Law of Iwan for cyclic granular material behavior

Summary:

The model of behavior of Iwan [1], [2] is an elastoplastic model classical multimecanism for the description of the behavior deviatoric cyclic of materials and particularly the géomatériaux one. Work hardening is kinematic linear for each mechanism and the rule of flow considered is associated. By construction, the model checks the rules of Masing automatically.

The model allows an easy description of the behavior viaS parameters directly obtained from the curves of variation of the modulus of secant rigidity. The user must simply provide the parameters of a hyperbolic function describing the variation of the secant module according to the shearing strain.

It is implemented in the MFront environment, which facilitates the implementation and maintainability of the model.

Contents

1 Theoretical formulation

1.1 Vision 1D of the behavior model of Iwan

A description purely 1D of the model of Iwan consists in an arrangement in parallel or series of elements of Jenkin, each element being composed of a spring and a perfectly plastic shoe rubbing (criterion of Coulomb) in series (Figure 1). The arrangement of the elements and the parameters of the model determine the form of the answer in cyclic shearing.

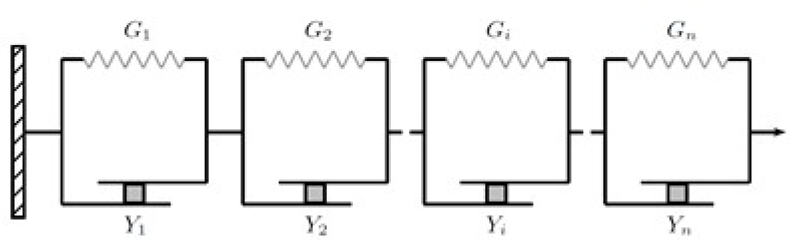	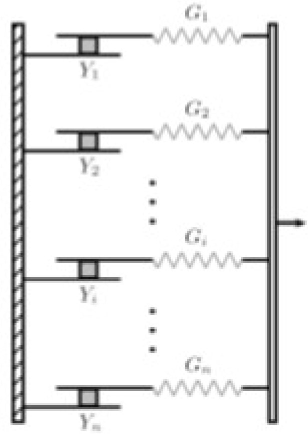
Model parallel series	Model parallel-series

Figure 1.1-1: Rheological models of Iwan 1D [3]

1.2 Formulation 3D model of Iwan

At the time DE description 3D of the behavior, each element is associated with a surface of load, defined starting from the deviatoric constraint associated with each mechanism. One thus passes from a description 1D of behaviour in shearing to a deviatoric description 3D of the behavior starting from the invariants of constraint and deformation, in the following way definite:

Average pressure: $$p=\frac{tr(\boldsymbol{\sigma})}{3} \quad (1)$$

Deviative tensor of constraints: $$\mathbf{S}=\boldsymbol{\sigma}-p\mathbf{I} \quad (2)$$

Diverter of constraints: $$q=\|\mathbf{S}\|_{VM}^{3D}=\sqrt{\frac{3}{2}\mathbf{S}:\mathbf{S}} \quad (3)$$

Voluminal deformation: $$\varepsilon_v=tr(\boldsymbol{\varepsilon}) \quad (4)$$

Deviatoric deformation: $$q_n=|(\mathbf{S}-\mathbf{X}_n)|_{VM}^{3D}=\sqrt{\frac{3}{2}(\mathbf{S}-\mathbf{X}_n):(\mathbf{S}-\mathbf{X}_n)} \quad (5)$$

For an arrangement of type parallel series, when the pressure applied exceeds the ultimate stress Y_n mechanism n , there is work hardening of this mechanism and the associated constraint remains equal to the ultimate stress, i.e only a kinematic work hardening is associated with each mechanism of the model. The model allows a great flexibility in the description of the behavior, in particular by the use potential direct of the data of test laboratory. The error associated with the description of the cyclic behavior depends directly amongst surfaces on load and their layout. However, more deep east the number of mechanisms, plus the system to be solved is numerically expensive, in spite of the passage of a tensorial system to a scalar system (section 2).

