Elements with internal discontinuity, behavior CZM_EXP and piloting of the loading

Summary:

While being based on a briefly exposed energy formulation in a first part we present a digital model of two-dimensional starting and propagation of cracks. This one is pressed on a finite element with an internal discontinuity of displacement, near to the models known by the name “embedded discontinuity finite element” in the literature.

For this model, we detail law of behavior used (of the Cohesive type exponential Model Zone: keyword CZM_EXP), properties of the finite element as well as the resolution of the problem of minimization of total energy. In addition we describe a technique of piloting of the loading allowing to follow possible instabilities in the total answer of the structure.
These elements can be used in 2D plan or axisymmetric only with modelings PLAN_ELDI and AXIS_ELDI. They are validated on the cases tests ssnp128a (documentation [V6.03.128]) and ssna115a (documentation [V6.01.115]).

Remarks :

- This documentation is largely inspired by work of thesis [6]. The interested reader will be able to refer to it to have a more complete vision of the link of this model with the energy approach “Frankfurt Marigo” like a certain number of digital results.
- The purpose of the choice to gather in the same documentation the presentation of a finite element and a law of behavior is related to the specificity of the model and is to facilitate its comprehension.

Contents

1 Modèle energy of cracking

The goal of this part is to draw up the framework general of a model of starting and propagation of crack based on a principle of minimization of energy. This model takes as a starting point those developed in the literature under name “Models of Cohesive Forces” in the direction where one seeks to take into account a residual interaction between the lips of the crack. However, the major difference in our approach is the formulation of the problem within a broader framework by adopting the energy point of view introduced by Frankfurt and Marigo [3].

The idea consists in taking into account the process of dissipation of energy during cracking thanks to an energy defined on a surface of discontinuity and depend on the jump of displacement through the latter. We will be satisfied here to pose the problem in its continuous form in space and semi-discretized in time without seeking to clarify the form of the energy of surface.

The following part will be devoted to the implementation of the finite element with internal discontinuity being based on this energy formulation.

1.1 Principle general

A structure is considered *rubber band* defined by the field Ω and a crack defined as a surface of noted discontinuity Γ through which displacement $\boldsymbol{u}$ a jump admits $\boldsymbol{\delta}$:

$$\boldsymbol{\delta} = [\![\boldsymbol{u}]\!]_{\Gamma} \tag{1}$$

Total energy is defined E_T of this structure as the sum of its noted elastic energy Φ , of an energy of noted surface Ψ and of W^{ext} potential of the external efforts:

$$E_T = \Phi + \Psi - W^{\text{ext}} \tag{2}$$

One notes $\Phi = \int_{\Omega} \phi \, . \, d\Omega$ and $\Psi = \int_{\Gamma} \psi \, . \, d\Gamma$, where ϕ and ψ indicate respectively the density of energy elastic and the density of energy of surface. Total energy is a function of displacement $\boldsymbol{u}$ and of the jump of displacement $\boldsymbol{\delta}$, it **problem of minimization** is written:

$$\text{Chercher } (\boldsymbol{u}^*, \boldsymbol{\delta}^*) \text{ minimum local de l'énergie totale } E_T(\boldsymbol{u}, \boldsymbol{\delta}) \tag{3}$$

Let us recall that one carries out a search for minimum *room* because total energy is not limited in a lower position in the presence of external forces. Now, let us see which assumptions one can formulate on the energy of surface in order to take into account the condition of noninterpenetration of the lips of the crack and according to whether one or not regards cracking as a reversible process.

1.2 Reversible model

Let us present a first simple, but not very realistic model from a physical point of view on a macroscopic scale, where it is supposed that the cracking of a material is a reversible process. It is considered that the total refermeture of the lips of a crack makes it possible to find a healthy material. The energy of surface can be written as the sum of a function of the euclidian norm of the jump of displacement and of an indicatrix giving an account of the condition of not-interpenetration of the lips:

$$\Psi(\boldsymbol{\delta}) = \Psi_{rev}(\|\boldsymbol{\delta}\|) + I_{\mathbb{R}^+}(\delta_n) \tag{4}$$

with $\delta_n = \boldsymbol{\delta} \cdot \boldsymbol{n}$ where $\boldsymbol{n}$ is the normal with discontinuity and $I_{\mathbb{R}^+}(\delta_n)$ is the indicating function:

$$I_{\mathbb{R}^+}(\delta_n) = \begin{cases} +\infty & \text{si } \delta_n < 0 \\ 0 & \text{si } \delta_n \geq 0 \end{cases} \tag{5}$$

The model developed in *Code_Aster* allows as for him to take into account the irreversibility of cracking, it is presented in the following section.

1.3 Model with memory

While being based on the writing of the preceding reversible model, one introduces a noted internal variable κ allowing to memorize the state of cracking at a given moment and to thus translate its irreversible character. One defines his evolution in the following way:

$$\begin{cases} \kappa(0) &= \kappa_0 \\ \kappa(t) &= \sup_{\tau \le t} \{ \|\delta(\tau)\|, \kappa_0 \} \quad \forall t > 0 \end{cases} \tag{6}$$

One can in an equivalent way express this law of evolution by means of a function threshold f defined by:

$$f(\kappa, \delta) = \|\delta\| - \kappa \tag{7}$$

The law of evolution then takes the shape of a classical condition of coherence:

$$f \le 0, \quad \dot{\kappa} \ge 0, \quad f.\kappa = 0 \tag{8}$$

The energy of surface is defined Ψ like a function of the jump of displacement and variable κ :

$$\Psi(\delta, \kappa) = H(\|\delta\| - \kappa).\Psi_{dis}(\|\delta\|) + \left[1 - H(\|\delta\| - \kappa)\right].\Psi_{lin}(\|\delta\|, \kappa) + I_{\mathbb{R}^+}(\delta_n) \tag{9}$$

with H function of Heaviside defined by:

$$H(x) = \begin{cases} 1 & \text{si} \quad x \ge 0 \\ 0 & \text{si} \quad x < 0 \end{cases} \tag{10}$$

The energy of surface Ψ will thus be worth Ψ_{dis} if the threshold is positive, representing dissipation when the crack evolves. And it will be worth Ψ_{lin}, representing an evolution without dissipation of energy, if the threshold is negative. Ψ_{lin} takes into account the case where the crack is closed; one makes the choice of a linear discharge, therefore the density of energy of corresponding surface Ψ_{lin}, will be form:

$$\Psi_{lin}(\|\delta\|, \kappa) = \frac{1}{2}.R(\kappa).\|\delta\|^2 + C_0 \tag{11}$$

where $R(\kappa)$ and C_0 are selected to ensure the continuously derivable character of energy. Density of energy of surface Ψ_{dis} when cracking evolves can take many forms; we will give of it an example for the model finite element of the chapter 5. From the densities of energy of surface Ψ_{dis} and Ψ_{lin} one can define the vector forced $\vec{\sigma}$ through discontinuity:

$$\begin{cases} \vec{\sigma} = \dfrac{\partial \Psi_{lin}}{\partial \delta} & \text{si} \quad \|\delta\| < \kappa \\ \vec{\sigma} = \dfrac{\partial \Psi_{dis}}{\partial \delta} & \text{si} \quad \|\delta\| > \kappa \end{cases} \tag{12}$$

It is the law of interface which we will adopt.

1.4 Temporal discretization

Let us present the temporal discretization now. Let us specify first of all that one considers only quasi-static evolutions, time is thus parameterized by increments of loading. By carrying out a semi-discretization in time, one defines the energy of surface Ψ at one moment i like a function of the jump of displacement and of κ^{i-1} variable interns at the moment $i-1$:

$$\Psi(\delta, \kappa^{i-1}) = H(\|\delta\| - \kappa^{i-1}).\Psi_{dis}(\|\delta\|) + \left[1 - H(\|\delta\| - \kappa^{i-1})\right].\Psi_{lin}(\|\delta\|, \kappa^{i-1}) + I_{\mathbb{R}^+}(\delta_n) \tag{13}$$

The problem of minimization is written then at the moment i :

$$\min_{\boldsymbol{u}, \boldsymbol{\delta}} E_T(\boldsymbol{u}, \boldsymbol{\delta}; \kappa^{i-1}) \tag{14}$$

The variable interns at the moment i once the jump at the moment is brought up to date i known:

$$\kappa^i = \max\left(\|\delta\|, \kappa^{i-1}\right) \quad (15)$$

2 Model with internal discontinuity

This model makes it possible to take into account the starting and the propagation of cracks in a structure for a given direction. This one is based on finite element particular which one calls *elements with internal discontinuity* (of English embedded discontinuity finite elements). The principal idea rests on the introduction of a discontinuity included into the element and of an internal variable of threshold managing the dissipative process as well as the irreversible character of cracking. Moreover, the jump of displacement will be considered constant by element what will facilitate the digital resolution. One will be able to adopt a static technique of condensation where the calculation of the jump during minimization will be done at the elementary level.

The element with discontinuity was developed with an aim of freeing itself from the regularization of energy into zero (penalized adherence) from the law CZM_EXP_REG developed for the element of joint (see documentation [R3.06.09] for the element and [R7.02.11] the law). Indeed, this one is prejudicial for the models based on a principle of minimum of energy since it results in cancelling the derivative of the density of energy of surface in the zero i.e. critical stress of the model. In such a situation, the starting of a crack occurs as of loading, also weak is it (let us note however that does not prevent from getting results interesting with such a law, since one sets the way of cracking).

