Law of Rankine

Summary:

This document presents the method of resolution of the law of Rankine.

Contents

Notations

$\sigma_1 \geq \sigma_2 \geq \sigma_3$	Principal constraints (in this order)
E	Young modulus
ν	Poisson's ratio
$K=\frac{E}{3(1-2\nu)}$	Elastic module of compressibility
$G=\frac{E}{2(1+\nu)}$	Elastic modulus of rigidity
σ_t	Limit of traction material
$p=\frac{I_1}{3}=\frac{trace(\boldsymbol{\sigma})}{3}$	Average constraint
$p<0$	Convention of sign for the constraint in compression
$\boldsymbol{\sigma}^e$	Tensor of elastic prediction constraints
$\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}^e+\boldsymbol{\varepsilon}^p$	Tensors of the deflections total, elastic and increment of plastic deformation
$\varepsilon_v^p=trace(\boldsymbol{\varepsilon}^p)$	Increment of the voluminal plastic deformation
$\tilde{\boldsymbol{\varepsilon}}^p=\boldsymbol{\varepsilon}^p-\frac{\varepsilon_v^p}{3}\mathbf{1}$	Increment of the deviatoric plastic deformation
$\varepsilon^p=\|\tilde{\boldsymbol{\varepsilon}}^p\|=\sqrt{\frac{3}{2}\tilde{\boldsymbol{\varepsilon}}^p:\tilde{\boldsymbol{\varepsilon}}^p}$	Increment of deviatoric plastic deformation, or increment of equivalent deformation normalizes

1 Introduction

The law of Rankine is Formulée in terms of principal constraints. This formulation suppose itisotropy of material (see [1] and [2]) Indeed, this condition is necessary to guarantee that the method of radial return preserves the principal directions. Its interest lies in the fact that it simplifies the writing of the equations and authorizes of this fact of the very powerful methods of resolution (bus quasi-analytical).

The elastic behavior is purely linear.

The surface of charge is characterized by three plans within the space of principal constraints $\sigma_1 \geq \sigma_2 \geq \sigma_3$. Each one of these plans is characterized by an equation of the type:

$$R_{i=1,2,3}(\boldsymbol{\sigma}^{+}) = \sigma_i^{+} - \sigma_t = 0 \qquad (1)$$

Where σ_t is the single data of material and characterizes limit in traction material. The law is associated.

2 Local integration of the law of Rankine

The rate of plastic deformation is given using the formula of Koiter:

$$d\boldsymbol{\varepsilon}^p=\sum_{j=1}^{m} d\mu_j \frac{\partial R_j}{\partial \boldsymbol{\sigma}}=\sum_{j=1}^{m} d\mu_j \boldsymbol{n}_j \quad (2)$$

Where $d\mu_i \geq 0$ are the plastic multipliers associated with the mechanisms i , and:

$$\frac{\partial R_i}{\partial \sigma_j}=n_{ij}=\delta_{ij} \quad (3)$$

And Où m characterize the number of active mechanisms, equal to one, two or three according to the following situations:

- the final constraint is inside the surface of load, the point is regular and $m=1$;
- the final constraint is on an edge of the cone, the point is singular and $m=2$;
- the final constraint is neither inside the surface of load nor on an edge. It is then projected at the top of the cone, the point is singular and $m=3$;

The final constraint $\boldsymbol{\sigma}^+$ is calculated starting from a noted elastic prediction $\boldsymbol{\sigma}^e$ and of a correction $\Delta\boldsymbol{\sigma}_C=\boldsymbol{C}.\Delta\boldsymbol{\varepsilon}^p$ so that:

$$\boldsymbol{\sigma}^+=\boldsymbol{\sigma}^e-\Delta\boldsymbol{\sigma}_C=\boldsymbol{\sigma}^e-\sum_{j=1}^{m}\Delta\mu_j \boldsymbol{C}.\boldsymbol{n}_j \quad (4)$$

Plastic multipliers $d\mu_j$ are calculated by injecting the equation (**4**) in the equation (**1**), which gives:

$$\sum_{j=1}^{m} d\mu_j (\boldsymbol{C}.\boldsymbol{n}_j)_i=\sigma_i^e-\sigma_t \quad (5)$$

In what follows, one details the expressions corresponding to the various situations mentioned above.

The plastic deformations voluminal and equivalent are written:

$$\begin{cases}\varepsilon_v^p=\varepsilon_1+\varepsilon_2+\varepsilon_3\\ \tilde{\boldsymbol{\varepsilon}}^p=\|\tilde{\boldsymbol{\varepsilon}}^p\|=\sqrt{\frac{2}{3}\tilde{\boldsymbol{\varepsilon}}^p:\tilde{\boldsymbol{\varepsilon}}^p}\end{cases} \quad (6)$$

AveC $\tilde{\boldsymbol{\varepsilon}}^p=\boldsymbol{\varepsilon}^p-\frac{\varepsilon_v^p}{3}\mathbf{1}$.

2.1 Case where only one mechanism is active

The mechanism R_1 is active. From where:

$$d\mu\left(K+\frac{4}{3}G\right)=R_1(\boldsymbol{\sigma}^e)=\sigma_1^e-\sigma_t \quad (7)$$

With σ_t limit of traction of material. From where:

$$d\mu=\frac{\langle R_1(\boldsymbol{\sigma}^e)\rangle_+}{K+\frac{4}{3}G} \quad (8)$$

Where the operator $\langle\ \rangle_+$ indicate the positive part of the associated size.

