\right)}\quad\text{and}\quad\Phi=A_v\left(\frac{f^{vp}\left(\sigma^e,\xi_{vp}^-\right)}{Pa}\right)^{n_v}$$

$$\frac{\partial\Delta\lambda}{\partial\Delta\varepsilon_{kl}}=\frac{\frac{\partial f^d}{\partial\sigma_{ij}}.\left[D_{ijkl}^e-D_{ijmn}^e G_{mn}^{visc}\left(\sigma^-,\xi_v^-\right)\frac{\partial\Phi}{\partial\Delta\varepsilon_{kl}}\Delta t\right]}{\left(\frac{\partial f^d}{\partial\sigma_{ij}}D_{ijmn}^e G_{mn}\left(\sigma^-,\xi_p^-\right)-\frac{\partial f^d}{\partial\xi_p}\sqrt{\frac{2}{3}}\tilde{G}_{II}\left(\sigma^-,\xi_p^-\right)\right)}$$

$$\frac{\partial\Phi}{\partial\Delta\varepsilon_{kl}}=\frac{A_v.n_v}{Patm}\left(\frac{f^{vp}\left(\sigma^e,\xi_{vp}^-\right)}{Patm}\right)^{n_v-1}.\frac{\partial f^{vp}}{\partial\sigma_{ij}^e}.\frac{\partial\sigma_{ij}^e}{\partial\Delta\varepsilon_{kl}}=\frac{A_v.n_v}{Patm}\left(\frac{f^{vp}\left(\sigma^e,\xi_{vp}^-\right)}{Patm}\right)^{n_v-1}.\frac{\partial f^{vp}}{\partial\sigma_{ij}^e}.D_{ijkl}^e$$

$$\frac{\partial \sigma_{ij}}{\partial \Delta\varepsilon_{kl}} = D^e_{ijkl} - D^e_{ijmn} . G^-_{mn} \frac{\frac{\partial f^d}{\partial \sigma_{ij}} . \left[D^e_{ijkl} - D^e_{ijmn} G^{visc}_{mn}\left(\sigma^-, \xi^-_p\right) \frac{\partial \Phi}{\partial \Delta\varepsilon_{kl}} \Delta t \right]}{\left(\frac{\partial f^d}{\partial \sigma_{ij}} D^e_{ijkl} G_{kl}\left(\sigma^-, \xi^-_p\right) - \frac{\partial f^d}{\partial \xi_p} \sqrt{\frac{2}{3}} \tilde{G}_{II}\left(\sigma^-, \xi^-_p\right) \right)} -$$

$$D^e_{ijmn} . G^{visc-}_{mn} \frac{A_v . n_v}{Patm} \left(\frac{f^{vp}\left(\sigma^e, \xi^-_{vp}\right)}{Patm} \right)^{n_v - 1} . \frac{\partial f^{vp}}{\partial \sigma^e_{ij}} . D^e_{ijkl} \Delta t$$

- $\Delta \xi_p = \Delta \gamma_p + \Delta \gamma_v$ (dilating case: the state of stresses exceeds the characteristic threshold)

$$\Delta\lambda = \frac{f^d\left(\sigma^-_{ij}, \xi^-_p\right) + \frac{\partial f^d}{\partial \xi_p} \Delta\gamma_{vp} + \frac{\partial f^d}{\partial \sigma_{ij}} \left[D^e_{ijkl} \Delta\varepsilon_{kl} - D^e_{ijkl} \langle \Phi \rangle G^{visc}_{kl}\left(\sigma^-, \xi^-_{vp}\right) \Delta t \right]}{\left(\frac{\partial f^d}{\partial \sigma_{ij}} D^e_{ijkl} G_{kl}\left(\sigma^-, \xi^-_p\right) - \frac{\partial f^d}{\partial \xi_p} \sqrt{\frac{2}{3}} \tilde{G}_{II}\left(\sigma^-, \xi^-_p\right) \right)}$$

$$\frac{\partial \Delta\lambda}{\partial \Delta\varepsilon_{kl}} = \frac{\frac{\partial f^d}{\partial \sigma_{ij}} . \left[D^e_{ijkl} - D^e_{ijmn} G^{visc}_{mn}\left(\sigma^-, \xi^-_p\right) \frac{\partial \Phi}{\partial \Delta\varepsilon_{kl}} \Delta t \right] + \frac{\partial f^d}{\partial \xi_p} . \frac{\partial \Delta\gamma_{vp}}{\partial \Delta\varepsilon^{vp}_{kl}} . \frac{\partial \Delta\varepsilon^{vp}_{kl}}{\partial \Delta\sigma^e_{ij}} . D^e_{ijkl}}{\left(\frac{\partial f^d}{\partial \sigma_{ij}} D^e_{ijmn} G_{mn}\left(\sigma^-, \xi^-_p\right) - \frac{\partial f^d}{\partial \xi_p} \sqrt{\frac{2}{3}} \tilde{G}_{II}\left(\sigma^-, \xi^-_p\right) \right)}$$

where:

$$\frac{\partial \Delta\gamma_{vp}}{\partial \Delta\varepsilon^{vp}_{kl}} = \frac{1}{2} \left(\frac{2}{3} \Delta\tilde{\varepsilon}^{vp}_{ij} . \Delta\tilde{\varepsilon}^{vp}_{ij} \right)^{-\frac{1}{2}} . \frac{2}{3} . 2 . \Delta\tilde{\varepsilon}^{vp}_{ij} . \frac{\partial \Delta\tilde{\varepsilon}^{vp}_{ij}}{\partial \Delta\varepsilon^{vp}_{kl}} = \frac{2}{3} . \frac{\Delta\tilde{\varepsilon}^{vp}_{ij}}{\Delta\gamma_{vp}} . \left(\delta_{ik} . \delta_{jl} - \frac{1}{3} \delta_{ij} . \delta_{kl} \right)$$

$$\frac{\partial \Delta\varepsilon^{vp}_{kl}}{\partial \Delta\sigma^e_{ij}} = \frac{n_v . A_v}{Patm} . \left(\frac{f^{vp}\left(\sigma^e, \xi^-_v\right)}{Patm} \right)^{n_v - 1} . \frac{\partial f^{vp}}{\partial \sigma^e_{ij}} . G^{visc-}_{kl}\left(\sigma^-, \xi^-_{vp}\right) \Delta t$$

4.2.3 Algorithm of resolution in Code_Aster

- Change of sign of the constraints to the state $-$ and of the increase in deformation:
  $\sigma^-_{L \wedge K} = -\sigma^-$

  $\Delta\varepsilon_{L \wedge K} = -\Delta\varepsilon$
- Calculation of the elastic constraint of prediction σ^e : $\sigma^e = \sigma^- + D^e \Delta\varepsilon$
- Checking of the sign of the viscous criterion with the viscous variable max: $f^{vp}\left(\sigma^e, \xi_{vp-\max}\right)$
- If $f^{vp}\left(\sigma^e, \xi_{vp-\max}\right) > 0$: dilating case and coupled work hardening $V_5 = 1$. One regards as variable of work hardening of the plastic criterion the office plurality between the plastic variable of work hardening and the viscous variable: $\Delta \xi_p = \Delta \gamma_p + \Delta \gamma_{vp}$
- If $f^{vp}\left(\sigma^e, \xi_{vp-\max}\right) < 0$: case contracting and work hardening not coupled $V_5 = 0$. The variable of work hardening of the plastic criterion is the plastic variable: $\Delta \xi_p = \Delta \gamma_p$

For the viscous criterion: $u^{vp}=A^{vp}(\xi_{vp})s_{II}H(\theta)+B^{vp}(\xi_{vp})I_1+D^{vp}(\xi_{vp})$

If $u^{vp}(\sigma^e,\xi_{vp}^-)<0$:

- if $-\frac{D^{vp}}{B^{vp}}<-\frac{D^d}{B^d}$ (Figure 4.2.3-a) then recutting of the step of time

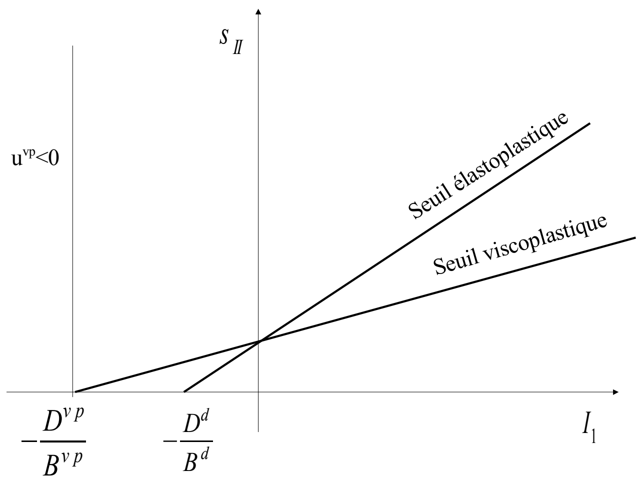


Figure 4.2.3-a. Schematic representation in the case $u^{vp}(\sigma^e,\xi_{vp}^-)<0$ **,** $-\frac{D^{vp}}{B^{vp}}<-\frac{D^d}{B^d}$

If not (Figure 4.2.3-b) two cases arise:

- if σ^e is in the zone A then $f^{vp}(\sigma^e)$ is not defined but one can make a geometrical projection (cf notices paragraph 3.3.3)
- if σ^e is in the zone B then it is necessary to make a projection at the top.