To take into account discontinuity one carries out an enrichment of displacements and deformations which makes it possible to ensure their compatibility.

In this part one will initially present the law of behavior which controls the element with discontinuity. One will be brought to define a vector forced in the element which will be useful during minimization of energy. One will detail then the properties of the finite element then the digital resolution of the problem of minimization of energy.

2.1 Law of behavior of the element with discontinuity : CZM_EXP

The elements with discontinuity have a mixed behavior: rubber band and dissipative. The total energy of the element is written as the sum of an elastic energy and an energy of surface of the type CZM. The following parts will be devoted to the presentation of the latter like to the constraint which in drift.

2.1.1 Energy total in the element with discontinuity

Total energy is chosen E_T like the sum of an elastic energy definite on the noted element Ω_e and of an energy of surface defined on the discontinuity of the element noted Γ_e :

$$E_T(\boldsymbol{U}, \delta; \kappa) = \Phi(\boldsymbol{U}, \delta) + \Psi(\delta, \kappa) \quad (16)$$

where Φ corresponds to elastic energy. Let us note that this one depends on nodal displacement $\boldsymbol{U}$ and of the jump of displacement $\boldsymbol{\delta}$ constant (in the element) which one will see that it induces an additional deformation. The energy of surface Ψ depends on the jump of displacement and on κ internal variable allowing to treat the irreversibility of cracking.

Notice : The way of cracking is known *a priori*, the normal with discontinuity will thus be fixed by the orientation of the elements in the grid (see figure 1).

2.1.1.1 Energy rubber band

To define elastic energy we will need to introduce as of now the interpolation of the field of displacement into the element with discontinuity, the properties of this last will be however detailed in a part to follow (see §2.2). One represents on the figure 1 the element with discontinuity noted Ω_e, it is a quadrangle with four nodes with an internal discontinuity which one notes Γ_e.

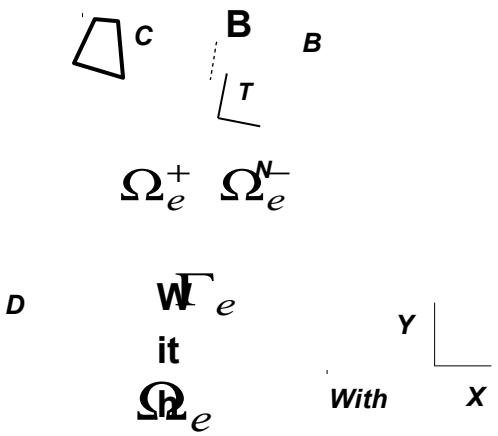

Figure 1: Element with discontinuity

The interpolation of the field of displacement $\boldsymbol{u}(x)$ in the element with discontinuity is written:

$$\boldsymbol{u}(x)=\boldsymbol{N}(x).\begin{bmatrix}\boldsymbol{U}_A\\ \boldsymbol{U}_B\\ \boldsymbol{U}_C-\delta\\ \boldsymbol{U}_D-\delta\end{bmatrix}+H_{\Gamma_e}(x).\delta \tag{17}$$

with $\boldsymbol{N}$ matrix of the functions of classical bilinear form of type $Q1$, H_{Γ_e} the function of Heaviside on the discontinuity of the element Γ_e being worth 1 if $x\in\Omega_e^+$ and 0 if $x\in\Omega_e^-$. The vector δ corresponds to the jump of displacement on discontinuity. One has well $[\![\boldsymbol{u}]\!]=\delta$ since $\boldsymbol{N}$ is a continuous function and $H_{\Gamma_e}=1$ on Γ_e. Moreover, them $\boldsymbol{U}$ (subscripted by the tops) correspond to nodal displacements, indeed one can write:

$$\begin{Bmatrix}\boldsymbol{U}_A\\ \boldsymbol{U}_B\\ \boldsymbol{U}_C\\ \boldsymbol{U}_D\end{Bmatrix}=\begin{Bmatrix}\boldsymbol{U}_A\\ \boldsymbol{U}_B\\ \boldsymbol{U}_C+\delta-\delta\\ \boldsymbol{U}_D+\delta-\delta\end{Bmatrix} \tag{18}$$

From the approximate field (17) one can define the deformation associated on the element ***out of discontinuity*** :

$$\epsilon=\nabla^s\boldsymbol{N}(x).\begin{Bmatrix}\boldsymbol{U}_A\\ \boldsymbol{U}_B\\ \boldsymbol{U}_C-\delta\\ \boldsymbol{U}_D-\delta\end{Bmatrix}+\underbrace{\nabla^s H_{\Gamma_e}(x)\cdot\delta}_{=0} \tag{19}$$

∇^s indicate the symmetrized gradient. One can récrire (19) in the synthetic form:

$$\epsilon=\boldsymbol{B}.\boldsymbol{U}-\boldsymbol{D}.\delta \tag{20}$$

where $\boldsymbol{B}$ is the matrix of the symmetrized gradients of the functions of form, $\boldsymbol{U}=(\boldsymbol{U}_A,\boldsymbol{U}_B,\boldsymbol{U}_C,\boldsymbol{U}_D)^T$ and $\boldsymbol{D}.\delta=\boldsymbol{B}(\boldsymbol{0},\boldsymbol{0},\delta,\delta)^T$. In the expression (20) one distinguishes part of the deformation related to displacements $\boldsymbol{B}.\boldsymbol{U}$ and another part $-\boldsymbol{D}.\delta$ dependent on the jump. The constraint for an elastic law of behavior is written:

$$\sigma=\boldsymbol{E}.\epsilon \tag{21}$$

With $\boldsymbol{E}$ the tensor of elasticity. Elastic energy in the element is thus defined by:

$$\Phi(\boldsymbol{U},\delta)=\frac{1}{2}.\int_{\Omega_e}\left(\boldsymbol{B}.\boldsymbol{U}-\boldsymbol{D}.\delta\right)^T.\boldsymbol{E}.\left(\boldsymbol{B}.\boldsymbol{U}-\boldsymbol{D}.\delta\right).d\Omega \quad (22)$$

2.1.1.2 Energy of surface

As we presented in the ideal model (see §1), one chooses an energy of surface depending on the standard of the jump of displacement $\|\delta\|$, of κ positive internal variable (its initial value κ_0 is worthless) and orientation of the crack (only for the condition of contact). One takes:

$$\Psi(\delta,\kappa)=H(\|\delta\|-\kappa).\Psi_{dis}(\|\delta\|)+\left[1-H(\|\delta\|-\kappa)\right].\Psi_{lin}(\|\delta\|,\kappa)+I_{\mathbb{R}^+}(\delta\cdot\boldsymbol{n}) \quad (23)$$

The indicating function $I_{\mathbb{R}^+}$ allows to take into account the condition of noninterpenetration of the lips of the crack:

$$I_{\mathbb{R}^+}(\delta_n)=\begin{cases}+\infty & \text{si } \delta\cdot\boldsymbol{n}<0\\ 0 & \text{si } \delta\cdot\boldsymbol{n}\geq 0\end{cases} \quad (24)$$

According to the value of the threshold, the energy of surface will be worth Ψ_{dis} or Ψ_{lin} (more the indicating function). In the first case one will speak about dissipative mode, in the second linear case of mode. One can write two energies in the form:

$$\Psi_{dis}(\|\delta\|)=\int_{\Gamma_e}\psi_{dis}(\|\delta\|).d\Gamma \text{ and } \Psi_{lin}(\|\delta\|,\kappa)=\int_{\Gamma_e}\psi_{lin}(\|\delta\|,\kappa).d\Gamma \quad (25)$$

with ψ_{dis} and ψ_{lin} densities of energy of surface. Let us present in detail the values of these densities.

Density of energy of surface in linear mode

If an existing crack evolves (opening or refermeture) without dissipating energy i.e. $\|\delta\|<\kappa$, the element will be in a linear phase (load or discharge). One thus chooses a density of quadratic energy function of the standard of the jump:

$$\Psi_{lin}(\|\delta\|,\kappa)=\frac{1}{2}.P(\kappa).\|\delta\|^2+C_0 \quad (26)$$

where $P(\kappa)=\frac{\sigma_c}{\kappa}.\exp\left(-\frac{\sigma_c}{G_c}.\kappa\right)$ is selected in order to ensure the continuity of the derivative of Ψ in κ (i.e the continuity of the constraint in the element from one mode to another) and C_0 constant allowing to ensure the continuity of Ψ in κ.

Density of energy in dissipative mode

While taking as a starting point the idea by Barenblatt [1] to take account of the process of rupture of the interatomic connections, it is supposed that the energy of surface is worthless into zero and grows towards the energy of Griffith when the value of the jump becomes important in front of the length characteristic of the atomic scale. Moreover while being based on the form of the interatomic potentials one chooses a concave increasing energy and with convex derivative. It is known in particular that concavity plays a crucial role in dimension 1 to limit the number of cracks (see Charlotte *et al.* [2]). Thus, if $\|\delta\|<\kappa$, one chooses a density of energy of surface of the following form, illustrated on the figure 2 :

$$\Psi_{dis}(\|\delta\|)=G_c.\left[1-\exp\left(-\frac{\sigma_c}{G_c}.\|\delta\|\right)\right] \quad (27)$$

G_c represent the rate of refund of energy critical (or tenacity) of material and $\sigma_c=\Psi'_{rev}(0)$ the critical stress.

