Calculation of the factors of intensity of the constraints by extrapolation of the field of displacements

Summary:

One describes a method of calculating here of $K1$, $K2$ and $K3$ in 2D (plan and axisymmetric) and 3D by extrapolation of the jumps of displacements on the lips of the crack. It is usable using the order `POST_K1_K2_K3`, as well for a crack with a grid (classical finite elements) as for a crack nonwith a grid (finite elements nouveau riches: method X-FEM).

If the crack is with a grid, it must necessarily be plane; if the crack is not with a grid (method X-FEM), it can be nonplane (but sufficiently regular). In both cases, the method is applicable only for linear, homogeneous and isotropic materials elastic.

The method used is theoretically less precise than calculation starting from the bilinear form rate of refund of energy and displacements singular [R7.02.01 and R7.02.05] (operator `CALC_G`). It however makes it possible to easily obtain relatively reliable values of the factors of intensity of the constraints. The comparison of the various methods of calculating is useful to estimate the precision of the got results.

The precision of the results of the method of extrapolation of the jumps of displacement is clearly improved if the grid is quadratic. For a crack with a grid, it is recommended to use elements known as of "Barsoum" in bottom of crack (elements whose nodes mediums are located at the quarter of the edges). For a crack nonwith a grid, it is recommended to enrich several layers by elements around the bottom of crack.

1 Position of the problem

The method of calculating of the factors of intensity of the constraints by extrapolation of displacement is based on the asymptotic development of the field of displacement in bottom of crack.

In 2D, in a springy medium, linear, isotropic and homogeneous, the displacement and stress fields known for the modes of opening of the crack (are analytically characterized by $K1$), of slip plan ($K2$) and of slip antiplan ($K3$), cf [bib1] and [bib2]. In the case general in 3D, one can show that the asymptotic behavior of displacements and constraints is the sum of the solutions correspondents to modes 1 and 2 (in plane deformations) and to mode 3 (antiplan), and of four other particular solutions, but which are more regular than the preceding ones [bib3].

In all the cases, the singularity is thus the same one and one can write the following relations in the normal plan at the bottom of crack, in a point M :

$$K_1(M)=\lim_{r\to 0}\left(\frac{E}{8\left(1-\nu^2\right)}[U_m]\sqrt{\frac{2\pi}{r}}\right)$$

$$K_2(M)=\lim_{r\to 0}\left(\frac{E}{8\left(1-\nu^2\right)}[U_n]\sqrt{\frac{2\pi}{r}}\right)$$

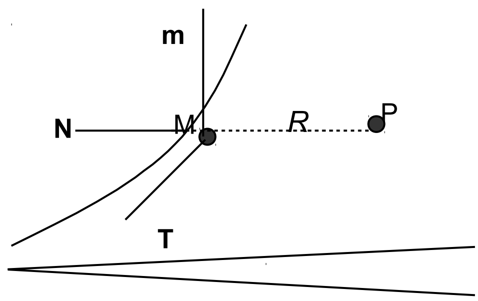
$$K_3(M)=\lim_{r\to 0}\left(\frac{E}{8\left(1+\nu\right)}[U_t]\sqrt{\frac{2\pi}{r}}\right)$$

with:

$\mathbf{t},\mathbf{n}$ in the plan of the crack in M,
$\mathbf{t}$ tangent vector at the bottom of crack in M,
$\mathbf{n}$ normal vector at the bottom of crack in M,
$\mathbf{m}$ normal vector with the plan of the crack in M,
$[U]$ jump of displacement enters the lips of crack:

$$[U_{\mathbf{m}}]=\left(U^{\text{lèvre supérieure}}-U^{\text{lèvre inférieure}}\right)\cdot\mathbf{m}$$

$r=\|\mathbf{MP}\|$ where P is a point of the normal plan at the bottom of crack in M, located on one of the lips.



