Élément Hydraulic coupled with XFEM

Summary:

In this document we present the formulation of a new type of finite elements. It is about a hydraulic element crossed by a discontinuity (an interface or a crack) whose design rests on the use of the finite element method extended in the formulation of the discretized equations of the model poro-mechanics for a porous environment saturated with fluid.

Contents

1 Introduction

The formulation of the model coupled HM-XFEM is based on the equations of the model of joint [R7.02.15] like on those of the model of behavior THHM [R7.01.11] in the saturated case. Classically, the model of joint is used to model in 2D the behavior of a water seal or a discontinuity in the presence of a flow of fluid in her centre, thus generating a pressure of fluid. The model of joint makes it possible to take into account:

- the preferential flow of the fluid in discontinuity conditioned by the opening of this one,
- the exchange of fluid enters the porous environment and discontinuity the beam,
- propagation of discontinuity,
- deformation of the porous environment induced by the pressure of fluid.

Within the framework of the classical finite element method, the use of such a model presents a major drawback. Indeed it is necessary to explicitly represent discontinuity in the grid and to make agree the lips of this one with the edges of the elements constituting the grid. That implies that for its evolution, it is necessary to resort to algorithms of projection worked out for the actualization of its geometry. This stage can prove very expensive in computing times for complex geometries.

In order to free itself from this constraint related to the grid, we plan the introduction of a new hydraulic element (HM) coupled with the wide finite element method (XFEM) [1,2,3]. This method, based on the principle of partition of the unit [4], ensures a greater flexibility for modelings utilizing more or less complex geometries. Indeed with this method, discontinuity is not represented any more physically in the grid but in manner symbolic system while enriching with additional degrees of freedom, the approximated fields. For more details on the finite element method extended in *Code_Aster*, the user can refer to documentation [R7.02.12] (mechanical case only).

In the literature certain authors already considered the coupling of method XFEM with the model poro-mechanics coupled HM. It is the case of [5] in the case of the dynamic analysis of the porous environments in unsaturated conditions. The extension to case THM for a saturated medium crossed by an impermeable interface is considered in [6]. Other authors as [7] proposed a model which can take into account the singularity of the field of pressure at a peak of discontinuity, by adapting the expression of the singular functions for the bottom of discontinuity. However, the models developed by these authors take into account neither the phenomena of exchange which can exist between the surrounding medium (named massive in the continuation of this document) and discontinuity, nor the propagation of the latter within the porous environment. This point will be taken into account in the model developed in present documentation by the introduction of the cohesive laws regularized into the weak formulation of the mechanical equilibrium equation (see § 3.1.2).

From a point of view practises, the main difficulty in the formulation of element HM-XFEM is the construction of various spaces of approximation of the mechanical magnitudes and hydraulic. The respect of the stability condition LBB is essential in order to obtain a single and convergent solution [8.9]. The violation of this condition (i.e. of not choosing the good degree of interpolation of the various fields approximated to see § 5.2.1) involve oscillations of the solution. The cohabitation of elements HM-XFEM and the classical elements HM (those whose documentations refer [R7.01.11] and [R7.01.10]) is also a delicate point, in particular with regard to the distribution of the degrees of freedom to the various nodes (tops or mediums) of each type of element (see § 5.2.1).

In the continuation of this documentation we will recall the framework of study of the problem, then in the second time the fundamental equations of the model poro-mechanics implied in the formulation of model HM-XFEM. Finally we will proceed to the discretization of the variational forms of the equilibrium equations, at the same time in time (thanks to one θ - diagram) and spaces some (thanks to method XFEM).

The presentation of model HM-XFEM and its validation were the object of a scientific publication [15].

2 Presentation of the problem

2.1 Definition of the field of study

That is to say $\Omega \subset \mathbb{R}^d$ with $d \in \{2,3\}$ a field entirely crossed by a crack or a permeable interface. That is to say $\boldsymbol{n}$ the normal external with the border $\partial\Omega$ field, and $\boldsymbol{n}_c$ that of the interface Γ_c.

It is possible to break up:

- the border of the field Ω in $\partial\Omega = \Gamma_u \cup \Gamma_t \cup \Gamma_p \cup \Gamma_F$ where the boundary conditions are imposed (of type Dirichlet and Neumann) for the hydrodynamics (on Γ_p and Γ_F) and for mechanics (on Γ_u and Γ_t),
- the interface in $\Gamma_c = \Gamma_f \cup \Gamma_1 \cup \Gamma_2$ where Γ_1 and Γ_2 the lips of discontinuity represent. One impose conditions of flow on Γ_1, Γ_2 and Γ_f and of the cohesive surface efforts on Γ_1 and Γ_2.

Figure 2.1-1 give a notation symbolic of the conditions imposed on the border of the field.

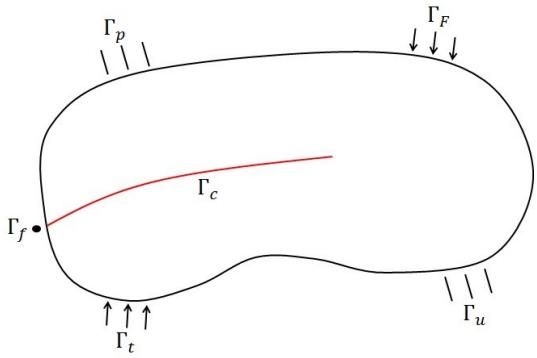

Figure 2.1-1: Imposition of the boundary conditions

2.2 Assumptions and notations

One considers a porous environment saturated with liquid (in general of water). The rate mixing associated in *Code_Aster* is thus LIQU_SATU (for more details to refer to the note of model THM [U2.04.05]). In addition there exists a longitudinal flow of fluid on the level of the interface. By supposing that this one is permeable, the exchanges of fluid take place between the solid mass (left Ω who is not the interface) and interfaces it. They are represented on Figure 2.2-1.

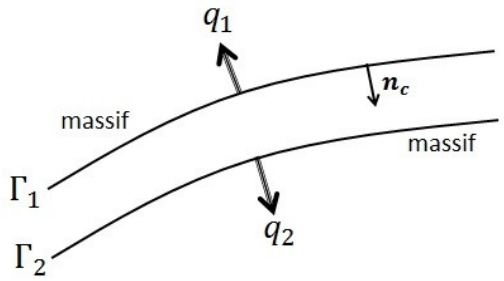

Figure 2.2-1: Orientation of exchanges between the solid mass and discontinuity

q_1 and q_2 are flows due to the exchanges between the solid mass and the interface, and are expressed in $kg.m^{-2}.s^{-1}$. These flows come from the interface and are directed respectively towards the parts higher and lower of the solid mass than the level of the lips Γ_1 and Γ_2 interface. They are directed discontinuity towards the solid mass.

The field of pressure on the level of the solid mass is noted p and that on the level of the interface is noted p_f (field induced by the fluid circulating on the level of the interface). The field of displacements is noted $\boldsymbol{u}$ and the jump of displacement on the level of the interface is noted $[\![\boldsymbol{u}]\!]$.

That is to say P_1 a point of Γ_1 and P_2 a point of Γ_2 and $\boldsymbol{n_c}=\boldsymbol{n_c^1}$ the normal external with Γ_1 and $\boldsymbol{n_c^2}$ the normal external with Γ_2. The jump of normal displacement (taken negative or null in opening and positive in interpenetration of the lips) is thus defined in the following way:

$$[\![\boldsymbol{u}]\!].\boldsymbol{n_c}=(\boldsymbol{u}(P_1)-\boldsymbol{u}(P_2)).\boldsymbol{n_c}\leqslant 0$$

On Figure 2.2-2, one indicates the conventions adopted for the taking into account of the jump of displacement to the level of the interface.

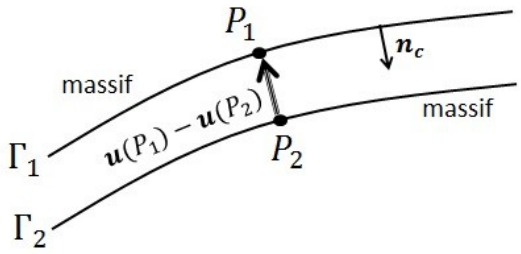

Figure 2.2-2: Definition of the jump of displacement on the level of discontinuity

For the solid mass, as for the interface, the assumption of the effective constraints is taken into account. As follows:

- the total constraint in the solid mass is noted $\boldsymbol{\sigma}$,
- the total constraint (related to the cohesive efforts) on the level of the interface is noted $\boldsymbol{t_c}$.

The assumption of the small disturbances is allowed. In addition it is considered that the sizes mechanics and hydraulics are isotropic.

3 Équations constitutive of model HM-XFEM

3.1 Équations for mechanics

3.1.1 Équation of balance (case of the solid mass)

In the case of the solid mass the conservation equation of the momentum (by taking account of the voluminal efforts) can be put in the form:

$$\mathbf{Div}(\boldsymbol{\sigma})+r\boldsymbol{F^m}=\mathbf{0}$$

with:

- $\boldsymbol{\sigma}=\boldsymbol{\sigma}'-bp\mathbf{1}$ (under the assumption of the effective constraints) where b is the coefficient of Biot,
- r density homogenized such as in the saturated case $r=r_0+m_w$ where r_0 is the density homogenized in the configuration of reference and m_w mass water contributions,
- $r\boldsymbol{F^m}$ represent the voluminal efforts acting on Ω (in practice it is the forces of gravity).

3.1.2 Model of cohesive zones (case of the interface)

In order to take into account the propagation of discontinuity, the irreversibility of the fracturing and the not-interpenetration of the lips of discontinuity, one chooses to model the behavior of the interface or the crack using a cohesive law.
For more information concerning the establishment of these laws, the user can refer to documentation [R7.02.11]. In what concerns us, on the level of discontinuity, it is possible to distinguish 3 zones:

- a completely open zone, on the level of which the total constraint on the lips of discontinuity is equal to $p_f \boldsymbol{n_c}$ on Γ_2 and equalizes with $-p_f \boldsymbol{n_c}$ on Γ_1. In this zone the value of the constraint is due mainly to the there circulating fluid,
- a cohesive zone (or Processing Fractures Zones (FPZ)) whose opening depends on the value of the total constraint which is then equal to $\boldsymbol{t_c}=\boldsymbol{t_c}'-p_f \boldsymbol{n}$, $\boldsymbol{n}$ being the normal external to surface concerned Γ_1 or Γ_2. Beyond a certain opening w_c, the constraint corresponds to that of the preceding zone,
- and an adherent healthy zone on the level of which the lips of discontinuity are in contact and do not interpenetrate.

On Figure 3.1.2-1 one locates the various zones of constraints associated with the model of cohesive zones. The face of crack is then naturally localised at the border between the cohesive zone and the adherent zone.

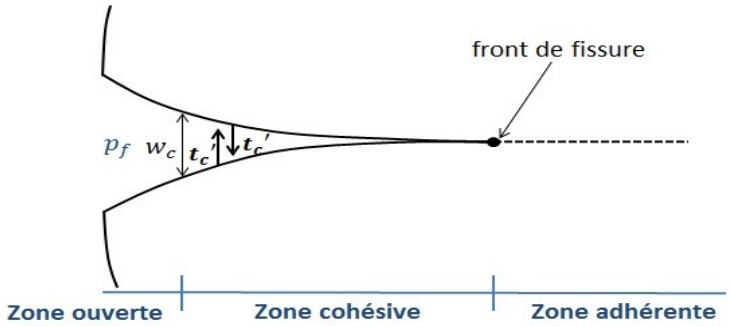

Figure 3.1.2-1: Distribution of the constraints on the level of discontinuity

The law of behavior used for the cohesive laws takes the shape of a lenitive relation between the cohesive constraint $\boldsymbol{t_c}'$ and the jump of displacement $[\![\boldsymbol{u}]\!]$ on the level as of lips of discontinuity.
Thus one poses:

$$\boldsymbol{t_c}' = \frac{\partial \psi}{\partial [\![\boldsymbol{u}]\!]}$$

with ψ the energy of surface, whose expression depends on the cohesive law used.

The cohesive law adopted for hydraulic elements XFEM is the law CZM_LIN_MIX detailed in [R7.02.19]. It is about a not regularized cohesive law (thus with an infinite initial slope). σ_c is the critical stress from which the damage starts in the cohesive zone. $G_c=\frac{\sigma_c w_c}{2}$ corresponds to the quantity of energy required to break a unit of area completely.

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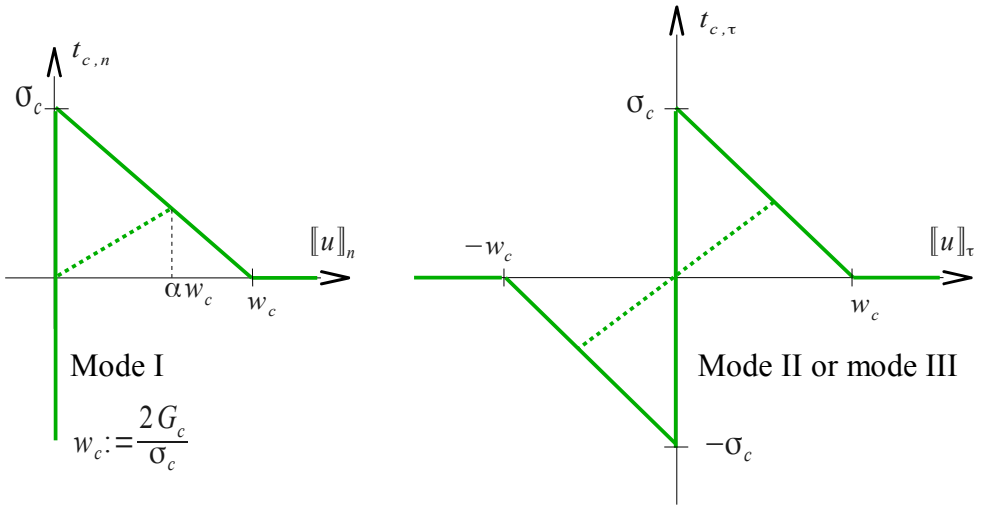


Figure 3.1.2-2 : Mixed cohesive law.