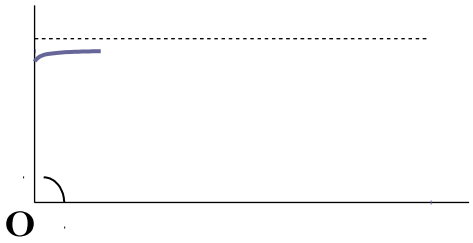

Figure 2: Density of energy of surface according to the standard of the jump

2.1.2 Vector forced in the element with discontinuity

That is to say $\vec{\sigma}=(\sigma_n,\sigma_t)$ the vector forced in the element. When the element is healthy $(\kappa=0)$, the energy of surface is worthless. The vector forced is defined starting from the tensor of the constraints of the elastic law of behavior and the normal to the line of discontinuity:

$$\vec{\sigma}=\boldsymbol{\sigma}\cdot\boldsymbol{n} \tag{28}$$

One will see thereafter (see §2.3.1.1 on the criterion of starting) that the jump remains null as long as the standard of the vector forced does not reach the critical stress σ_c. On the other hand, if the threshold κ is not null, the vector not forced in the element is given by the derivative of the density of energy of surface compared to the jump (the vector forced remains equal to $\boldsymbol{\sigma}\cdot\boldsymbol{n}$ since the element ensures the continuity of the normal constraint, to see §2.2.2).

2.1.2.1 Linear mode

In linear mode ($\|\delta\|<\kappa$), the vector forced is worth:

$$\vec{\sigma}=\frac{\partial\Psi_{lin}}{\partial\delta}=P(\kappa).\delta \tag{29}$$

And its derivative compared to the jump:

$$\frac{\partial\vec{\sigma}}{\partial\delta}=\boldsymbol{Id}.P(\kappa) \tag{30}$$

2.1.2.2 Dissipative mode

In dissipative mode ($\|\delta\|>\kappa$), the vector forced is worth:

$$\vec{\sigma}=\frac{\partial\Psi_{dis}}{\partial\delta}=\sigma_c.\frac{\delta}{\|\delta\|}.\exp\left(-\frac{\sigma_c}{G_c}.\|\delta\|\right) \tag{31}$$

And its derivative compared to the jump:

$$\frac{\partial\vec{\sigma}}{\partial\delta}=\sigma_c.\left[\frac{\boldsymbol{Id}}{\|\delta\|}-\frac{\delta}{\|\delta\|}\otimes\frac{\delta}{\|\delta\|}.\left(\frac{\sigma_c}{G_c}+\frac{1}{\|\delta\|}\right)\right].\exp\left(-\frac{\sigma_c}{G_c}.\|\delta\|\right) \tag{32}$$

2.1.2.3 Graphic illustration

One represents on the figure 3 evolution of the standard of the vector forced in the element according to the standard of the jump. The arrows represent the possible evolutions of the vector forced following the case (healthy element, linear mode or dissipative mode).

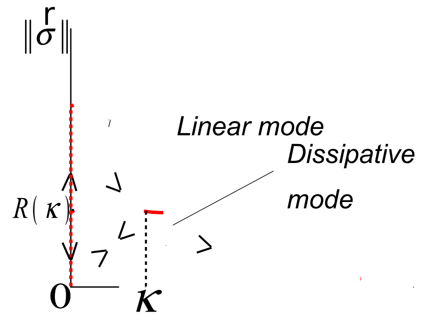

Figure 3: Vector normalizes forced according to the standard of the jump

With the threshold in jump κ a threshold in constraint corresponds which one notes $R(\kappa)$. This last will determine starting from which level of constraint the crack will dissipate energy. This threshold will evolve with the opening of the crack, it depends on κ and is defined by:

$$R(\kappa)=\sigma_c.\exp\left(-\frac{\sigma_c}{G_c}.\kappa\right) \quad (33)$$

Let us note that when $\kappa=0$ this threshold corresponds to the criterion of starting which we will present to the §2.3.1.1.

2.2 Properties of the element with discontinuity

The element with discontinuity is a quadrangle with four nodes with a jump of internal displacement. A first part will be devoted to the geometrical description of the element like to the presentation of a parameter setting while being pressed on the element of reference. Then, we will see that the element ensures the continuity of the normal constraint through the jump of displacement. Then, we will show the unicity of the jump of displacement provided that the size of the element is sufficiently small. To finish, we will see that the choice of a constant jump introduces a parasitic energy of surface which tends towards zero when the grid is refined.

2.2.1 Geometry of the element and parameter setting

2.2.1.1 Geometry

That is to say (X, Y) the Cartesian base constituting a total reference mark of $\mathbb{R}^2$. The element with discontinuity, noted Ω_e is a finite element quadrangle with four nodes. It consists of an elastic area and a discontinuity Γ_e passing by the center of the element (segment passing by the mediums on the sides $[AD]$ and $[BC]$) and length l (see figure 4).

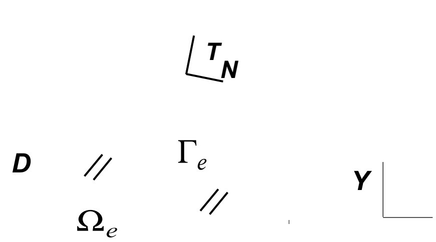

Figure 4: Element with discontinuity

The orientation of discontinuity defines a local reference mark in the element $(\boldsymbol{n},\boldsymbol{t})$. The corresponding element of reference is a square $\hat{\Omega}_e$ defined by the field $[-1,1]\times[-1,1]$ (see figure 5). Each element has four points of Gauss. Those of the element of reference have as coordinates $\left(\pm\sqrt{(3)}/3,\pm\sqrt{(3)}/3\right)$. Let us note $\boldsymbol{\omega}_g$ and $\hat{\boldsymbol{\omega}}_g$ weights of the points of Gauss respectively in the real configuration and the configuration of reference. Lastly, let us recall that the approximation of the field of displacement in the element was presented in the § part2.1.1.1.

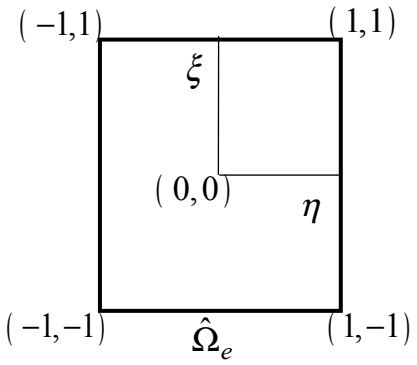

Figure 5: Element of reference

2.2.1.2 Parameter setting

That is to say T the geometrical transformation which at a point (x,y) element with discontinuity associates a point (η,ξ) of element of reference. Let us seek to parameterize the position of a point M unspecified of the element with discontinuity by (η,ξ) its coordinates in the reference mark of the element of reference.

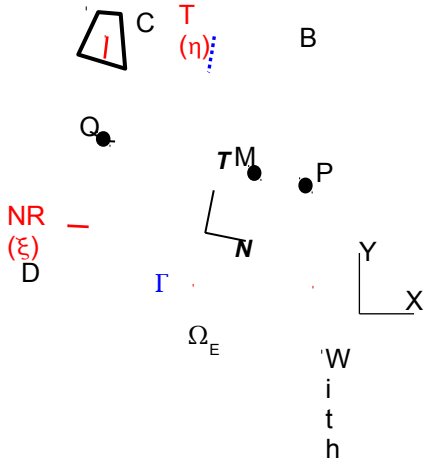

Figure 6: Parameter setting of the element with discontinuity

Are P, Q and M points (see figur

$$\begin{cases}\overrightarrow{AP}=\frac{1}{2}.(1+\xi).\overrightarrow{AB}\\ \overrightarrow{DQ}=\frac{1}{2}.(1+\xi).\overrightarrow{DC}\\ \overrightarrow{PM}=\frac{1}{2}.(1-\eta).\overrightarrow{PQ}\end{cases} \quad (34)$$

By using the sum of the vectors, one a:

$$\overrightarrow{AM}=\overrightarrow{AP}+\overrightarrow{PM}=\frac{1}{2}.(1+\xi).\overrightarrow{AB}+\frac{1}{2}.(1-\eta).\overrightarrow{PQ} \quad (35)$$

The vector is rewritten $\overrightarrow{PQ}$:

$$\overrightarrow{PQ}=\overrightarrow{PA}+\overrightarrow{AD}+\overrightarrow{DQ} \quad (36)$$

While developing with the expressions of (34):

$$\overrightarrow{PQ}=\frac{1}{2}.(1+\xi).\overrightarrow{BA}+\overrightarrow{AD}+\frac{1}{2}.(1+\xi).\overrightarrow{DC} \quad (37)$$

By gathering the terms, one obtains finally:

$$\overrightarrow{AM}=\frac{1+\eta}{2}.\frac{1+\xi}{2}.\overrightarrow{AB}+\frac{1-\eta}{2}.\overrightarrow{AD}+\frac{1-\eta}{2}.\frac{1+\xi}{2}.\overrightarrow{DC} \quad (38)$$