If the crack is not plane, the three vectors are locally defined at the point M bottom considered. The relations precedents thus provide a method to identify numerically $K1$, $K2$ and $K3$. From the factors of intensity of the constraints, the formula of Irwin then makes it possible to calculate the rate of refund of energy G :

$$G=\frac{1}{E}\left(K_1^2+K_2^2\right) \quad \text{in plane constraints}$$

$$G=\frac{1-\nu^2}{E}\left(K_1^2+K_2^2\right)\left(+\frac{1+\nu}{E}K_3^2\right) \quad \text{in plane deformations (and in 3D)}$$

Note:

- *One can note that the signs of $K2$ and $K3$ depend on the orientation on $\mathbf{t}$ and $\mathbf{n}$. This is not too awkward insofar as the criteria of rupture or tiredness use only the absolute values of $K2$ and $K3$.*
- *One can also give expressions according to the stress fields, but the values of the vectors forced on the lips of the crack are less precise than displacements (because resulting from a transport of the points of Gauss to the nodes).*
- *The expression of the asymptotic fields is valid for the nonplane cracks (curved cracks for example), but those must nevertheless be sufficiently regular. The user must take care has minimum so that a normal can be defined in any point of the bottom.*
- *The method used here is theoretically less precise than calculation starting from the bilinear form rate of refund of energy and displacements singular [R7.02.01 and R7.02.05] (operator `CALC_G`). It however makes it possible to easily obtain relatively reliable values of the factors of intensity of the constraints. The comparison of the various methods of calculating is always useful to estimate the precision of the got results.*

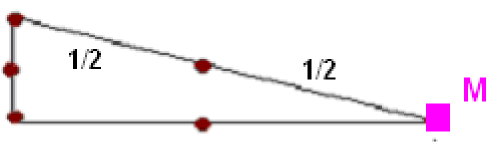
2 Implementation of the methods of extrapolation

The methods of extrapolation of displacements are put in work in the operator `POST_K1_K2_K3`, starting from the field of displacement calculated on all the structure. The definitions of the factors of intensity of the constraints are not true that asymptotically; extrapolation is thus done while being restricted in the vicinity of the bottom of crack limited by a maximum distance $dmax$ at the bottom. $dmax$ is the parameter `ABSC_CURV_MAXI` of the operator. In the case of a crack with a grid `ABSC_CURV_MAXI` is optional. If it is not noted, $dmax$ is calculated automatically in `POST_K1_K2_K3` and four times the maximum size of the meshs connected to the nodes of the bottom is worth.

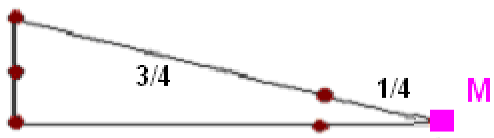
The principle general of calculation is the following:

Buckle on the nodes of the bottom of crack (not running: M)
    Definition of the plan Γ normal with the crack and the bottom of crack, the point M (plan of normal $\mathbf{t}$)
    Identification of the nodes of the two lips which belong to Γ : P_i^{sup} and P_i^{inf}
        Buckle on these nodes:
            If $r_i^{sup} = \|MP_i^{sup}\| dmax$: extraction of displacement in P_i^{sup}
            If $r_i^{inf} = \|MP_i^{inf}\| dmax$: extraction of displacement in P_i^{inf}
        Calculation of the jump of displacement in the three directions
    Extrapolation of the jump of displacement

Three methods of extrapolation are programmed. They are illustrated in this paragraph for a crack with a grid (quadratic grid), with or without elements of the type "Barsoum". The elements of Barsoum are such as the nodes not tops on the sides of the quadratic elements concerning the bottom of crack are moved with the quarter of with dimensions [bib4]. They make it possible to better collect the singularity of the stress field in bottom of crack