3.1.3 Boundary conditions for mechanics

The writing of the boundary conditions for mechanics on the border of the field Ω and on the level of the interface are written:

Boundary conditions for mechanics

- $\boldsymbol{u} = \boldsymbol{0}$ on Γ_u (imposed displacements)
- $\boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{t}$ on Γ_t (imposed surface efforts)
- $\boldsymbol{\sigma} \cdot \boldsymbol{n_c^1} = \boldsymbol{t_c^1}$ on Γ_1 (imposed cohesive surface efforts)
- $\boldsymbol{\sigma} \cdot \boldsymbol{n_c^2} = \boldsymbol{t_c^2}$ on Γ_2 (efforts surface cohesive imposed)

3.2 É quations for the hydrodynamics

3.2.1 Équation of conservation of the mass (case of the solid mass)

In the case of the solid mass the conservation equation of the mass is put in the form:

$$\frac{\partial m_w}{\partial t} + Div(\boldsymbol{M}) = 0$$

with:

- m_w mass contributions (by unit of volume) expressed S in $kg.m^{-3}$,
- $\boldsymbol{M}$ expressed mass flows in $kg.m^{-2}.s^{-1}$.

3.2.2 Équation of conservation of the mass (case of the interface)

In the case of the interface the conservation equation of the mass is put in the form:

$$\frac{\partial w}{\partial t} + Div(\boldsymbol{W}) = 0$$

with:

- w mass contributions (by unit of area) in the crack expressed in $kg.m^{-2}$,

- $\boldsymbol{W}$ mass flows S in the crack expressed in $kg.m^{-1}.s^{-1}$,

3.2.3 Boundary conditions for the hydrodynamics

The writing of the boundary conditions for the hydrodynamics on the border of the field Ω and the level of the interface allows to obtain:

Boundary conditions for the hydrodynamics

- $p = 0$ on Γ_p (pressure imposed)
- $\boldsymbol{M}\cdot\boldsymbol{n} = M_{\text{ext}}$ on Γ_F (for the solid mass)
- $\boldsymbol{W}\cdot\boldsymbol{n_c} = W_{\text{ext}}$ on Γ_f (for discontinuity)
- $\boldsymbol{M}\cdot\boldsymbol{n_c^1} = q_1$ on Γ_1 (equality of massive flow/discontinuity)
- $\boldsymbol{M}\cdot\boldsymbol{n_c^2} = q_2$ on Γ_2 (equality of massive flow/discontinuity)

An additional boundary condition to take into account is the continuity of the pressure p_f on the level of each lip of the interface. This condition is necessary because of very low thickness of the interface. It results in the following linear relation:

- $p^{\text{inf}} = p_f$ on Γ_1
- $p^{\text{sup}} = p_f$ on Γ_2

with:

- p^{sup} pressure of fluid above the interface (field enriched by XFEM),
- p^{inf} pressure of fluid below the interface (field enriched by XFEM).

3.3 Équations of the model poro-mechanics

In this part one does nothing but point out the useful equations of the model poro-mechanics developed in documentation [R7.01.11].

3.3.1 Expression of the mass contributions

- Case of the solid mass
  The mass contributions can be put in the form (with $S_{lq}=1$):

$$m_w = \rho\varphi\left(1+\varepsilon_v\right)$$

with:

- ρ, φ respectively density of water (in $kg.m^{-3}$) , the variable of porosity eulérienne,
- $\varepsilon_v = Tr(\boldsymbol{\varepsilon}) = Tr(\boldsymbol{\nabla}_s \boldsymbol{u})$ voluminal deformation (where Tr is the linear application traces).

- Case of the interface
  The mass contributions can be put in the form:

$$w = \rho[\![\boldsymbol{u}]\!].\boldsymbol{n_c}$$

with:

- ρ respectively density of water,
- $[\![\boldsymbol{u}]\!].\boldsymbol{n_c}$ the normal opening of discontinuity (in m) .

3.3.2 Expression of mass flows

- Case of the solid mass
  Mass flow in the solid mass follows the law of Darcy. Thus one poses:

$$\boldsymbol{M}=\lambda\rho\left(-\nabla p+\rho\boldsymbol{F}^{\boldsymbol{m}}\right)$$

with $\lambda=\frac{K^{\text{int}}}{\mu}$ L mobility has liquid where K^{int} is the intrinsic permeability of the solid mass (in m^2) and μ the dynamic viscosity of the fluid (in $Pa.s$).

Note:

expression of mobility utilizes actually the relative permeability of the fluid $k_w^{\text{rel}}(S_{\text{w}})$ (which is a function of water saturation and is given by the law of Mualem/Van Genuchten [1 0]) i.e. $\lambda=\frac{K^{\text{int}}k_w^{\text{rel}}}{\mu}$. É as well given as the medium is saturated ($S_{\text{w}}=1$) , the permeability relating to water is thus equal to 1 .

- Case of the interface
Mass flow in discontinuity can be written according to the cubic law [11] (one neglects the effects of gravity). Thus one poses:

$$\boldsymbol{W}=-\frac{\rho(\llbracket\boldsymbol{u}\rrbracket.n_c)^3}{12\mu}\nabla p_f$$

with μ the dynamic viscosity of the fluid (in $Pa.s$) .

3.3.3 Evolution of the variable of porosity

The evolution of the variable of porosity (eulérienne) characterizing the solid mass is given in the isothermal case saturated by:

$$d\varphi=(b-\varphi)\left(d\varepsilon_v+\frac{dp}{K_s}\right)$$

with K_s the module of compressibility of the solid matrix (in Pa) and b the coefficient of Biot.

3.3.4 Evolution of the density of the fluid

- Case of the solid mass
The evolution of the density of the fluid in the solid mass is given in the isothermal case saturated by:

$$\frac{d\rho}{\rho}=\frac{dp}{K_w}$$

with K_w the coefficient of compressibility of the liquid (in Pa).

- Case of the interface
The evolution of the density of the fluid on the level of the interface is given in the isothermal case saturated by:

$$\frac{d\rho}{\rho}=\frac{dp_f}{K_w}$$

3.3.5 Derived from the mass contributions

Éas well given as differential mass contributions intervenes in the expression of the tangent operator associated with the system linearized (see § 6), its expression is pointed out here in the isothermal saturated case.

- Case of the solid mass
  First of all differential of the contributions mass can be written:

$$dm_w = \rho\varphi d\,\varepsilon_v + \rho(1+\varepsilon_v)d\,\varphi + \varphi(1+\varepsilon_v)\,d\rho$$

  Taking into account the assumption of the small disturbances, by preoccupation with a simplification, one supposes that $(1+\varepsilon_v)\approx 1$. With final (while replacing $d\,\varphi$ and $d\,\rho$ by their expressions):

$$d\,m_w = \rho b\,d\,\varepsilon_v + \left(\left(\frac{\rho(b-\varphi)}{K_s} + \frac{\rho\varphi}{K_w}\right)\right)dp$$

- Case of the interface
  Differential of the contributions mass is written:

$$dw = \rho d\left(\llbracket \boldsymbol{u} \rrbracket \cdot \boldsymbol{n_c}\right) + \left(\llbracket \boldsymbol{u} \rrbracket \cdot \boldsymbol{n_c}\right)\rho\,\frac{d\,p_f}{K_w}$$

3.3.6 Derived from mass flows

Since differential flows mass intervenes in the expression of the tangent operator associated with the system linearized (see § 6) its expression is pointed out here in the isothermal saturated case.

- Case of the solid mass
  The differential of mass flows in the case of the solid mass is written:

$$d\,\boldsymbol{M} = \left(\frac{\boldsymbol{M}}{\rho} + \rho\lambda\,\boldsymbol{F^m}\right)\rho\,\frac{d\,p}{K_w} + \frac{\boldsymbol{M}}{\lambda}d\,\lambda - \rho\lambda\,d\left(\nabla\,p\right)$$

- Case of the interface
  The differential of mass flows in the case of the interface is written:

$$d\,\boldsymbol{W} = \left(\frac{\boldsymbol{W}}{\rho}\right)\rho\,\frac{d\,p}{K_w} - \frac{\rho}{12\mu}\nabla\,p_f\left(3\left(\llbracket \boldsymbol{u} \rrbracket \cdot \boldsymbol{n_c}\right)^2\right)d\left(\llbracket \boldsymbol{u} \rrbracket \cdot \boldsymbol{n_c}\right) - \frac{\rho\left(\llbracket \boldsymbol{u} \rrbracket \cdot \boldsymbol{n_c}\right)^3}{12\mu}\,d\left(\nabla\,p_f\right)$$

4 Variational formulation

Before giving the expression of the variational formulations of the equilibrium equations presented above, we will give the definition of spaces of approximation of the fields of displacements, of pressures (of the solid mass and the interface), of flows q_1 and q_2 and of the multipliers of Lagrange λ, μ and of the jump of displacement w useful for the model of cohesive zone:

- the space of the fields of displacements kinematically acceptable on the border of the field Ω is such as:

$$U_0 = \{u^* \in H_1(\Omega) / u^* \text{ discontinu à travers } \Gamma_c, u^* = 0 \text{ sur } \Gamma_u\}$$

- the space of the acceptable fields of pressure on the border of the field Ω is such as:

$$P_0=\{p^*\in H_1(\Omega)/\ p^* \text{ discontinue à travers } \Gamma_c, p^*=0 \text{ sur } \Gamma_p\}$$

- the space of the unknown factors q_1 and q_2 is such as:

$$Q_1=\{q_1^*\in H^{-1/2}(\Gamma_c)/q_1^*\in\Gamma_1\} \text{ and } Q_2=\{q_2^*\in H^{-1/2}(\Gamma_c)/q_2^*\in\Gamma_2\}$$

- the space of the unknown factors of pressure p_f is such as:

$$F_0=\{p_f^*\in H^{-1/2}(\Gamma_c)/\ p_f^* \text{ continue sur } \Gamma_c\}$$

- the space of the unknown factors of the multipliers of Lagrange $\boldsymbol{\lambda},\boldsymbol{\mu}$ and of the jump of displacement $\boldsymbol{w}$ is such as:

$$L_0=\{\lambda^*\in H^{-1/2}(\Gamma_c)/\lambda^* \text{ continue sur } \Gamma_c\}$$

4.1 Weak formulation of the mechanical problem

As explained within [R7.02.19], the framework of the formulation of type “mortar” for the model of cohesive zone, the jump of displacement $\boldsymbol{w}$ is introduced like a new unknown factor of the problem, which will not be discretized like $[\![\boldsymbol{u}]\!]$ but will be a projection on a reduced space $\boldsymbol{M}_h$ (see § 5.2.2). The total energy of the problem is written then:

$$E(\boldsymbol{u},\boldsymbol{\lambda},\boldsymbol{w})=\frac{1}{2}\int_\Omega \epsilon(\boldsymbol{u}):C:\epsilon(\boldsymbol{u})d\Omega-\int_{\Gamma_t}\boldsymbol{t}\cdot\boldsymbol{u}\,d\Gamma_t+\int_{\Gamma_c}\Pi(\boldsymbol{w},\boldsymbol{\lambda})d\Gamma_c$$

$\Pi(\boldsymbol{w},\boldsymbol{\lambda})$ is the density of energy of surface and $\boldsymbol{t}$ surface efforts imposed on Γ_t . The multiplier of Lagrange $\boldsymbol{\lambda}$ will be discretized on same space as $\boldsymbol{w}$ (confer [R7.02.19]).