One is satisfied then with the message: " stop for coefficients non-cohesive materials of the law ".

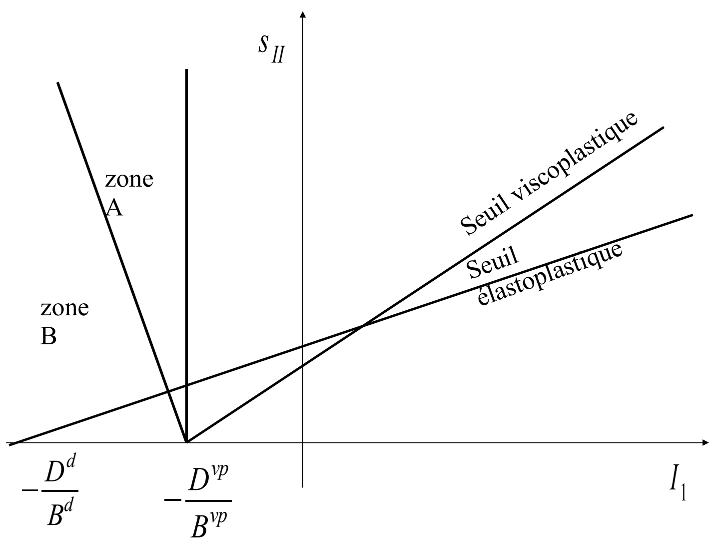


Figure 4.2.3-b. Schematic representation in the case $u^{vp}\left(\sigma^e, \xi_{vp}^-\right)<0$ **,** $-\frac{D^d}{B^d}<-\frac{D^{vp}}{B^{vp}}$

For the elastoplastic criterion:

$$u^d=A^d\left(\xi_p\right)s_{II}\,H\left(\theta\right)+B^d\left(\xi_p\right)I_1+D^d\left(\xi_p\right)$$

If $u^d\left(\sigma^e, \xi_p^-\right)<0$ then there is recutting of the step of time

For the viscous criterion:

If $f^{vp}\left(\sigma^e, \xi_{vp}^-\right)<0$ then not of creep and $\Delta\varepsilon_{vp}=0$

- If $f^{vp}\left(\sigma^e, \xi_{vp}^-\right)>0$ then creep develops according to the following form:

$$\Delta\varepsilon_{vp}=A\left[\frac{\langle f^{vp}\left(\sigma^e, \xi_{vp}^-\right)\rangle}{Patm}\right]^{n_v} G^{visc}\left(\sigma^-, \xi_{vp}^-\right)\Delta t \quad \text{with} \quad G^{visc}=\frac{\partial f^{vp}}{\partial\sigma}-\left(\frac{\partial f^{vp}}{\partial\sigma}n\right)n$$

$\langle\ \rangle$: Hooks of Macauley

where $\sin\left(\Psi\right)=\mu_{0,v}\left(\frac{\sigma_{\max}-\sigma_{\lim}}{\xi_{0,v}\sigma_{\max}+\sigma_{\lim}}\right)$ (see § 3.6.1) , β' and n are deduced

one can deduce $\Delta\gamma_{vp}=\sqrt{\frac{2}{3}\Delta\tilde{\varepsilon}_{vp}.\Delta\tilde{\varepsilon}_{vp}}>0$ where $\Delta\tilde{\varepsilon}_{vp}=\Delta\varepsilon_{vp}-\frac{tr\left(\Delta\varepsilon_{vp}\right)}{3}I^d$

reactualization of the variable of work hardening of the viscous criterion:

$\xi_{vp}=\xi_{vp}^-+\Delta\xi_{vp}$ with $\Delta\xi_{vp}=Min\left[\Delta\gamma_{vp}, \xi_{v-\max}-\xi_{vp}^-\right]$

reactualization of the constraints: $\sigma=\sigma^e-D^e\,\Delta\varepsilon_{vp}$

reactualization of the internal variables:

$$V_1=\xi_p=\xi_p^-+\Delta\gamma_{vp}$$
$$V_2=\gamma_p=\gamma_p^-$$
$$V_3=\xi_{vp}=\xi_{vp}^-+\Delta\xi_{vp}$$
$$V_4=\gamma_{vp}=\gamma_{vp}^-+\Delta\gamma_{vp}$$

For the elastoplastic criterion:

- If $f^d\left(\sigma^e-D^e\,\Delta\xi_{vp}, \xi_p^-+\Delta\gamma_{vp}\right)\le 0$ then: $\Delta\gamma=\Delta\gamma_p=\Delta\xi_p=0$
- If $f^d\left(\sigma^e-D^e\,\Delta\varepsilon_{vp}, \xi_p^-+\Delta\gamma_{vp}\right)>0$ then: $\Delta\tilde{\varepsilon}_p=\Delta\lambda G\left(\sigma^-, \xi_p^-\right)$ with

$\sin(\Psi)=\mu_{0,v}\left(\frac{\sigma_{max}-\sigma_{lim}}{\xi_{0,v}\sigma_{max}+\sigma_{lim}}\right)$ (see § 3.6.1) if $0\le\xi_p^-<\xi_{pic}$

$\sin(\Psi)=\mu_1\left(\frac{\alpha-\alpha_{res}}{\xi_1\alpha-\alpha_{res}}\right)$ (see § 3.6.2) if $\xi_p^->\xi_{pic}$

$G=\frac{\partial f^d}{\partial\sigma}-\left(\frac{\partial f^d}{\partial\sigma}n\right)n$ β' and n are deduced

One seeks $\Delta\lambda>0$ such as $f^d\left(\sigma,\xi_p^-\right)=0$

One deduces $\Delta\gamma_p>0$

If work hardening not coupled (contractance) $f^d\left(\sigma^e-D^e\Delta\xi_{vp},\xi_p^-+\Delta\gamma_{vp}\right)\le 0$ then: $\Delta\xi_p=\Delta\gamma_p$

if not coupled work hardening (dilatancy) $f^d\left(\sigma^e-D^e\Delta\varepsilon_{vp},\xi_p^-+\Delta\gamma_{vp}\right)>0$ then: $\Delta\xi_p=\Delta\gamma_p+\Delta\gamma_{vp}$

One the table of the internal variables supplements:

$V_1=\xi_p=\xi_p^-+\Delta\xi_p$
$V_2=\gamma_p=\gamma_p^-+\Delta\gamma_p$

Update of the constraints:

$\Delta\varepsilon^{irr}=\Delta\varepsilon^{vp}+\Delta\varepsilon^p$
$\Delta\varepsilon^e=\Delta\varepsilon-\Delta\varepsilon^{irr}$
$\Delta\sigma=D^e\Delta\varepsilon^e$
$\sigma=\sigma^-+\Delta\sigma$

Summary of the algorithm

- $\sigma^e=\sigma^-+D^e\Delta\varepsilon$
- if $f^{vp}\left(\sigma^e,\xi_{v\max}\right)<0$ then contractance ($VARV=0$) and the plastic variable is $\Delta\xi^p=\Delta\gamma^p$
- if $f^{vp}\left(\sigma^e,\xi_{v\max}\right)>0$ then dilatancy ($VARV=1$) and the plastic variable is $\Delta\xi^p=\Delta\gamma^p+\Delta\gamma^{vp}$