The plastic deformation is written:

$$\delta \boldsymbol{\varepsilon}^p = \delta\mu \frac{\partial R_1}{\partial \boldsymbol{\sigma}} = \delta\mu \begin{pmatrix}1\\0\\0\end{pmatrix} \quad (9)$$

And correction $\Delta\boldsymbol{\sigma}_C$ is written:

$$\Delta\boldsymbol{\sigma}_C = \boldsymbol{C}.\delta\boldsymbol{\varepsilon}^p = \delta\mu\,\boldsymbol{C}.\begin{pmatrix}1\\0\\0\end{pmatrix} = \delta\mu\begin{pmatrix}K+\frac{4}{3}G\\ K-\frac{2}{3}G\\ K-\frac{2}{3}G\end{pmatrix} \quad (10)$$

The plastic deformations voluminal and equivalent are written:

$$\begin{cases}\delta\varepsilon_v^p = \delta\mu\\ \delta\tilde{\boldsymbol{\varepsilon}}^p = \sqrt{\frac{2}{3}.\frac{1}{9}\begin{pmatrix}2\delta\mu\\-\delta\mu\\-\delta\mu\end{pmatrix}.\begin{pmatrix}2\delta\mu\\-\delta\mu\\-\delta\mu\end{pmatrix}} = \frac{2}{3}\delta\mu\end{cases} \quad (11)$$

2.2 Case where two mechanisms are active

Mechanisms R_1 and R_2 are active. From where:

$$\begin{cases} d\mu_1 \underbrace{\left(K+\frac{4}{3}G\right)}_{A} + d\mu_2 \underbrace{\left(K-\frac{2}{3}G\right)}_{B} = R_1(\boldsymbol{\sigma}^e) = \sigma_1^e - \sigma_t \\ d\mu_1\left(K-\frac{2}{3}G\right) + d\mu_2\left(K+\frac{4}{3}G\right) = R_2(\boldsymbol{\sigma}^e) = \sigma_2^e - \sigma_t \end{cases} \quad (12)$$

While posing det the determinant of the system, one a:

$$det = A^2 - B^2 = 4G\left(K+\frac{G}{3}\right) > 0 \quad (13)$$

The determinant being always strictly null, IL exists always a solution which is written:

$$\begin{cases} d\mu_1 = \frac{\langle AR_1(\boldsymbol{\sigma}^e) - BR_2(\boldsymbol{\sigma}^e)\rangle_+}{det} \\ d\mu_2 = \frac{\langle AR_2(\boldsymbol{\sigma}^e) - BR_1(\boldsymbol{\sigma}^e)\rangle_+}{det} \end{cases} \quad (14)$$

The plastic deformation is written:

$$\delta\boldsymbol{\varepsilon}^P = \delta\mu_1 \frac{\partial R_1}{\partial\boldsymbol{\sigma}} + \delta\mu_2\frac{\partial R_2}{\partial\boldsymbol{\sigma}} = \delta\mu_1\begin{pmatrix}1\\0\\0\end{pmatrix} + \delta\mu_2\begin{pmatrix}0\\1\\0\end{pmatrix} \quad (15)$$

And correction $\Delta\boldsymbol{\sigma}_C$ is written:

$$\Delta\boldsymbol{\sigma}_C=\boldsymbol{C}.\delta\boldsymbol{\varepsilon}^P=\delta\mu_1\boldsymbol{C}.\begin{pmatrix}1\\0\\0\end{pmatrix}+\delta\mu_2\boldsymbol{C}.\begin{pmatrix}0\\1\\0\end{pmatrix}=\delta\mu_1\begin{pmatrix}K+\frac{4}{3}G\\K-\frac{2}{3}G\\K-\frac{2}{3}G\end{pmatrix}+\delta\mu_2\begin{pmatrix}K-\frac{2}{3}G\\K+\frac{4}{3}G\\K-\frac{2}{3}G\end{pmatrix} \tag{16}$$

The plastic deformations voluminal and equivalent are written:

$$\begin{cases}\delta\varepsilon_v^P=\delta\mu_1+\delta\mu_2\\ \delta\tilde{\boldsymbol{\varepsilon}}^p=\sqrt{\frac{2}{3}.\frac{1}{9}\begin{pmatrix}2\delta\mu_1-\delta\mu_2\\2\delta\mu_2-\delta\mu_1\\-\delta\mu_1-\delta\mu_2\end{pmatrix}\cdot\begin{pmatrix}2\delta\mu_1-\delta\mu_2\\2\delta\mu_2-\delta\mu_1\\-\delta\mu_1-\delta\mu_2\end{pmatrix}}=\frac{2}{3}\sqrt{\delta\mu_1^2+\delta\mu_2^2-\delta\mu_1\delta\mu_2}\end{cases} \tag{17}$$

2.3 Case of projection at the top of the cone

In this case, one has directly:

$$\begin{cases}p^+=p^e-K\,\Delta\varepsilon_v^p=\sigma_t\\ \boldsymbol{\sigma}^+=p^+\mathbf{1}\end{cases} \tag{18}$$

The voluminal plastic deformation directly is obtained:

$$\Delta\varepsilon_v^p=\frac{p^e-\sigma_t}{K} \tag{19}$$

Mechanisms R_1, R_2 and R_3 Shave active. From where:

$$\begin{cases}d\mu_1A+(d\mu_2+d\mu_3)B=R_1(\boldsymbol{\sigma}^e)=\sigma_1^e-\sigma_t\\ d\mu_2A+(d\mu_1+d\mu_3)B=R_2(\boldsymbol{\sigma}^e)=\sigma_2^e-\sigma_t\\ d\mu_3A+(d\mu_1+d\mu_2)B=R_3(\boldsymbol{\sigma}^e)=\sigma_3^e-\sigma_t\end{cases} \tag{20}$$

It is shown that after some algebraic handling, one obtains:

$$\begin{cases}d\mu_1=\frac{(A+B)R_1-B(R_2+R_3)}{6KG}\\ d\mu_2=\frac{(A+B)R_2-B(R_1+R_3)}{6KG}\\ d\mu_3=\frac{(A+B)R_3-B(R_1+R_2)}{6KG}\end{cases} \tag{21}$$

The equivalent plastic deformation is written then:

$$\delta\tilde{\boldsymbol{\varepsilon}}^p=\sqrt{\frac{2}{3}\cdot\frac{1}{9}\begin{pmatrix}2\delta\mu_1-\delta\mu_2-\delta\mu_3\\2\delta\mu_2-\delta\mu_1-\delta\mu_3\\2\delta\mu_3-\delta\mu_1-\delta\mu_2\end{pmatrix}\cdot\begin{pmatrix}2\delta\mu_1-\delta\mu_2-\delta\mu_3\\2\delta\mu_2-\delta\mu_1-\delta\mu_3\\2\delta\mu_3-\delta\mu_1-\delta\mu_2\end{pmatrix}}$$

$$\delta\tilde{\boldsymbol{\varepsilon}}^p=\frac{2}{3}\sqrt{\delta\mu_1^2+\delta\mu_2^2+\delta\mu_3^2-\delta\mu_1\delta\mu_2-\delta\mu_1\delta\mu_3-\delta\mu_2\delta\mu_3} \quad (22)$$

2.4 Internal variables of the model

The internal variables of the model are with the number of nine :

- `V1` is the voluminal plastic deformation ε_v^p ;
- `V2` is the equivalent plastic deformation (deviatoric) $\tilde{\varepsilon}^p=\|\tilde{\boldsymbol{\varepsilon}}^p\|$;
- `V3` is the indicator of plasticity;
- `V4` with `V9` are the six components of the tensor of the plastic deformations $\boldsymbol{\varepsilon}^p$.