The solution of the continuous problem consists of a minimization under constraints of equality $(\boldsymbol{u},\boldsymbol{w},\boldsymbol{\lambda})=\underset{\boldsymbol{w}^*=[\![\boldsymbol{u}^*]\!]}{argmin}E(\boldsymbol{u}^*,\boldsymbol{\lambda}^*,\boldsymbol{w}^*)$. We can write the Lagrangian associated one like:

$$L(\boldsymbol{u},\boldsymbol{w},\boldsymbol{\lambda},\boldsymbol{\mu})=\frac{1}{2}\int_\Omega \epsilon(\boldsymbol{u}):C:\epsilon(\boldsymbol{u})d\Omega-\int_{\Gamma_t}\boldsymbol{t}\cdot\boldsymbol{u}\,d\Gamma_t+\int_{\Gamma_c}\Pi(\boldsymbol{w},\boldsymbol{\lambda})d\Gamma_c+\int_{\Gamma_c}\boldsymbol{\mu}\cdot([\![\boldsymbol{u}]\!]-\boldsymbol{w})d\Gamma_c$$

The multiplier of Lagrange $\boldsymbol{\mu}$ will be also discretized on reduced space $\boldsymbol{M}_h$ .The writing of the conditions of optimality of this Lagrangian led to the following variational formulation:

Equilibrium equation
$$\forall\,\boldsymbol{u}^*\in U_0,\int_\Omega\sigma(\boldsymbol{u}):\epsilon(\boldsymbol{u}^*)d\Omega-\int_{\Gamma_t}\boldsymbol{t}\cdot\boldsymbol{u}^*d\Gamma_t+\int_{\Gamma_c}\boldsymbol{\mu}\cdot[\![\boldsymbol{u}^*]\!]d\Gamma_c=0$$

Projection of the jump of displacement
$$\forall\,\boldsymbol{\mu}^*\in L_0,\int_{\Gamma_c}([\![\boldsymbol{u}]\!]-\boldsymbol{w})\cdot\boldsymbol{\mu}^*d\Gamma_c=0$$

Expression of the cohesive force
$$\forall\,\boldsymbol{w}^*\in L_0,-\int_{\Gamma_c}\left[\boldsymbol{\mu}-\boldsymbol{t}_c(\boldsymbol{\lambda}+r\,\boldsymbol{w})\right]\cdot\boldsymbol{w}^*d\Gamma_c=0$$

Law of interface
$$\forall\,\boldsymbol{\lambda}^*\in L_0,-\int_{\Gamma_c}\frac{\left[\boldsymbol{\lambda}-\boldsymbol{t}_c(\boldsymbol{\lambda}+r\,\boldsymbol{w})\right]}{r}\cdot\boldsymbol{\lambda}^*d\Gamma_c=0$$

r ESt the parameter of increase (confer [R7.02.19]). Let us recall that:

- $\boldsymbol{\sigma}=\boldsymbol{\sigma}^{'}-bp\mathbf{1}$
- $\boldsymbol{t_c}=\boldsymbol{t_c}^{'}-p_f\boldsymbol{n}$

4.2 Weak formulations of the hydrodynamic problem

4.2.1 Weak formulation for the solid mass

The weak formulation of the conservation equation of the mass in the case of the solid mass is written:

$$\begin{aligned}&-\int_\Omega \frac{\partial m_w}{\partial t} p^* d\Omega + \int_\Omega \boldsymbol{M}\cdot\boldsymbol{\nabla} p^* d\Omega \\ &= \int_{\Gamma_F} M_{\text{ext}} p^* d\Gamma_F - \int_{\Gamma_1} q_1 p^* d\Gamma_1 - \int_{\Gamma_2} q_2 p^* d\Gamma_2\end{aligned} \quad \forall\, p^* \in P_0$$

with M_{ext} the normal flows imposed on the part Γ_F of $\partial\Omega$.

4.2.2 Weak formulation for the interface

The weak formulation of the conservation equation of the mass in the case of the interface is written:

$$\begin{aligned}&-\int_{\Gamma_c} \frac{\partial w}{\partial t} p_f^* d\Gamma_c + \int_{\Gamma_c} \boldsymbol{W}\cdot\boldsymbol{\nabla} p_f^* d\Gamma_c \\ &= \int_{\Gamma_f} W_{\text{ext}} p_f^* d\Gamma_f + \int_{\Gamma_1} q_1 p_f^* d\Gamma_1 + \int_{\Gamma_2} q_2 p_f^* d\Gamma_2\end{aligned} \quad \forall\, p_f^* \in F_0$$

with W_{ext} the normal flows imposed on the part Γ_f of Γ_c.

The weak formulation of the condition of continuity of the pressure p_f on the level of the interface is written:

$$\int_{\Gamma_1} \left(p^{\sup} - p_f\right) q_1^* d\Gamma_1 = 0 \quad \forall\, q_1^* \in Q_1$$

$$\int_{\Gamma_2} \left(p^{\inf} - p_f\right) q_2^* d\Gamma_2 = 0 \quad \forall\, q_2^* \in Q_2$$

Note:

> *The condition of continuity of the pressure p_f to the level of each lip of the interface brings concerned two linear relations of the type $p^{\sup}-p_f=0$ on Γ_1 and $p^{\inf}-p_f=0$ on Γ_2. In Code_Aster, in order to manage this kind of relation (which is in fact a boundary condition and not an equilibrium equation, the such conservation equation of the mass), we resort to the introduction of fields of multipliers of Lagrange. In fact, the multipliers (which one names in the continuation hydraulic multipliers of Lagrange) concerned in these two variational formulations are in fact virtual flows q_1^* and q_2^*.*

5 Discretization of the problem

5.1 Temporal discretization

The conservation equations of the mass in the case of the solid mass and the interface utilize explicitly in their formulations the temporal variable t. In order to discretize these equations, we use one θ - diagram.

In term of notation a subscripted variable by one $+$ is a variable taken at the end of the step of time and a subscripted variable by one $-$ is a variable taken at the beginning of step of time which is *a priori* known. One poses $\Delta t = t^+ - t^-$.

5.1.1 Discretization of the mechanical equation

The discretization in time of the equations for mechanics does not utilize of θ - diagram. They are expressed at the moment $+$ (i.e. after the phase of prediction) and are written:

Equilibrium equation
$$\forall \boldsymbol{u}^* \in U_0, \int_\Omega \boldsymbol{\sigma}^+(\boldsymbol{u}) : \epsilon(\boldsymbol{u}^*) d\Omega - \int_{\Gamma_t} \boldsymbol{t}^+ \cdot \boldsymbol{u}^* d\Gamma_t + \int_{\Gamma_c} \boldsymbol{\mu}^+ \cdot [\![\boldsymbol{u}^*]\!] d\Gamma_c = 0$$

Projection of the jump of displacement
$$\forall \boldsymbol{\mu}^* \in L_0, \int_{\Gamma_c} \left([\![\boldsymbol{u}^+]\!] - \boldsymbol{w}^+\right) \cdot \boldsymbol{\mu}^* d\Gamma_c = 0$$

Expression of the cohesive force
$$\forall \boldsymbol{w}^* \in L_0, -\int_{\Gamma_c} \left[\boldsymbol{\mu}^+ - \boldsymbol{t}_c^+(\boldsymbol{\lambda}^+ + r\,\boldsymbol{w}^+)\right] \cdot \boldsymbol{w}^* d\Gamma_c = 0$$

Law of interface
$$\forall \boldsymbol{\lambda}^* \in L_0, -\int_{\Gamma_c} \frac{\left[\boldsymbol{\lambda}^+ - \boldsymbol{t}_c^+(\boldsymbol{\lambda}^+ + r\,\boldsymbol{w}^+)\right]}{r} \cdot \boldsymbol{\lambda}^* d\Gamma_c = 0$$

5.1.2 Discretization of the equations of the hydrodynamics

5.1.2.1 Case of the solid mass

Discretization in time of the conservation equation of the mass using one θ - diagram is written:

$$\begin{aligned} &-\int_\Omega \frac{m_w^+ - m_w^-}{\Delta t} p^* d\Omega + \theta \int_\Omega \boldsymbol{M}^+ \cdot \nabla p^* d\Omega + (1-\theta) \int_\Omega \boldsymbol{M}^- \cdot \nabla p^* d\Omega \\ &= \theta \int_{\Gamma_F} M_{ext}^+ p^* d\Gamma_F + (1-\theta) \int_{\Gamma_F} M_{ext}^- p^* d\Gamma_F - \theta \int_{\Gamma_1} q_1^+ p^* d\Gamma_1 \qquad \forall p^* \in P_0 \\ &- (1-\theta) \int_{\Gamma_1} q_1^- p^* d\Gamma_1 - \theta \int_{\Gamma_2} q_2^+ p^* d\Gamma_2 - (1-\theta) \int_{\Gamma_2} q_2^- p^* d\Gamma_2 \end{aligned}$$

5.1.2.2 Case of the interface

Discretization in time of the conservation equation of the mass using one θ - diagram is written:

$$\begin{aligned} &-\int_{\Gamma_c} \frac{w^+ - w^-}{\Delta t} p_f^* d\Gamma_c + \theta \int_{\Gamma_c} \boldsymbol{W}^+ \cdot \nabla p_f^* d\Gamma_c + (1-\theta) \int_{\Gamma_c} \boldsymbol{W}^- \cdot \nabla p_f^* d\Gamma_c \\ &= \theta \int_{\Gamma_f} W_{ext}^+ p_f^* d\Gamma_f + (1-\theta) \int_{\Gamma_f} W_{ext}^- p_f^* d\Gamma_f + \theta \int_{\Gamma_1} q_1^+ p_f^* d\Gamma_1 \qquad \forall p_f^* \in F_0 \\ &+ (1-\theta) \int_{\Gamma_1} q_1^- p_f^* d\Gamma_1 + \theta \int_{\Gamma_2} q_2^+ p_f^* d\Gamma_2 \\ &+ (1-\theta) \int_{\Gamma_2} q_2^- p_f^* d\Gamma_2 \end{aligned}$$

5.1.3 Discretization of the equations of the model poro-mechanics

The equations presented in this part correspond to the equations of the § 3.3 expressed in an incremental way. These equations are developed because they are affected by the discretization with XFEM.

5.1.3.1 Case of the mass contributions

- Case of the solid mass
  The mass contributions are written then in an incremental way:

$$m_w^+ - m_w^- = \rho^+ \varphi^+ \left(1+\varepsilon_v^+\right) - \rho^- \varphi^- \left(1+\varepsilon_v^-\right)$$

- Case of the interface
  The mass contributions are written then in an incremental way:

$$w^+ - w^- = \rho^+ \left(\left[\!\left[\boldsymbol{u} \right]\!\right] \cdot \boldsymbol{n_c}\right)^+ - \rho^- \left(\left[\!\left[\boldsymbol{u} \right]\!\right] \cdot \boldsymbol{n_c}\right)^-$$

5.2 Discretization with XFEM

5.2.1 Representation of associated element HM-XFEM and ddls

To represent element HM-XFEM, we chose to use quadratic elements which can be either quadrangles with 8 nodes (QUAD8) or triangles with 6 nodes (TRIA6), or hexahedrons with 20 nodes (HEXA20), or pentahedrons with 15 nodes (PENTA15), or pyramids with 13 nodes (PYRA13), or tetrahedrons with 10 nodes (TETRA10). We consider that any element of the grid crossed by the interface is of type HM-XFEM. This element bathes around elements HM not nouveau riches. They are those used classically for modelings HM.

The Figure 5.2.1-1 represent element HM-XFEM with a mesh QUAD8.

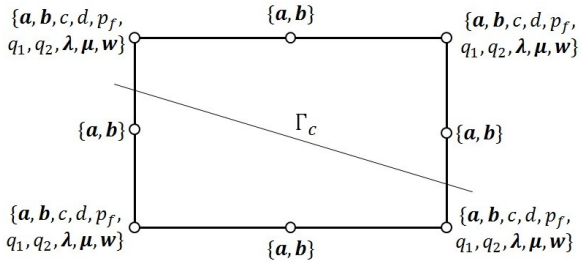


Figure 5.2.1-1 : Representation of element HM-XFEM with QUAD8

On each figure presented previously, the lists in each node of the element contain the degrees of freedom (ddls) associated with each category of nodes (nodes tops or nodes mediums).

Degrees of freedom:

- $\boldsymbol{a}$ and $\boldsymbol{b}$ are respectively associated with the part classical and enriched by the approximation of the field of displacements $\boldsymbol{u^h}$,
- c and d are respectively associated with the part classical and enriched by the approximation of the field of pressure in the solid mass p^h,
- p_f are associated with the approximation of the field of pressure p_f^h on the level of the interface,
- q_1 and q_2 are associated with the approximation of the fields of hydraulic multipliers of Lagrange q_1^h and q_2^h.
- $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$ and $\boldsymbol{w}$ are respectively associated with the approximation of the multipliers of Lagrange $\boldsymbol{\lambda^h}$ and $\boldsymbol{\mu^h}$ and of the jump of displacement $\boldsymbol{w^h}$.

Hydraulic elements present in *Code_Aster* use a “mixed interpolation”, in order to reduce the oscillations of the digital solution [14]. The field of displacements is thus interpolated in a quadratic way while the field of pressure of pore is interpolated in a linear way. Degrees of freedom c and d are thus carried only by the nodes tops while the degrees of freedom $\boldsymbol{a}$ and $\boldsymbol{b}$ are carried at the

same time by the nodes mediums and the nodes tops. With regard to the fields associated with discontinuities: p_f^h, q_1^h, q_2^h, $\boldsymbol{\lambda}^h$, $\boldsymbol{\mu}^h$ and $\boldsymbol{w}^h$, degrees of freedoms are carried only by the nodes tops. In addition, their space of approximation is reduced (see § 5.2.2) in order to observe the stability condition LBB [8.9]. This condition indeed imposes a hierarchy between spaces of approximation, without what one observes the appearance of oscillations and it not unicity of the solution of the coupled problem.

In addition it is important to note that the elements HM-XFEM which share nodes tops with classical elements HM, must undergo an additional treatment. It is indeed necessary to put at zero degrees of freedom nouveau riches (for the fields of pressure in the solid mass and displacements) but also to put at zero degrees of freedom p_f, q_1, q_2, $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$ and $\boldsymbol{w}$ on the level of the nodes joint tops between these two types of element. The procedure of elimination used is that described in the § 4.4 of documentation [R5.03.54].

On Figure 5.2.1-1 one summarizes the problematic points associated with the cohabitation between the elements HM-XFEM and elements HM classical.