Checking of creep:

- calculation of $f^{vp}\left(\sigma^e,\xi_{vp}^-\right)$
- if $f^{vp}\left(\sigma^e,\xi_{vp}^-\right)<0$ **(Pas de creep)**

  $\Delta\xi_{vp}=\Delta\gamma_{vp}=0$
  $\xi_{vp}=\xi_{vp}^-$

$\gamma_{vp}=\gamma_{vp}^{-}$

- if $f^{vp}\left(\sigma^{e},\xi_{vp}^{-}\right)>0$ (**creep**)

  calculation of $\Delta\varepsilon_{vp}$ and of $\Delta\gamma_{vp}$ according to σ^{e} and of ξ_{vp}^{-}

  $\Delta\xi_{vp}=\min\left(\Delta\gamma_{vp},\xi_{v-\max}-\xi_{vp}^{-}\right)$
  $\xi_{vp}=\xi_{vp}^{-}+\Delta\xi_{vp}$
  $\gamma_{vp}=\gamma_{vp}^{-}+\Delta\gamma_{vp}$

- Adjustment of the elastic prediction: $\sigma_{n}^{e}=\sigma^{e}-D^{e}\Delta\varepsilon_{vp}$

Checking of plasticity :

- calculation of $f^{d}\left(\sigma_{n}^{e},\xi_{p}^{-}+\Delta\gamma_{vp}\right)$
- if $f^{d}\left(\sigma_{n}^{e},\xi_{p}^{-}+\Delta\gamma_{vp}\right)<0$ (Elasticity**)**

  $\Delta\varepsilon_{p}=\Delta\gamma_{p}=0$
  $\gamma_{p}=\gamma_{p}^{-}$
  $\xi_{p}=\xi_{p}^{-}+\Delta\xi_{p}$ with
  $\Delta\xi_{p}=0$ if $VARV=0$
  $\Delta\xi_{p}=\Delta\gamma_{vp}$ if $VARV=1$

  update of the constraints:
  $\sigma=\sigma^{e}-D^{e}\Delta\varepsilon_{vp}$

- if $f^{d}\left(\sigma_{n}^{e},\xi_{p}^{-}+\Delta\gamma_{vp}\right)>0$ (**Plasticity)**

  calculation of $\Delta\lambda$, $\Delta\gamma_{p}$ and $\Delta\varepsilon_{p}$

  $\Delta\xi_{p}=\Delta\gamma_{p}$ if $VARV=0$
  $\Delta\varepsilon_{p}=\Delta\gamma_{p}+\Delta\gamma_{vp}$ if $VARV=1$
  $\xi_{p}=\xi_{p}^{-}+\Delta\xi_{p}$

  update of the constraints:
  $\sigma=\sigma^{-}+D^{e}(\Delta\varepsilon-\Delta\varepsilon_{vp}-\Delta\varepsilon_{p})$

  table of the internal variables:
  $V1=\xi_{p}$
  $V2=\gamma_{p}$
  $V3=\xi_{vp}$
  $V4=\gamma_{vp}$
  $V5=VARV$ (0 if contractance or 1 if dilatancy)
  $V6=$ indicator of viscosity
  $V7=$ indicator of plasticity

4.3 Implicit diagram of integration

The integration of the model LETK according to the implicit diagram of integration is realized under environment PLASTI . The integration of model LETK under implicit scheme is currently available only by calculation of a disturbed matrix jacobienne local ('NEWTON_PERT').

The algorithm of resolution follows following logic. It uses an elastic prediction then iterations of correction if the viscous and/or plastic thresholds are requested. The purpose of the diagram is to produce the variation of the constraints and variables of work hardening under the effect of an increment of deformation.

The local subdivision of the model is activable under this diagram of integration by the keyword ITER_INTE_PAS keyword factor BEHAVIOR , cf [U4.51.11]).

4.3.1 Elastic phase of prediction

This phase is similar to that presented in section 4.2.1 .
The going beyond the thresholds of plasticity and viscosity is tested compared to this state of stresses. The expression of the thresholds tested is clarified in the § 3.2 .

- If none the thresholds is requested, the prediction is regarded as valid compared to the models. An update of the internal variables is undertaken to display the state of activation of the various thresholds.
- If a threshold among both to consider (plasticity and/or viscosity) is requested, the resolution of a local system of nonlinear equations must be initiated. The defined mechanisms of dissipation as potentially active must lead to the going beyond the associated thresholds (plasticity and/or viscosity)

4.3.2 Phase of correction: nonlinear equations to solve

This stage consists in solving the system of nonlinear local equations established on the basis of viscous and/or plastic mechanism. After convergence, the constraints and internal variables of the model are put up to date.

The unknown factors of the system of nonlinear equations are the constraints σ_{n+1} , the plastic multiplier λ^p_{n+1} , the plastic variable of work hardening ξ^p_{n+1} and the viscous variable of work hardening ξ^{vp}_{n+1} .
The vector of the inconuu be thus comprises to the maximum for modelings $3D$ 9 unknown factors.

The nonlinear equations to solve are the following ones:

- The incremental equation of state, $E1$:

$$\underline{\sigma}_{n+1}-\underline{\sigma}_n-C^e\left(\underline{\sigma}_{n+1}\right):\left(\Delta\underline{\epsilon}-\Delta\lambda\,\underline{G}^p-\Phi\left(f^{vp}\right)\cdot\underline{G}^{vp}\right)=0$$

- The condition of Kuhn-Tucker, $E2$:

$$\begin{cases}\text{Si} \quad f^d\leqslant 0 \quad \text{alors} \quad \Delta\lambda=0\\ \text{Si} \quad f^d=0 \quad \text{alors} \quad \Delta\lambda>0\end{cases}$$

- Incremental evolution of the variable of work hardening plastics, $E3$:

$\xi^p_{n+1}-\xi^p_n-\Delta\xi^p=0$, with $\Delta\xi^p$ evolving according to the conditions specified with the § 3.4.2 .

- Incremental evolution of the viscoplastic variable of work hardening, $E4$:

$\xi_{n+1}^{vp}-\xi_n^{vp}-\Delta\xi^{vp}=0$, with $\Delta\xi^{vp}$ evolving according to the conditions specified with the § 3.4.1 .

These equations constitute a square system $R(\Delta Y)$, where the unknown factors are $\Delta Y=(\Delta\underline{\underline{\sigma}},\Delta\lambda,\Delta\xi^p,\Delta\xi^{vp})$. With the iteration j loop of on-the-spot correction of Newton, one abstr. O C the following matric equation:

$$\frac{dR(\Delta Y^j)}{d(\Delta Y^j)}\cdot\delta(\Delta Y^{j+1})=-R(\Delta Y^j)$$

The matrix jacobienne $\frac{dR(\Delta Y^j)}{d(\Delta Y^j)}$, nonsymmetrical, builds itself in the following way:

$$\frac{dR(\Delta Y^j)}{d(\Delta Y^j)}=\begin{bmatrix}\frac{\partial E1}{\partial\underline{\underline{\sigma}}_{n+1}^j} & \frac{\partial E1}{\partial\Delta\lambda^j} & \frac{\partial E1}{\partial\xi_p^j} & \frac{\partial E1}{\partial\xi_{vp}^j}\\ \frac{E2}{\partial\underline{\underline{\sigma}}_{n+1}^j} & \frac{E2}{\partial\Delta\lambda^j} & \frac{E2}{\partial\xi_p^j} & \frac{E2}{\partial\xi_{vp}^j}\\ \frac{E3}{\partial\underline{\underline{\sigma}}_{n+1}^j} & \frac{E3}{\partial\Delta\lambda^j} & \frac{E3}{\partial\xi_p^j} & \frac{E3}{\partial\xi_{vp}^j}\\ \frac{E4}{\partial\underline{\underline{\sigma}}_{n+1}^j} & \frac{E4}{\partial\Delta\lambda^j} & \frac{E4}{\partial\xi_p^j} & \frac{E4}{\partial\xi_{vp}^j}\end{bmatrix}$$

This matrix is evaluated today analytically or by disturbance (`ALGO_INTE = 'NEWTON'` or `ALGO_INTE = 'NEWTON_PERT'`).