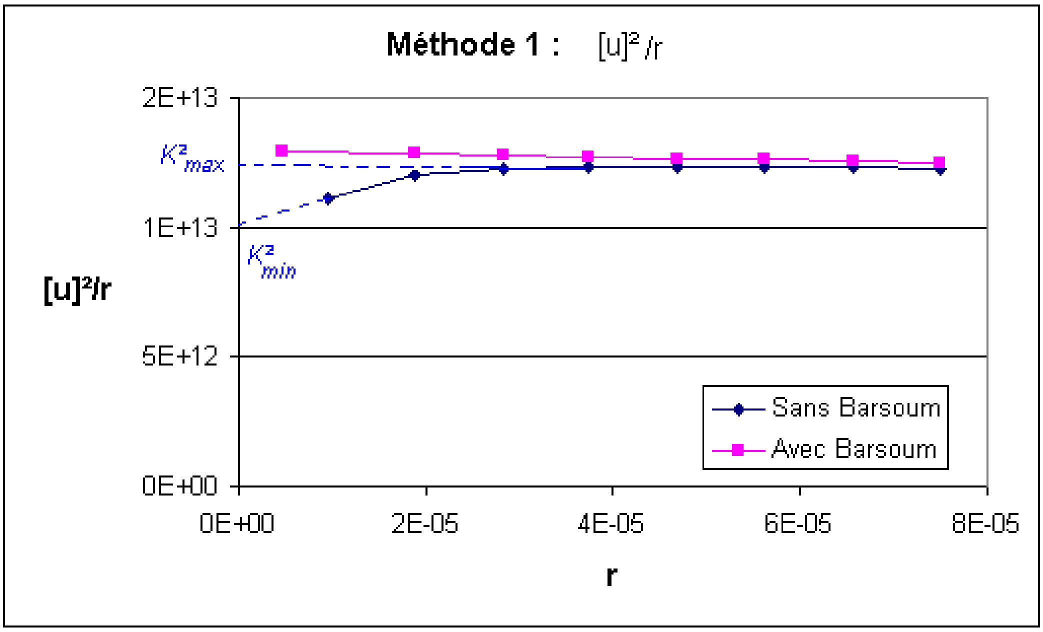


Classical finite element (M being a node of the bottom of crack)

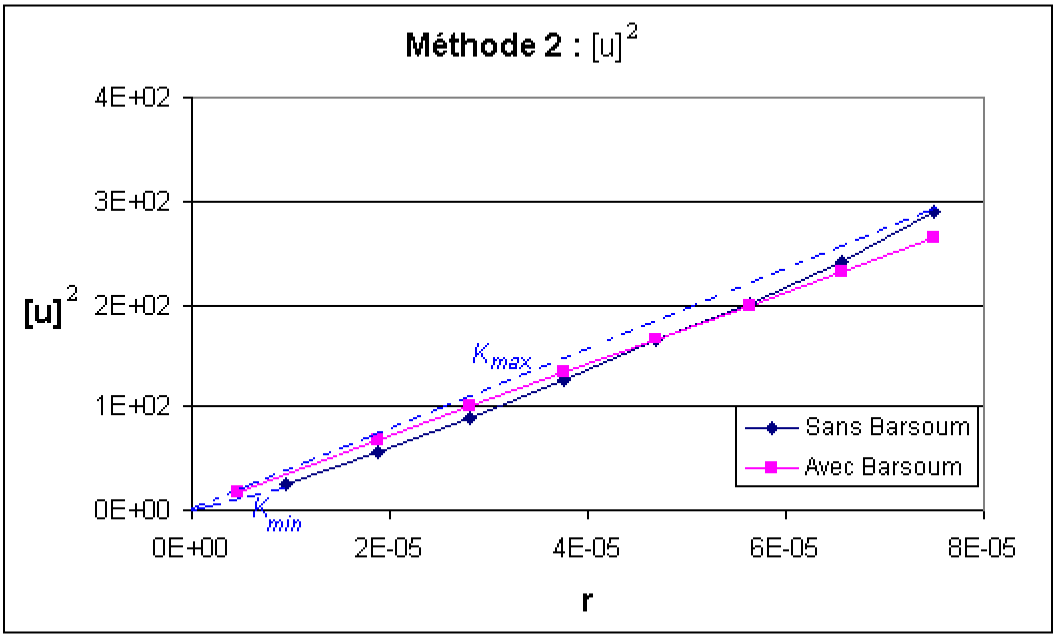


Finite element of Barsoum type

- **Method 1:** one calculates the jump of the field of displacements squared and one divides it by r. Various values of K^2 are obtained (except for a multiplicative factor) by extrapolation in $r=0$ segments of right-hand sides thus obtained. If the solution were perfect (analytical asymptotic field everywhere), one should obtain a line. Actually, one obtains almost a line with a grid of the type "Barsoum", and a nonright curve if not:



- **Method 2:** one traces the jump of the field of displacements squared according to r. Approximations of K are (always with a multiplicative factor near) equal to the root of the slope of the segments connecting the origin to the various points of the curve.



- **Method 3:** one identifies the stress intensity factor K starting from the jump of displacement $[U]$ by a method of least squares. Retiming is done on a segment length $dmax$, where $dmax$ is the parameter fixed in the operand ABSC_CURV_MAXI of the operator POST_K1_K2_K3 or in the case of a crack with a grid, if ABSC_CURV_MAXI is not indicated, $dmax$ four times the maximum size of the meshs connected to the nodes of the bottom is worth :

$$K \text{ minimize } J(k)=\frac{1}{2}\int_0^{\text{dmax}} \left([U(r)]-k\sqrt{r}\right)^2 dr$$

That is to say thus the formula clarifies to calculate K :

$$K=\frac{2}{r_m^2}\int_0^{\text{dmax}} [U(r)]\sqrt{r}\,dr=\frac{1}{r_m^2}\sum_{i=0}^{nbno-1} (r_{i+1}-r_i)\left([U]_{i+1}\sqrt{r_{i+1}}-[U]_i\sqrt{r_i}\right)$$

where $nbno$ is the number of nodes on the segment of retiming $[0, dmax]$. One notices that in this expression K is, for one $dmax$ fixed, the linear shape of the field of displacement.

3 Precision of the methods suggested

The method of extrapolation of the jumps of displacement was validated on tests whose analytical solutions are known. One has certain results below of them, in 2D and 3D, for a crack with a grid or not. One also compares the results with the method theoretically more precise founded on calculation of the rate of refund of energy and on the singular functions (operator CALC_G : method theta).

3.1 Test SSLP313: 2D C_PLAN (crack with a grid)

It is about a crack inclined in an infinite medium subjected to a uniform stress field in a direction (analytical in plane constraints, exact reference solution in infinite medium). The crack opens in mixed mode ($K1$ and $K2$) [V3.02.313].

For the test, the crack is with a grid in a rather large plate. The quadratic grid is very fine. The results are the following:

Reference solution (analytical solution)

$K1$	$K2$	G
3.58E+06	2.69E+06	1.00E+02

Calculation with the method theta (CALC_G)

	$K1$	$K2$	G
CALC_G without node with the quarter	3.60E+06	2.70E+06	1.01E+02
Variation/ref.	*0.8%*	*0.2%*	*1.1%*
CALC_G with nodes with the quarter	3.60E+06	2.70E+06	1.01E+02
Variation/ref.	*0.8%*	*0.2%*	*1.2%*

POST_K1_K2_K3 : grid without node of edges to the quarter

method	$K1_{max}$	$K1_{min}$	$K2_{max}$	$K2_{min}$	G_{max}	G_{min}	**Variation** G_{max} **/ ref.**	**Variation** G_{min} **/ ref.**
1	3.54E+06	3.19E+06	2.63E+06	1.92E+06	9.73E+01	6.94E+01	– 3.33%	– 30.70%
2	3.51E+06	3.33E+06	2.61E+06	2.25E+06	9.57E+01	8.08E+01	– 4.50%	– 19.32%
3	3.50E+06		2.59E+06		9.47E+01		-5.47%	

`POST_K1_K2_K3` *: grid with nodes of edges to the quarter*

method	$K1_{max}$	$K1_{min}$	$K2_{max}$	$K2_{min}$	G_{max}	G_{min}	**Variation** G_{max} **/ ref.**	**Variation** G_{min} **/ ref.**
1	3.61E+06	3.60E+06	2.70E+06	2.69E+06	1.01E+02	1.01E+02	1.29%	1.07%
2	3.60E+06	3.53E+06	2.69E+06	2.65E+06	1.01E+02	9.75E+01	1.02%	– 2.67%
3	3.56E+06		2.66E+06		9.88E+01		-1.42%	