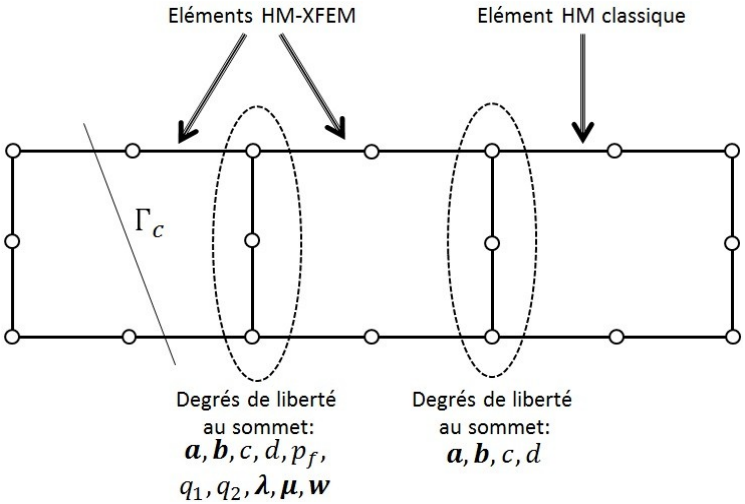


Figure 5.2.1-1: Cohabitation of element HM-XFEM with classical element HM

5.2.2 Space reduced for the discretization of the fields associated with discontinuity

For a detailed description of the discretization of the unknown factors of contact (and a fortiori of all the unknown factors associated with discontinuity), the reader can refer to documentation [R5.03.54], §5. In short, them initial components multiplier are defined on Nœuds tops K elements parents intersected (see fig. 5.2.2-1). Implementation such elements of contact is detailed in [R5.03.54], §4. One imposes then relations of equality between some of these initial components in order to lead to a lower number N_λ of degrees of freedom indeed independent. These relations are carried by certain intersected edges V, known as vital edges: a degree of freedom I really independent is divided by a group of Nœuds of K (see fig. 5.2.2-1), which produces a function of wide form of contact $\psi_I := \sum_{i \in I} N_i$ (see fig. 5.2.2-1). The algorithm of selection of such vital edges, and thus of construction of reduced space, is detailed in documentation [R5.03.54], §6. The field of multipliers is then obtained by interpolation on the elements parents and the discrete multiplier is the trace of this field on the interface:

$$\boldsymbol{M}_h := \left\{ \sum_I \boldsymbol{\mu}_I \psi_I|_\Gamma, \boldsymbol{\mu}_I \in \mathbb{R}^d \right\}$$

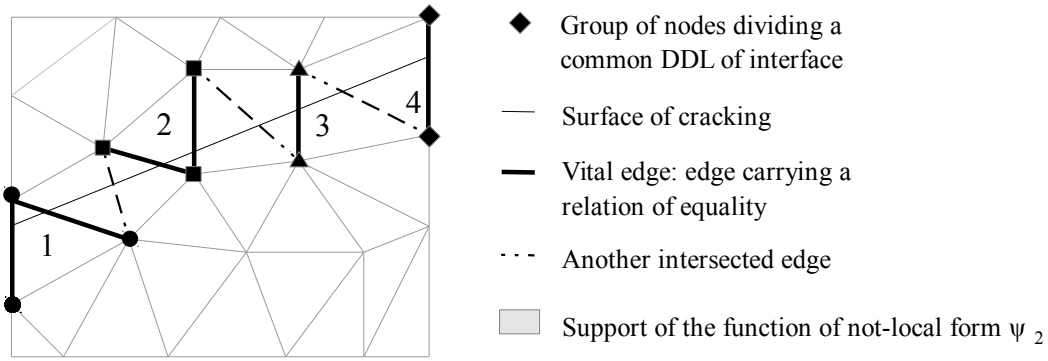


Figure 5.2.2-1 : Grid nonin conformity with the crack and space reduced for the interface

In our case, the fields associated with discontinuity are not only the multipliers of Lagrange λ, μ and the jump of displacement w useful for the cohesive law CZM_LIN_MIX but also hydraulic fields associated with discontinuity: pressure of fluid in discontinuity p_f , and flows q_1 and q_2 . These fields all are discretized on reduced space $\boldsymbol{M}_h$.

5.2.3 Approximations of the fields with XFEM

The approximation of the involved fields can thus be put in the form (the fields tests used in the weak forms of the equilibrium equations are approximated same manner):

- for the field of displacements:

$$\boldsymbol{u}^{\boldsymbol{h}}(x)=\sum_{i=1}^{nn} \boldsymbol{a}_{\boldsymbol{i}}\,\phi_i(x) + \sum_{j=1}^{nne} \boldsymbol{b}_{\boldsymbol{j}}\,\phi_j(x)\,\widetilde{H}\left(lsn(x)\right)$$

- for the field of pressure in the solid mass:

$$p^h(x)=\sum_{i=1}^{nns} c_i\,\psi_i(x) + \sum_{j=1}^{nnse} d_j\,\psi_j(x)\,\widetilde{H}\left(lsn(x)\right)$$

- for the field of pressure in the interface:

$$p_f^h(x)=\sum_{i=1}^{nnc} (p_f)_i\,\widetilde{\psi}_i(x)$$

- for the fields of hydraulic multipliers of Lagrange:

$$q_1^h(x)=\sum_{i=1}^{nnc} (q_1)_i\,\widetilde{\psi}_i(x)$$

$$q_2^h(x)=\sum_{i=1}^{nnc} (q_2)_i\,\widetilde{\psi}_i(x)$$

- for the fields of multipliers of Lagrange and the useful jump of displacement for the model of cohesive zone:

$$\boldsymbol{\lambda}^{\boldsymbol{h}}(x)=\sum_{i=1}^{nnc} \boldsymbol{\lambda}_i\,\widetilde{\psi}_i(x)$$

$$\boldsymbol{\mu}^{\boldsymbol{h}}(x)=\sum_{i=1}^{nnc} \boldsymbol{\mu}_i\,\widetilde{\psi}_i(x)$$

$$\boldsymbol{w}^{\boldsymbol{h}}(x)=\sum_{i=1}^{nnc}\boldsymbol{w}_i\,\widetilde{\psi}_i(x)$$

with:

- lsn *level set* normal whose Iso-zero represents the interface. For more precise details on the use of *level set* with XFEM in *Code_Aster*, the reader will be able to refer to documentation [R7.02.13],
- $\widetilde{H}$ the discontinuous function through the discontinuity located by $lsn(x)=0$ (confer [R7.02.12])
- $\phi_i(x)$ functions of form of the quadratic element used for the interpolation of the field of displacements,
- $\psi_i(x)$ the function of form of the linear element relative used for the interpolation of the field of pressure in the solid mass.
- $\widetilde{\psi}_i(x)$ linear functions of form used for the interpolation of the fields associated with discontinuity. They differ from the functions of form of the linear element relative $\psi_i(x)$ if nodes tops of the linear element relative do not belong to any edge intersected by discontinuity (see § 5.4 and [R5.03.54] for more details). This situation appears only for the nonsimpliciaux elements.
- nn the whole of the nodes of the grid and nne the whole of the nodes nouveau riches of the grid.
- nn the whole of the nodes tops of the grid and nne the whole of the nodes tops nouveau riches of the grid.
- nnc the nodes tops belonging to an edge intersected by discontinuity Γ_c .

Note:

As we can notice it in the definition of the approximations of the fields of displacements and pressure of the solid mass, enrichment due to the presence of a point of crack is not taken into account. In the formulation of the mechanical problem, we base ourselves on a model of cohesive zones regularized. The advantage of such a model (historically introduced by Barenblatt [12] in order to improve the theory of Griffith concerning the fractured mediums [13]) is of stage the fact that the constraints are infinite at a peak of crack. Consequently, enrichment by singular functions of the approximation of the field of displacements is inappropriate here, but completely possible with regard to the enrichment of the approximation of the field of pressure in the solid mass as suggests it [7].

5.3 Extension to the multi-fissured case

5.3.1 Representation of the element multi-fissured associated HM-XFEM and ddls

In *Code_Aster*, elements XFEM can support to 4 cracks per element (see [R7.02.12]). It is also possible to have connected cracks. These features are in particular available for elements HM-XFEM. With each additional crack, one adds a set of degrees of freedom nouveau riches for the field of displacement and of pressure of pore as well as a set of degrees of freedom associated with additional discontinuity. On the Figure 5.3.1-1, one represents an element HM-XFEM QUAD8 intersected by two interfaces (off-line on the left and connected on the right) as well as the degrees of freedom carried by each node.

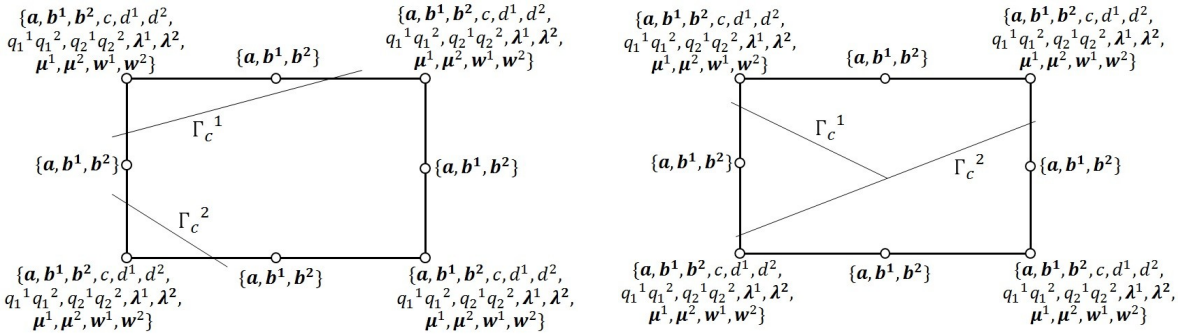


Figure 5.3.1-1 : Representation of an element HM-XFEM QUAD8 intersected by two discontinuities off-line (on the left) and connected (on the right)

5.3.2 Approximations of the fields with XFEM in the multi-fissured case

If several cracks intersect the same mesh, for each node, a correspondence is established between each interface and the degrees of freedom nouveau riches and interface (see [R7.02.12]). Thus the approximation of the involved fields can be put in the following form:

- for the field of displacements:

$$\boldsymbol{u}^{\boldsymbol{h}}(x)=\sum_{i=1}^{nn} \boldsymbol{a_i}\,\phi_i(x) + \sum_{ifiss=1}^{nfiss}\sum_{j=1}^{nne} \boldsymbol{b_j^{\alpha(ifiss,j)}}\,\phi_j(x)\,\widetilde{H}\left(lsn_{ifiss}(x)\right)$$

- for the field of pressure in the solid mass:

$$p^h(x)=\sum_{i=1}^{nns} c_i\,\psi_i(x) + \sum_{ifiss}^{nfiss}\sum_{j=1}^{nnse} d_j^{\alpha(ifiss,j)}\,\psi_j(x)\,\widetilde{H}\left(lsn_{ifiss}(x)\right)$$

- for the degrees of freedom of interface, one has for example for the pressure of fluid p_f in the interface $ifiss$:

$$p_f^{h,ifiss}(x)=\sum_{i=1}^{nnc}\left(p_f^{\alpha(ifiss,i)}\right)_i\,\widetilde{\psi}_i(x)$$

with:

- α the function which with each interface associates the number of associated degree of freedom enriched or interface for each Nœud.

5.3.3 Junction of hydraulic fractures

The fields associated with each interface are thus discretized with sets of distinct degrees of freedom. The interfaces thus function independently, even if they exert their influence one on the other via the porous matrix. In the case of a hydraulic junction of interface, it is however appropriate to impose a hydraulic connection, in order to allow the exchanges of fluid the level of the junction (see Figure Figure 5.3.3-1) . For this purpose, one can is to impose the conservation of the mass on the level of the junction (law of the nodes on flows $\boldsymbol{W}$ in each branch of the junction), that is to say to impose the continuity of the pressure p_f in the cracks on the level of the junction.

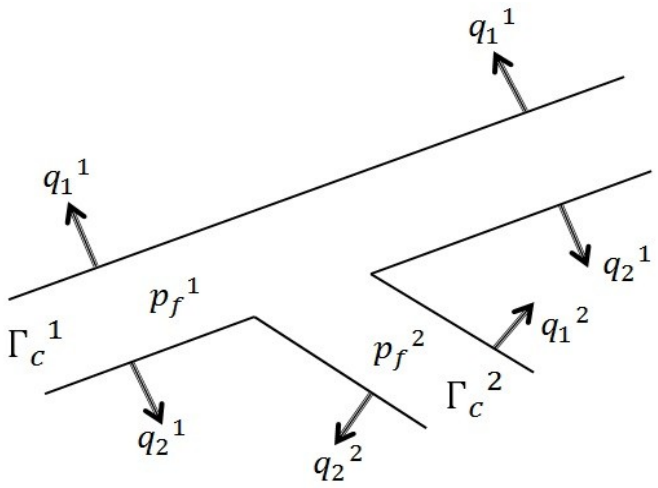
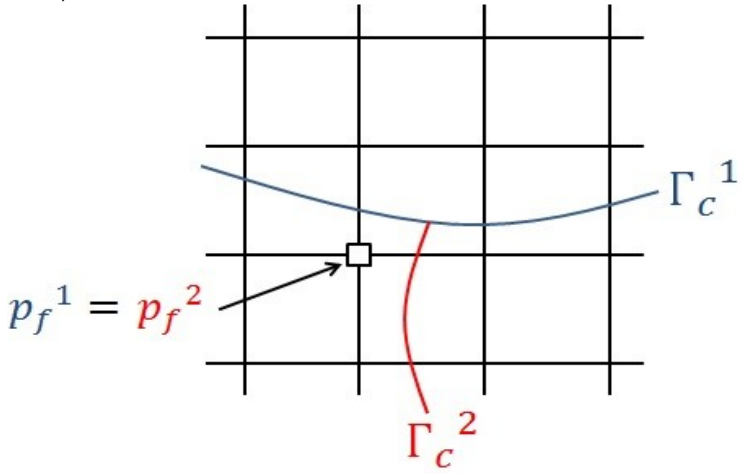


Figure 5.3.3-1: Hydraulic junction of crack

Being given the reduced space of approximation which we have for the field p_f , it is preferable to impose the continuity of the pressure p_f on the level of the junction, rather than to impose an equality on flows W who utilize the gradient of p_f . For this purpose, one identifies a node carrying the degree of freedom p_f at the same time for the principal crack and the connected crack and one forces the equality of these two degrees of freedom (see Figure Figure 5.3.3-2).