With an aim of standardizing the scales between the various equations to be solved, one makes the choice to put at the level of deformations bearing the E1 equation on the incremental equation of state. One applies for that the reverse of the modulus of nonlinear rigidity of elasticity. This choice makes it possible to ensure a more uniform convergence on the whole of the system.

Convergence famous is acquired since $\|R(\Delta Y^j)\|<$ `RESI_INTE_RELA`. One also makes sure when the mechanism of plasticity is active that the plastic multiplier is strictly positive. If it is not the case, local integration is started again without taking account of the mechanism of plasticity. Only the mechanism of viscosity can then be considered.

4.3.2.1 Expression of the terms of the matrix jacobienne

Derived terms associated with (R_1) are:

$$\frac{d(R_1)_{ij}}{d((\Delta Y_1)_{mn})}=I_{ilkl}-\frac{\partial C_{ijkl}^e}{\partial\sigma_{mn}}:[\Delta\epsilon_{kl}-\Delta\lambda\cdot G_{kl}^p-\Delta\epsilon_{kl}^{vp}]+\Delta\lambda C_{ijkl}^e:\frac{\partial G_{kl}^p}{\partial\sigma_{mn}}+$$
$$C_{ijkl}^e:G_{kl}^{vp}\otimes\frac{\partial(\langle\phi(f^{vp})\rangle^+)}{\partial\sigma_{mn}}\Delta t+\langle\phi(f^{vp})\rangle^+\cdot\Delta t\cdot C_{ijkl}^e:\frac{\partial G_{kl}^{vp}}{\partial\sigma_{mn}}$$

$$\frac{d(R_1)_{ij}}{d(\Delta Y_2)}=C_{ijkl}^e:G_{kl}^p$$

$$\frac{d(R_1)_{ij}}{d(\Delta Y_3)}=\Delta\lambda\cdot C_{ijkl}^e:\frac{\partial G_{kl}^p}{\partial\xi^p}$$

$$\frac{d(R_1)_{ij}}{d(\Delta Y_4)}=C^e_{ijkl}:\left[\frac{\partial\langle\phi(f^{vp})\rangle^+}{\partial\xi^{vp}}G^{vp}_{kl}+\langle\phi(f^{vp})\rangle^+\cdot\frac{\partial G^{vp}_{kl}}{\partial\xi^{vp}}\right]\cdot\Delta t$$

Derived terms associated with (R_2) s.e distinguishes according to the expression taken to satisfy the condition with Kuhn-Tucker:

If $(R_2)=\Delta\lambda$:

$$\frac{d(R_2)}{d((\Delta Y_1)_{ij})}=0_{ij}$$

$$\frac{d(R_2)}{d(\Delta Y_2)}=1$$

$$\frac{d(R_2)}{d(\Delta Y_3)}=0$$

$$\frac{d(R_2)}{d(\Delta Y_4)}=0$$

S I $(R_2)=f^p$:

$$\frac{d(R_2)}{d((\Delta Y_1)_{ij})}=\frac{\partial f^p}{\partial\sigma_{ij}}$$

$$\frac{d(R_2)}{d(\Delta Y_2)}=0$$

$$\frac{d(R_2)}{d(\Delta Y_3)}=\frac{\partial f^p}{\partial\xi^p}$$

$$\frac{d(R_2)}{d(\Delta Y_4)}=0$$

Derived terms associated with (R_3) are:

$$\frac{d(R_3)}{d(\Delta Y_1)_{ij}}=-\Delta\lambda\cdot\sqrt{\frac{2}{3}}\cdot\frac{\partial\tilde{G}^p_{II}}{\partial\sigma_{ij}}$$ (case contract ant: the state of stresses is below characteristic threshold)

$$\frac{d(R_3)}{d(\Delta Y_1)_{ij}}=-\Delta\lambda\cdot\sqrt{\frac{2}{3}}\cdot\frac{\partial\tilde{G}^p_{II}}{\partial\sigma_{ij}}-\sqrt{\frac{2}{3}}\cdot\Delta t\cdot\left[\left(\tilde{G}^{vp}_{II}\cdot\frac{\partial\langle\phi(f^{vp})\rangle^+}{\partial\sigma_{ij}}+\langle\phi(f^{vp})\rangle^+\cdot\frac{\partial\tilde{G}^{vp}_{II}}{\partial\sigma_{ij}}\right)\right]$$ (dilating case: the state of stresses exceeds the characteristic threshold)

$$\frac{d(R_3)}{d(\Delta Y_2)}=-\sqrt{\frac{2}{3}}\cdot\tilde{G}^p_{II}$$

$$\frac{d(R_3)}{d(\Delta Y_3)}=1-\Delta\lambda\cdot\sqrt{\frac{2}{3}}\cdot\frac{\partial\tilde{G}^p_{II}}{\partial\xi^p}$$

$$\frac{d(R_3)}{d(\Delta Y_4)}=-\sqrt{\frac{2}{3}}\cdot\Delta t\cdot\left(\frac{\partial\langle\phi(f^{vp})\rangle^+}{\xi^{vp}}\cdot\tilde{G}^{vp}_{II}+\langle\phi(f^{vp})\rangle^+\cdot\frac{\partial\tilde{G}^{vp}_{II}}{\xi^{vp}}\right)$$ (dilating case: the state of stresses exceeds the characteristic threshold)

Derived terms associated with (R_4) are:

$$\frac{d(R_4)}{d(\Delta Y_1)_{ij}}=-\sqrt{\frac{2}{3}}\cdot\Delta t\cdot\left(\frac{\partial\langle\phi(f^{vp})\rangle^+}{\sigma_{ij}}\cdot\tilde{G}^{vp}_{II}+\langle\phi(f^{vp})\rangle^+\cdot\frac{\partial\tilde{G}^{vp}_{II}}{\sigma_{ij}}\right)$$ if $\dot{\gamma}^{vp}\leqslant\xi^{vp}_{\max}-\xi^{vp}(t^-)$

$$\frac{d\left(R_4\right)}{d\left(\Delta Y_2\right)}=0$$

$$\frac{d\left(R_4\right)}{d\left(\Delta Y_3\right)}=0$$

$$\frac{d\left(R_4\right)}{d\left(\Delta Y_4\right)}=1-\sqrt{\frac{2}{3}}\cdot\Delta t\cdot\left(\frac{\partial\langle\phi\left(f^{vp}\right)\rangle^{+}}{\xi^{vp}}\cdot\tilde{G}_{II}^{vp}+\langle\phi\left(f^{vp}\right)\rangle^{+}\cdot\frac{\partial\tilde{G}_{II}^{vp}}{\xi^{vp}}\right)$$

The detailed expression of the whole of the put ends concerned depends on the derivative principal following:

$$\frac{dC_{ijkl}^{e}}{d\sigma_{mn}}\;;\;\frac{dG_{ij}^{p}}{d\sigma_{kl}}\;;\;\frac{d\langle\phi\left(f^{vp}\right)\rangle^{+}}{d\sigma_{ij}}\;;\;\frac{dG_{ij}^{vp}}{d\sigma_{kl}}\;;\;\frac{d\tilde{G}_{II}^{p}}{d\sigma_{ij}}\;;\;\frac{d\tilde{G}_{II}^{vp}}{d\sigma_{ij}}\;;$$

$$\frac{dG_{ij}^{p}}{d\xi^{p}}\;;\;\frac{d\tilde{G}_{II}^{p}}{d\xi^{p}}\;;$$

$$\frac{d\langle\phi\left(f^{vp}\right)\rangle^{+}}{d\xi^{vp}}\;;\;\frac{dG_{ij}^{vp}}{d\xi^{vp}}\;;\;\frac{d\tilde{G}_{II}^{vp}}{d\xi^{vp}}\;.$$

The quantities mentioned above are presented in appendix of the document.

4.3.3 Phase of update

The update of the vector solution is carried out according to the following operation:
$\Delta Y=\Delta Y^{j+1}=\Delta Y^{j}+\delta\Delta Y^{j+1}$
This phase of update consists in deferring the evolution of the constraints, plastic deformations, viscoplastic deformations and parameters of plastic and viscoplastic work hardening.