And thus:

$$d\,\boldsymbol{M}=\begin{Bmatrix}dx\\dy\end{Bmatrix}=\frac{\boldsymbol{N}(\xi)}{2}.d\,\eta+\frac{\boldsymbol{T}(\eta)}{2}.d\,\xi \quad (39)$$

With the two vectors of the reference mark on discontinuity:

$$\begin{cases}\boldsymbol{N}(\xi)=\begin{Bmatrix}N_x\\N_y\end{Bmatrix}=\frac{(1+\xi)}{2}.\overrightarrow{AB}-\frac{(1-\xi)}{2}.\overrightarrow{DC}-\overrightarrow{AD}\\ \boldsymbol{T}(\eta)=\begin{Bmatrix}T_x\\T_y\end{Bmatrix}=\frac{(1+\eta)}{2}.\overrightarrow{AB}+\frac{(1-\eta)}{2}.\overrightarrow{DC}\end{cases} \quad (40)$$

There is then the parameter setting:

$$\begin{Bmatrix}dx\\dy\end{Bmatrix}=[\boldsymbol{J}].\begin{Bmatrix}d\,\eta\\d\,\xi\end{Bmatrix} \quad (41)$$

Where $[\boldsymbol{J}]$ is the matrix jacobienne of the transformation:

$$[\boldsymbol{J}]=\frac{1}{2}.\begin{bmatrix}N_x & T_x\\N_y & T_y\end{bmatrix} \quad (42)$$

Notice : Discontinuity being at the center of the element, one can define the vectors of the local reference mark in this one like its length starting from the vector $\boldsymbol{T}$ in $\eta=0$:

$$\begin{cases}\boldsymbol{n}=R_{-\pi/2}\,\boldsymbol{T}(0)/\|\boldsymbol{T}(0)\|\\ \boldsymbol{t}=\boldsymbol{T}(0)/\|\boldsymbol{T}(0)\|\\ l=\|\boldsymbol{T}(0)\|\end{cases} \quad (43)$$

2.2.2 Continuity of the normal constraint

While being based on the preceding parameter setting, one shows in [6] how the element with discontinuity ensures the continuity of the normal constraint through the jump of displacement when the constraint is homogeneous in the element. Known as differently, it is shown that the vector forced $\vec{\sigma}=\psi'(\delta)$ law of behavior CZM_EXP (see §2.1.2) is equal to the normal constraint in the element $\sigma.\boldsymbol{n}$.

2.2.3 Condition of existence and unicity of the jump in the element

From a digital point of view it is important to be placed in a case where the research of the jump in the element led to a single solution. In other words it is necessary that the solution of the problem of minimization (44) have only one solution with $\boldsymbol{U}$ and κ fixed.

$$\min_{\delta \in \mathbb{R}^2} E_T(\boldsymbol{U}, \boldsymbol{\delta}, \kappa) \tag{44}$$

In [6] it is shown that the existence and the unicity of the jump are assured as soon as the following condition, bearing on the geometry like on the parameters material of the element, is assured:

$$\frac{\mu}{16} \cdot \sum_g \left(\frac{1}{\omega_g} \cdot \min_{\eta_g} \|\boldsymbol{T}(\eta_g)\|^2 \right) > l \cdot \frac{\sigma_c^2}{G_c} \tag{45}$$

Notice : In the typical case where the element with discontinuity is rectangular, the weights of the points of Gauss are equal to a quarter of the surface of the element. Moreover, the length of the element is equal to the length of discontinuity l. If one notes e the width of the element, one has for all g :

$$\omega_g = \frac{l \cdot e}{4} \tag{46}$$

Moreover $\|\boldsymbol{T}\| = l$ on each point of Gauss, therefore the condition of unicity becomes a condition over the width of the element:

$$e < \mu \cdot \frac{G_c}{\sigma_c^2} \tag{47}$$

It is this last condition which makes it possible to ensure the unicity of the jump for the elements discontinuity in *Code_Aster*. It is thus necessary to handle with precaution the latter when they are not rectangular form and to secure only the condition given by (45) is well checked.

2.2.4 Parasitic energy

The constant jump in the element with discontinuity results in introducing a parasitic energy with the interface between two adjacent elements. One shows in [6] that this energy tends towards zero when the grid is refined.

2.3 Minimization of total energy

By adopting the principle of minimization of energy, the goal of this part is to present the calculation of the jump of displacement in the elements to discontinuity like that of the field of displacement. Total energy (see §2.1.1) is not convex with respect to the couple $(\boldsymbol{U}, \boldsymbol{\delta})$. The search for a total minimum for such a functional calculus is not possible with the digital method which we will use. Moreover, with imposed force, total energy not being limited in a lower position, the total minimum does not exist. These two arguments bring us has to make the choice to search a local minimum. The problem of minimization of total energy is written:

$$\text{Chercher } (\boldsymbol{U}^*, \boldsymbol{\delta}^*) \text{ minimum local de l'énergie totale } E_T(\boldsymbol{U}, \boldsymbol{\delta}, \kappa) \tag{48}$$

In a first section (§2.3.1) one will seek to calculate $\boldsymbol{\delta}^*$ local minimum of total energy with $\boldsymbol{U}$ fixed:

$$\boldsymbol{\delta}^*(\boldsymbol{U}) = \operatorname{argmin}_{\delta} E_T(\boldsymbol{U}, \boldsymbol{\delta}, \kappa) \tag{49}$$

In the second section (§2.3.2), one will be satisfied to seek a requirement so that $\boldsymbol{U}^*$ that is to say minimum room of total energy with $\boldsymbol{\delta} = \boldsymbol{\delta}^*(\boldsymbol{U})$:

$$\boldsymbol{U}^* = \operatorname{argmin}_{\boldsymbol{U}} E_T(\boldsymbol{U}, \boldsymbol{\delta}^*(\boldsymbol{U}), \kappa) \tag{50}$$

This second stage amounts solving the equilibrium equations by taking account of possible work of the external efforts and the displacements imposed on the structure.

Notice : Digital processing of the indicatrix $I_{\mathbb{R}^+}(\boldsymbol{\delta} \cdot \boldsymbol{n})$ translating the condition of not-interpenetration of the lips of the crack, will be carried out by adding the constraint $\boldsymbol{\delta} \cdot \boldsymbol{n} \geq 0$ with the problem of minimization.

2.3.1 Minimization of total energy compared to the jump

The object of this section is to calculate the jumps of displacement, on each element with discontinuity of a grid, which minimizes total energy. If one index by i elements with discontinuity and which one notes $\boldsymbol{\delta}_i$ the jumps of displacement on each one of these elements, the problem of minimization is written:

$$\text{Chercher les } \boldsymbol{\delta}_i \text{ minima locaux de l'énergie totale, solutions de } \min_{\boldsymbol{\delta}_i, \boldsymbol{\delta}_i \cdot \boldsymbol{n} \geq 0} \sum_i E_T\left(\boldsymbol{U}, \boldsymbol{\delta}_i, \kappa\right) \tag{51}$$

Two important points will make it possible to simplify the problem appreciably. First of all, the choice of a jump of displacement independent from one element to another makes it possible to minimize total energy compared to the jump on an elementary level (technique of static condensation). There is the following equality:

$$\min_{\boldsymbol{\delta}_i, \boldsymbol{\delta}_i \cdot \boldsymbol{n} \geq 0} \sum_i E_T\left(\boldsymbol{U}, \boldsymbol{\delta}_i, \kappa\right) = \sum_i \min_{\boldsymbol{\delta}_i, \boldsymbol{\delta}_i \cdot \boldsymbol{n} \geq 0} E_T\left(\boldsymbol{U}, \boldsymbol{\delta}_i, \kappa\right) \tag{52}$$

In addition, one saw in the §2.2.3 that, for sufficiently small elements, the jump of displacement is single. Thus, the problem is brought back to a search for total minimum on each element:

$$\text{Sur chaque élément chercher } \boldsymbol{\delta} \in S \text{ minimum global de } E_T\left(\boldsymbol{U}, \boldsymbol{\delta}_i, \kappa\right) \tag{53}$$

with S the whole of the jump acceptable:

$$S = \{\boldsymbol{s} / \boldsymbol{s} \text{ constant par élément et } \boldsymbol{s} \cdot \boldsymbol{n} \geq 0\} \tag{54}$$

The equilibrium condition known as condition of optimality of order one becomes necessary and sufficient to determine the solution of the problem. This one is written in the form:

$$E_T(\boldsymbol{U}, \boldsymbol{\delta}, \kappa) \leq E_T(\boldsymbol{U}, \boldsymbol{\delta} + h\boldsymbol{\phi}, \kappa) \tag{55}$$

For all $\boldsymbol{\phi}$ constant by element and for h sufficient small so that $\boldsymbol{\delta} + h\boldsymbol{\phi} \in S$. While passing in extreme cases when h tends towards zero, becomes a condition on the directional derivative:

$$\lim_{h \to 0} \frac{1}{h} \cdot \left(E_T\left(\boldsymbol{U}, \boldsymbol{\delta} + h\boldsymbol{\phi}, \kappa\right) - E_T\left(\boldsymbol{U}, \boldsymbol{\delta}, \kappa\right)\right) \geq 0 \tag{56}$$

For all $\boldsymbol{\phi}$ constant by element such as $\boldsymbol{\delta} + h\boldsymbol{\phi} \in S$. Generally, the directional derivative is a function positively homogeneous of degree one in $\boldsymbol{\phi}$.