3 Form of the tangent matrix in the principal base

3.1 Case where only one mechanism is active

One a:

$$d\boldsymbol{\sigma}^{+}=\boldsymbol{C}.\delta\boldsymbol{\varepsilon}-\frac{\delta\sigma_1}{K+\frac{4}{3}G}\begin{pmatrix}K+\frac{4}{3}G\\K-\frac{2}{3}G\\K-\frac{2}{3}G\end{pmatrix} \quad (23)$$

While posing $\begin{cases}A=K+\frac{4}{3}G\\B=K-\frac{2}{3}G\end{cases}$, ON a:

$$\frac{\delta\sigma_1}{A}\begin{pmatrix}A\\B\\B\end{pmatrix}=\frac{1}{A}\begin{pmatrix}A\\B\\B\end{pmatrix}\begin{pmatrix}1&0&0\end{pmatrix}.\boldsymbol{C}.\delta\boldsymbol{\varepsilon}=\frac{1}{A}\begin{pmatrix}A\\B\\B\end{pmatrix}\begin{pmatrix}A&B&B\end{pmatrix}.\delta\boldsymbol{\varepsilon}=\begin{bmatrix}A&B&B\\B&\frac{B^2}{A}&\frac{B^2}{A}\\B&\frac{B^2}{A}&\frac{B^2}{A}\end{bmatrix}.\delta\boldsymbol{\varepsilon} \quad (24)$$

That is to say the following expression of the tangent matrix $\boldsymbol{T}$ (see 57 for the matrix of elasticity $\boldsymbol{C}$) :

$$\boldsymbol{T}=\frac{\delta\boldsymbol{\sigma}^{+}}{\delta\boldsymbol{\varepsilon}}=\boldsymbol{C}-\begin{bmatrix}A&B&B\\B&\frac{B^2}{A}&\frac{B^2}{A}\\B&\frac{B^2}{A}&\frac{B^2}{A}\end{bmatrix}=\begin{bmatrix}0&0&0\\0&\frac{A^2-B^2}{A}&\frac{B(A-B)}{A}\\0&\frac{B(A-B)}{A}&\frac{A^2-B^2}{A}\end{bmatrix} \quad (25)$$

3.2 Case where two mechanisms are active

One a:

$$d\boldsymbol{\sigma}^{+}=\boldsymbol{C}.\delta\boldsymbol{\varepsilon}-\underbrace{\left(\frac{A\delta\sigma_1-B\delta\sigma_2}{det}\begin{pmatrix}A\\B\\B\end{pmatrix}+\frac{A\delta\sigma_2-B\delta\sigma_1}{det}\begin{pmatrix}B\\A\\B\end{pmatrix}\right)}_{\delta\Delta\sigma_C} \quad (26)$$

One obtains :

$$\delta\Delta\boldsymbol{\sigma}_C=\delta\sigma_1\begin{pmatrix}1\\0\\\frac{B}{A+B}\end{pmatrix}+\delta\sigma_2\begin{pmatrix}0\\1\\\frac{B}{A+B}\end{pmatrix}=\left[\begin{pmatrix}1\\0\\\frac{B}{A+B}\end{pmatrix}\begin{pmatrix}A&B&B\end{pmatrix}+\begin{pmatrix}0\\1\\\frac{B}{A+B}\end{pmatrix}\begin{pmatrix}B&A&B\end{pmatrix}\right].\delta\boldsymbol{\varepsilon} \quad (27)$$

That is to say:

$$\delta\Delta\boldsymbol{\sigma}_C=\left[\begin{bmatrix} A & B & B \\ 0 & 0 & 0 \\ \frac{AB}{A+B} & \frac{B^2}{A+B} & \frac{B^2}{A+B} \end{bmatrix}+\begin{bmatrix} 0 & 0 & 0 \\ B & A & B \\ \frac{B^2}{A+B} & \frac{AB}{A+B} & \frac{B^2}{A+B} \end{bmatrix}\right].\delta\boldsymbol{\varepsilon}$$

$$\delta\Delta\boldsymbol{\sigma}_C=\begin{bmatrix} A & B & B \\ B & A & B \\ B & B & \frac{2B^2}{A+B} \end{bmatrix}.\delta\boldsymbol{\varepsilon} \tag{28}$$

That is to say the following expression of the tangent matrix $\boldsymbol{T}$:

$$\boldsymbol{T}=\frac{\delta\boldsymbol{\sigma}^+}{\delta\boldsymbol{\varepsilon}}=\boldsymbol{C}-\begin{bmatrix} A & B & B \\ B & A & B \\ B & B & \frac{2B^2}{A+B} \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{A(A+B)-2B^2}{A+B} \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{3KG}{K+\frac{G}{3}} \end{bmatrix} \tag{29}$$

3.3 Case of projection at the top of the cone

It is shown that itone has crudely $\boldsymbol{T}=\boldsymbol{0}$.

4 Form of the tangent matrix in the total base

The paragraph **§3** allows to build the consistent tangent matrix in the principal base, noted $\boldsymbol{T}$. It is advisable from now on to bring back this matrix in the total base (Cartesian), that one will note $\bar{\boldsymbol{T}}$.