Figure 5.3.3-2: Imposition of the continuity of the pressure p_f in the hydraulic cracks on the level of a junction.

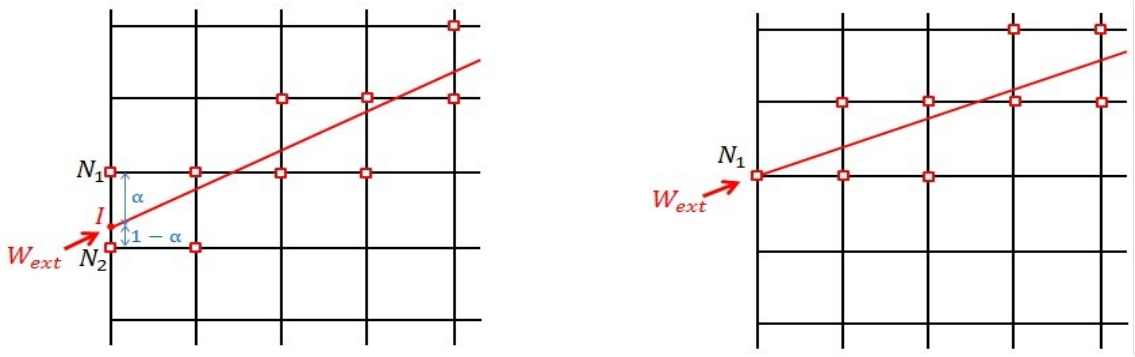


5.4 Imposition of a flow in an interface hydraulics

With regard to the surface flows of fluid injected into the porous matrix, the integration of the second member is done naturally on the edge Γ_F on which flow is imposed M_{ext} (see § 4.2.1), of dimension $ndim-1$ if the dimension of the field Ω is $ndim$. On the other hand, when it is a question of imposing a flow of fluid directly in the fracture, the dimension of the mouth Γ_f is $ndim-2$ if the dimension of the field Ω is $ndim$. The integration of the loading flow W_{ext}

requires thus more attention. Let us interest initially if the dimension of Ω is 2. Flow W_{ext} express yourself then in $kg.s^{-1}$ and Γ_f is tiny room to a point. Let us recall that the space of approximation of the fields associated with the cohesive interface like p_f be based only on the nodes tops of the edges strictly intersected by the interface and on the nodes on which the interface passes. On the Figure Figure 5.4-1, nodes indeed carrying the degree of freedom p_f are marked by white squares and reds. The integration of the term $\int_{\Gamma_f} W_{ext} p_f^* d\Gamma_f$ will thus be done on the restriction of this space of approximation at the edge of the field. If the cohesive interface is in conformity at the edge of the field, one will impose directly W_{ext} on the node N_1 (see Figure 5.4-1 right-hand side). In the case nonin conformity, one will have to determine the relative distance α between the mouth I and the nodes tops of the edge of edge intersected N_1 and N_2 (see Figure 5.4-1 left). One will impose then $(1-\alpha)W_{ext}$ on the node N_1 and αW_{ext} on the node N_2.

Figure 5.4-1: Imposition of a flow in a fracture for the models 2D plan in the case nonin conformity (on the left) and in the case conforms (on the right).

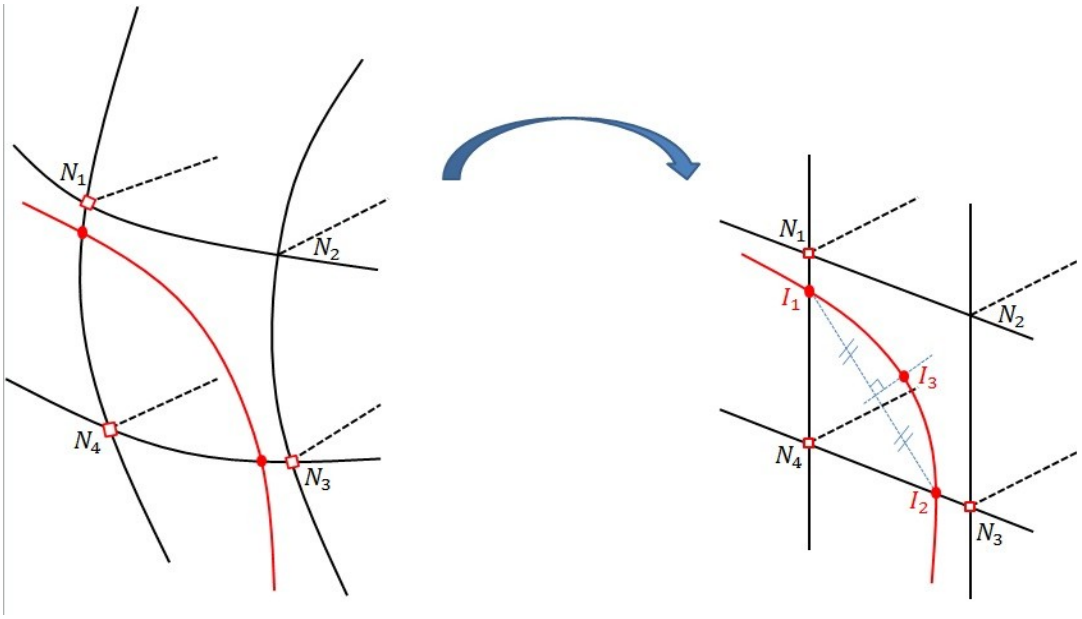


If the dimension of Ω is 3, flow W_{ext} express yourself in $kg.m^{-1}.s^{-1}$ and Γ_f is a curve. In order to integrate the term $\int_{\Gamma_f} W_{ext} p_f^* d\Gamma_f$, it is necessary to have a support which approximates the curve Γ_f. For this purpose, one rebuilds an approximation of the interface on the edge of the field like a chain of segments to 3 nodes. In each face of edge, the interface Γ_c is discretized by a segment with 3 nodes. With this intention, one uses the same procedure as for the creation of under elements of integration XFEM (see [R7.02.12]). Faces of edge intersected by Γ_c are rocked within the space of reference (see Figure 5.4-2). The intersections then are determined I_1 and I_2 with the edges of the face which will constitute the nodes ends of the segment with 3 nodes. Then one determines the position of the node medium I_3 on the mediator of the segment $[I_1 I_2]$. One then obtains a quadratic approximation of the interface at the edge of the field Ω. One is then capable to evaluate $\int_{\Gamma_f} W_{ext} p_f^* d\Gamma_f$ by carrying out an integration on the chain of the segments with 3 Nœuds which approximates Γ_f. It is necessary however well to take care of the space of approximation of p_f on the intersected faces. As recalled previously, the space of approximation of the field p_f be based only on the nodes tops of the edges strictly intersected by discontinuity and the nodes on which discontinuity passes. On the FigureFigure 5.4-2, only nodes N_1, N_3 and N_4 face quadrangle of edge carry the degree of freedom p_f. In order to satisfy all the same the partition with the unit in the face quadrangle, one uses functions of form modified for the field p_f :

$$\widetilde{\psi}_{N_1} = \psi_{N_1} + \frac{\psi_{N_2}}{3}$$
$$\widetilde{\psi}_{N_2} = 0$$
$$\widetilde{\psi}_{N_3} = \psi_{N_3} + \frac{\psi_{N_2}}{3}$$
$$\widetilde{\psi}_{N_4} = \psi_{N_4} + \frac{\psi_{N_2}}{3}$$

with ψ functions of form of the linear element relative and $\widetilde{\psi}$ functions of form modified to adapt to the space of approximation of the field p_f .

Figure 5.4-2: Imposition of a flow in a fracture for the models 3D



6 Resolution of the coupled problem

6.1 Linearization of the coupled problem

6.1.1 Linearization by the method of Newton-Raphson

The coupled problem being nonlinear (it not linearity of the problem is due under the terms of mass of the variational formulations of the conservation equations of the mass for the solid mass and the interface and under the cohesive terms for mechanics) we carry out its linearization using the method of Newton-Raphson.

That is to say $\boldsymbol{F}$ the nonlinear system associated with the variational formulations of the conservation equations of the mass (for the solid mass and the interface), of the mechanical equilibrium equation and the condition of continuity of the pressure p_f on the level of the interface. That is to say $\boldsymbol{x}^k$ the vector of the nodal unknown factors to the iteration of Newton k such as:

$$\boldsymbol{x}^k = \{\boldsymbol{u}^k \ p^k \ p_f^k \ q_1^k \ q_2^k \ \boldsymbol{\lambda} \ \boldsymbol{\mu} \ \boldsymbol{w}\}^T$$

With the iteration $k+1$ (the vector of the nodal unknown factors $\boldsymbol{x}^{k+1}$) one is not known poses:

$$\boldsymbol{F}(\boldsymbol{x}^{k+1})=\boldsymbol{0}$$

In order to be able to determine $\boldsymbol{x}^{k+1}$, we resort to a development of Taylor of $\boldsymbol{F}$ (which is a presumedly continuous and derivable vector function) in the vicinity of $\boldsymbol{x}^{k}$ (then known with the iteration $k+1$). Thus the linear system with the iteration $k+1$ is written:

$$-\boldsymbol{F}(\boldsymbol{x}^{k}) = \frac{\partial \boldsymbol{F}(\boldsymbol{x}^{k})}{\partial \boldsymbol{x}^{k}}\cdot\delta\boldsymbol{x}^{k}$$

with $\delta\boldsymbol{x}^{k}=\boldsymbol{x}^{k+1}-\boldsymbol{x}^{k}$ the increment of the values of the nodal unknown factors between two successive iterations (which is an unknown factor with the iteration $k+1$), $\frac{\partial \boldsymbol{F}(\boldsymbol{x}^{k})}{\partial \boldsymbol{x}^{k}}$ the tangent matrix and $\boldsymbol{F}(\boldsymbol{x}^{k})$ the second member. These the last two terms are known with the iteration $k+1$ and are functions of $\boldsymbol{x}^{k}$.

6.1.2 É integral criture of the linearized problem

In the continuation one considers that the unknown factors are noted with one δ and the fields tests with one $*$ while exposing.

The linear system with the iteration of Newton $k+1$ is written (for a step of time):

- mechanical equilibrium equation
  $\forall \boldsymbol{u}^{*}\in U_0$:

$$\int_\Omega \boldsymbol{\sigma}'^{+}(\delta\boldsymbol{u}):\boldsymbol{\varepsilon}(\boldsymbol{u}^{*})\,d\Omega - \int_\Omega b\,\delta p\,(\mathbf{1}:\boldsymbol{\varepsilon}(\boldsymbol{u}^{*}))\,d\Omega - \int_\Omega \rho^{+} b\,\boldsymbol{F}^{m+}\,Tr(\nabla(\delta\boldsymbol{u}))\,\boldsymbol{u}^{*}\,d\Omega$$
$$- \int_\Omega \left(\frac{\rho^{+}(b-\varphi^{+})}{K_s} + \frac{\rho^{+}\varphi^{+}}{K_w}\right)\delta p\,\boldsymbol{F}^{m+}\boldsymbol{u}^{*}\,d\Omega + \int_{\Gamma_c}\delta\boldsymbol{\mu}\cdot[\![\boldsymbol{u}^{*}]\!]\,d\Gamma_c$$
$$= -\int_\Omega \boldsymbol{\sigma}'^{+}(\boldsymbol{u}):\boldsymbol{\varepsilon}(\boldsymbol{u}^{*})\,d\Omega + \int_\Omega b\,p^{+}(\mathbf{1}:\boldsymbol{\varepsilon}(\boldsymbol{u}^{*}))\,d\Omega + \int_\Omega r^{+}\boldsymbol{F}^{m+}\boldsymbol{u}^{*}\,d\Omega + \int_{\Gamma_t}\boldsymbol{t}^{+}\boldsymbol{u}^{*}\,d\Gamma_t$$
$$- \int_{\Gamma_c}\boldsymbol{\mu}\cdot[\![\boldsymbol{u}^{*}]\!]\,d\Gamma_c$$