Thereafter we will present the calculation of the jump displacement on an element, while being based on the condition of optimality. One will start by highlighting a requirement and sufficient, which one will call criterion of starting, so that the null jump is solution of the problem. Then we will detail the calculation of the jump after starting, for the two types of behavior of the element: linear and dissipative.

2.3.1.1 Criterion of starting

In this part we will seek to determine in which condition the jump of null displacement is solution of the problem (53). Let us note that the choice to make depend energy on surface of the standard of the jump implies that total energy does not admit a derivative into zero. This fact the derivative of energy in the direction $\boldsymbol{\phi}$ will not be a function linear but positively homogeneous of degree one in $\boldsymbol{\phi}$. By using the definition of total energy (see §2.1.1) when the element is healthy (i.e when $\kappa = 0$), one calculates his derivative in the direction $\boldsymbol{\phi}$ in zero:

$$\lim_{h \to 0} \frac{1}{h} \cdot \left(E_T\left(\boldsymbol{U}, h\boldsymbol{\phi}, \kappa\right) - E_T\left(\boldsymbol{U}, 0, \kappa\right)\right) = -\sum_g \omega_g \cdot \phi^t \cdot \boldsymbol{D}_{\boldsymbol{g}}^{\boldsymbol{t}} \cdot \boldsymbol{E} \cdot \boldsymbol{B}_{\boldsymbol{g}} \cdot \boldsymbol{U} + l \cdot \sigma_c \cdot \|\phi\| \tag{57}$$

The condition of optimality (55) conduit thus with the inequality:

$$-\sum_g \omega_g \cdot \phi^t \cdot \boldsymbol{D}_{\boldsymbol{g}}^{\boldsymbol{t}} \cdot \boldsymbol{E} \cdot \boldsymbol{B}_{\boldsymbol{g}} \cdot \boldsymbol{U} + l \cdot \sigma_c \cdot \|\phi\| \geq 0 \tag{58}$$

For all $\boldsymbol{\phi} \in S$. However, to show the continuity of the normal constraint (see [6]), it was seen that:

$$\sum_g \omega_g \cdot \boldsymbol{D}_{\boldsymbol{g}}^{\boldsymbol{t}} \cdot \boldsymbol{E} \cdot \boldsymbol{B}_{\boldsymbol{g}} \cdot \boldsymbol{U} = l \cdot \vec{\boldsymbol{\sigma}} \tag{59}$$

Moreover $\boldsymbol{\phi}$ is constant on the element, therefore (58) becomes:

$$-\vec{\boldsymbol{\sigma}} \cdot \boldsymbol{\phi} + \sigma_c \cdot \|\boldsymbol{\phi}\| \geq 0 \tag{60}$$

While posing $\begin{cases}\phi_n=\rho.\cos\theta\\ \phi_t=\rho.\sin\theta\end{cases}$ with $\rho>0$ and $\frac{-\pi}{2}\leq\theta\leq\frac{\pi}{2}$ so that $\phi_n\geq 0$, and by using the definition $\vec{\sigma}=(\sigma_n,\sigma_t)$, the condition is obtained:

$$\sigma_n.\cos\theta+\sigma_t.\sin\theta\leq\sigma_c \quad \forall\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right] \tag{61}$$

A study of the function $f(\theta)=\sigma_n.\cos\theta+\sigma_t.\sin\theta$ give:

$$\sup_{-\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2}} f(\theta)=\sqrt{<\sigma_n>_+^2+\sigma_t^2} \tag{62}$$

With the definition of the positive part of a quantity $<.>_+=max(.,0)$. The inequality (61) conduit thus with the criterion of starting in constraint according to:

$$\sqrt{<\sigma_n>_+^2+\sigma_t^2}\leq\sigma_c \tag{63}$$

One represents this criterion in the plan (σ_n,σ_t) on the figure 7.

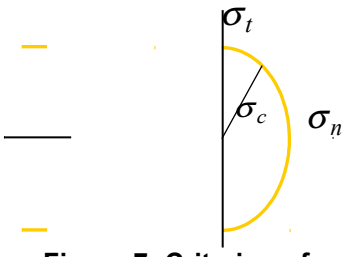

Figure 7: Criterion of starting in constraint of the element with discontinuity

Remarks : One places figure in the case of where orientation of discontinuity $\boldsymbol{n}$ is fixed (by the grid). That explains why one obtains a condition on the vector forced and not on the principal components of the constraint.

2.3.1.2 Calculation of the jump

Let us seek to solve the problem of minimization if the jump is not null. In this case, the directional derivative is a linear function of $\boldsymbol{\phi}$, the condition of optimality (55) is written:

$$\lim_{h\to 0}\frac{1}{h}.\left(E_T(\boldsymbol{U},\boldsymbol{\delta}+h\boldsymbol{\phi},\kappa)-E_T(\boldsymbol{U},\boldsymbol{\delta},\kappa)\right)=\frac{\partial E_T}{\partial\boldsymbol{\delta}}\cdot\boldsymbol{\phi}\geq 0 \tag{64}$$

For all $\boldsymbol{\phi}$ constant by element such as $\boldsymbol{\delta}+h\boldsymbol{\phi}\in S$. While exploiting the acceptable directions one from of deduced the conditions:

$$\begin{cases}\frac{\partial E_T}{\partial\delta_n}.\phi_n\geq 0 \quad \forall\phi_n\geq 0 \quad \text{si} \quad \delta_n=0\\ \frac{\partial E_T}{\partial\delta_n}.\phi_n=0 \quad \forall\phi_n \quad\quad \text{si} \quad \delta_n>0\\ \frac{\partial E_T}{\partial\delta_t}.\phi_t=0 \quad \forall\phi_t\end{cases} \tag{65}$$

In addition, by approaching the integral by a discrete sum on the points of Gauss, the derivative of total energy in the direction $\boldsymbol{\phi}$ is written:

$$\frac{\partial E_T}{\partial \boldsymbol{\delta}}\cdot\boldsymbol{\phi}=-\sum_g \omega_g.\phi^t.\boldsymbol{D}_{\boldsymbol{g}}^{\boldsymbol{t}}.\boldsymbol{E}.\left(\boldsymbol{B}_{\boldsymbol{g}}.\boldsymbol{U}-\boldsymbol{D}_{\boldsymbol{g}}\boldsymbol{\delta}\right).\boldsymbol{\phi}+l.\frac{\partial \psi}{\partial \boldsymbol{\delta}}\cdot\boldsymbol{\phi} \tag{66}$$

With ψ the density of energy of surface, equalizes with ψ_{lin} in linear mode and with ψ_{dis} in dissipative mode. To reduce the notations one defines $\boldsymbol{S}$ in the following way:

$$\boldsymbol{S}=\begin{Bmatrix}S_n\\S_t\end{Bmatrix}=-\sum_g \omega_g.\boldsymbol{D}_{\boldsymbol{g}}^{\boldsymbol{t}}.\boldsymbol{E}.\boldsymbol{B}_{\boldsymbol{g}}.\boldsymbol{U} \tag{67}$$

and $\boldsymbol{Q}$:

$$\boldsymbol{Q}=\begin{bmatrix}Q_{nn} & Q_{nt}\\Q_{nt} & Q_{tt}\end{bmatrix}=-\sum_g \omega_g.\boldsymbol{D}_{\boldsymbol{g}}^{\boldsymbol{t}}.\boldsymbol{E}.\boldsymbol{D}_{\boldsymbol{g}} \tag{68}$$

There is thus the writing simplified of (66):

$$\frac{\partial E_T}{\partial \boldsymbol{\delta}}\cdot\boldsymbol{\phi}=(\boldsymbol{S}+\boldsymbol{Q}.\boldsymbol{\delta})\cdot\boldsymbol{\phi}+l.\frac{\partial \psi}{\partial \boldsymbol{\delta}}\cdot\boldsymbol{\phi} \tag{69}$$

Now let us seek to calculate the jump displacement starting from the conditions by using the derivative of the energy written in the form (69). One will distinguish calculation in the linear mode from that in the dissipative mode and for each one of them one will distinguish the case or the normal jump is null of that where it is not it.

Calculation of the jump in linear mode

In linear mode density of energy of surface east given by:

$$\psi=\psi_{lin}=\frac{1}{2}.P(\kappa).\boldsymbol{\delta}\cdot\boldsymbol{\delta} \tag{70}$$

The derivative of total energy compared to the jump (69) becomes:

$$\frac{\partial E_T}{\partial \boldsymbol{\delta}}\cdot\boldsymbol{\phi}=(\boldsymbol{S}+\boldsymbol{Q}.\boldsymbol{\delta})\cdot\boldsymbol{\phi}+l.P(\kappa).\boldsymbol{\delta}\cdot\boldsymbol{\phi} \tag{71}$$

In the case of a normal jump no one $(0,\delta_t)$, Lbe conditions of optimality (65) lead to:

$$\begin{cases}S_n+Q_{nt}.\delta_t\geq 0\\ \delta_t=-\dfrac{S_t}{Q_t+l.P(\kappa)}\end{cases} \tag{72}$$

The tangent jump is given explicitly by the second condition. The first imposes that the normal constraint in the element is negative or worthless. If such were not the case, there would be necessarily a normal jump. Known as differently, the jump in the element will be null as soon as the element is put in compression. This translated the taking into account of the not-interpenetration of the lips of the crack.
In the case of a jump normalizedL strictly positive $(\delta_n>0,\delta_t)$, Lbe conditions of optimality (65) give the jump of displacement directly:

$$\boldsymbol{\delta}=-\left(\boldsymbol{Q}+\boldsymbol{Id}.l.P(\kappa)\right)^{-1}.\boldsymbol{S} \tag{73}$$

In linear mode the internal variable κ do not evolve.