The model is based on the theory of elastoplasticity multimecanism. In a description parallel series, he assumes a decomposition additive and independent of the deformations on mechanisms of work hardening:

$$\boldsymbol{\varepsilon}=\sum_{n=1}^{N}\boldsymbol{\varepsilon}_n \quad (6)$$

For each mechanism, the model considers an additive decomposition of the tensor of total deflections $\boldsymbol{\varepsilon}$ in an elastic part $\boldsymbol{\varepsilon}_n^e$ and a plastic part $\boldsymbol{\varepsilon}_n^p$:

$$\boldsymbol{\varepsilon}_n=\boldsymbol{\varepsilon}_n^e+\boldsymbol{\varepsilon}_n^p \quad (7)$$

The relation stress-strain is written directly starting from the tensor of elasticity of order 4, $\boldsymbol{C}$. Elasticity is thus linear and the model does not make it possible to take into account the effect of containment in constant the rubber bands.

$$\boldsymbol{\sigma}=\boldsymbol{C}\boldsymbol{\varepsilon}^e \quad (8)$$

The model describing only the behavior deviatoric, the voluminal answer is supposed to be elastic linear:

$$\varepsilon_v=\frac{p}{3K} \quad (9)$$

where K is the module of compression.

In what follows, the various ingredients necessary to describe the law of behavior within an elastoplastic framework are described, namely:

- Surface of load f : defined the zone within the space of constraints inside which the behavior is linear rubber band ($f<0$ or $f=0$ and $df<0$) and on which the behavior is plastic ($f=0$ and $df>0$).
- Law of flow: rule defining the increment of plastic deformation, in a generic way $\dot{\boldsymbol{\varepsilon}}^p=\dot{\lambda}\boldsymbol{\Psi}$, where $\dot{\lambda}$ is it plastic multiplier and $\boldsymbol{\Psi}$ the law of flow.

1.2.1 Description of the surface of load

The surface of load of each mechanism is written in the following way:

$$f_n=q_n-Y_n \quad (10)$$

Withvec Y_n a constant limit from which the mechanism n hammer-hardened. This value is selected starting from a pure shear test, while associating with each mechanism of work hardening n a couple of values (γ_n,τ_n) with $\gamma=2\varepsilon_{ij}$, $i\neq j$. Y_n is thus directly obtained from τ_n by the following relation:

$$Y_n=\sqrt{\frac{3}{2}}\tau_n \quad (11)$$

Lvalue has of q_n is the deviatoric constraint associated with the mechanism n , in the following way definite:

$$q_n=\|(\mathbf{S}-\mathbf{X}_n)\|_{VM}^{3D}=\sqrt{\frac{3}{2}(\mathbf{S}-\mathbf{X}_n):(\mathbf{S}-\mathbf{X}_n)} \quad (12)$$

The tensor $\mathbf{X}_n$ represent the kinematic work hardening of the mechanism n . It is obtained starting from the expression defined by Prager:

$$\boldsymbol{X}_n=C_n\dot{\boldsymbol{\varepsilon}}_n^p \quad (13)$$

Withvec C_n a parameter of the model determined starting from a monotonous shear test. Owing to the fact that C_n that is to say constant, one considers regards work hardening as being linear.

1.2.2 Description of the Loi of flow

Increments of plastic deformation of each mechanism n are obtained starting from the plastic multiplier $\dot{\lambda}_n$ and of the law of flow $\mathbf{\Psi}_n$:

$$\dot{\boldsymbol{\varepsilon}}_n^p = \dot{\lambda}_n \mathbf{\Psi}_n \quad (14)$$

LE model supposing an associated flow, the law of flow is obtained directly from surface of load:

$$\mathbf{\Psi}_n = \frac{\partial f_n}{\partial \boldsymbol{\sigma}} = \frac{3}{2} \frac{(\boldsymbol{S} - \boldsymbol{X}_n)}{q_n} \quad (15)$$