4.3.4 Tangent operator of speed

The tangent operator of speed was introduced right now within the framework of the explicit diagram of integration. This operator of rigidity is used at the time of the predictions on a total scale of Newton-Raphson out of tangent matrix (`PREDICTION=' TANGENTE'`). Sources FORTRAN associated with the onstruction with this operator are common to both diagrams of integration (`ALGO_INTE = ('NEWTON', 'SPECIFIC')`).

4.3.5 Consistent tangent operator

On the basis of analytical development specified in the document [R5.03.12], it is possible to determine the tangent operator $\boldsymbol{M}_c=\frac{\partial\sigma}{\partial\epsilon}$ starting from the terms of the matrix jacobienne definite above, § 4.3.2 ($J=\frac{dR}{dY}$).

Indeed, the system $\Phi\left(\Delta Y\right)=0$ is checked at the end of the increment and for a small variation of Φ , by considering this time ϵ like a variable, the system remains with balance and thus one checks $d\Phi=0$.
By differentiation, one obtains:

$$\frac{\partial\Phi}{\partial\Delta\epsilon}d\left(\Delta\epsilon\right)+\frac{\partial\Phi}{\partial\Delta\sigma}d\left(\Delta\sigma\right)+\frac{\partial\Phi}{\partial\Delta\lambda}d\left(\Delta\lambda\right)+\frac{\partial\Phi}{\partial\Delta\xi_p}d\left(\Delta\xi_p\right)+\frac{\partial\Phi}{\partial\Delta\xi_{vp}}d\left(\Delta\xi_{vp}\right)=0$$

One rewrites the system by putting the ends in ϵ in the member of right-hand side:

$$\frac{\partial\Phi}{\partial\Delta\epsilon}d(\Delta\epsilon)+\frac{\partial\Phi}{\partial\Delta\sigma}d(\Delta\sigma)+\frac{\partial\Phi}{\partial\Delta\lambda}d(\Delta\lambda)+\frac{\partial\Phi}{\partial\Delta\xi_p}d(\Delta\xi_p)=-\frac{\partial\Phi}{\partial\Delta\xi_{vp}}d(\Delta\xi_{vp})$$

This system can then be written in the following form:

$J\cdot d(\Delta Y)=-\frac{\partial\Phi}{\partial(\Delta\epsilon)}d(\Delta\epsilon)$ with $\frac{\partial\Phi}{\partial(\Delta\epsilon)}=\{-C^e(\sigma),0,0,0\}$

Finally, one obtains: $J\cdot d(\Delta Y)=\{C^e(\sigma):\Delta\epsilon,\,0,0,0\}$

One writes then the system per blocks while separating $d(\Delta\sigma)$ other variables $Z=(\Delta\lambda,\Delta\xi_p,\Delta\xi_{vp})$, which gives:

$$\begin{bmatrix}J_{\sigma\sigma} & J_{\sigma Z}\\ J_{Z\sigma} & J_{ZZ}\end{bmatrix}.\begin{pmatrix}\Delta\sigma\\ Z\end{pmatrix}=\begin{pmatrix}C^e(\sigma)d(\Delta\epsilon)\\ 0\end{pmatrix}$$

The expression of the tangent operator becomes:

$$\boldsymbol{M}_c=\frac{\partial\sigma}{\partial\epsilon}=\frac{d(\Delta\sigma)}{d(\Delta\epsilon)}=\left[J_{\sigma\sigma}-J_{\sigma Z}(J_{ZZ})^{-1}J_{Z\sigma}\right]^{-1}C^e(\sigma)$$

Notice : The Jacobienne matrix not being symmetrical, the tangent operator $\boldsymbol{M}_c$ is not it either.

5 References

R 1 . Model “L&K” for Code_Aster. IH-HAVL-SIO-00015-A.

R 2 . Model “L&K” for Code_Aster. IH-HAVL-SIO-00015-B.

R3. Réunion on viscoplastic model L&K of the CIH simplified for Code_Aster. CR-AMA-2007-142.

R 4 . Digital modeling of the behavior of the underground works by a viscoplastic approach. Thesis presented to the INPL by A. Kleine, November 2007.

6 Features and checking

This document relates to the law of behavior `LETK` (keyword `BEHAVIOR` of `STAT_NON_LINE`) and the associated material `LETK` (order `DEFI_MATERIAU`).

This behavior is checked by the two cases tests:

- SSNV206 - Triaxial compression test with model LETK of the CIH - [V6.04.206]
- WTNV135 - Triaxial compression test drained with model LETK of the CIH - [V7.31.135]

7 Description of the versions of the document

Version Aster	Author (S) Organization (S)	Description of the modifications
9.2	J.El-Gharib, C. Chavant, EDF-R&D/AMA F.Laigle, A.Kleine EDF-CIH	Initial text
11.2	A.Foucault	Algorithm of integration by implicit scheme

11.3	A.Foucault	Analytical development of the matrix jacobienne DR/DY

8 Appendices: Terms of the matrix jacobienne

Evaluation of the terms relative to $\dfrac{d\left(R_1\right)_{ij}}{d\left(\left(\Delta Y_1\right)_{mn}\right)}$:

$$\frac{d\left(R_1\right)_{ij}}{d\left(\left(\Delta Y_1\right)_{mn}\right)}=I_{ilkl}-\frac{\partial C^e_{ijkl}}{\partial\sigma_{mn}}:\left[\Delta\epsilon_{kl}-\Delta\lambda\cdot G^p_{kl}-\Delta\epsilon^{vp}_{kl}\right]+\Delta\lambda C^e_{ijkl}:\frac{\partial G^p_{kl}}{\partial\sigma_{mn}}+$$

$$C^e_{ijkl}:G^{vp}_{kl}\otimes\frac{\partial\left(\left\langle\phi\left(f^{vp}\right)\right\rangle^+\right)}{\partial\sigma_{mn}}\Delta t+\left\langle\phi\left(f^{vp}\right)\right\rangle^+\cdot\Delta t\cdot C^e_{ijkl}:\frac{\partial G^{vp}_{kl}}{\partial\sigma_{mn}}$$

$$\frac{\partial C^e_{ijkl}}{\partial\sigma_{mn}}=\frac{n_{elas}}{I_1}\cdot C^e_{ijkl}\otimes\delta_{mn}$$

$$\frac{\partial G^p_{ij}}{\partial\sigma_{kl}}=\frac{\partial}{\sigma_{kl}}\left(\frac{\partial f^p}{\partial\sigma_{ij}}\right)-\left(\frac{\partial}{\sigma_{kl}}\left(\frac{\partial f^p}{\partial\sigma_{ij}}\right):n_{mn}\right)\otimes n_{ij}-\left(\frac{\partial f^p}{\partial\sigma_{mn}}:\frac{\partial n_{mn}}{\partial\sigma_{kl}}\right)\otimes n_{ij}-\left(\frac{\partial f^p}{\partial\sigma_{mn}}:n_{mn}\right)\cdot\frac{\partial n_{ij}}{\partial\sigma_{kl}}$$

with $\dfrac{\partial f^d}{\partial\sigma_{ij}}=\dfrac{\partial\left(s_{II}H(\theta)\right)}{\partial\sigma_{ij}}-a^d\left(\xi_p\right)\sigma_c H^c_0\left[A^d\left(\xi_p\right)s_{II}H(\theta)+B^d\left(\xi_p\right)I_1+D^d\left(\xi_p\right)\right]^{a^d\left(\xi_p\right)-1}\left(A^d\left(\xi_p\right)\dfrac{\partial\left(s_{II}H(\theta)\right)}{\partial\sigma_{ij}}+B^d\left(\xi_p\right)I_d\right)$ that is to say

$$\frac{\partial}{\sigma_{kl}}\left(\frac{\partial f^p}{\partial\sigma_{ij}}\right)=\frac{\partial}{\partial\sigma_{kl}}\left(\frac{\partial s_{II}H(\theta)}{\partial\sigma_{ij}}\right)-a^d\left(\xi^p\right)\sigma_c H^c_0\left[A^d\left(\xi^p\right)s_{II}H(\theta)+B^d\left(\xi^p\right)I_1+D^d\left(\xi^p\right)\right]^{a^d\left(\xi^p\right)-1}$$

$$\cdot A^d\left(\xi^p\right)\frac{\partial}{\partial\sigma_{kl}}\left(\frac{\partial s_{II}H(\theta)}{\partial\sigma_{ij}}\right)-a^d\left(\xi^p\right)\left(a^d\left(\xi^p\right)-1\right)\sigma_c H^c_0\left(A^d\left(\xi^p\right)s_{II}H(\theta)+B^d\left(\xi^p\right)I_1+D^d\left(\xi^p\right)\right)^{a^d\left(\xi^p\right)-2}$$

$$\left(A^d\left(\xi^p\right)\frac{\partial s_{II}H(\theta)}{\partial\sigma_{ij}}+B^d\left(\xi^p\right)\delta_{ij}\right)\otimes\left(A^d\left(\xi^p\right)\frac{\partial s_{II}H(\theta)}{\partial\sigma_{kl}}+B^d\left(\xi^p\right)\delta_{kl}\right)$$