Calculation of the jump in dissipative mode

In dissipative mode density of energy of surface east given by:

$$\psi=\psi_{dis}=G_c.\left[1-\exp\left(-\frac{\sigma_c}{G_c}.\|\boldsymbol{\delta}\|\right)\right] \tag{74}$$

The derivative of total energy compared to the jump (69) becomes:

$$\frac{\partial E_T}{\partial \boldsymbol{\delta}}\cdot\boldsymbol{\phi}=(\boldsymbol{S}+\boldsymbol{Q}.\boldsymbol{\delta})\cdot\boldsymbol{\phi}+l.\sigma_c.\frac{\boldsymbol{\delta}\cdot\boldsymbol{\phi}}{\|\boldsymbol{\delta}\|}.\exp\left(-\frac{\sigma_c}{G_c}.\|\boldsymbol{\delta}\|\right) \tag{75}$$

In the case of a normal jump no one $(0,\delta_t)$, Lbe conditions of optimality (65) lead to:

$$\begin{cases} S_n+Q_{nt}.\delta_t\geq 0 \\ S_t+Q_{tt}.\delta_t+l.\sigma_c.\operatorname{sign}(\delta_t).\exp\left(-\frac{\sigma_c}{G_c}.|\delta_t|\right)=0 \end{cases} \tag{76}$$

The first condition implies that the normal jump is null when the normal constraint in the element is negative. What translates the taking into account of the not-interpenetration of the lips of the crack. Second condition, one deduces that $\operatorname{sign}(\delta_t)=-\operatorname{sign}(S_t)$, therefore $\delta_t=-\operatorname{sign}(S_t).\beta$ with $\beta>0$ is solution of the following nonlinear scalar equation:

$$|S_t|+Q_n.\beta+l.\sigma_c.\exp\left(-\frac{\sigma_c}{G_c}.\beta\right)=0 \tag{77}$$

that one solves by an algorithm of Newton.
In the case of a jump normalizedL strictly positive $(\delta_n>0,\delta_t)$, Lbe conditions of optimality (65) lead to the following nonlinear equation:

$$\boldsymbol{S}+\boldsymbol{Q}.\boldsymbol{\delta}+l.\sigma_c.\frac{\boldsymbol{\delta}_t}{\|\boldsymbol{\delta}\|}.\exp\left(-\frac{\sigma_c}{G_c}.\|\boldsymbol{\delta}\|\right)=0 \tag{78}$$

One notes $\boldsymbol{\delta}=r.\tilde{\boldsymbol{\delta}}$ with $r>0$ and $\tilde{\boldsymbol{\delta}}=(\cos\theta,\sin\theta)$ where $\frac{-\pi}{2}\leq\theta\leq\frac{\pi}{2}$ so that $\tilde{\boldsymbol{\delta}}\cdot\boldsymbol{n}\geq 0$. According to (78) one a:

$$\tilde{\boldsymbol{\delta}}=-\left(r.\boldsymbol{Q}+l.\sigma_c.\boldsymbol{Id}.\exp\left(-\frac{\sigma_c}{G_c}.r\right)\right)^{-1}.\boldsymbol{S} \tag{79}$$

It remains to calculate r. For that it is enough to solve the nonlinear scalar equation $\|\tilde{\boldsymbol{\delta}}\|=1$ by an algorithm of Newton. In dissipative mode the internal variable κ evolve, it corresponds to the standard of the jump calculated $\kappa=\|\tilde{\boldsymbol{\delta}}\|$.

2.3.1.3 Calculation algorithm

In this part, one presents the calculation algorithm of the jump $\boldsymbol{\delta}^*$ on an element given as well as the evolution of the internal variable κ.

1. **Beginning of calculation**
2. Test of existence and unicity of the solution (see §2.2.3)
3. If ($\kappa=0$ and $\sqrt{<\sigma_n>_+^2+\sigma_t^2}\leq\sigma_c$) then
   - $\boldsymbol{\delta}^*=\boldsymbol{0}$, the threshold κ remain null → **End of calculation**
4. If $\kappa>0$ then
   - Calculation of the threshold in linear mode (see 2.3.1.2) → $\boldsymbol{\delta}^c$
     - If $\|\boldsymbol{\delta}^c\|<\kappa$, then $\boldsymbol{\delta}^*=\boldsymbol{\delta}^c$, the threshold κ do not evolve → **End of calculation**
5. Calculation of the jump in dissipative mode (see § 2.3.1.2) → $\boldsymbol{\delta}^c$
   - $\boldsymbol{\delta}^*=\boldsymbol{\delta}^c$, actualization of the threshold $\kappa=\|\boldsymbol{\delta}^*\|$ → **End of calculation**
6. **End of calculation**

2.3.2 Minimization of total energy by report with displacements

After having calculated the jumps of displacement $\delta^*(U)$ on each element with discontinuity, the objective is to calculate the field of displacements which minimizes total energy. The problem is formulated in the following way:

$$\text{Chercher } U^* \text{ minimum local de l'énergie totale } E_T(U,\delta^*(U),\kappa) \tag{80}$$

The functional calculus $E_T(U,\delta^*(U),\kappa)$ is not convex in U. To be minimum room, the solution must check the stability and equilibrium conditions. The second condition relates to the derivative second of the functional calculus and requires the use of a method specifically dedicated to minimization. The algorithm of Newton used does not make it possible to check this second condition. One will thus be satisfied to check the equilibrium condition, requirement so that displacement is solution of the problem (80).
Let us consider a structure Ω with the following loading:

- f density of force voluminal on Ω ;
- F density of force surface on Γ_N ;
- U^d displacements imposed on Γ_D

Γ_D and Γ_N are disjoined parts of the border of Ω. The work of the external efforts is written:

$$W^{ext}(U)=\int_\Omega f.U.d\Omega+\int_{\Gamma_N} F.U.d\Gamma_N \tag{81}$$

The total energy of the structure is written then:

$$E_T(U,\delta^*(U),\kappa)=\Phi(U,\delta^*(U))+\Psi(\delta^*(U),\kappa)-W^{ext}(U) \tag{82}$$

With U belonging to the space of the fields of displacement kinematically acceptable. The condition of optimality of order one, requirement so that U that is to say minimum room, is written:

$$\frac{\partial E_T}{\partial U}(U,\delta^*(U),\kappa).V\geq 0 \tag{83}$$

For any field test V acceptable. It is known that:

$$\frac{\partial E_T}{\partial \delta}(U,\delta^*(U),\kappa)=0 \tag{84}$$

What gives:

$$\int_\Omega B^t.E.\left(B.U-D.\delta^*(U)\right).V.d\Omega-\int_\Omega f.V.d\Omega-\int_{\Gamma_N} F.V.d\Gamma_N\geq 0 \quad \forall V \tag{85}$$

This inequality becomes an equality while taking V chosen well, and, as this equality is true for all V acceptable, one obtains balance between the interior and external efforts:

$$F^{\text{int}}(U)=F^{\text{ext}} \tag{86}$$

With:

$$\begin{cases} F^{\text{int}}(U)=\int_\Omega B^t.E.\left(B.U-D.\delta^*\right).d\Omega \\ F^{\text{ext}}=\int_\Omega f.d\Omega+\int_{\Gamma_N} F.d\Gamma_N \end{cases} \tag{87}$$

Let us note C the linear operator translating the imposed conditions of displacement. The dualisation of these conditions leads to the following system:

$$\begin{cases} F^{\text{int}}(U)+C^t.\lambda=F^{\text{ext}} \\ C.U=U^d \end{cases} \tag{88}$$

The unknown factors are now at any moments the couple (U,λ) where λ represent the multipliers of Lagrange associated with the conditions with Dirichlet. The method used to solve is a method of Newton (see documentation of STAT_NON_LINE [R5.03.01}). The calculation of the tangent matrix is explained in appendix.

3 Méthode of piloting of the loading

The purpose of the method of piloting of the loading is to take into account instabilities of the structure, and to thus follow possible “back return” of the total answer forces displacement. A typical case of this method, initially developed by Lorentz and Badel [7], was adapted to the preceding model of cracking.
The principle of the method of piloting as well as the resolution of the new total system which results from this is explained in detail in [R5.03.80]. We will detail the part of piloting specific to the law CZM_EXP by explaining the choice of the equation of control of piloting as well as the method to solve it.