4.1 Notice preliminary

It should be noted that the construction of this consistent tangent matrix is a crucial stage at the same time for the robustness and the performance of the algorithm:

- Firstly, it is perfectly known that such a matrix allows a quadratic rate of convergence for the process of Newton;
- Secondly, this matrix gives an account of the rotation of the principal directions during an increment. Without it, the formulation of the law of Rankine in terms of principal constraints described in the paragraph **§1** would not be complete, since principal constraints, maintained fixed during the local integration of the law (**§2**), could not turn on the total level of the structure.

In this paragraph, one describes in detail the method allowing to build $\bar{\boldsymbol{T}}$ from $\boldsymbol{T}$. In appendix (§5), one will find the elements of theory necessary to the transformation of the tensorial quantities from one base to another.

4.2 Application to the case of Rankine

The transposition of the formulas appendix §5 with the numeric work implementation deserves some precise details. There are first of all the following correspondences:

- $\boldsymbol{X}=\hat{\boldsymbol{\varepsilon}}^{pred}$ and $x_\alpha=\varepsilon_\alpha^{pred}$;
- $\boldsymbol{Y}=\hat{\boldsymbol{\sigma}}^{+}$ and $y_\alpha=\sigma_\alpha^{+}$;
- $\boldsymbol{E}_\alpha=\hat{\boldsymbol{d}}_\alpha^{pred}\otimes\hat{\boldsymbol{d}}_\alpha^{pred}$;
- $(\boldsymbol{T})_{\alpha\beta}=\dfrac{\partial y_\alpha}{\partial x_\beta}$ is the consistent tangent matrix in the principal base calculated in the paragraph **§3** ;

The notation ^{pred} indicate that one works with “predicted” sizes given as starter by the process of Newton, the notation $^{+}$ with sizes resulting from the local resolution of the law of behavior, and the notation $\hat{}$ with the base of Voigt. It will be noted that the predicted principal directions $\hat{\boldsymbol{d}}_\alpha^{pred}$ are fixed during the local resolution, which is coherent with the assumption of isotropy adopted (see the explanations of the paragraph **§5**).

Having all this information at the conclusion of the local resolution of the law of behavior, one from of deduced the consistent tangent matrix $\bar{\boldsymbol{T}}=\bar{\boldsymbol{D}}$ expressed in the base of projection $\bar{\boldsymbol{b}}$ defined in the paragraph **§6 :**

- The equation (**40**) in the two-dimensional case in plane constraints (`C_PLAN`);
- The equation (**44**) in the two-dimensional case in deformation planes (`D_PLAN`) or axisymmetric (`AXIS`);
- The equation (**52**) in the three-dimensional case (3D);

The second important information relates to the convention of writing of the various tensors. Indeed, by preoccupation with general information, a notation used for the tensors in all the paragraph **§4** is the classical notation, revealing of the tensors until the order four. This writing is unsuitable with the digital resolution, where one prefers to use condensed notations made possible by the fact that one works with tensors symmetrical of order two (constraints and deformations are it always). One distinguishes two forms from notations condensed correspondent at two bases of projection (see **§6**):

- The orthonormal base $\bar{\boldsymbol{b}}$ symmetrical tensors of order two. It is in this base that the constraints and the deformations are given to the entry and the exit of the local resolution of the behavior;
- The base known as of Voigt $\hat{\boldsymbol{b}}$, much more convenient to use at the time of the local digital resolution of the behavior because it avoids having to handle coefficients in $\sqrt{2}$ at the time of the matric operations;

4.3 Diagram of resolution law of behavior

Entries:

- Constraints expressed in the total base $\bar{\boldsymbol{\sigma}}^{-}$;
- Increment of deformation expressed in the total base $\Delta\bar{\boldsymbol{\varepsilon}}$;

Calculations:

- The elastic constraints are evaluated $\hat{\boldsymbol{\sigma}}^{e}$ and elastic strain $\hat{\boldsymbol{\varepsilon}}^{e}$ in the base of Voigt;
- One transforms them in the principal base, one obtains $\boldsymbol{\sigma}^{e}$ and $\boldsymbol{\varepsilon}^{e}$;
- One integrates the law of behavior and one obtains the increment of plastic deformation $\Delta\boldsymbol{\varepsilon}^{p}$ and constraints in the base of Voigt $\hat{\boldsymbol{\sigma}}^{+}$;
- Calculation of the tangent matrix in the principal base:
  - $\left(\hat{\boldsymbol{\sigma}}^{+},\hat{\boldsymbol{\varepsilon}}^{e}\right)\rightarrow\hat{\boldsymbol{E}}_{\alpha}$ then $T_{\alpha\beta}=\frac{\partial\sigma_{\alpha}}{\partial\varepsilon_{\beta}}$
- Transfer of the tangent matrix in the base of Voigt:
  - $\left(\hat{\boldsymbol{\sigma}}^{+},\hat{\boldsymbol{\varepsilon}}^{e},\hat{\boldsymbol{E}}_{\alpha},\boldsymbol{T}\right)\rightarrow\hat{T}_{ij}=\frac{\partial\sigma_{i}}{\partial\varepsilon_{j}}$
- Transfer of the tangent matrix in the base of total:
  - $\hat{\boldsymbol{T}}\rightarrow\bar{\boldsymbol{T}}$

5 Appendix: Q uelques results on the isotropic symmetrical tensors of order two

5.1 Definition of an isotropic symmetrical tensor

One defines by S^3 the space of symmetrical tensors of order two in the vector space of dimension $n=3$, and tensors $\boldsymbol{Y}\in S^3$ and $\boldsymbol{X}\in S^3$ such as:

$$\boldsymbol{Y}(\boldsymbol{X}):S^3\rightarrow S^3 \quad (30)$$

The tensorial function $\boldsymbol{Y}(\boldsymbol{X})$ is known as isotropic if:

$$\boldsymbol{R}.\boldsymbol{Y}(\boldsymbol{X}).\boldsymbol{R}^t=\boldsymbol{Y}(\boldsymbol{R}.\boldsymbol{X}.\boldsymbol{R}^t) \quad (31)$$

Whatever the rotation $\boldsymbol{R}$. The assumption of isotropy implies that $\boldsymbol{Y}$ and $\boldsymbol{X}$ are coaxial, i.e. qu'they have the same principal directions $\boldsymbol{d}_{\alpha=1,2,3}$. One notes:

$$\begin{aligned}\boldsymbol{X}&=\sum_{\alpha=1}^{3} x_\alpha(\boldsymbol{d}_\alpha\otimes\boldsymbol{d}_\alpha)=\sum_{\alpha=1}^{3} x_\alpha\boldsymbol{E}_\alpha\\ \boldsymbol{Y}(\boldsymbol{X})&=\sum_{\alpha=1}^{3} y_\alpha(\boldsymbol{d}_\alpha\otimes\boldsymbol{d}_\alpha)=\sum_{\alpha=1}^{3} y_\alpha\boldsymbol{E}_\alpha\end{aligned} \quad (32)$$

Where $y_\alpha=y_\alpha(x_1,x_2,x_3)$ and x_α they represent eigenvalues of $\boldsymbol{Y}$ and $\boldsymbol{X}$, respectively.