- projection of the jump of displacement
  $\forall \boldsymbol{\mu}^{*}\in L_0$:

$$\int_{\Gamma_c}([\![\delta\boldsymbol{u}]\!]-\delta\boldsymbol{w})\cdot\boldsymbol{\mu}^{*}\,d\Gamma_c = -\int_{\Gamma_c}([\![\boldsymbol{u}]\!]-\boldsymbol{w})\cdot\boldsymbol{\mu}^{*}\,d\Gamma_c$$

- constraint cohesive
  $\forall \boldsymbol{w}^{*}\in L_0$:

$$-\int_{\Gamma_c}\left[\delta\boldsymbol{\mu}-\frac{\partial \boldsymbol{t}'_c}{\partial(\boldsymbol{\lambda}+r\,\boldsymbol{w})}\cdot(\delta\boldsymbol{\lambda}+r\,\delta\boldsymbol{w})\right]\cdot\boldsymbol{w}^{*}\,d\Gamma_c - \int_{\Gamma_c}\delta p_f\,\boldsymbol{n_c}\cdot\boldsymbol{w}^{*}\,d\Gamma_c$$
$$= \int_{\Gamma_c}\left[\boldsymbol{\mu}-\boldsymbol{t}'_c(\boldsymbol{\lambda}+r\,\boldsymbol{w})+p_f\boldsymbol{n_c}\right]\cdot\boldsymbol{w}^{*}\,d\Gamma_c$$

- law of interface
  $\forall \boldsymbol{\lambda}^* \in L_0$:

$$-\int_{\Gamma_c}\left[\frac{\delta\boldsymbol{\lambda}}{r}-\frac{\partial \boldsymbol{t}'_c}{\partial(\boldsymbol{\lambda}+r\,\boldsymbol{w})}\cdot\left(\frac{\delta\boldsymbol{\lambda}}{r}+\delta\boldsymbol{w}\right)\right]\cdot\boldsymbol{\lambda}^* d\Gamma_c$$
$$=\int_{\Gamma_c}\frac{\left[\boldsymbol{\lambda}-\boldsymbol{t}'_c(\boldsymbol{\lambda}+r\,\boldsymbol{w})\right]}{r}\cdot\boldsymbol{\lambda}^* d\Gamma_c$$

- conservation equation of the mass (case of the solid mass)
  Thus $\forall p^* \in P_0$:

$$-\int_\Omega \rho^+ b\,Tr(\nabla(\delta\boldsymbol{u}))p^* d\Omega-\int_\Omega\left(\left(\frac{\rho^+(b-\varphi^+)}{K_s}+\frac{\rho^+\varphi^+}{K_w}\right)\delta p\right)p^* d\Omega+$$
$$\Delta t\theta\left[\int_\Omega\left(\lambda^+\left(-\nabla p^+ + \rho^+\boldsymbol{F}^{m+}\right)+\lambda^+\rho^+\boldsymbol{F}^{m+}\right)\frac{\rho^+}{K_w}\delta p\nabla p^* d\Omega\right]+$$
$$\Delta t\theta\left[\int_\Omega\rho^+\left(-\nabla p^+ + \rho^+\boldsymbol{F}^{m+}\right)\frac{\partial\lambda^+}{\partial p^+}\delta p\nabla p^* d\Omega-\int_\Omega\rho^+\lambda^+\nabla(\delta p)\nabla p^* d\Omega\right]+$$
$$\Delta t\theta\left[\int_\Omega\rho^+\left(-\nabla p^+ + \rho^+\boldsymbol{F}^{m+}\right)\frac{\partial\lambda^+}{\partial\varepsilon_v^+}Tr(\nabla(\delta\boldsymbol{u}))\nabla p^* d\Omega\right]$$
$$+\Delta t\theta\int_{\Gamma_1}\delta q_1 p^* d\Gamma_1+\Delta t\theta\int_{\Gamma_2}\delta q_2 p^* d\Gamma_2=\int_\Omega\left(m_w^+-m_w^-\right)p^* d\Omega$$
$$-\Delta t\theta\int_\Omega\boldsymbol{M}^+\nabla p^* d\Omega-\Delta t(1-\theta)\int_\Omega\boldsymbol{M}^-\nabla p^* d\Omega$$
$$+\Delta t\theta\int_{\Gamma_F}M_{\text{ext}}^+ p^* d\Gamma_F+\Delta t(1-\theta)\int_{\Gamma_F}M_{\text{ext}}^- p^* d\Gamma_F$$
$$-\Delta t\theta\int_{\Gamma_1}q_1^+ p^* d\Gamma_1-\Delta t(1-\theta)\int_{\Gamma_1}q_1^- p^* d\Gamma_1$$
$$-\Delta t\theta\int_{\Gamma_2}q_2^+ p^* d\Gamma_2-\Delta t(1-\theta)\int_{\Gamma_2}q_2^- p^* d\Gamma_2$$

- conservation equation of the mass (case of the interface)
  Thus $\forall p_f^* \in F_0$:

$$
\begin{aligned}
&-\int_{\Gamma_c}\rho^{+}\delta[\![\boldsymbol{u}]\!]\cdot\boldsymbol{n}_c\,p_f^{*}\,d\Gamma_c-\int_{\Gamma_c}\left(\frac{\rho^{+}[\![\boldsymbol{u}^{+}]\!]\cdot\boldsymbol{n}_c}{K_w}\delta p_f\right)p_f^{*}\,d\Gamma_c\\
&+\Delta t\theta\left[\int_{\Gamma_c}-\frac{\rho^{+}\nabla p_f^{+}}{12\mu}\left(3\left([\![\boldsymbol{u}^{+}]\!]\cdot\boldsymbol{n}_c\right)^2\right)\delta[\![\boldsymbol{u}]\!]\cdot\boldsymbol{n}_c\nabla p_f^{*}\,d\Gamma_c\right]\\
&+\Delta t\theta\left[\int_{\Gamma_c}-\frac{\left([\![\boldsymbol{u}^{+}]\!]\cdot\boldsymbol{n}_c\right)^3\nabla p_f^{+}}{12\mu}\frac{\rho^{+}}{K_w}\delta p_f\nabla p_f^{*}\,d\Gamma_c\right]\\
&+\Delta t\theta\left[\int_{\Gamma_c}-\frac{\rho^{+}\left([\![\boldsymbol{u}^{+}]\!]\cdot\boldsymbol{n}_c\right)^3}{12\mu}\nabla(\delta p_f)\nabla p_f^{*}\,d\Gamma_c\right]\\
&-\Delta t\theta\left[\int_{\Gamma_1}\delta q_1 p_f^{*}\,d\Gamma_1\right]-\Delta t\theta\left[\int_{\Gamma_2}\delta q_2 p_f^{*}\,d\Gamma_2\right]\\
&=\int_{\Gamma_c}(w^{+}-w^{-})p_f^{*}\,d\Gamma_c-\Delta t\theta\int_{\Gamma_c}\boldsymbol{W}^{+}\nabla p_f^{*}\,d\Gamma_c-\Delta t(1-\theta)\int_{\Gamma_c}\boldsymbol{W}^{-}\nabla p_f^{*}\,d\Gamma_c\\
&+\Delta t\theta\left[\int_{\Gamma_1}q_1^{+}p_f^{*}\,d\Gamma_1\right]+\Delta t(1-\theta)\left[\int_{\Gamma_1}q_1^{-}p_f^{*}\,d\Gamma_1\right]\\
&+\Delta t\theta\left[\int_{\Gamma_2}q_2^{+}p_f^{*}\,d\Gamma_2\right]+\Delta t(1-\theta)\left[\int_{\Gamma_2}q_2^{-}p_f^{*}\,d\Gamma_2\right]\\
&+\Delta t\theta\int_{\Gamma_f}W_{ext}^{+}p_f^{*}\,d\Gamma_f+\Delta t(1-\theta)\int_{\Gamma_f}W_{ext}^{-}p_f^{*}\,d\Gamma_f
\end{aligned}
$$

- equation of continuity of the pressure p_f on the level of the interface
  Thus $\forall q_1^{*}\in Q_1$ and $\forall q_2^{*}\in Q_2$

$$\int_{\Gamma_1}\delta p^{inf}q_1^{*}\,d\Gamma_1-\int_{\Gamma_1}\delta p_f q_1^{*}\,d\Gamma_1=-\int_{\Gamma_1}\left(p^{inf}-p_f\right)q_1^{*}\,d\Gamma_1 \text{ on } \Gamma_1$$

$$\int_{\Gamma_2}\delta p^{\sup}q_2^{*}\,d\Gamma_2-\int_{\Gamma_2}\delta p_f q_2^{*}\,d\Gamma_2=-\int_{\Gamma_2}\left(p^{\sup}-p_f\right)q_2^{*}\,d\Gamma_2 \text{ on } \Gamma_2$$

6.2 Writing of the elementary terms with XFEM

6.2.1 Writing in matric form of the coupled problem

The system of equations previously discretized to the iteration of Newton $k+1$ can put itself in the form (where $\delta\boldsymbol{u}$, δp , δp_f , δq_1 , δq_2 , $\delta\boldsymbol{\lambda}$, $\delta\boldsymbol{\mu}$ and $\delta\boldsymbol{w}$ are the unknown factors of the problem to be solved):

Équilibre mechanical	$\{\boldsymbol{u}^{*}\}\left[K_{meca}^{1}\right](\delta\boldsymbol{u})+\{\boldsymbol{u}^{*}\}\left[K_{meca}^{2}\right](\delta p)+\{\boldsymbol{u}^{*}\}[A](\delta\boldsymbol{u})+\{\boldsymbol{u}^{*}\}[B](\delta p)+\{[\![\boldsymbol{u}]\!]^{*}\}\left[C^{1}\right]\{\delta\boldsymbol{\mu}\}=\{\boldsymbol{u}^{*}\}(L_{meca})+\{[\![\boldsymbol{u}]\!]^{*}\}(L^{1})$
Projection of the jump of displacement	$\{\boldsymbol{\mu}^{*}\}\left[K^{\mu u}\right](\delta\boldsymbol{u})+\{\boldsymbol{\mu}^{*}\}\left[-K^{w\mu}\right]^{T}(\delta w)=\{\boldsymbol{\mu}^{*}\}(L_u)-\{\boldsymbol{\mu}^{*}\}(L_w)$
Cohesive constraint	$\{\boldsymbol{w}^{*}\}\left[-K^{w\mu}\right](\delta\boldsymbol{\mu})+\{\boldsymbol{w}^{*}\}\left[D^{ww}\right](\delta w)+\{\boldsymbol{w}^{*}\}\left[K^{wp}\right](\delta p_f)+\{\boldsymbol{w}^{*}\}\left[D^{\lambda w}\right]^{T}(\delta\boldsymbol{\lambda})=-\{\boldsymbol{w}^{*}\}(L_{\mu}^{2})+\{\boldsymbol{w}^{*}\}(L_{cohe}^{1})+\{\boldsymbol{w}^{*}\}(L_p)$
Law of interface	$\{\boldsymbol{\lambda}^{*}\}\left[D^{\lambda w}\right](\delta w)+\{\boldsymbol{\lambda}^{*}\}\left[D^{\lambda\lambda}\right](\delta\boldsymbol{\lambda})=-\{\boldsymbol{\lambda}^{*}\}(L_{\mu}^{1})+\{\boldsymbol{\lambda}^{*}\}(L_{cohe}^{2})$

Conservation of mass (case of the solid mass)

$$\{p^*\}[M^1_{hydro}](\delta \boldsymbol{u}) + \{p^*\}[M^2_{hydro}](\delta p) + \Delta t\theta\{p^*\}[K^1_{hydro}](\delta p) + \Delta t\theta\{p^*\}[K^2_{hydro}](\delta p) + \Delta t\theta\{p^*\}[K^3_{hydro}](\delta p) + \Delta t\theta\{p^*\}[K^4_{hydro}](\delta \boldsymbol{u}) + \Delta t\theta\{p^*\}[E^1](\delta q_1) + \Delta t\theta\{p^*\}[E^2](\delta q_2 = \{p^*\}(L^1_{hydro}) + \Delta t\{p^*\}(L^2_{hydro})_\theta + \Delta t\{p^*\}(L^3_{hydro})_\theta + \Delta t\{p^*\}(L^4_{hydro})_\theta + \Delta t\{p^*\}(L^5_{hydro})_\theta$$

Conservation of mass (case of the interface)

$$\{p^*_f\}[W^1_{hydro}](\delta[\![\boldsymbol{u}]\!]) + \{p^*_f\}[W^2_{hydro}](\delta p_f) + \Delta t\theta\{p^*_f\}[H^1_{hydro}](\delta[\!\![\Delta t\theta\{p^*_f\}[H^2_{hydro}](\delta p_f) + \Delta t\theta\{p^*_f\}[H^3_{hydro}](\delta p_f) + \Delta t\theta\{p^*_f\}[D^1](\delta q_1) + \Delta t\theta\{p^*_f\}[D^2](\delta[\![\boldsymbol{u}]\!]) \Delta t\theta\{p^*_f\}[D^3](\delta q_2) + \Delta t\theta\{p^*_f\}[D^4](\delta[\![\boldsymbol{u}]\!]) = \{p^*_f\}(L^6_{hydro}) + \Delta t\{p^*_f\}(L^7_{hydro})_\theta + \Delta t\{p^*_f\}(L^8_{hydro})_\theta + \Delta t\{p^*_f\}(L^9_{hydro})_\theta + \Delta t\{p^*_f\}(L^{10}_{hydro})_\theta$$

Continuity of the pressure

$$\{q^*_1\}[D^1](\delta p) + \{q^*_1\}[D^2](\delta p_f) = \{q^*_1\}(J^1_{cont})$$

$$\{q^*_2\}[D^3](\delta p) + \{q^*_2\}[D^4](\delta p_f) = \{q^*_2\}(J^2_{cont})$$

K^1_{meca} is the elementary matrix of mechanical rigidity classically met in mechanics,

K^2_{meca} is due to the decomposition of the tensor of the constraints of the solid mass (assumption of the effective constraints),

A and B are due to the taking into account of the mass contributions in the expression of the homogenized density intervening in the expression of the voluminal efforts on Ω,

C^1 is the elementary matrix of rigidity for the interface,

$K^{\mu u}$ and $K^{w\mu}$ are matrices discretizing the operators “mortar”, the last one also managing the basic change.