3.1 Equation of control of piloting

The goal of this part is to present the equation of control of piloting like its resolution for the model to internal discontinuity CZM_EXP. The unknown factor of this equation is the intensity of the loading η_i^n at the moment of calculation i and with the iteration of Newton n that one will note henceforth η to simplify the notation. As we saw with the §2.1 the law of behavior of the element with discontinuity is controlled by a threshold. Let us note F_{el} the function threshold in following constraint:

$$F_{el}(\sigma_n,\sigma_t)=\sqrt{<\sigma_n>_+^2+\sigma_t^2}-R(\kappa) \tag{89}$$

With the vector jump of constraint which is worth:

$$\vec{\sigma}=\begin{Bmatrix}\sigma_n\\ \sigma_t\end{Bmatrix}=\frac{1}{l}.\sum_g \omega_g.\boldsymbol{D_g^t}.\boldsymbol{E}.\left(\boldsymbol{B_g}.\boldsymbol{U}-\boldsymbol{D_g}\delta\right) \tag{90}$$

And $R(\kappa)$ the variable of threshold in constraint defined by (33):

$$R(\kappa)=\sigma_c.\exp\left(-\frac{\sigma_c}{G_c}.\kappa\right) \tag{91}$$

Let us note G_{el} the function threshold in jump:

$$G_{el}(\delta_n,\delta_t)=\sqrt{\delta_n^2+\delta_t^2}-\kappa \tag{92}$$

With $\delta_n\geq 0$ and κ the threshold in jump. Let us choose an equation of control of piloting so that the intensity of the loading η makes leave the criterion at least an element to discontinuity of the structure of a quantity proportional to $\Delta\tau$. The functions thresholds depend then on η and will note we them $\tilde{F}_{el}(\eta)$ and $\tilde{G}_{el}(\eta)$. The equation of control of piloting is written:

- For the criterion in constraint:

$$\tilde{P}(\eta)\underset{\text{def}}{=}\max_j \tilde{F}_{el}^j(\eta)=R(\kappa^-).\Delta\tau \tag{93}$$

- For the criterion in jump:

$$\tilde{P}(\eta)\underset{\text{def}}{=}\max_j \tilde{G}_{el}^j(\eta)=\kappa^-.\Delta\tau \tag{94}$$

Where j indicate the index of the elements with discontinuity of the grid and κ^- the variable threshold at the moment $i-1$.

Remarks :

- If $\kappa^-=0$, the jump being null, we will use the equation of control of the piloting expressed with the criterion in constraint (93).
- In the case where, $\kappa^->0$ we will use the equation of control of the piloting expressed with the criterion in jump (94). Indeed, the jump not being known, one cannot use the constraints (σ_n,σ_t) who depend explicitly on this one.

3.1.1 Resolution of the equation with the criterion in constraint

We thus place ourselves if $\kappa^{-}=0$. The goal here is to explain the resolution of (93) for a given element, without taking account of max on all the elements (the taking into account of max is explained in [R5.03.80]). One thus seeks to solve:

$$\tilde{F}_{el}(\eta)=R(0).\Delta\tau \tag{95}$$

Initially let us reveal explicitly η in the expression of the front criterion $\tilde{F}_{el}(\eta)$ to explain the resolution of the equation. At the moment i displacement is sought U_i who expresses himself with the iteration of Newton n :

$$\mathbf{U}_i=\mathbf{U}_{i-1}+\Delta\mathbf{U}_i^n+\delta\mathbf{U}_{impo,i}^n+\eta_i.\delta\mathbf{U}_{pilo,i}^n \tag{96}$$

One separates the known part (within the meaning of non-pilotéee) of displacement:

$$\mathbf{U}_{impo}=\mathbf{U}_{i-1}+\Delta\mathbf{U}_i^n+\delta\mathbf{U}_{impo,i}^n \tag{97}$$

And the controlled part:

$$\mathbf{U}_{pilo}=\delta\mathbf{U}_{pilo,i}^n \tag{98}$$

Directly:

$$\mathbf{U}_i=\mathbf{U}_{impo}+\eta.\mathbf{U}_{pilo} \tag{99}$$

The jump of displacement being no one a:

$$\begin{Bmatrix}\sigma_n\\ \sigma_t\end{Bmatrix}=\frac{1}{l}.\sum_g \omega_g.\boldsymbol{D}_g^t.\boldsymbol{E}.\boldsymbol{B}_g.\boldsymbol{U}_i \tag{100}$$

By using the decomposition (99):

$$\begin{Bmatrix}\sigma_n\\ \sigma_t\end{Bmatrix}=\mathbf{S}_{impo}+\eta.\mathbf{S}_{pilo} \tag{101}$$

With:

$$\begin{aligned}\mathbf{S}_{impo}&=\frac{1}{l}.\sum_g \omega_g.\boldsymbol{D}_g^t.\boldsymbol{E}.\boldsymbol{B}_g.\boldsymbol{U}_{impo}\\ \mathbf{S}_{pilo}&=\frac{1}{l}.\sum_g \omega_g.\boldsymbol{D}_g^t.\boldsymbol{E}.\boldsymbol{B}_g.\boldsymbol{U}_{pilo}\end{aligned} \tag{102}$$

The equation of control of piloting is thus written, by taking account owing to the fact that $\kappa^{-}=0$:

$$\tilde{F}_{el}(\eta)\underset{\text{def}}{=}\sqrt{\langle(\mathbf{S}_{\text{impo}}+\eta.\mathbf{S}_{\text{pilo}})\cdot\boldsymbol{n}\rangle_+^2+((\mathbf{S}_{\text{impo}}+\eta.\mathbf{S}_{\text{pilo}})\cdot\boldsymbol{t})^2}-\sigma_c=\sigma_c.\Delta\tau \tag{103}$$

This equation corresponds to a polynomial of degree two, which one notes p_1 , if $(\mathbf{S}_{\text{impo}}+\eta.\mathbf{S}_{\text{pilo}})\cdot\boldsymbol{n}>0$ and with a polynomial of degree two which one notes p_2 in the contrary case.

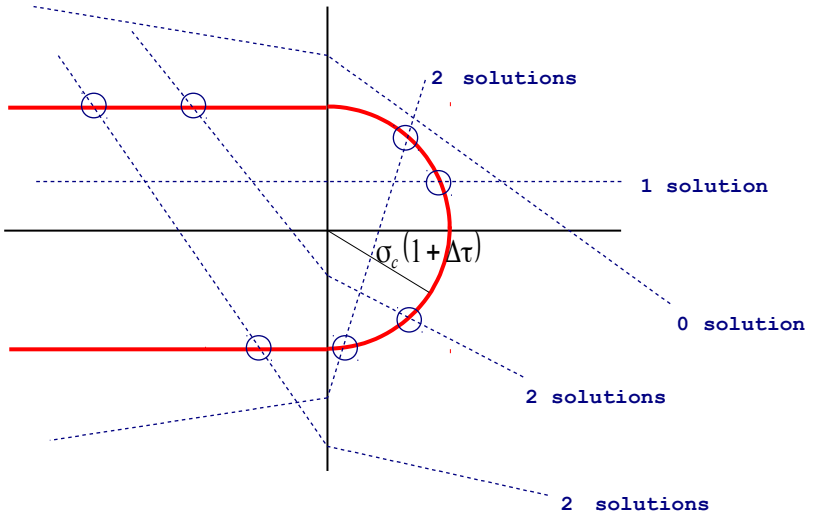

Figure 8: Exit of the criterion in constraint

To find η , these two polynomials are solved. That returns, in space (σ_n, σ_t) , to find the intersection of two half-lines with the criterion in constraint. The figure (8) represent " exit of criterion " controlled by the parameter $\Delta\tau$.
Each one of these two polynomials has 0.1 or 2 solutions, one will admit the solutions of p_1 who check $(\mathbf{S}_{\text{impo}}+\eta.\mathbf{S}_{\text{pilo}})\cdot\boldsymbol{n}>0$ and solutions of p_2 who check $(\mathbf{S}_{\text{impo}}+\eta.\mathbf{S}_{\text{pilo}})\cdot\boldsymbol{n}\leq 0$. The allowed solutions will be to more both, one notes them $\eta_{k=0,1,2}$.