However $\dfrac{\partial\left(s_{II}H(\theta)\right)}{\partial\sigma_{ij}}=\left(\left(\dfrac{H^c_0-H^e_0}{h^c_0-h^e_0}\right)\dfrac{\partial h(\theta)}{\partial s_{kl}}s_{II}+H(\theta)\dfrac{s_{kl}}{s_{II}}\right).\left(\delta_{ik}.\delta_{jl}-\dfrac{1}{3}\delta_{ij}.\delta_{kl}\right)$ from where

$$\frac{\partial}{\partial\sigma_{kl}}\left(\frac{\partial s_{II}H(\theta)}{\partial\sigma_{ij}}\right)=\left(\frac{H^c_0-H^e_0}{h^c_0-h^e_0}\right)\frac{\partial}{\partial\sigma_{kl}}\left(\frac{\partial h(\theta)}{\partial s_{mn}}\right)s_{II}\frac{\partial s_{mn}}{\partial\sigma_{ij}}+\left(\frac{H^c_0-H^e_0}{h^c_0-h^e_0}\right)\frac{\partial h(\theta)}{\partial s_{mn}}\frac{\partial s_{mn}}{\partial\sigma_{ij}}\frac{\partial s_{II}}{\partial s_{pq}}\frac{\partial s_{pq}}{\partial\sigma_{kl}}+$$

$$\frac{\partial H(\theta)}{\partial\sigma_{kl}}\frac{s_{mn}}{s_{II}}\frac{\partial s_{mn}}{\partial\sigma_{ij}}+\frac{H(\theta)}{s_{II}}\frac{\partial s_{mn}}{\partial\sigma_{kl}}\frac{\partial s_{mn}}{\partial\sigma_{ij}}-\frac{H(\theta)s_{mn}}{s^2_{II}}\frac{\partial s_{II}}{\partial s_{pq}}\frac{\partial s_{pq}}{\partial\sigma_{kl}}\frac{\partial s_{mn}}{\partial\sigma_{ij}}$$

One has moreover $\dfrac{\partial h(\theta)}{\partial s_{kl}}=\dfrac{\gamma\cos(3\theta)}{6\mathrm{h}(\theta)^5}\dfrac{3\mathrm{s}_{kl}}{s^2_{II}}-\dfrac{\gamma\sqrt{54}}{6\mathrm{h}(\theta)^5 s^3_{II}}\left(\dfrac{\partial\det(\underline{s})}{\partial s_{kl}}\right)$

$$\frac{\partial}{\partial \sigma_{kl}}\left(\frac{\partial h(\theta)}{\partial s_{mn}}\right)=\frac{\partial}{\partial s_{pq}}\left(\frac{\partial h(\theta)}{\partial s_{mn}}\right)\frac{\partial s_{pq}}{\partial \sigma_{kl}}$$

$$\begin{aligned}\frac{\partial}{\partial s_{pq}}\left(\frac{\partial h(\theta)}{\partial s_{mn}}\right)=&\frac{\sqrt{54}\gamma}{2h^5(\theta)s_{II}^5}s_{mn}\otimes\frac{\partial det(s_{ij})}{\partial s_{pq}}-\frac{3\gamma\sqrt{54}det(s_{ij})}{2h^5(\theta)s_{II}^7}s_{mn}\otimes s_{pq}+\frac{\gamma\cos 3\theta}{2h^5(\theta)s_{II}^2}I_{mnpq}-\\&\frac{\gamma\cos 3\theta}{h^5(\theta)s_{II}^4}s_{mn}\otimes s_{pq}-\frac{5\gamma\cos 3\theta}{2h^6(\theta)s_{II}^2}s_{mn}\otimes\frac{\partial h(\theta)}{\partial s_{pq}}+\frac{5\gamma\sqrt{54}}{6s_{II}^3h^6(\theta)}\frac{\partial det(s_{ij})}{\partial s_{mn}}\otimes\frac{\partial h(\theta)}{\partial s_{pq}}+\\&\frac{\gamma\sqrt{54}}{2h^5(\theta)s_{II}^5}\frac{\partial det(s_{ij})}{\partial s_{mn}}\otimes s_{pq}-\frac{\gamma\sqrt{54}}{6h^5(\theta)s_{II}^3}\frac{\partial^2 det(s_{ij})}{\partial s_{mn}\partial s_{pq}}\end{aligned}$$

with

$$\frac{\partial^2 det(s_{ij})}{\partial s_{mn}\partial s_{pq}}=\begin{bmatrix}0 & s_{33} & s_{22} & 0 & 0 & -\sqrt{2}s_{23}\\ s_{33} & 0 & s_{11} & 0 & -\sqrt{2}s_{13} & 0\\ s_{22} & s_{11} & 0 & -\sqrt{2}s_{12} & 0 & 0\\ 0 & 0 & -\sqrt{2}s_{12} & -s_{33} & s_{13} & s_{23}\\ 0 & -\sqrt{2}s_{13} & 0 & s_{13} & -s_{11} & s_{12}\\ -\sqrt{2}s_{23} & 0 & 0 & s_{23} & s_{12} & -s_{22}\end{bmatrix}$$

It is pointed out that $n_{ij}=\frac{\beta'\frac{s_{ij}}{s_{II}}-\delta_{ij}}{\sqrt{\beta'^2+3}}$ and $\beta'=-\frac{2\sqrt{6}\sin(\Psi)}{3-\sin(\Psi)}$

$$\frac{\partial n_{ij}}{\partial \sigma_{kl}}=\frac{\left[\frac{\partial\beta'}{\partial\sigma_{kl}}\frac{s_{ij}}{s_{II}}+\frac{\beta'}{s_{II}}\frac{\partial s_{ij}}{\partial\sigma_{kl}}-\frac{\beta' s_{ij}}{s_{II}^2}\frac{\partial s_{II}}{\partial\sigma_{kl}}\right](\beta'^2+3)-\beta'\left(\beta'\frac{s_{ij}}{s_{II}}-\delta_{ij}\right)\otimes\frac{\partial\beta'}{\partial\sigma_{kl}}}{(\beta'^2+3)\sqrt{\beta'^2+3}}$$

$$\frac{\partial\beta'}{\partial\sigma_{kl}}=\frac{\partial\beta'}{\partial s_{mn}}\frac{\partial s_{mn}}{\partial\sigma_{kl}}+\frac{\partial\beta'}{\partial I_1}\frac{\partial I_1}{\partial\sigma_{kl}}$$

$$\frac{\partial\beta'}{\partial s_{mn}}=\frac{-6\sqrt{6}}{(3-\sin\psi)^2}\frac{\partial\sin\psi}{\partial s_{mn}}\quad\text{and}\quad\frac{\partial\beta'}{\partial I_1}=\frac{-6\sqrt{6}}{(3-\sin\psi)^2}\frac{\partial\sin\psi}{\partial I_1}$$

Distinction of the expressions between the viscoplastic behavior pre-peak or and post-peak