$K^{w\,p}$ is an elementary matrix of rigidity for the interface,

Matrices D are all diagonal per blocks: for I and J two distinct DDL of Lagrange, they check $D_{IJ}=0$,

M^1_{hydro} and M^2_{hydro} are the elementary matrices of mass in the case of the solid mass for the hydrodynamics,

W^1_{hydro} and W^2_{hydro} are the elementary matrices of mass in the case of the interface for the hydrodynamics ,

K^1_{hydro}, K^2_{hydro}, K^3_{hydro} and K^4_{hydro} are the elementary matrices of rigidity for the hydrodynamics in the case of the solid mass,

H^1_{hydro}, H^2_{hydro} and H^3_{hydro} are the elementary matrices of rigidity for the hydrodynamics in the case of the interface,

E^1 and E^2 are the matrices of exchanges in the case of the solid mass,

D^1 and D^2 are the matrices of exchanges in the case of the interface,

F^1, F^2, F^3 and F^4 are the matrices of continuity of the pressure on the level of the interface,

L_{meca} is the second member of the forces of volumes and surface applied to the field and to its border,
L^1 is the second member for the interface,

L_u , L_w and L_p are second members for the projection of the jump of displacements,
L_{μ}^2 , and L_{cohe}^1 are second members for the cohesive constraint,
L_{μ}^1 and L_{cohe}^2 are second members for the law of interface,
L_{hydro}^i with $i \in [\![1,5]\!]$ second members due to the mass contributions and flows in the case of solid mass for the hydrodynamics,
L_{hydro}^i with $i \in [\![6,10]\!]$ second members due to the mass contributions and flows in the case of the interface for the hydrodynamics,
J_{hydro}^1 and J_{hydro}^2 second members due to the exchanges on Γ_1 and Γ_2 .

Note:

As we can notice it in L be forms of elementary matrices definite Ci afterwards , quantities ρ^+ , φ^+ (functions displacements and of the pressure), p^+ and $\boldsymbol{u}^+(ou[\![\boldsymbol{u}^+]\!])$ are left in the state (not discretized), because they are sizes obtained during the preceding iteration of Newton (for the step of current time +) . They are thus a priori known. In the form of the elementary matrices (for the mechanical and hydrodynamic case) we will not voluntarily indicate the number of the iteration of Newton on these quantities to avoid overloading the expressions.

6.2.2 Form of the elementary matrices for mechanics

Matrices elementary of mechanical rigidity to the iteration of Newton $k+1$ is written:

$$\{\boldsymbol{u}^*\}\left[K_{meca}^1\right](\delta \boldsymbol{u}) = \int_\Omega (\boldsymbol{a_i^*} + \widetilde{H}\,\boldsymbol{b_i^*}) \nabla \phi_i C_{ij} \nabla \phi_j (\delta \boldsymbol{a}_j + \widetilde{H}\,\delta \boldsymbol{b}_j)\, d\Omega$$

The elementary matrices due to the decomposition of the tensor of the constraints of the solid mass (assumption of the effective constraints) is written with the iteration of Newton $k+1$:

$$\{\boldsymbol{u}^*\}\left[K_{meca}^2\right](\delta p) = -\int_\Omega b(\boldsymbol{a_i^*} + \widetilde{H}\,\boldsymbol{b_i^*}) \nabla \phi_i [\boldsymbol{Id}] \psi_j (\delta c_j + \widetilde{H}\,\delta d_j)\, d\Omega$$

The matrix A with the iteration of Newton $k+1$ is written:

$$\{\boldsymbol{u}^*\}[A](\delta \boldsymbol{u}) = -\int_\Omega \rho^+ b(\boldsymbol{a_i^*} + \widetilde{H}\,\boldsymbol{b_i^*}) \phi_i [\boldsymbol{Id}] \nabla \phi_j (\delta \boldsymbol{a}_j + \widetilde{H}\,\delta \boldsymbol{b}_j) \boldsymbol{F}^{\boldsymbol{m}+} d\Omega$$

The matrix B with the iteration of Newton $k+1$ is written:

$$\{\boldsymbol{u}^*\}[B](\delta p) = -\int_\Omega \left(\left(\frac{\rho^+ (b - \varphi^+)}{K_s} + \frac{\rho^+ \varphi^+}{K_w} \right) \right) (\boldsymbol{a_i^*} + \widetilde{H}\,\boldsymbol{b_i^*}) \phi_i \psi_j (\delta c_j + \widetilde{H}\,\delta d_j) \boldsymbol{F}^{\boldsymbol{m}+} d\Omega$$

The matrix associated with projection with the jump with displacements to the iteration with Newton $k+1$ is written:

$$\{[\![\boldsymbol{u}]\!]^*\}\left[C^1\right](\delta \boldsymbol{\mu}) = \int_{\Gamma_c} \left(2\,\widetilde{H}\,\boldsymbol{b_i^*}\right) \phi_i \widetilde{\psi}_j (\delta \boldsymbol{\mu}_j)\, d\Gamma_c$$

Components of the unknown factors $\boldsymbol{u}$ and $\boldsymbol{\mu}$ are defined in a fixed base $(\boldsymbol{e}_X, \boldsymbol{e}_Y, \boldsymbol{e}_Z)$, while components of $\boldsymbol{w}$ and $\boldsymbol{\lambda}$ are defined in the local base $(\boldsymbol{n}, \boldsymbol{\tau}_1, \boldsymbol{\tau}_2)$ on fissured surface Γ_c in each point $\boldsymbol{x} \in \Gamma_c$, so that :

$$\boldsymbol{w}(\boldsymbol{x}) = \sum_{i=1}^{N_\lambda} \psi_I(\boldsymbol{x}) \left(w_{I,n} \boldsymbol{n}(\boldsymbol{x}) + w_{I,\tau 1} \boldsymbol{\tau}_1(\boldsymbol{x}) + w_{I,\tau 2} \boldsymbol{\tau}_2(\boldsymbol{x}) \right)$$

A similar definition is worth for $\boldsymbol{\lambda}$. For a degree of freedom I reduced space (see §5.2.2), it is possible to determine the components $t'^I_{c,n}, t'^I_{t,\tau 1}, t'^I_{c,\tau 2}$ cohesive force from $(w_{I,n}, w_{I,\tau 1}, w_{I,\tau 2})$, $(\lambda_{I,n}, \lambda_{I,\tau 1}, \lambda_{I,\tau 2})$ and of the cohesive law. These components are not intended to be associated with a particular direction I degree of freedom, but intended to be connected to the weak direction for the total constraint $\boldsymbol{\mu}$ written in fixed base (confer [R7.02.19]).

$$\{\boldsymbol{u}\}^* \left[K^{\mu u}\right](\delta \boldsymbol{\mu}) = \int_{\Gamma_c} \widetilde{\psi}_j \phi_i 2 \widetilde{H} \boldsymbol{b}_i^* \cdot (\delta \boldsymbol{\mu}_j) d\Gamma_c$$

$$\{\boldsymbol{w}\}^* \left[K^{w\mu}\right](\delta \boldsymbol{\mu}) = \int_{\Gamma_c} \widetilde{\psi}_i \widetilde{\psi}_j \boldsymbol{w_j}^* \cdot \boldsymbol{Q} \cdot (\delta \boldsymbol{\mu}_J) d\Gamma_c \quad \text{with } \boldsymbol{Q} \text{ the matrix of basic change orthonormal definite like previously.}$$

$$\{\boldsymbol{w}\}^* \left[K^{wp}\right](\delta p_f) = \int_{\Gamma_c} \widetilde{\psi}_j \widetilde{\psi}_i \boldsymbol{w}_i^* \cdot (\delta p_f)_j \boldsymbol{n_c} d\Gamma_c$$

$$\{\boldsymbol{w}\}^* \left[D^{ww}\right](\delta \boldsymbol{w}) = \int_{\Gamma_c} \boldsymbol{w_i}^* (\delta \boldsymbol{w}_i) r \frac{\partial \boldsymbol{t}'_c}{\partial(\boldsymbol{\lambda} + r \boldsymbol{w})} (\boldsymbol{\lambda_i} + r \boldsymbol{w}_i) \widetilde{\psi}_i^2 d\Gamma_c$$

$$\{\boldsymbol{\lambda}\}^* \left[D^{\lambda w}\right](\delta \boldsymbol{w}) = \int_{\Gamma_c} \boldsymbol{\lambda_i}^* (\delta \boldsymbol{w}_i) \frac{\partial \boldsymbol{t}'_c}{\partial(\boldsymbol{\lambda} + r \boldsymbol{w})} (\boldsymbol{\lambda_i} + r \boldsymbol{w}_i) \widetilde{\psi}_i^2 d\Gamma_c$$

$$\{\boldsymbol{\lambda}\}^* \left[D^{\lambda\lambda}\right](\delta \boldsymbol{\lambda}) = \int_{\Gamma_c} \boldsymbol{\lambda_i}^* (\delta \boldsymbol{\lambda}_i) \frac{1}{r} \left(\frac{\partial \boldsymbol{t}'_c}{\partial(\boldsymbol{\lambda} + r \boldsymbol{w})} (\boldsymbol{\lambda_i} + r \boldsymbol{w}_i) - \boldsymbol{1} \right) \widetilde{\psi}_i^2 d\Gamma_c$$

6.2.3 Expression of the second members for mechanics

In the expressions of the second members introduced here one indicates the number of the preceding iteration of Newton k .

$$\{\boldsymbol{u}^*\}(L_{meca}) = -\int_\Omega (\boldsymbol{a_i}^* + \widetilde{H} \boldsymbol{b_i}^*) \boldsymbol{\nabla} \phi_i \, (\boldsymbol{\sigma}'^+(\boldsymbol{u}))^k d\Omega + \int_\Omega b (\boldsymbol{a_i}^* + \widetilde{H} \boldsymbol{b_i}^*) \boldsymbol{\nabla} \phi_i [\boldsymbol{Id}] (p^+)^k d\Omega$$
$$+ \int_\Omega (\boldsymbol{a_i}^* + \widetilde{H} \boldsymbol{b_i}^*) \phi_i (r_0 + (m_w^+)^k) \boldsymbol{F^{m+}} d\Omega + \int_{\Gamma_t} (\boldsymbol{a_i}^* + \widetilde{H} \boldsymbol{b_i}^*) \phi_i (t^+)^k d\Gamma_t$$

$$\{[\![\boldsymbol{u}]\!]^*\}(L^1) = -\int_{\Gamma_c} \left(2 \widetilde{H} \boldsymbol{b_i}^*\right) \phi_i \widetilde{\psi}_j \, \boldsymbol{\mu_j^k} d\Gamma_c$$

$$\{u\}_i^* \left(L_\mu^1\right)_i = -\boldsymbol{b}_i^* \cdot \int_{\Gamma_c} 2 \phi_i \boldsymbol{\mu} \, d\Gamma_c$$

$$\{\boldsymbol{\mu}\}^* (L_u) = -\boldsymbol{\mu}_i^* \cdot \int_{\Gamma_c} \widetilde{\psi}_i [\![\boldsymbol{u}]\!] d\Gamma_c$$

$$\{\boldsymbol{w}\}^* (L_p) = -\boldsymbol{w}_i^* \cdot \int_{\Gamma_c} \widetilde{\psi}_i p_f \boldsymbol{n_c} d\Gamma_c$$

$$\{\boldsymbol{\mu}\}^{*}\left(L_{w}\right)=-\boldsymbol{\mu}_{i}^{*}\cdot\int_{\Gamma_{c}}\widetilde{\psi}_{i}\boldsymbol{Q}^{T}\cdot\boldsymbol{w}\,d\Gamma_{c}$$

$$\{\boldsymbol{w}\}^{*}\left(L_{\mu}^{2}\right)=\boldsymbol{w}_{i}^{*}\cdot\int_{\Gamma_{c}}\widetilde{\psi}_{i}\boldsymbol{Q}\cdot\boldsymbol{\mu}\,d\Gamma_{c}$$

$$\{\boldsymbol{w}\}^{*}\left(L_{coh}^{1}\right)=\boldsymbol{w}_{\boldsymbol{i}}^{*}\cdot\boldsymbol{t}'_{c}(\boldsymbol{\lambda}_{\boldsymbol{i}}+r\,\boldsymbol{w}_{\boldsymbol{i}})\int_{\Gamma_{c}}\widetilde{\psi}_{i}d\Gamma_{c}$$

$$\{\boldsymbol{w}\}^{*}\left(L_{coh}^{2}\right)_{I}=\frac{\boldsymbol{w}_{\boldsymbol{i}}^{*}}{r}\boldsymbol{t}'_{c}(\boldsymbol{\lambda}_{\boldsymbol{i}}+r\,\boldsymbol{w}_{\boldsymbol{i}})\int_{\Gamma_{c}}\widetilde{\psi}_{I}d\Gamma_{c}$$