The plastic multiplier associated with each mechanism is obtained starting from the equation of consistency, i.e. $\dot{f}_n = 0$:

$$\dot{\lambda}_n = \frac{\dfrac{\partial f_n}{\partial \boldsymbol{\sigma}} : C_n : \dot{\boldsymbol{\varepsilon}}}{\dfrac{\partial f_n}{\partial \boldsymbol{\sigma}} : C_n : \mathbf{\Psi}_n - C_n \dfrac{\partial f_n}{\partial \boldsymbol{X}_n} : \mathbf{\Psi}_n} \quad (16)$$

Withvec:

$$\frac{\partial f_n}{\partial \boldsymbol{X}_n} = \frac{3}{2} \frac{(\boldsymbol{S} - \boldsymbol{X}_n)}{q_n} \quad (17)$$

The calculation of multiplier plastic is clarified in the section 2.

1.2.3 Calculation of the parameters of work hardening C_n

Determination of the parameters of kinematic work hardening C_n is made starting from a way of pure stresses shear ($\Delta p = 0$). Let us suppose the answer known for this way of constraints starting from an initial state to worthless deviatoric constraint and worthless plastic deformation. In this thatS:

$$\dot{\boldsymbol{\gamma}} = \left(\frac{1}{2G} + \sum_{n=1}^{N} \frac{1}{C_n} \right) \dot{\boldsymbol{\tau}} \quad (18)$$

From N couples (γ_n, τ_n), one can estimate the value of C_n in a recursive way with the following expression:

$$\frac{1}{C_n} = \frac{\gamma_{k+1} - \gamma_k}{\tau_{k+1} - \tau_k} - \frac{1}{2G} - \sum_{m=1}^{k-1} \frac{1}{C_m} \quad (19)$$

The calculation ofS parameters of kinematic work hardening C_n is made in the block `@InitLocalVars` of Mfront.

2 Digital integration of the law of behavior

The choice in MFront is related to the integration of in the following way written systems of differential equations:

$$\dot{Y}=G(\boldsymbol{Y},t) \quad (20)$$

O ù G is a function a priori nonlinear and supposed has minimum continuously derivable. An implicit integration of type theta method is selected for the digital integration of the non-linear system. In this case the system is written in the following way:

$$\boldsymbol{Y}|_{t+\Delta t}=\boldsymbol{Y}|_t+\Delta t\,\boldsymbol{G}\left((1-\theta)\boldsymbol{Y}|_t+\theta\boldsymbol{Y}|_{t+\Delta t},t+\theta\Delta t\right) \quad (21)$$

Or in an equivalent way:

$$\boldsymbol{F}(\Delta\boldsymbol{Y})=\Delta\boldsymbol{Y}-\Delta t\,\boldsymbol{G}\left((1-\theta)\boldsymbol{Y}|_t+\theta\boldsymbol{Y}|_{t+\Delta t},t+\theta\Delta t\right)=0 \quad (22)$$

The method of Newton-Raphson is selected for the calculation of the zero of the function $F(\Delta Y)$. That is to say:

$$\boldsymbol{F}(\Delta\boldsymbol{Y})=\boldsymbol{F}(\Delta\boldsymbol{Y}^-)+\left(\frac{d\boldsymbol{F}}{d\Delta\boldsymbol{Y}}\right)_{\Delta\boldsymbol{Y}=\Delta\boldsymbol{Y}^-}(\Delta\boldsymbol{Y}+\Delta\boldsymbol{Y}^-) \quad (23)$$

Where $\boldsymbol{J}=\frac{d\boldsymbol{F}}{d\Delta Y}$ is the matrix jacobienne of the function $\boldsymbol{F}$. The approximation $\Delta\boldsymbol{Y}^+$ is obtained while considering $\boldsymbol{F}(\Delta\boldsymbol{Y}^+)=0$, that is to say:

$$\Delta\boldsymbol{Y}^+=\Delta\boldsymbol{Y}^--\left(J^{-1}(\Delta\boldsymbol{Y}^-)\right)\boldsymbol{F}(\Delta\boldsymbol{Y}^-) \quad (24)$$

The difficulty of the method of Newton-Raphson rests on the calculation of the matrix jacobienne, whose analytical calculation is provided in the section 2.4. A purely implicit diagram is selected for the integration of the equations ($\theta=1$), in order to give physically satisfactory results.