5.2 Derived from an isotropic tensorial function of order two

It is supposed that the isotropic tensorial function $\boldsymbol{Y}(\boldsymbol{X})$ is differentiable compared to $\boldsymbol{X}$, and his derivative is defined $\boldsymbol{D}$ such as:

$$\boldsymbol{D}(\boldsymbol{X})\stackrel{\text{def}}{=}\frac{d\,\boldsymbol{Y}(\boldsymbol{X})}{d\,\boldsymbol{X}} \quad (33)$$

Applied to the equation (**32**), the following expression is obtained:

$$\boldsymbol{D}(\boldsymbol{X})=\sum_{\alpha=1}^{3}\left(\boldsymbol{E}_\alpha\otimes\frac{d\,y_\alpha}{d\,\boldsymbol{X}}+y_\alpha\frac{d\,\boldsymbol{E}_\alpha}{d\,\boldsymbol{X}}\right)=\sum_{\alpha=1}^{3}\left(y_\alpha\frac{d\,\boldsymbol{E}_\alpha}{d\,\boldsymbol{X}}+\sum_{\beta=1}^{3}\frac{\partial y_\alpha}{\partial x_\beta}\boldsymbol{E}_\alpha\otimes\frac{d\,x_\beta}{d\,\boldsymbol{X}}\right) \quad (34)$$

5.2.1 Two-dimensional case of type forced plane (`C_PLAN`)

In dimension two (cases `C_PLAN`), the characteristic equation $det(\boldsymbol{X}-x_\alpha\boldsymbol{I})=0$ give a quadratic equation of the eigenvalues $x_{\alpha=1,2}$ of $\boldsymbol{X}$ following type:

$$x_\alpha^2-I_1x_\alpha+I_2=0 \text{ with } \alpha=1,2 \quad (35)$$

With:

$$\begin{cases}I_1=trace(\boldsymbol{X})=X_{11}+X_{22}\\ I_2=det(\boldsymbol{X})=X_{11}X_{22}-X_{12}X_{21}\end{cases} \quad (36)$$

The resolution of the spectral problem easily gives the following solutions for the eigenvalues:

$$\begin{cases} x_1 = \dfrac{I_1 + \sqrt{I_1^2 - 4 I_2}}{2} \\ x_2 = \dfrac{I_1 - \sqrt{I_1^2 - 4 I_2}}{2} \end{cases} \quad (37)$$

And clean vectors, taking account of the multiplicity of the eigenvalues:

$$\begin{cases} \boldsymbol{E}_\alpha = \dfrac{\boldsymbol{X} + (x_\alpha - I_1)\boldsymbol{I}}{2 x_\alpha - I_1} & si \quad x_1 \neq x_2 \\ \boldsymbol{E}_1 = \boldsymbol{I} & si \quad x_1 = x_2 \end{cases} \quad (38)$$

In particular, Carlson and Hoger show that if $x_1 \neq x_2$, one a:

$$\frac{d x_\alpha}{d \boldsymbol{X}} = \boldsymbol{E}_\alpha \quad (39)$$

By using the equations (**37**), (**38**) and (**39**) in (**34**), the expression of the derivative is obtained $\boldsymbol{D}(\boldsymbol{X})$, taking account of the multiplicity of the eigenvalues:

$$\boldsymbol{D}(\boldsymbol{X}) = \begin{cases} \dfrac{y_1 - y_2}{x_1 - x_2} [\boldsymbol{I}_S - \boldsymbol{E}_1 \otimes \boldsymbol{E}_1 - \boldsymbol{E}_2 \otimes \boldsymbol{E}_2] + \displaystyle\sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} \frac{\partial y_\alpha}{\partial x_\beta} \boldsymbol{E}_\alpha \otimes \boldsymbol{E}_\beta & si \quad x_1 \neq x_2 \\ \left(\dfrac{\partial y_1}{\partial x_1} - \dfrac{\partial y_1}{\partial x_2} \right) \boldsymbol{I}_S + \dfrac{\partial y_1}{\partial x_2} \boldsymbol{I} \otimes \boldsymbol{I} & si \quad x_1 = x_2 \end{cases} \quad (40)$$

With the matrix identity $\boldsymbol{I}$:

$$(\boldsymbol{I})_{ijkl} = \delta_{ik} \delta_{jl} \quad (41)$$

The matrix of transposition $(\boldsymbol{I}_t)_{ijkl} = \delta_{il} \delta_{jk}$ and symmetrization stamps it $\boldsymbol{I}_S$, such as:

$$(\boldsymbol{I}_S)_{ijkl} = \frac{1}{2}(\boldsymbol{I} + \boldsymbol{I}_t) = \frac{1}{2}(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \quad (42)$$

Note:

It is noticed that the term $\dfrac{y_1 - y_2}{x_1 - x_2} [\boldsymbol{I}_S - \boldsymbol{E}_1 \otimes \boldsymbol{E}_1 - \boldsymbol{E}_2 \otimes \boldsymbol{E}_2]$ in the derivative $\boldsymbol{D}(\boldsymbol{X})$ first equation of (**40**) express the rotation of the principal directions in the plan.

