

SSL14 - Gantry plan articulated in foot

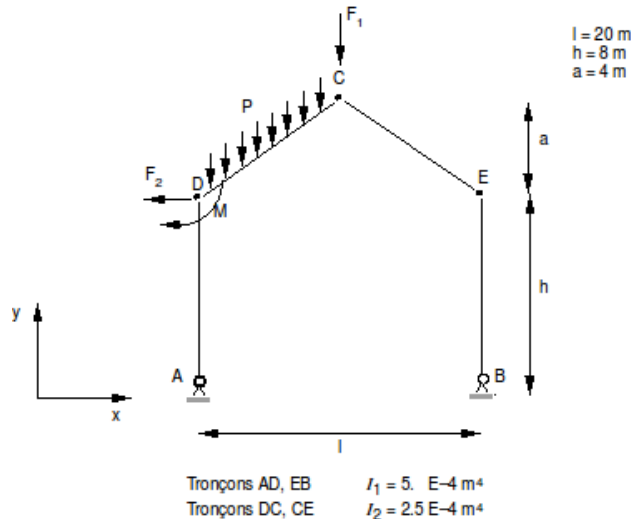
Summary

This test relates to the study of a gantry made up of hurred beams, articulated in foot, in linear static analysis.

The gantry is modelled with linear elements `SEG2` and subjected to four loadings (distributed or specific).

1 Problem of reference

1.1 Geometry



Geometry of the gantry (m) :

- $l = 20$
- $h = 8$
- $a = 4$

Quadratic moments of the beams (m^4) :

- Sections AD, EB : $I_1 = 5.0 \text{ E-4}$
- Sections DC, CE : $I_2 = 2.5 \text{ E-4}$

The gantry consists of symmetrical beams of sections, so that $IY = IZ$.

One takes account only of the energy of inflection, because the beams are very slim. This is why the other characteristics of section of beam do not intervene.

1.2 Material properties

Isotropic linear elastic material: $E = 2.1 \text{ E11 Pa}$

1.3 Boundary conditions and loadings

Feet of posts A and B articulated.

Loadings

Nodal force in C :	$F_y = -2000 \text{ N} = F_1$
Nodal force in D :	$F_x = -10000 \text{ N} = F_2$
Moment in D :	$M_x = -100000 \text{ N.m} = M$
Force distributed on the section DC :	$P_z = -3000 \text{ N/m}$

2 Reference solution

2.1 Method of calculating used for the reference solution

The method of calculating and the solution were determined by F. Voltaire (EDF R & D/AMA) and are exposed in the appendix.

2.2 Results of reference

Horizontal reactions F_x and verticals F_y at the point A .

Bending moment M_z in C .

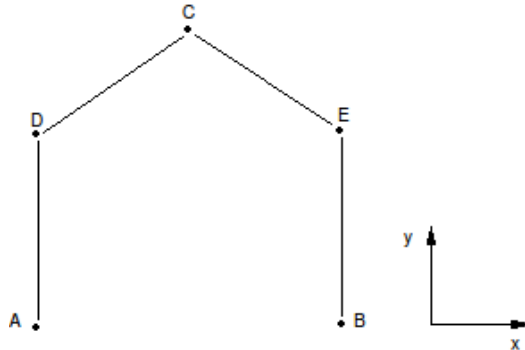
Displacements horizontal D_x and vertical D_y point C .

2.3 Uncertainty on the solution

Analytical solution.

3 Modeling A

3.1 Characteristics of modeling



- Modeling POU_D_E
- 10 elements section by section, is 40 elements $SEG2$
- Displacement in the plan: $DZ = 0$ on all the grid
- Feet of posts A and B articulated: $DX = DY = 0$

3.2 Sizes tested and results

3.2.1 Values tested

Loading	Node	Value tested	Reference
p	C	$Dx (m)$	0.0110476
		$Dy (m)$	- 0.012422374
		$Mz (N.m)$	18672.994
	A	$Fx (N)$	5175.37
		$Fy (N)$	24233.24
F_1	C	$Dx (m)$	0.00000
		$Dy (m)$	- 0.01497330
		$Mz (N.m)$	41422.161
	A	$Fx (N)$	4881.487
		$Fy (N)$	10000.00
F_2	C	$Dx (m)$	- 0.03000956
		$Dy (m)$	- 0.00299466
		$Mz (N.m)$	8284.432
	A	$Fx (N)$	5976.297
		$Fy (N)$	4000.00
C	C	$Dx (m)$	0.0273532
		$Dy (m)$	- 0.001215646
		$Mz (N.m)$	4916.724
	A	$Fx (N)$	4576.394
		$Fy (N)$	5000.00