According to the approach presented in the document [4], for a linear kinematic work hardening it is possible to reduce the tensorial system to a scalar system, grace to the colinearity the increment of plastic deformation enters $\dot{\boldsymbol{\varepsilon}}_n^p$ and the difference $\boldsymbol{S}-\boldsymbol{X}_n$ in absence of plastic deformation, $\boldsymbol{S}_n^e$. That can be written in the following way for a multiple kinematic work hardening:

$$\boldsymbol{S}_n^e=\boldsymbol{S}^-+\dot{\boldsymbol{S}}-\boldsymbol{X}_n^-=\boldsymbol{S}^-+2\mu\dot{\boldsymbol{\varepsilon}}_n^e-\boldsymbol{X}_n^- \quad (25)$$

Thus:

$$\boldsymbol{S}-\boldsymbol{X}_n=\boldsymbol{S}_n^e-\dot{\boldsymbol{X}}_n=\boldsymbol{S}_n^e-C_n\dot{\boldsymbol{\varepsilon}}_n^p \quad (26)$$

The increment of plastic deformation of the mechanism n is thus written:

$$\dot{\boldsymbol{\varepsilon}}_n^p=\frac{3}{2}\frac{\dot{\lambda}_n}{q_n}\left(\boldsymbol{S}_n^e-C_n\dot{\boldsymbol{\varepsilon}}_n^p\right)=\frac{3}{2}\frac{\dot{\lambda}_n}{q_n^e}\boldsymbol{S}_n^e \quad (27)$$

Withvec q_n^e the standard of $\boldsymbol{S}_n^e$. While replacing $\dot{\boldsymbol{\varepsilon}}_n^p$ in the expression (26), one obtains:

$$\boldsymbol{S}-\boldsymbol{X}_n=\boldsymbol{S}_n^e\left(1-\frac{\frac{3}{2}C_n\dot{\lambda}_n}{q_n^e}\right) \quad (28)$$

In injecting the expression of $\boldsymbol{S}-\boldsymbol{X}_n$ under the condition $f=0$, one obtains:

$$\left(q_n^e-\frac{3}{2}C_n\dot{\lambda}_n\right)-Y_n=0 \quad (29)$$

Insofar as $Y_n>0$ and $q_n^e>0$, this equation admits the following solution for the plastic multiplier:

$$\dot{\lambda}_n = \frac{q_n^e - Y_n}{\frac{3}{2} C_n} \quad (30)$$

Therefore, inevitably:

$$0 < \dot{\lambda}_n < \frac{q_n^e}{\frac{3}{2} C_n} \quad (31)$$

The system to be solved is thus directly based on the plastic multiplier and not on the tensor of work hardening, which is an auxiliary variable (`@AuxiliaryStateVariable`) in the routine MFront.

Insofar as the update of the tensor of constraints in the block `@ComputeStress` be carried out before the iteration of Newton of the algorithm of resolution, in the MFront routine q_n^e is obtained directly starting from the up to date put value of the tensor deviatoric $\mathbf{S}$, q_n .

2.1 Stage of prediction

stage of prediction is managed starting from the keyword factor `PREDICTION` keyword `NEWTON` in the operator of resolution nonlinear, it block `@Predictor` on MFront not needing to be well informed for a resolution with **code_hasster**.

2.2 Stage of correction by algorithm of Newton

The vector of unknown factors being given by $\boldsymbol{Y} = \left(\boldsymbol{\varepsilon}^e, \dot{\lambda}_n\right)$, the iterations of the algorithm of Newton consist in solving the following system:

$$\begin{cases} f_{\varepsilon}^e = \dot{\boldsymbol{\varepsilon}}^e - \dot{\boldsymbol{\varepsilon}} + \sum_{n=1}^{N} \dot{\lambda}_n \boldsymbol{\Psi}_n = 0 \\ f_{\dot{\lambda}_n} = \dot{\lambda}_n - \frac{q_n - Y_n}{\frac{3}{2} C_n} \end{cases} \quad (32)$$

The writing of the equations in scalar form allows to reduce the size of the system of equations to be solved and this manner of improving the performance of the model.