6.2.4 Form of the elementary matrices for the hydrodynamics

6.2.4.1 Case of the solid mass

Elementary matrices of mass with the iteration of Newton $k+1$ are written:

$$\{p^{*}\}\left[M_{hydro}^{1}\right](\delta\boldsymbol{u})=-\int_{\Omega}b\rho^{+}(c_{i}^{*}+\widetilde{H}\,d_{i}^{*})\,\psi_{i}[\boldsymbol{Id}]\nabla\phi_{j}(\delta\boldsymbol{a}_{j}+\widetilde{H}\,\delta\boldsymbol{b}_{j})\,d\Omega$$

$$\{p^{*}\}\left[M_{hydro}^{2}\right](\delta p)=-\int_{\Omega}\left(\left(\frac{\rho^{+}(b-\varphi^{+})}{K_{s}}+\frac{\rho^{+}\varphi^{+}}{K_{w}}\right)\right)(c_{i}^{*}+\widetilde{H}\,d_{i}^{*})\,\psi_{i}\,\psi_{j}(\delta c_{j}+\widetilde{H}\,\delta d_{j})\,d\Omega$$

Elementary matrices of rigidity to the iteration of Newton $k+1$ are written:

$$\{p^{*}\}[K_{hydro}^{1}](\delta p)=\int_{\Omega}\left(\lambda^{+}\left(-\nabla p^{+}+\rho^{+}\boldsymbol{F}^{\boldsymbol{m}+}\right)+\rho^{+}\lambda^{+}\boldsymbol{F}^{\boldsymbol{m}+}\right)\frac{\rho^{+}}{K_{w}}(c_{i}^{*}+\widetilde{H}\,d_{i}^{*})\nabla\,\psi_{i}\,\psi_{j}(\delta c_{j}+\widetilde{H}\,\delta d_{j})\,d\Omega$$

$$\{p^{*}\}[K_{hydro}^{2}](\delta p)=\int_{\Omega}\rho^{+}\left(-\nabla p^{+}+\rho^{+}\boldsymbol{F}^{\boldsymbol{m}+}\right)\frac{\partial\lambda^{+}}{\partial p^{+}}(c_{i}^{*}+\widetilde{H}\,d_{i}^{*})\nabla\,\psi_{i}\,\psi_{j}(\delta c_{j}+\widetilde{H}\,\delta d_{j})\,d\Omega$$

$$\{p^{*}\}[K_{hydro}^{3}](\delta p)=-\int_{\Omega}\rho^{+}\lambda^{+}(c_{i}^{*}+\widetilde{H}\,d_{i}^{*})\nabla\,\psi_{i}\nabla\,\psi_{j}(\delta c_{j}+\widetilde{H}\,\delta d_{j})\,d\Omega$$

$$\{p^{*}\}[K_{hydro}^{4}](\delta\boldsymbol{u})=\int_{\Omega}\rho^{+}\left(-\nabla p^{+}+\rho^{+}\boldsymbol{F}^{\boldsymbol{m}+}\right)\frac{\partial\lambda^{+}}{\partial\varepsilon_{v}^{+}}(c_{i}^{*}+\widetilde{H}\,d_{i}^{*})\nabla\,\psi_{i}[\boldsymbol{Id}]\nabla\phi_{j}(\delta\boldsymbol{a}_{\boldsymbol{j}}+\widetilde{H}\,\delta\boldsymbol{b}_{\boldsymbol{j}})\,d\Omega$$

Elementary matrices of exchange for the solid mass with the iteration of Newton $k+1$ are written:

$$\{p^{*}\}[E^{1}](\delta q_{1})=\int_{\Gamma_{1}}(c_{i}^{*}+\widetilde{H}\,d_{i}^{*})\,\psi_{i}\,\widetilde{\psi}_{j}(\delta q_{1})_{j}\,d\Gamma_{1}$$

$$\{p^{*}\}[E^{2}](\delta q_{2})=\int_{\Gamma_{2}}(c_{i}^{*}+\widetilde{H}\,d_{i}^{*})\,\psi_{i}\,\widetilde{\psi}_{j}(\delta q_{2})_{j}\,d\Gamma_{2}$$

6.2.4.2 Case of the interface

Elementary matrices of mass with the iteration of Newton $k+1$ are written:

$$\{p_{f}^{*}\}[W_{hydro}^{1}](\delta[\![\boldsymbol{u}]\!])=-\int_{\Gamma_{c}}\rho^{+}(p_{f}^{*})_{i}\,\widetilde{\psi}_{i}\,\phi_{j}\left(2\widetilde{H}\,\delta\boldsymbol{b}_{\boldsymbol{j}}\right)\boldsymbol{n}_{\boldsymbol{c}}\,d\Gamma_{c}$$

$$\{p_f^*\}[W_{hydro}^2](\delta p_f) = -\int_{\Gamma_c} \frac{\rho^+ [\![\boldsymbol{u}]\!]\cdot \boldsymbol{n_c}}{K_w} (p_f^*)_i \widetilde{\psi}_i \widetilde{\psi}_j (\delta p_f)_j d\Gamma_c$$

Elementary matrices of rigidity to the iteration of Newton $k+1$ are written:

$$\{p_f^*\}[H_{hydro}^1](\delta [\![\boldsymbol{u}]\!]) = -\int_{\Gamma_c} \frac{\rho^+ \left(3\left([\![\boldsymbol{u}]\!]\cdot \boldsymbol{n_c}\right)^2\right) \nabla p_f^+}{12\mu} (p_f^*)_i \nabla \widetilde{\psi}_i \phi_j \left(2\widetilde{H}\, \delta \boldsymbol{b_j}\right) \boldsymbol{n_c} d\Gamma_c$$

$$\{p_f^*\}[H_{hydro}^2](\delta p_f) = -\int_{\Gamma_c} \frac{\left([\![\boldsymbol{u}]\!]\cdot \boldsymbol{n_c}\right)^3 \nabla p_f^+}{12\mu} \frac{\rho^+}{K_w} (p_f^*)_i \nabla \widetilde{\psi}_i \widetilde{\psi}_j (\delta p_f)_j d\Gamma_c$$

$$\{p_f^*\}[H_{hydro}^3](\delta p_f) = -\int_{\Gamma_c} \frac{\rho^+ \left([\![\boldsymbol{u}]\!]\cdot \boldsymbol{n_c}\right)^3}{12\mu} (p_f^*)_i \nabla \widetilde{\psi}_i \nabla \widetilde{\psi}_j (\delta p_f)_j d\Gamma_c$$

Elementary matrices of exchange for the interface with the iteration of Newton $k+1$ are written:

$$\{p_f^*\}[D^1](\delta q_1) = -\int_{\Gamma_1} (p_f^*)_i \widetilde{\psi}_i \widetilde{\psi}_j (\delta q_1)_j d\Gamma_1$$

$$\{p_f^*\}[D^2](\delta q_2) = -\int_{\Gamma_2} (p_f^*)_i \widetilde{\psi}_i \widetilde{\psi}_j (\delta q_2)_j d\Gamma_2$$

6.2.4.3 Continuity of the pressure

Elementary matrices of exchange (for the equation of continuity of the pressure on the level of the interface) to the iteration of Newton $k+1$ are written:

$$\{q_1^*\}[F^1](\delta p^{inf}) = \int_{\Gamma_1} (q_1^*)_i \widetilde{\psi}_i \psi_j (\delta c_j + \widetilde{H}\, \delta d_j) d\Gamma_1$$

$$\{q_1^*\}[F^2](\delta p_f) = -\int_{\Gamma_1} (q_1^*)_i \widetilde{\psi}_i \widetilde{\psi}_j (\delta p_f)_j d\Gamma_1$$

$$\{q_2^*\}[F^3](\delta p^{\sup}) = \int_{\Gamma_2} (q_2^*)_i \widetilde{\psi}_i \psi_j (\delta c_j + \widetilde{H}\, \delta d_j) d\Gamma_2$$

$$\{q_2^*\}[F^4](\delta p_f) = -\int_{\Gamma_2} (q_2^*)_i \widetilde{\psi}_i \widetilde{\psi}_j (\delta p_f)_j d\Gamma_2$$

Note:

In the form of the elementary matrices defined in this part, one can note the presence of two additional "unknown factors", δp^{inf} and $\delta p^{\sup}$. They correspond in fact to the unknown factor relating to the fields of pressure δp defined respectively on Γ_1 (i.e. δp^{inf}) and on Γ_2 (i.e. $\delta p^{\sup}$).

6.2.5 Expression of the second elementary members for the hydrodynamics

6.2.5.1 Case of the solid mass

In the expressions of the second members introduced here one indicates the number of the preceding iteration of Newton:

$$\{p^*\}(L^1_{hydro}) = \int_\Omega (c_i^* + \widetilde{H}\, d_i^*)\, \psi_i \left((m_w^+)^k - (m_w^-)^k\right) d\Omega$$

$$\{p^*\}(L^2_{hydro})_\theta = -\theta \int_\Omega (c_i^* + \widetilde{H}\, d_i^*) \nabla\, \psi_i (\boldsymbol{M}^+)^k\, d\Omega - (1-\theta) \int_\Omega (c_i^* + \widetilde{H}\, d_i^*) \nabla\, \psi_i (\boldsymbol{M}^-)^k\, d\Omega$$

$$\{p^*\}(L^3_{hydro})_\theta = \theta \int_{\Gamma_F} (c_i^* + \widetilde{H}\, d_i^*)\, \psi_i (M^+_{ext})^k\, d\Gamma_F + (1-\theta) \int_{\Gamma_F} (c_i^* + \widetilde{H}\, d_i^*)\, \psi_i (M^-_{ext})^k\, d\Gamma_F$$

$$\{p^*\}(L^4_{hydro})_\theta = -\theta \int_{\Gamma_1} (c_i^* + \widetilde{H}\, d_i^*)\, \psi_i (q_1^+)^k\, d\Gamma_1 - (1-\theta) \int_{\Gamma_1} (c_i^* + \widetilde{H}\, d_i^*)\, \psi_i (q_1^-)^k\, d\Gamma_1$$

$$\{p^*\}(L^5_{hydro})_\theta = -\theta \int_{\Gamma_2} (c_i^* + \widetilde{H}\, d_i^*)\, \psi_i (q_2^+)^k\, d\Gamma_2 - (1-\theta) \int_{\Gamma_2} (c_i^* + \widetilde{H}\, d_i^*)\, \psi_i (q_2^-)^k\, d\Gamma_2$$

6.2.5.2 Case of the interface

In the expressions of the second members introduced here one indicates the number of the preceding iteration of Newton.

$$\{p_f^*\}(L^6_{hydro}) = \int_{\Gamma_c} (p_f^*)_i\, \widetilde{\psi}_i \left((w^+)^k - (w^-)^k\right) d\Gamma_c$$

$$\{p_f^*\}(L^7_{hydro})_\theta = -\theta \int_{\Gamma_c} (p_f^*)_i \nabla\, \widetilde{\psi}_i (\boldsymbol{W}^+)^k\, d\Gamma_c - (1-\theta) \int_{\Gamma_c} (p_f^*)_i \nabla\, \widetilde{\psi}_i (\boldsymbol{W}^-)^k\, d\Gamma_c$$

$$\{p_f^*\}(L^8_{hydro})_\theta = \theta \int_{\Gamma_f} (p_f^*)_i\, \widetilde{\psi}_i (W^+_{ext})^k\, d\Gamma_f + (1-\theta) \int_{\Gamma_f} (p_f^*)_i\, \widetilde{\psi}_i (W^-_{ext})^k\, d\Gamma_f$$

$$\{p_f^*\}(L^9_{hydro})_\theta = \theta \int_{\Gamma_1} (p_f^*)_i\, \widetilde{\psi}_i (q_1^+)^k\, d\Gamma_1 + (1-\theta) \int_{\Gamma_1} (p_f^*)_i\, \widetilde{\psi}_i (q_1^-)^k\, d\Gamma_1$$

$$\{p_f^*\}(L^{10}_{hydro})_\theta = \theta \int_{\Gamma_2} (p_f^*)_i\, \widetilde{\psi}_i (q_2^+)^k\, d\Gamma_2 + (1-\theta) \int_{\Gamma_2} (p_f^*)_i\, \widetilde{\psi}_i (q_2^-)^k\, d\Gamma_2$$

6.2.5.3 Continuity of the pressure

In the expressions of the second members introduced here (for the equation of continuity of the pressure on the level of the interface), one indicates the number of the preceding iteration of Newton.

$$\{q_1^*\}(J^1_{cont}) = -\int_{\Gamma_1} (q_1^*)_i\, \widetilde{\psi}_i \left((p^{inf})^k - (p_f^+)^k\right) d\Gamma_1$$

$$\{q_2^*\}(J^2_{cont}) = -\int_{\Gamma_1} (q_2^*)_i\, \widetilde{\psi}_i \left((p^{\sup})^k - (p_f^+)^k\right) d\Gamma_2$$

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8 Description of the versions of the document

Version Aster	Author (S) Organization (S)	Description of the modifications
12	Maxime Faivre (ENSG)	Initial version
13	Bertrand PAUL (IFPEN)	Version 2