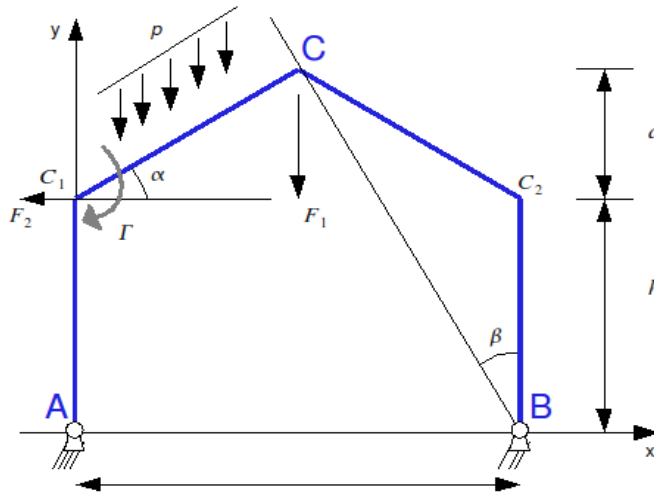
4 Summary of the results

Results got with modeling `POU_D_E` are in very good agreement with the analytical solution and thus validate the calculation of lattice of beams subjected to efforts specific or distributed.

5 Appendix

5.1 Presentation

One considers the gantry opposite, subjected to various loads.



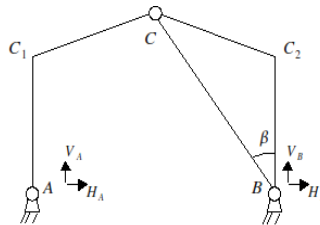
One considers the gantry opposite, subjected to various loads.

Hyperstaticity of degree 1.
Hyperstatic unknown factor: X :
moment out of C .

Top-load distributed p on C_1C_2 .
Two forces F_1 , F_2 and a couple in C_1 .

Hyperstaticity of degree 1.
Hyperstatic unknown factor: X
Loads applied:

- moment in C ,
- top-load distributed p on C_1C_2 ,
- force F_1 , F_2 applied in C_1 ,
- couples Γ applied in C_1



$$\tan(\alpha) = \frac{2a}{l} = 0.4 \Rightarrow (\cos(\alpha))^{-1} = \sqrt{1.16} = 1.077033$$

$$\tan(\beta) = \frac{l}{2(a+h)} = 1.2$$

$$b = \frac{l}{2\cos(\alpha)} ; \sin(\alpha) = \frac{a}{b}$$

5.2 Isostatic requests under real load distributed p on C_1C

5.2.1 Isostatic reactions of supports

$$H_A + H_B = 0 \quad V_A + V_B = \frac{pl}{2\cos(\alpha)} \quad lV_B = \frac{pl^2}{8\cos(\alpha)}$$

The part CB is articulated and charged only at its ends

$$\begin{pmatrix} H_B \\ V_B \end{pmatrix} \wedge BC = 0 \Leftrightarrow H_B = -V_B \tan(\beta)$$

From where isostatic reactions

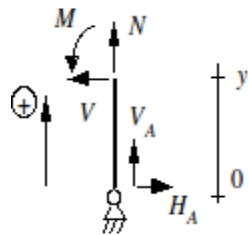
$$H_A = \frac{pl}{8\cos(\alpha)} \tan(\beta) \quad ; \quad V_A = \frac{3pl}{8\cos(\alpha)} \quad ; \quad H_B = \frac{-pl}{8\cos(\alpha)} \tan(\beta) \quad ; \quad V_B = \frac{pl}{8\cos(\alpha)}$$

Note:

$$\frac{l \tan(\beta)}{8 \cos(\alpha)} = \frac{bl}{8(a+h)}$$

5.2.2 Requests

Beam AC_1

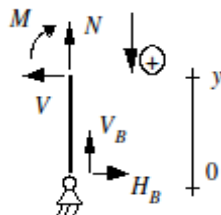


$$N_{iso} = \frac{-3pl}{8\cos(\alpha)}$$

$$V_{iso} = \frac{pl}{8\cos(\alpha)} \tan(\beta)$$

$$M_{iso} = \frac{-pl}{8\cos(\alpha)} y \cdot \tan(\beta)$$

Beam C_2B

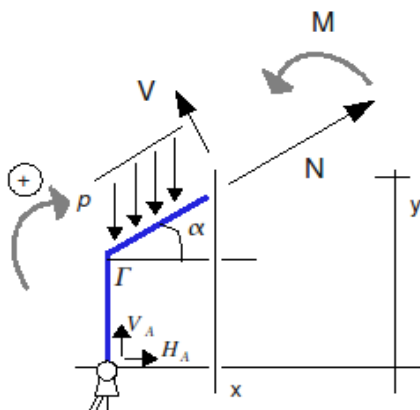


$$N_{iso} = \frac{-pl}{8\cos(\alpha)}$$

$$V_{iso} = \frac{-pl}{8\cos(\alpha)} \tan(\beta)$$

$$M_{iso} = \frac{-pl}{8\cos(\alpha)} y \cdot \tan(\beta)$$

Beam C_1C



$$N_{iso} = -H_A \cos(\alpha) - V_A \sin(\alpha) + \frac{px}{\cos(\alpha)} \sin(\alpha)$$