On this test one notes that the grid of the type “Barsoum” is essential if one wants results precise. With “Barsoum” method 1 is more stable. It provides values of G (from $K1$ and $K2$) to approximately 1% of the analytical solution. Methods 2 and 3 lead to errors from 1 to 2,5%. It is noted that in this case, the method by extrapolation of displacements is as precise as the method theta.

On the other hand, with a normal grid, the results of the method by extrapolation vary much (between – 3% and -30% of the solution). It is the same with linear elements. In the case of a grid without elements of “Barsoum”, method 3 is most precise.

3.2 Test SSLV134: 3D (crack with a grid)

It is about a plane crack in the shape of disc in an infinite medium 3D subjected to a uniform stress field in a direction (known analytical reference solution under the name of “penny shape ace”). The crack opens in pure mode 1, and it $K1$ is constant along the bottom of crack [V3.04.134].

For this test, the crack is with a grid in a block parallelepiped. The grid is relatively coarse.

Analytical reference solution:

$K1$	G room
$1.59\ 10^{6}$	11.59

Calculation with the method theta (`CALC_G` *)*

	G
`CALC_G` with nodes with the quarter	11.75
Variation/ref.	*1.3%*

`POST_K1_K2_K3` *: grid without nodes of edges to the quarter*

method	$K1_{max}$	$K1_{min}$	G_{max}	G_{min}	**Variation** G_{max} **/ ref.**	**Variation** G_{min} **/ ref.**
1	1.56E+06	1.45E+06	1.11E+01	9.63E+00	– 4.32%	– 16.91%
2	1.53E+06	1.49E+06	1.06E+01	1.01E+01	– 8.35%	– 13.08%
3	1.52E+06		1.05E+01		– 9.51%	

`POST_K1_K2_K3` : grid with nodes of edges to the quarter

method	$K1_{max}$	$K1_{min}$	G_{max}	G_{min}	**Variation** G_{max} **/ ref.**	**Variation** G_{min} **/ ref.**
1	1.61E+06	1.59E+06	1.18E+01	1.16E+01	1.32%	– 0.06%
2	1.59E+06	1.53E+06	1.15E+01	1.07E+01	– 0.42%	– 7.87%
3	1.55E+06		1.10E+01		– 5.16%	

On this test one still notes that the grid of the type "Barsoum" is essential if one wants results precise. With "Barsoum" method 1 is most stable, with a variation with the solution of reference lower than 1,5% for G . The grid is relatively coarse, which explains why the method theta is more precise.

3.3 Test SSLV134: 3D (crack nonwith a grid)

The case considered is the same one as that of the preceding paragraph, but this time the crack is not with a grid. It is directly defined in the command file, by using method X-FEM [R7.02.12]. Grid not being regular with respect to the crack, the values of K and of G calculated vary along the bottom of crack. For the comparison below, one retains the value corresponding to an arbitrarily selected particular point (medium of the bottom of crack represented).

The grid is **linear** and relatively **coarse**. In the method X-FEM, the user can choose the zone on which the elements around the bottom of crack are nouveau riches with asymptotic displacements (keywords RAYON_ENRI and NB_COUCHES of DEFI_FISS_XFEM). This enrichment aims at improving the precision of calculation. One compares here the results got with an enrichment limited to the only elements containing the bottom of crack and with an enrichment on four layers of elements around the bottom of crack.