2.3 Stage of update

Once the convergence of the completed local resolution, the vector of the unknown factors is updated classically in the following way:

$$\boldsymbol{Y}^{+} = \boldsymbol{Y}_{j+1} = \boldsymbol{Y}_j + \delta \boldsymbol{Y}_{j+1} \quad (33)$$

One also profits to update the auxiliary variables $\boldsymbol{X}_n$ and $\boldsymbol{\varepsilon}^p$ in the block `@UpdateAuxiliaryStateVars` .

2.4 Calculation of the terms of the matrix jacobienne

In order to calculate the coherent tangent operator (directive **`@TangentOperator`** of MFront), it is necessary to have the terms of the matrix jacobienne $\boldsymbol{J}$ system of equations to be solved. In this case, the matrix jacobienne is written in the following way:

$$J=\begin{bmatrix} \frac{\partial f_{\dot{\lambda}_n}}{\partial \dot{\lambda}_n} \boldsymbol{I_4} & \frac{\partial \boldsymbol{f}_{\varepsilon}^{e}}{\partial \dot{\lambda}_n} \otimes \boldsymbol{I} \\ \frac{\partial f_{\dot{\lambda}_n}}{\partial \dot{\boldsymbol{\varepsilon}}^{e}} \otimes \boldsymbol{I} & \frac{\partial \boldsymbol{f}_{\varepsilon}^{e}}{\partial \dot{\boldsymbol{\varepsilon}}^{e}} \end{bmatrix} \tag{34}$$

MFront makes it possible to numerically obtain these terms by the use of the algorithm **NewtonRaphson_NumericalJacobian** . But the version developed here uses an analytical matrix, more powerful. D years what follows, one proposes an analytical description of the derivative partial while considering $\theta=1$.

The terms to be calculated are the following: $\frac{\partial f_{\dot{\lambda}_n}}{\partial \dot{\lambda}_n}$ (scalar), $\frac{\partial f_{\dot{\lambda}_n}}{\partial \dot{\boldsymbol{\varepsilon}}^{e}}$ (tensor of order two), $\frac{\partial \boldsymbol{f}_{\varepsilon}^{e}}{\partial \dot{\lambda}_n}$ (tensor of order two) and $\frac{\partial \boldsymbol{f}_{\varepsilon}^{e}}{\partial \dot{\boldsymbol{\varepsilon}}^{e}}$ (tensor of order four). The derivative partial compared to the plastic multiplier $\dot{\lambda}_n$ for each mechanism n are immediate:

$$\frac{\partial f_{\dot{\lambda}_n}}{\partial \dot{\lambda}_n}=1 \tag{35}$$

And:

$$\frac{\partial f_{\varepsilon}^{e}}{\partial \dot{\lambda}_n}=\boldsymbol{\Psi}_n \tag{36}$$

It remains to calculate the derivative compared to the elastic strain $\dot{\boldsymbol{\varepsilon}}^{e}$. One obtains the following expression for $\frac{\partial \boldsymbol{f}_{\varepsilon}^{e}}{\partial \dot{\boldsymbol{\varepsilon}}^{e}}$:

$$\frac{\partial \boldsymbol{f}_{\varepsilon}^{e}}{\partial \dot{\boldsymbol{\varepsilon}}^{e}}=\frac{2\mu\dot{\lambda}_n}{q_n}\left(\boldsymbol{M}-\boldsymbol{\Psi}\otimes\boldsymbol{\Psi}\right) \tag{37}$$

With the expression according to E for the tensor of order 4 $\boldsymbol{M}$:

$$\boldsymbol{M}=\frac{3}{2}\left(\boldsymbol{I_4}-\frac{1}{3}\boldsymbol{I}\otimes\boldsymbol{I}\right) \tag{38}$$

$\boldsymbol{I_4}$ being the tensor identity of order four and $\otimes$ the tensorial product. The calculation of $\frac{\partial f_{\dot{\lambda}_n}}{\partial \dot{\boldsymbol{\varepsilon}}^{e}}$ for each mechanism n is more delicate, because it depends if the mechanism is active or not during iterations of Newton.