Expression of the derivative in pre-peak or viscoplastic

$$\begin{aligned}\frac{\partial\sin\psi}{\partial s_{mn}}&=\frac{\partial\sin\psi}{\partial\sigma_{max}}\frac{\partial\sigma_{max}}{\partial s_{mn}}+\frac{\partial\sin\psi}{\partial\sigma_{\lim}}\frac{\partial\sigma_{\lim}}{\partial s_{mn}}\\&=\mu_{0,v}\left(\frac{(1+\xi_{0,v})\sigma_{\lim}}{(\xi_{0,v}\sigma_{max}+\sigma_{\lim})^2}\frac{\partial\sigma_{max}}{\partial s_{mn}}-\frac{(1+\xi_{0,v})\sigma_{max}}{(\xi_{0,v}\sigma_{max}+\sigma_{\lim})^2}\frac{\partial\sigma_{\lim}}{\partial s_{mn}}\right)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \sin\psi}{\partial I_1} &= \frac{\partial \sin\psi}{\partial \sigma_{max}}\frac{\partial \sigma_{max}}{\partial I_1}+\frac{\partial \sin\psi}{\partial \sigma_{\lim}}\frac{\partial \sigma_{\lim}}{\partial I_1}\\ &= \frac{\mu_{0,v}}{3}\left(\frac{(1+\xi_{0,v})\sigma_{\lim}}{(\xi_{0,v}\sigma_{max}+\sigma_{\lim})^2}-\frac{(1+m_{v,\max})(1+\xi_{0,v})\sigma_{\max}}{(\xi_{0,v}\sigma_{max}+\sigma_{\lim})^2}\right)\end{aligned}$$

$$\frac{\partial \sigma_{max}}{\partial s_{mn}}=\frac{1}{\sqrt{6}}\left[\frac{s_{II}}{H_0^c-H_0^e}\frac{\partial H(\theta)}{\partial s_{mn}}+\left(\frac{3}{2}+\frac{2H(\theta)-(H_0^c+H_0^e)}{2(H_0^c-H_0^e)}\right)\frac{s_{mn}}{s_{II}}\right] \text{ and } \frac{\partial \sigma_{max}}{\partial I_1}=\frac{1}{3}$$

$$\frac{\partial \sigma_{\lim}}{\partial s_{mn}}=\frac{(1+m_{v,\max})}{\sqrt{6}}\left[\frac{s_{II}}{H_0^c-H_0^e}\frac{\partial H(\theta)}{\partial s_{mn}}-\left(\frac{3}{2}-\frac{2H(\theta)-(H_0^c+H_0^e)}{2(H_0^c-H_0^e)}\right)\frac{s_{mn}}{s_{II}}\right] \text{ and } \frac{\partial \sigma_{\lim}}{\partial I_1}=\frac{(1+m_{v,\max})}{3}$$

Expression of the derivative in post-peak for the plastic law of dilatancy

$$\frac{\partial \sin\psi}{\partial s_{mn}}=\frac{\partial \sin\psi}{\partial \alpha}\left(\frac{\partial \alpha}{\partial \sigma_{min}}\frac{\partial \sigma_{min}}{\partial s_{mn}}+\frac{\partial \alpha}{\partial \sigma_{max}}\frac{\partial \sigma_{max}}{\partial s_{mn}}\right)$$

and

$$\frac{\partial \sin\psi}{\partial I_1}=\frac{\partial \sin\psi}{\partial \alpha}\left(\frac{\partial \alpha}{\partial \sigma_{min}}\frac{\partial \sigma_{min}}{\partial I_1}+\frac{\partial \alpha}{\partial \sigma_{max}}\frac{\partial \sigma_{max}}{\partial I_1}\right)$$

$$\frac{\partial \sin\psi}{\partial s_{mn}}=\mu_1\frac{(1+\xi_1)\alpha_{res}}{(\xi_1\alpha+\alpha_{res})^2}\left(-\frac{\sigma_{max}+\tilde{\sigma}}{(\sigma_{min}+\tilde{\sigma})^2}\frac{\partial \sigma_{min}}{\partial s_{mn}}+\frac{1}{\sigma_{min}+\tilde{\sigma}}\frac{\partial \sigma_{max}}{\partial s_{mn}}\right)$$

and

$$\frac{\partial \sin\psi}{\partial I_1}=\frac{\mu_1\alpha_{res}(1+\xi_1)(\sigma_{min}-\sigma_{max})}{3(\xi_1\alpha+\alpha_{res})^2(\sigma_{min}+\tilde{\sigma})^2}$$

$$\frac{\partial \sigma_{min}}{\partial s_{mn}}=\frac{1}{\sqrt{6}}\left[\frac{s_{II}}{H_0^c-H_0^e}\frac{\partial H(\theta)}{\partial s_{mn}}-\left(\frac{3}{2}-\frac{2H(\theta)-(H_0^c+H_0^e)}{2(H_0^c-H_0^e)}\right)\frac{s_{mn}}{s_{II}}\right] \text{ and } \frac{\partial \sigma_{min}}{\partial I_1}=\frac{1}{3}$$

The evaluation of the term $\frac{\partial G_{ij}^{vp}}{\partial \sigma_{kl}}$ is identical to that of $\frac{\partial G_{ij}^{p}}{\partial \sigma_{kl}}$ with the distinctions close already specified above.

One approaches now the evaluation of the term $\frac{\partial\left(\langle \phi(f^{vp})\rangle^+\right)}{\partial \sigma_{mn}}$ with $\Phi(f^{vp})=A_v\left(\frac{f^{vp}}{Pa}\right)^{n_v}$.

One obtains then:

$$\frac{\partial\left(\langle\phi\left(f^{vp}\right)\rangle^{+}\right)}{\partial\sigma_{mn}}=\frac{\partial f^{vp}}{\partial\sigma_{mn}}\cdot\frac{A_v n_v}{P_{\text{atm}}}\left(\frac{f^{vp}}{P_{\text{atm}}}\right)^{n_v-1}$$

Evaluation of the terms relative to $\frac{d\left(R_1\right)_{ij}}{d\left(\Delta Y_3\right)}=\Delta\lambda\cdot C^e_{ijkl}:\frac{\partial G^p_{kl}}{\partial\xi^p}$:

$$\frac{\partial G^p_{ij}}{\partial\xi^p}=\frac{\partial}{\xi^p}\left(\frac{\partial f^p}{\partial\sigma_{ij}}\right)-\left(\frac{\partial}{\xi^p}\left(\frac{\partial f^p}{\partial\sigma_{ij}}\right):n_{mn}\right)\otimes n_{ij}-\left(\frac{\partial f^p}{\partial\sigma_{mn}}:\frac{\partial n_{mn}}{\partial\xi^p}\right)\otimes n_{ij}-\left(\frac{\partial f^p}{\partial\sigma_{mn}}:n_{mn}\right)\cdot\frac{\partial n_{ij}}{\partial\xi^p}$$

with

$$\begin{aligned}\frac{\partial}{\xi^p}\left(\frac{\partial f^p}{\partial\sigma_{ij}}\right)=&\frac{\partial a^d}{\partial\xi^p}\sigma_c H^c_0\left(A^d s_{II}H(\theta)+B^d I_1+D^d\right)^{a^d-1}\cdot\left(A^d\frac{\partial\left(s_{II}H(\theta)\right)}{\partial\sigma_{ij}}+B^d\delta_{ij}\right)\\&-a^d\sigma_c H^c_0\left\{\frac{\partial a^d}{\xi^p}\ln\left(A^d s_{II}H(\theta)+B^d I_1+D^d\right)\ldots\right.\\&\left.\ldots+\frac{\left(a^d-1\right)}{A^d s_{II}H(\theta)+B^d I_1+D^d}\left(\frac{\partial A^d}{\partial\xi^p}s_{II}H(\theta)+\frac{\partial B^d}{\partial\xi^p}I_1+\frac{\partial D^d}{\partial\xi^p}\right)\right\}\cdot\\&\left(A^d s_{II}H(\theta)+B^d I_1+D^d\right)^{a^d-1}\left[A^d\frac{\partial\left(s_{II}H(\theta)\right)}{\partial\sigma_{ij}}+B^d\delta_{ij}\right]\\&-a^d\sigma_c H^c_0\left[A^d s_{II}H(\theta)+B^d I_1+D^d\right]^{a^d-1}\cdot\left(\frac{\partial A^d}{\partial\xi^p}\frac{\partial\left(s_{II}H(\theta)\right)}{\sigma_{ij}}+\frac{\partial B^d}{\partial\xi^p}\right)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial n_{kl}}{\xi^p}&=\frac{\partial n_{kl}}{\partial\beta'}\frac{\partial\beta'}{\partial\xi^p}\\&=\frac{\left(\frac{s_{kl}}{s_{II}}\left(\beta'^2+3\right)-2\beta'^2\frac{s_{kl}}{s_{II}}+2\beta'\delta_{kl}\right)}{\left(\beta'^2+3\right)^{3/2}}\frac{\partial\beta'}{\partial\xi^p}\\&=\frac{\left(\frac{s_{kl}}{s_{II}}\left(\beta'^2+3\right)-2\beta'^2\frac{s_{kl}}{s_{II}}+2\beta'\delta_{kl}\right)}{\left(\beta'^2+3\right)^{3/2}}\frac{-6\sqrt{6}\sin\psi}{\left(3-\sin\psi\right)^2}\frac{\partial\sin\psi}{\partial\xi^p}\end{aligned}$$