Calculation with the method theta (CALC_G – smoothing by default of the type LEGENDRE of degree 5)

	G
CALC_G with enrichment on a layer	11.42
Variation/ref.	*-1.4%*
CALC_G with enrichment on four layers	11.61
Variation/ref.	*0.2%*

POST_K1_K2_K3 : enrichment on only one sleeps

method	$K1_{max}$	$K1_{min}$	G_{max}	G_{min}	**Variation** G_{max} **/ ref.**	**Variation** G_{min} **/ ref.**
1	1.65E+06	1.43E+06	12.4	9.34	6.99%	– 19.41%
2	1.52E+06	1.44E+06	10.5	9.45	– 9.41%	– 18.46%
3	1.47E+06		9.81		– 15.35%	

POST_K1_K2_K3 : enrichment on four layers

method	$K1_{max}$	$K1_{min}$	G_{max}	G_{min}	**Variation** G_{max} **/ ref.**	**Variation** G_{min} **/ ref.**
1	1.58E+06	1.58E+06	11.3	11.3	– 2.51%	– 2.51%
2	1.55E+06	1.47E+06	10.9	9.88	– 5.95%	– 14.65%
3	1.51E+06		10.4		– 10.26%	

On this test, it is noted that it is essential to enrich on several layers by elements around the bottom by crack to have satisfactory results. To note that the grid used here is linear and relatively coarse: with a finer grid, the results are significantly improved. A study of convergence on a similar case is presented in [bib5].

With an enrichment on four layers, method 1 is that which leads to the most precise results. The maximum curvilinear X-coordinate corresponds, in both cases, with the distance from four elements approximately. The method theta is here as for it less sensitive to the parameter of enrichment.

4 Conclusion

The results got with the method of extrapolation of displacement are as a whole satisfactory, with less than 5% of error of G, especially if the elements of the bottom of crack are of type Barsoum (case fissures with a grid) or if several layers of elements are nouveau riches around the bottom (case fissures nonwith a grid, method X-FEM). In both cases, it is a question as well as possible of collecting the asymptotic behavior of displacement.

It should indeed be noticed that the asymptotic expression of displacements is valid only for r tending towards 0. This is why it is necessary to take care not to choose a too large field of extrapolation (distance $dmax$ of the operator `POST_K1_K2_K3` about 4 to 5 elements).

On the tests presented for a crack with a grid, method 1 gives the most precise results and most stable, that it is in 2D or 3D, if there are elements of Barsoum. If the grid does not comprise elements of Barsoum, one then advises to use the results of method 3. For a crack nonwith a grid, method 1 seems also most precise.

On a study for which one does not know a reference solution, it is possible to estimate the quality of calculation a posteriori. Indeed, `POST_K1_K2_K3` systematically provides for the first two methods the values maximum and the values minimum (on the whole of the calculated points) of the factors of intensity of the constraints, as well as the value of G recomputed by the formula of Irwin. Method 3 provides as for it only one value for each stress intensity factor. This method is a weighted average of the factors of intensity of the constraints extrapolated in each node.

A result can be regarded as satisfactory if the 5 values thus provided (min and max of the methods 1 and 2, and method 3) are close. One can also recommend to compare the results got with this method with those resulting from calculation from the rate of refund from energy and the singular functions (operator `CALC_G`).

5 Bibliography

[1] H.D.BUI: "Breaking process fragile "- Masson, 1978.

[2] J. LEMAITRE, J.L.CHABOCHE: "Mechanical of solid materials" - Dunod, 1996.

[3] J.B. LEBLOND: "Breaking process fragile and ductile" – Lavoisier, 2003.

[4] R.S. BARSOUM: "Triangular quarter-point elements ace elastic and perfectly-plastic ace tip elements" – Int. J. for Numerical Methods in Engineerig, vol. 11,85-98, 1977.

[5] S. GENIAUT: "Convergence in breaking process: validation of the classical finite elements and nouveau riches in Code_Aster" – Notes EDF R & D H-T64-2008-0047, 2008.

6 Description of the versions of the document

Index	Version Aster	Author (S) or contributor (S), organization	Description of the modifications
With	5	J.M.Proix EDF/R & D /AMA	Initial text
B	7.4	E.Galenne EDF/R & D /AMA	Method of least squares 7.2.24
C	9.4	E.Galenne EDF/R & D /AMA	Nonplane crack with XFEM 9.2.8