3.1.2 Resolution of the equation with the criterion in jump

The goal here is to solve the equation of control of piloting for a given element if $\kappa^{-}>0$, always without taking account of max on all the elements. One seeks to solve:

$$\tilde{G}_{el}(\eta)=\kappa^{-}.\Delta\tau \tag{104}$$

Initially let us reveal explicitly η in the expression of the criterion $\tilde{G}_{el}(\eta)$ then let us detail the calculation of (104). One proceeds like in the case of the criterion in constraint (§3.1.1). The difference relates to the writing of the jump of constraint in the case general (when the jump is not null):

$$\begin{Bmatrix}\sigma_n\\ \sigma_t\end{Bmatrix}=\frac{1}{l}.\sum_g \omega_g.\boldsymbol{D}_g^t.\boldsymbol{E}.(\boldsymbol{B}_g.\boldsymbol{U}_i-\boldsymbol{D}_g.\delta) \tag{105}$$

That one breaks up into three parts:

$$\begin{Bmatrix}\sigma_n\\ \sigma_t\end{Bmatrix}=\mathbf{S}_{impo}+\eta.\mathbf{S}_{pilo}+\boldsymbol{Q}.\delta \tag{106}$$

With:

$$\begin{aligned}\mathbf{S}_{impo}&=\frac{1}{l}.\sum_g \omega_g.\boldsymbol{D}_g^t.\boldsymbol{E}.\boldsymbol{B}_g.\boldsymbol{U}_{impo}\\ \mathbf{S}_{pilo}&=\frac{1}{l}.\sum_g \omega_g.\boldsymbol{D}_g^t.\boldsymbol{E}.\boldsymbol{B}_g.\boldsymbol{U}_{pilo}\\ \mathbf{Q}&=\frac{1}{l}.\sum_g \omega_g.\boldsymbol{D}_g^t.\boldsymbol{E}.\boldsymbol{D}_g\end{aligned} \tag{107}$$

In addition for a threshold in jump $\kappa^{-}>0$ fixed one has, according to the law of behavior `CZM_EXP` :

$$\begin{Bmatrix}\sigma_n\\ \sigma_t\end{Bmatrix}=P(\kappa^-).\boldsymbol{\delta} \tag{108}$$

With $P(\kappa^-)=\frac{\sigma_c}{\kappa^-}.\exp\left(-\frac{\sigma_c}{G_c}.\kappa^-\right)$. Of (106) and (108) one deduces:

$$\mathbf{S}_{impo}+\eta.\mathbf{S}_{pilo}+\boldsymbol{Q}.\boldsymbol{\delta}=P(\kappa^-).\boldsymbol{\delta} \tag{109}$$

What makes it possible to know $\boldsymbol{\delta}$ according to the parameter of piloting η :

$$\boldsymbol{\delta}=\begin{Bmatrix}\delta_n\\ \delta_t\end{Bmatrix}=\boldsymbol{\delta}_{\text{impo}}+\eta.\boldsymbol{\delta}_{\text{pilo}} \tag{110}$$

With:

$$\begin{aligned}\boldsymbol{\delta}_{\text{impo}}&=(P(\kappa^-).\boldsymbol{Id}-\boldsymbol{Q})^{-1}.\boldsymbol{S}_{\text{impo}}\\ \boldsymbol{\delta}_{\text{pilo}}&=(P(\kappa^-).\boldsymbol{Id}-\boldsymbol{Q})^{-1}.\boldsymbol{S}_{\text{pilo}}\end{aligned} \tag{111}$$

The equation of control of piloting is thus written:

$$\tilde{G}_{el}(\eta)\underset{\text{def}}{=}\sqrt{((\boldsymbol{\delta}_{\text{impo}}+\eta.\boldsymbol{\delta}_{\text{pilo}})\cdot\boldsymbol{n})^2+((\boldsymbol{\delta}_{\text{impo}}+\eta.\boldsymbol{\delta}_{\text{pilo}})\cdot\boldsymbol{t})^2}-\kappa^-=\kappa^-.\Delta\tau \tag{112}$$

With the condition of positive normal jump:

$$(\boldsymbol{\delta}_{\text{impo}}+\eta.\boldsymbol{\delta}_{\text{pilo}})\cdot\boldsymbol{n}\geq 0 \tag{113}$$

The criterion is reduced to a half rings in space (δ_n,δ_t). To be able to use piloting when the normal jump tends to being negative (compression of the element) we let us authorize a small exit of criterion for such jumps. " exit of criterion ", controlled by the parameter $\Delta\tau$, can be represented in space (δ_n,δ_t) by an arc of a circle and a segment (see figure 9).

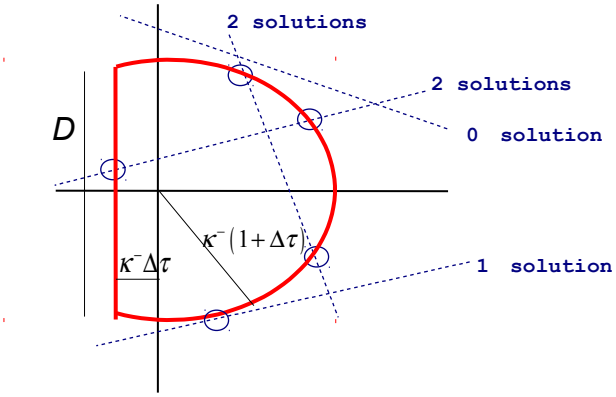

Figure 9: Exit of the criterion in jump

Notice : It is understood that only the prediction of the algorithm of Newton will leave the criterion, therefore in particular could violate the condition of contact. The solution of the problem, as for it, will respect it.
The resolution of the equation of control of piloting thus amounts finding the intersection between a line and an arc of a circle or the intersection between a line and a segment.

In short, one solves:

- The polynomial (112), of degree two in η (right intersection/circle): they are admitted η who check $(\boldsymbol{\delta}_{\text{impo}}+\eta.\boldsymbol{\delta}_{\text{pilo}})\cdot\boldsymbol{n}\geq-\kappa^-.\Delta\tau$;
- The polynomial $(\boldsymbol{\delta}_{\text{impo}}+\eta.\boldsymbol{\delta}_{\text{pilo}})\cdot\boldsymbol{n}=-\kappa^-.\Delta\tau$, of degree one in η : $(\boldsymbol{\delta}_p+\eta\boldsymbol{\delta}_d)\cdot\boldsymbol{n}=-\kappa^-\Delta\tau$ (right intersection/segment), they are admitted η who check $\left|(\boldsymbol{\delta}_{\text{impo}}+\eta.\boldsymbol{\delta}_{\text{pilo}})\right|\cdot\boldsymbol{t}\leq d$ with $d=\kappa^-.\sqrt{1+2.\Delta\tau}$.

One will note $\eta_{k=0,1,2}$ allowed solutions. As regards choice among these solutions one will refer to [R5.03.80].

4 APPENDIX: Calculation of the tangent matrix for the element with discontinuity

In this part we will detail the calculation of the derivative of the interior efforts compared to displacement: $\frac{\partial \boldsymbol{F}^{int}}{\partial \boldsymbol{U}}$ who appears in K_T tangent matrix of the system intervening in the algorithm of Newton. One a:

$$\boldsymbol{F}^{\text{int}}(\boldsymbol{U})=\int_{\Omega} \boldsymbol{B}^{t}.\boldsymbol{E}.\left(\boldsymbol{B}.\boldsymbol{U}-\boldsymbol{D}.\delta^{*}\right).d\Omega \tag{114}$$

Thus:

$$\frac{\partial \boldsymbol{F}^{\text{int}}(\boldsymbol{U})}{\partial \boldsymbol{U}}=\int_{\Omega} \boldsymbol{B}^{t}.\boldsymbol{E}.\left(\boldsymbol{B}-\boldsymbol{D}.\frac{\partial \delta^{*}}{\partial \boldsymbol{U}}\right).d\Omega \tag{115}$$

The term should be calculated $\frac{\partial \delta^{*}}{\partial \boldsymbol{U}}$ on each element with discontinuity Ω_e (static condensation). Minimization compared to δ total energy led us to calculate δ^{*} solution of the equation:

$$\int_{\Omega_e} \boldsymbol{D}^{t}.\boldsymbol{E}.\left(\boldsymbol{B}.\boldsymbol{U}-\boldsymbol{D}.\delta^{*}\right).d\Omega=l.\psi'\left(\delta^{*}\right) \tag{116}$$

Let us derive this equation compared to $\boldsymbol{U}$:

$$\int_{\Omega_e} \boldsymbol{D}^{t}.\boldsymbol{E}.\left(\boldsymbol{B}.\boldsymbol{U}-\boldsymbol{D}.\frac{d\delta^{*}}{d\boldsymbol{U}}\right).d\Omega=l.\psi''\left(\delta^{*}\right).\frac{d\delta^{*}}{d\boldsymbol{U}} \tag{117}$$

Thus:

$$\frac{d\delta^{*}}{dU}=\left[l.\Psi''(\delta*)+\int_{\Omega_e} \boldsymbol{D}^{t}.\boldsymbol{E}.\boldsymbol{D}.d\Omega\right]^{-1}.\int_{\Omega_e} \boldsymbol{D}^{t}.\boldsymbol{E}.\boldsymbol{B}.d\Omega \tag{118}$$

It is then enough to calculate Ψ'', three cases are distinguished:

1. If $\delta=\boldsymbol{0}$, then $\Psi''=0$;
2. If the element is in linear mode, $\Psi''=\Psi''_{lin}=\frac{\sigma_c}{\kappa}.\exp\left(-\frac{\sigma_c}{G_c}.\kappa\right).\boldsymbol{Id}_{2\times 2}$;
3. If the element is in dissipative mode, $\Psi''=\Psi''_{dis}=C.\boldsymbol{Id}_{2\times 2}+\tilde{C}.\delta\otimes\delta$ with

$$\begin{cases} C=\dfrac{\sigma_c}{\|\delta\|}.\exp\left(-\dfrac{\sigma_c}{G_c}.\|\delta\|\right) \\ \tilde{C}=-\left(\dfrac{1}{\|\delta\|}+\dfrac{\sigma_c}{G_c}\right).\dfrac{\sigma_c}{\|\delta\|^2}.\exp\left(-\dfrac{\sigma_c}{G_c}.\|\delta\|\right) \end{cases} \tag{119}$$

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6 Description of the versions of the document

Version Aster	Author (S) or contributor (S), organization	Description of the modifications
9.4	J.Laverne	Initial text
10.2	M.Abbas	Working of the piloting part