- if the law of followed dilatancy corresponds to the pre-peak field, $\frac{\partial\sin\psi}{\partial\xi^p}=0$
- if the law of followed dilatancy corresponds to the post-peak field, the following operations are necessary:

$$\begin{aligned}
\frac{\partial \sin\psi}{\partial \xi^p} &= \frac{\partial \sin\psi}{\partial \alpha}\frac{\partial \alpha}{\partial \xi^p}\\
&= \mu_1 \frac{(1+\xi_1)\alpha_{\text{res}}}{(\xi_1\alpha+\alpha_{\text{res}})^2}\frac{\partial \alpha}{\partial \xi^p}\\
&= \mu_1 \frac{(1+\xi_1)\alpha_{\text{res}}}{(\xi_1\alpha+\alpha_{\text{res}})^2}\frac{(\sigma_{\min}-\sigma_{\max})}{(\sigma_{\min}+\tilde{\sigma})^2}\frac{\partial \tilde{\sigma}}{\partial \xi^p}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \tilde{\sigma}}{\partial \xi^p} &= \frac{\partial \tilde{\sigma}}{\partial \tilde{c}}\frac{\partial \tilde{c}}{\partial \xi^p}+\frac{\partial \tilde{\sigma}}{\partial \tan(\phi)}\frac{\partial \tan(\phi)}{\partial \xi^p}\\
&= \frac{1}{\tan(\phi)}\frac{\partial \tilde{c}}{\partial \xi^p}-\frac{\tilde{c}}{\tan(\phi)^2}\frac{\partial \tan(\phi)}{\partial \xi^p}
\end{aligned}$$

with

$$\frac{\partial \tilde{c}}{\partial \xi^p}=\frac{\left(\sigma_c (s^d)^{a^d}\left[\frac{\partial a^d}{\partial \xi^p}\ln(s^d)+\frac{a^d}{s^d}\frac{\partial s^d}{\partial \xi^p}\right]\right)}{2\sqrt{1+a^d m^d (s^d)^{a^d-1}}}+\dots$$

$$\dots\frac{\sigma_c (s^d)^{a^d}\left(\frac{\partial a^d}{\partial \xi^p}m^d (s^d)^{a^d-1}+a^d\frac{\partial m^d}{\partial \xi^p}(s^d)^{a^d-1}+a^d m^d\left(\frac{\partial a^d}{\partial \xi^p}\ln s^d+\frac{a^d-1}{s^d}\frac{\partial s^d}{\partial \xi^p}\right)(s^d)^{a^d-1}\right)}{4\left(1+a^d m^d (s^d)^{a^d-1}\right)^{3/2}}$$

$$\begin{aligned}
\frac{\partial \tan\phi}{\partial \xi^p} &= (1+\tan^2\phi)\frac{\partial \phi}{\partial \xi^p}\\
&= \frac{(1+\tan^2\phi)\left[\frac{\partial a^d}{\partial \xi^p}m^d (s^d)^{a^d-1}+a^d\frac{\partial m^d}{\partial \xi^p}(s^d)^{a^d-1}+a^d m^d (s^d)^{a^d-1}\left(\frac{\partial a^d}{\partial \xi^p}\ln s^d+\frac{a^d-1}{s^d}\frac{\partial s^d}{\partial \xi^p}\right)\right]}{\left(2+a^d m^d (s^d)^{a^d-1}\right)\sqrt{\left(1+a^d m^d (s^d)^{a^d-1}\right)}}
\end{aligned}$$

Calculation of the terms relative to $\frac{d(R_1)_{ij}}{d(\Delta Y_4)}=C^e_{ijkl}:\left[\frac{\partial\langle\phi(f^{vp})\rangle^+}{\partial \xi^{vp}}G^{vp}_{kl}+\langle\phi(f^{vp})\rangle^+\cdot\frac{\partial G^{vp}_{kl}}{\partial \xi^{vp}}\right]\cdot\Delta t$:

The evaluation of the term $\frac{\partial G^{vp}_{kl}}{\partial \xi^{vp}}$ is identical in its form to preceding calculation for $\frac{\partial G^{p}_{kl}}{\partial \xi^{p}}$.

$$\frac{\partial\langle\phi(f^{vp})\rangle^+}{\partial \xi^{vp}}=\frac{A_v n_v}{P_{\text{atm}}}\left(\frac{f^{vp}}{P_{\text{atm}}}\right)^{n_v-1}\frac{\partial f^{vp}}{\partial \xi^{vp}}$$

The evaluation of the term $\frac{\partial f^{vp}}{\partial \xi^{vp}}$ identical in its form at the end is previously calculated $\frac{\partial f^{p}}{\partial \xi^{p}}$.

Evaluation of the terms relative to $\frac{d(R_3)}{d(\Delta Y_1)_{ij}}$ and $\frac{d(R_4)}{d(\Delta Y_1)_{ij}}$

$$\frac{\partial \tilde{G}_{II}^{p}}{\partial \sigma_{ij}} = \frac{\partial \tilde{G}_{II}^{p}}{\partial \tilde{G}_{kl}^{p}} \frac{\partial \tilde{G}_{kl}^{p}}{\partial G_{mn}^{p}} \frac{\partial G_{mn}^{p}}{\partial \sigma_{ij}} = \frac{\tilde{G}_{kl}^{p}}{\tilde{G}_{II}^{p}} \left(\delta_{mk} \delta_{nl} - \frac{1}{3} \delta_{mn} \delta_{kl} \right) \frac{\partial G_{mn}^{p}}{\partial \sigma_{ij}}$$

and

$$\frac{\partial \tilde{G}_{II}^{vp}}{\partial \sigma_{ij}} = \frac{\partial \tilde{G}_{II}^{vp}}{\partial \tilde{G}_{kl}^{vp}} \frac{\partial \tilde{G}_{kl}^{vp}}{\partial G_{mn}^{vp}} \frac{\partial G_{mn}^{vp}}{\partial \sigma_{ij}} = \frac{\tilde{G}_{kl}^{vp}}{\tilde{G}_{II}^{vp}} \left(\delta_{mk} \delta_{nl} - \frac{1}{3} \delta_{mn} \delta_{kl} \right) \frac{\partial G_{mn}^{vp}}{\partial \sigma_{ij}}$$

Evaluation of the terms relative to $\frac{d(R_3)}{d(\Delta Y_3)}$

$$\frac{\partial \tilde{G}_{II}^{p}}{\partial \xi^{p}} = \frac{\tilde{G}_{kl}^{p}}{\tilde{G}_{II}^{p}} \left(\delta_{mk} \delta_{nl} - \frac{1}{3} \delta_{mn} \delta_{kl} \right) \frac{\partial G_{mn}^{p}}{\partial \xi^{p}}$$

Evaluation of the terms relative to $\frac{d(R_3)}{d(\Delta Y_4)}$ and $\frac{d(R_4)}{d(\Delta Y_4)}$

$$\frac{\partial \tilde{G}_{II}^{vp}}{\partial \xi^{vp}} = \frac{\tilde{G}_{kl}^{vp}}{\tilde{G}_{II}^{vp}} \left(\delta_{mk} \delta_{nl} - \frac{1}{3} \delta_{mn} \delta_{kl} \right) \frac{\partial G_{mn}^{vp}}{\partial \xi^{vp}}$$