5.2.2 Two-dimensional case of plane deformations type (D_PLAN) and axisymmetric (AXIS)

The direction out-plan $\alpha = 3$ being fixed, the expression of the derivative $\boldsymbol{D}(\boldsymbol{X})$ is obtained starting from the preceding case. Indeed, by isolating the term $\alpha = 3$ in the equation (**34**), there is the following expression:

$$\boldsymbol{D}(\boldsymbol{X}) = \underbrace{\sum_{\alpha=1}^{2} \left(y_\alpha \frac{d \boldsymbol{E}_\alpha}{d \boldsymbol{X}} + \sum_{\beta=1}^{2} \frac{\partial y_\alpha}{\partial x_\beta} \boldsymbol{E}_\alpha \otimes \frac{d x_\beta}{d \boldsymbol{X}} \right)}_{\boldsymbol{D}_{2D}(\boldsymbol{X})} + \underbrace{\sum_{\alpha=1}^{2} \frac{\partial y_\alpha}{\partial x_3} \boldsymbol{E}_\alpha \otimes \frac{d x_3}{d \boldsymbol{X}} + \sum_{\beta=1}^{3} \frac{\partial y_3}{\partial x_\beta} \boldsymbol{E}_3 \otimes \frac{d x_\beta}{d \boldsymbol{X}}}_{\boldsymbol{D}_3(\boldsymbol{X})} \quad (43)$$

Where $\boldsymbol{D}_{2D}(\boldsymbol{X})$ is given by the equation (**34**). The complementary term $\boldsymbol{D}_3(\boldsymbol{X})$ is written, by taking account of the multiplicity of the eigenvalues, in the following way:

$$D_3(X)=\begin{cases}\sum_{\alpha=1}^{2}\left(\frac{\partial y_\alpha}{\partial x_3}E_\alpha\otimes E_3+\frac{\partial y_3}{\partial x_\alpha}E_3\otimes E_\alpha\right)+\frac{\partial y_3}{\partial x_3}E_3\otimes E_3 & si \quad x_1\neq x_2\\ \frac{\partial y_1}{\partial x_3}I_p\otimes E_3+\frac{\partial y_3}{\partial x_1}E_3\otimes I_p+\frac{\partial y_3}{\partial x_3}E_3\otimes E_3 & si \quad x_1=x_2\end{cases} \tag{44}$$

Where I_p is the matrix of the orthogonal projection of I_S in the plan (e_x, e_y) :

$$I_p=\begin{cases}\frac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) & \text{si } i,j,k,l\in\{1,2\}\\ 0 & \text{sinon}\end{cases} \tag{45}$$

5.2.3 Three-dimensional case

In dimension three, the characteristic equation $det(X-x_\alpha I)=0$ give a cubic equation of the eigenvalues $x_{\alpha=1,2,3}$ of X following type:

$$x_\alpha^3-I_1x_\alpha^2+I_2x_\alpha-I_3=0 \text{ with } \alpha=1,2,3 \tag{46}$$

With:

$$\begin{cases}I_1=trace(X)\\ I_2=\frac{1}{2}\left[trace(X)^2-trace(X^2)\right]\\ I_3=det(X)\end{cases} \tag{47}$$

The resolution of the spectral problem easily gives the following solutions for the eigenvalues:

$$\begin{cases}x_1=-2\sqrt{Q}\cos\left(\frac{\theta}{3}\right)+\frac{I_1}{3}\\ x_2=-2\sqrt{Q}\cos\left(\frac{\theta+2\pi}{3}\right)+\frac{I_1}{3}\\ x_3=-2\sqrt{Q}\cos\left(\frac{\theta-2\pi}{3}\right)+\frac{I_1}{3}\end{cases} \tag{48}$$

Where Q and θ are given by:

$$Q=\frac{I_1^2-3I_2}{9} \text{ and } \theta=\cos^{-1}\left(\frac{R}{\sqrt{Q^3}}\right) \tag{49}$$

With:

$$R=\frac{-2I_1^3+9I_1I_2-27I_3}{54} \tag{50}$$

And clean vectors, by taking account of the multiplicity of the eigenvalues:

$$\begin{cases} \boldsymbol{E}_\alpha = \dfrac{x_\alpha}{2x_\alpha^3 - I_1 x_\alpha^2 + I_3}\left[\boldsymbol{X}^2 + (x_\alpha - I_1)\boldsymbol{X} + \dfrac{I_3}{x_\alpha}\boldsymbol{I}\right] & si \quad x_1 \neq x_2 \neq x_3 \\ \boldsymbol{E}_\beta = \boldsymbol{I} - \boldsymbol{E}_\alpha & si \quad x_\alpha \neq x_\beta \\ \boldsymbol{E}_1 = \boldsymbol{I} & si \quad x_1 = x_2 = x_3 \end{cases} \tag{51}$$

In the second equation of (**51**), $\boldsymbol{E}_\alpha$ is calculated with the assistance the first equation. Without giving the intermediate stages of calculation, the derivative $\boldsymbol{D}(\boldsymbol{X})$, by taking account of the multiplicity of the eigenvalues, is written finally:

$$\boldsymbol{D}(\boldsymbol{X}) = \begin{cases} \displaystyle\sum_{\alpha=1}^{3} \frac{y_\alpha}{(x_\alpha - x_\beta)(x_\alpha - x_\gamma)}\left[\frac{d\boldsymbol{X}^2}{d\boldsymbol{X}} - (x_\beta + x_\gamma)\boldsymbol{I}_S \right. & \\ \left. -(2x_\alpha - x_\beta - x_\gamma)\boldsymbol{E}_\alpha \otimes \boldsymbol{E}_\alpha - (x_\beta - x_\gamma)(\boldsymbol{E}_\beta \otimes \boldsymbol{E}_\beta - \boldsymbol{E}_\gamma \otimes \boldsymbol{E}_\gamma)\right] & si \quad x_1 \neq x_2 \neq x_3 \\ \displaystyle + \sum_{a=1}^{3}\sum_{b=1}^{3} \frac{\partial y_a}{\partial x_b}\boldsymbol{E}_a \otimes \boldsymbol{E}_b & \\ s_1 \dfrac{d\boldsymbol{X}^2}{d\boldsymbol{X}} - s_2 \boldsymbol{I}_S - s_3 \boldsymbol{X}\otimes\boldsymbol{X} + s_4 \boldsymbol{X}\otimes\boldsymbol{I} + s_5 \boldsymbol{I}\otimes\boldsymbol{X} - s_6 \boldsymbol{I}\otimes\boldsymbol{I} & si \quad x_\alpha \neq x_\beta = x_\gamma \\ \left(\dfrac{\partial y_1}{\partial x_1} - \dfrac{\partial y_1}{\partial x_2}\right)\boldsymbol{I}_S + \dfrac{\partial y_1}{\partial x_2}\boldsymbol{I}\otimes\boldsymbol{I} & si \quad x_1 = x_2 = x_3 \end{cases} \tag{52}$$