SI the mechanism is not active ($f_{\dot{\lambda}_n}<0$):

$$\frac{\partial f_{\dot{\lambda}_n}}{\partial \dot{\boldsymbol{\varepsilon}}^{e}}=0 \tag{39}$$

If the mechanism is credit ($f_{\dot{\lambda}_n}>0$):

$$\frac{\partial f_{\dot{\lambda}_n}}{\partial \dot{\boldsymbol{\varepsilon}}^{e}}=-\frac{4}{3}\frac{\mu}{C_n}\boldsymbol{\Psi}_n:\boldsymbol{I}_4 \tag{40}$$

EN checking the valueS obtainedES for the digital jacobienne (option **@CompareToNumericalJacobian**), it was observed that for values of $f_{\dot{\lambda}_n}$ close relations of zero (positive or negative), $\frac{\partial f_{\dot{\lambda}_n}}{\partial \dot{\boldsymbol{\varepsilon}}^{e}}$ half of the expression obtained is worth analytically. The treatment in this case is made by considering a breaking value of the value of the surface of load of 10^{-6} .

3 Establishment in *Code_hasster*

The number of surfaces of kinematic work hardening of the model is fixed at twelve. The first eleven surfaces make it possible to cover the range of shearing strains enters 10^{-5} and $2{,}0\times10^{-2}$. One considers an elastic linear behavior for deformations smaller than 10^{-5} and a maximum shearing strain of $2{,}0\times10^{-2}$. Values of interpolation used are the following ones:

γ_1=1.00000000e-05
γ_2=2.15443469e-05
γ_3=4.64158883e-05
γ_4=1.00000000e-04
γ_5=2.15443469e-04
γ_6=4.64158883e-04
γ_7=1.00000000e-03
γ_8=2.15443469e-03
γ_9=4.64158883e-03
γ_{10}=1.00000000e-02
γ_{11}=2.00000000e-02

Beyond of a shearing strain of $2{,}0\times10^{-2}$, the results of the model will be approximate because the last surface, positioned with a shearing strain of $1{,}0\times10^{-1}$, simply allows to make so that the model provides an answer enters $2{,}0\times10^{-2}$ and $1{,}0\times10^{-1}$.

3.1 Internal variables

Internal variables (`StateVariable` and `AuxiliaryStateVariable` in the MFront language) following are available during a calculation with the law of behavior of Iwan:

`Eel (1-6)`	Tensor of elastic strain
`pp (7-18)`	Vector of scalar plastic multipliers $\dot{\lambda}_n$
`X (19-91)`	Vector of tensors of kinematic work hardening $\mathbf{X}_n$
`fn (92-103)`	Vector of values of the surface of load

3.2 Description model under MFront

The behavior is defined in the file `Iwan.will mfront`.

Parser/DSL	`Implicit`
Algorithm	`NewtonRaphson`

	`@Theta 1.`
	`@IterMax 50`
	`@Epsilon 1.E-12`

Internal variables (@StateVariable)	`real pp [1:12]`
Auxiliary internal variables (@AuxiliaryStateVariable)	`Stensor X [1:12]` `real fn [1:12]`
Variables of orders (@ExternalStateVariable)	***none***

Modelings	`'3D'`
	`'AXIS'`
	`'D_PLAN'`

Deformations	`'SMALL'`
	`'PETIT_REAC'`
	`'GDEF_LOG'`

4 Features and checking

Here the list of the cases of checking available:

SSNV205 B	[V6.04.205]	cyclic shear test controlled in deformation.
SSNV207 B	[V6.04.207]	cyclic shearing controlled in deformation with microphone-discharge.
COMP012 D	[V6.07.112]	Cyclic shear test with the order CALC_ESSAI_GEOMECA

5 Bibliography

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