Where (α, β, γ) corresponds to a cyclic permutation of $(1,2,3)$. $\boldsymbol{I}$ and $\boldsymbol{I}_S$ are given by the equations (**41**) and (**42**), respectively. By noticing that $\boldsymbol{X}$ is a tensor symmetrical, care should be taken to apply itsymmetrical operator of derivation for the evaluation of $\frac{d\boldsymbol{X}^2}{d\boldsymbol{X}}$, which gives the following form:

$$\begin{aligned} \left(\frac{d\boldsymbol{X}^2}{d\boldsymbol{X}}\right)_{ijkl} &= \frac{d(X_{im}X_{mj})}{dX_{kl}} = \frac{1}{2}(\delta_{ik}\delta_{lm} + \delta_{il}\delta_{km})X_{mj} + \frac{X_{im}}{2}(\delta_{mk}\delta_{jl} + \delta_{ml}\delta_{kj}) \\ &= \frac{1}{2}(\delta_{ik}X_{lj} + \delta_{il}X_{kj} + \delta_{jl}X_{ik} + \delta_{kj}X_{il}) \end{aligned} \tag{53}$$

Lastly, expressions of $s_{i=1,6}$ are the following ones:

$$\begin{aligned} s_1 &= \frac{y_\alpha - y_\gamma}{(x_\alpha - x_\gamma)^2} + \frac{1}{x_\alpha - x_\gamma}\left(\frac{\partial y_\gamma}{\partial x_\beta} - \frac{\partial y_\gamma}{\partial x_\gamma}\right) \\ s_2 &= 2x_\gamma \frac{y_\alpha - y_\gamma}{(x_\alpha - x_\gamma)^2} + \frac{x_\alpha + x_\gamma}{x_\alpha - x_\gamma}\left(\frac{\partial y_\gamma}{\partial x_\beta} - \frac{\partial y_\gamma}{\partial x_\gamma}\right) \\ s_3 &= 2\frac{y_\alpha - y_\gamma}{(x_\alpha - x_\gamma)^3} + \frac{1}{(x_\alpha - x_\gamma)^2}\left(\frac{\partial y_\alpha}{\partial x_\gamma} + \frac{\partial y_\gamma}{\partial x_\alpha} - \frac{\partial y_\alpha}{\partial x_\alpha} - \frac{\partial y_\gamma}{\partial x_\gamma}\right) \\ s_4 &= 2x_\gamma \frac{y_\alpha - y_\gamma}{(x_\alpha - x_\gamma)^3} + \frac{1}{x_\alpha - x_\gamma}\left(\frac{\partial y_\alpha}{\partial x_\gamma} - \frac{\partial y_\gamma}{\partial x_\beta}\right) + \frac{x_\gamma}{(x_\alpha - x_\gamma)^2}\left(\frac{\partial y_\alpha}{\partial x_\gamma} + \frac{\partial y_\gamma}{\partial x_\alpha} - \frac{\partial y_\alpha}{\partial x_\alpha} - \frac{\partial y_\gamma}{\partial x_\gamma}\right) \\ s_5 &= 2x_\gamma \frac{y_\alpha - y_\gamma}{(x_\alpha - x_\gamma)^3} + \frac{1}{x_\alpha - x_\gamma}\left(\frac{\partial y_\gamma}{\partial x_\alpha} - \frac{\partial y_\gamma}{\partial x_\beta}\right) + \frac{x_\gamma}{(x_\alpha - x_\gamma)^2}\left(\frac{\partial y_\alpha}{\partial x_\gamma} + \frac{\partial y_\gamma}{\partial x_\alpha} - \frac{\partial y_\alpha}{\partial x_\alpha} - \frac{\partial y_\gamma}{\partial x_\gamma}\right) \\ s_6 &= 2x_\gamma^2 \frac{y_\alpha - y_\gamma}{(x_\alpha - x_\gamma)^3} + \frac{x_\alpha x_\gamma}{(x_\alpha - x_\gamma)^2}\left(\frac{\partial y_\alpha}{\partial x_\gamma} + \frac{\partial y_\gamma}{\partial x_\alpha}\right) - \frac{x_\gamma^2}{(x_\alpha - x_\gamma)^2}\left(\frac{\partial y_\alpha}{\partial x_\alpha} + \frac{\partial y_\gamma}{\partial x_\gamma}\right) - \frac{x_\alpha + x_\gamma}{x_\alpha - x_\gamma}\frac{\partial y_\gamma}{\partial x_\beta} \end{aligned} \tag{54}$$

Where (α,β,γ) corresponds to a cyclic permutation of $(1,2,3)$.

Note:

One notices LE following term:

$$\sum_{\alpha=1}^{3}\frac{y_\alpha}{(x_\alpha-x_\beta)(x_\alpha-x_\gamma)}\left[\frac{d\boldsymbol{X}^2}{d\boldsymbol{X}}-(x_\beta+x_\gamma)\boldsymbol{I}_S-(2x_\alpha-x_\beta-x_\gamma)\boldsymbol{E}_\alpha\otimes\boldsymbol{E}_\alpha\right.\\ \left.-(x_\beta-x_\gamma)(\boldsymbol{E}_\beta\otimes\boldsymbol{E}_\beta-\boldsymbol{E}_\gamma\otimes\boldsymbol{E}_\gamma)\right] \quad (55)$$

This term appears in the derivative $\boldsymbol{D}(\boldsymbol{X})$ first equation of (**52**) and express the rotation of the principal directions in three-dimensional space.

6 Appendix: Convention on the tensorial notations

Vectors of the strains and the stresses in the principal base $(\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3)$ are noted:

$$\boldsymbol{\varepsilon}=\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\end{Bmatrix} \text{ and } \boldsymbol{\sigma}=\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{Bmatrix} \quad (56)$$

The tensor of elasticity $\boldsymbol{C}$ allowing to connect $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ in the principal base, such as $\boldsymbol{\sigma}=\boldsymbol{C}.\boldsymbol{\varepsilon}$ is written:

$$\boldsymbol{C}=\begin{bmatrix} K+\frac{4}{3}G & K-\frac{2}{3}G & K-\frac{2}{3}G\\ K-\frac{2}{3}G & K+\frac{4}{3}G & K-\frac{2}{3}G\\ K-\frac{2}{3}G & K-\frac{2}{3}G & K+\frac{4}{3}G \end{bmatrix} \quad (57)$$

With K the elastic module of compressibility and G the elastic modulus of rigidity. The strains and the stresses are of the symmetrical tensors of order two. One generally exploits this symmetry (six independent components) by representing them by vectors of dimension six resulting from the projection of these tensors in suitable bases.
The strains and the stresses given as starter and produced at exit of the resolution of the law of behavior are expressed in the orthonormal base of the symmetrical tensors of order two, noted $\bar{\boldsymbol{b}}$:

$$\bar{\boldsymbol{b}}=\begin{cases} \boldsymbol{e}_x\otimes\boldsymbol{e}_x\\ \boldsymbol{e}_y\otimes\boldsymbol{e}_y\\ \boldsymbol{e}_z\otimes\boldsymbol{e}_z\\ \dfrac{\boldsymbol{e}_x\otimes\boldsymbol{e}_y+\boldsymbol{e}_y\otimes\boldsymbol{e}_x}{\sqrt{2}}\\ \dfrac{\boldsymbol{e}_x\otimes\boldsymbol{e}_z+\boldsymbol{e}_z\otimes\boldsymbol{e}_x}{\sqrt{2}}\\ \dfrac{\boldsymbol{e}_y\otimes\boldsymbol{e}_z+\boldsymbol{e}_z\otimes\boldsymbol{e}_y}{\sqrt{2}} \end{cases} \quad (58)$$

Where $(\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z)$ represent the unit vectors of the total Cartesian base orthonormal, presumedly fixed. The condensed expression of the tensors of the strains and the stresses projected in the base $\bar{\boldsymbol{b}}$ is written:

$$\bar{\boldsymbol{\varepsilon}}=\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \sqrt{2}\,\varepsilon_{xy}\\ \sqrt{2}\,\varepsilon_{yz}\\ \sqrt{2}\,\varepsilon_{xz}\end{Bmatrix} \text{ and } \bar{\boldsymbol{\sigma}}=\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sqrt{2}\,\sigma_{xy}\\ \sqrt{2}\,\sigma_{yz}\\ \sqrt{2}\,\sigma_{xz}\end{Bmatrix} \quad (59)$$

This writing reveals a term in $\sqrt{2}$ in front of the cross components. It allows:

- To express the tensor of elasticity of order four of 81 components by a tensor of order two of 36 components;
- To symmetrize this tensor of elasticity.

Indeed, while noting $\sigma_{ij}=C_{ijkl}\varepsilon_{kl}$, its form projected in the base $\bar{\boldsymbol{b}}$ becomes $\bar{\sigma}_i=\bar{C}_{ij}\bar{\varepsilon}_j$, where one has the following expression for $\bar{\boldsymbol{C}}$:

$$\bar{C}=\begin{bmatrix} C_{xxxx} & C_{xxyy} & C_{xxzz} & \sqrt{2}C_{xxxy} & \sqrt{2}C_{xxxz} & \sqrt{2}C_{xxyz} \\ C_{xxyy} & C_{yyyy} & C_{yyzz} & \sqrt{2}C_{yyxy} & \sqrt{2}C_{yyxz} & \sqrt{2}C_{yyyz} \\ C_{zzxx} & C_{zzyy} & C_{zzzz} & \sqrt{2}C_{zzxy} & \sqrt{2}C_{zzxz} & \sqrt{2}C_{zzyz} \\ \sqrt{2}C_{xyxx} & \sqrt{2}C_{xyyy} & \sqrt{2}C_{xyzz} & 2C_{xyxy} & 2C_{xyxz} & 2C_{xyyz} \\ \sqrt{2}C_{xzxx} & \sqrt{2}C_{xzyy} & \sqrt{2}C_{xzzz} & 2C_{xzxy} & 2C_{xzxz} & 2C_{xzyz} \\ \sqrt{2}C_{yzxx} & \sqrt{2}C_{yzyy} & \sqrt{2}C_{yzzz} & 2C_{yzxy} & 2C_{yzxz} & 2C_{yzyz} \end{bmatrix} \quad (60)$$

The condensed form (**60**) is not convenient to use because of the need for handling the terms in $\sqrt{2}$ at the time of the matric operations. One prefers to him another writing, based on projection in base known as of Voigt, noted $\tilde{\boldsymbol{b}}$ and having the following expression:

$$\hat{\boldsymbol{b}}=\begin{cases} \boldsymbol{e}_x \otimes \boldsymbol{e}_x \\ \boldsymbol{e}_y \otimes \boldsymbol{e}_y \\ \boldsymbol{e}_z \otimes \boldsymbol{e}_z \\ \boldsymbol{e}_x \otimes \boldsymbol{e}_y \\ \boldsymbol{e}_x \otimes \boldsymbol{e}_z \\ \boldsymbol{e}_y \otimes \boldsymbol{e}_z \end{cases} \quad (61)$$

The condensed expression of the tensors of the strains and the stresses projected in the base of Voigt $\hat{\boldsymbol{b}}$ is written:

$$\hat{\boldsymbol{\varepsilon}}=\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{xz} \end{Bmatrix} \text{ and } \hat{\boldsymbol{\sigma}}=\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{xz} \end{Bmatrix} \quad (62)$$

This writing makes it possible to be freed from the terms in $\sqrt{2}$ in front of the crossed components, and is more convenient to use during the digital resolution of the law of behavior.

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