$$= -\frac{pl}{8} \left(\tan(\beta) + 3 \tan(\alpha) - 8 \tan(\alpha) \frac{x}{l} \right)$$

$$V_{iso} = H_A \sin(\alpha) - V_A \cos(\alpha) + \frac{px}{\cos(\alpha)} \cos(\alpha)$$

$$= \frac{pl}{8} \left(\tan(\beta) \tan(\alpha) - 3 + 8 \frac{x}{l} \right)$$

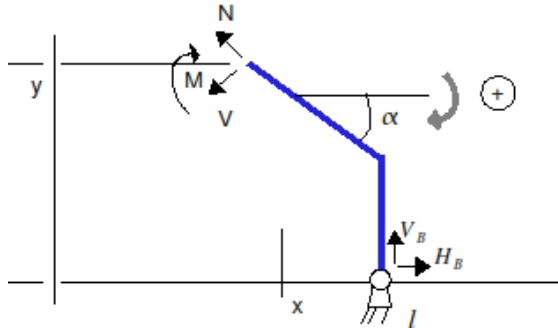
$$M_{iso} = -\frac{px^2}{2\cos(\alpha)} + V_A x - H_A y$$

$$= \frac{p}{\cos(\alpha)} \left(\frac{-x^2}{2} + \frac{3lx}{8} - \frac{ly \tan(\beta)}{8} \right)$$

with $M_{iso} = 0$ in C

$$M_{iso} = \frac{-pl}{8(a+h)} \left(2s^2 \left(\frac{a+h}{b} \right) - s(2a+3h) + bh \right) \quad \text{with } s = \frac{x}{\cos(\alpha)} \in [0, b]$$

Beam CC_2

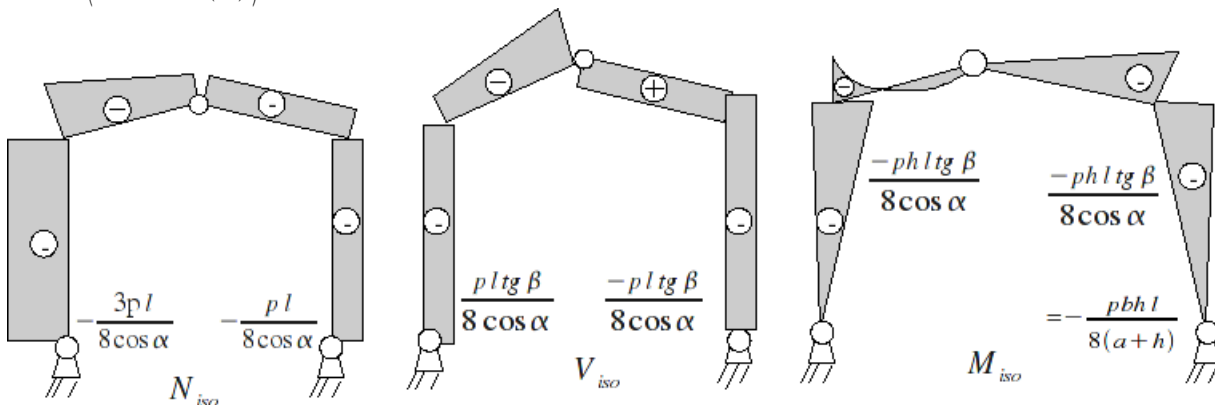


$$\begin{aligned}
 N_{iso} &= H_B \cos(\alpha) - V_B \sin(\alpha) \\
 &= -\frac{pl}{8} (\tan(\beta) + \tan(\alpha)) \\
 V_{iso} &= H_B \sin(\alpha) + V_B \cos(\alpha) \\
 &= -\frac{pl}{8} (\tan(\beta) \tan(\alpha) - 1) \\
 M_{iso} &= H_B y - V_B (l - x) \\
 &= -\frac{pl}{8 \cos(\alpha)} (y \cdot \tan(\beta) - (l - x))
 \end{aligned}$$

with $M_{iso} = 0$ in C

5.2.3 Diagrams

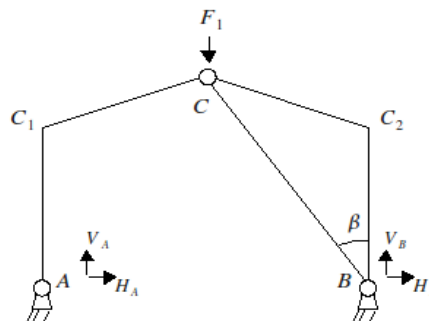
$$\left(b = \frac{l}{2 \cos(\alpha)} \right)$$



5.3 Requests under concentrated force F_1 (downwards)

5.3.1 Reactions of support

$$\begin{aligned}
 H_A + H_B &= 0; \\
 V_A + V_B &= F_1; \\
 \left(\begin{matrix} H_A \\ V_A \end{matrix} \right) \wedge AC &= 0 = \left(\begin{matrix} H_B \\ V_B \end{matrix} \right) \wedge BC;
 \end{aligned}$$



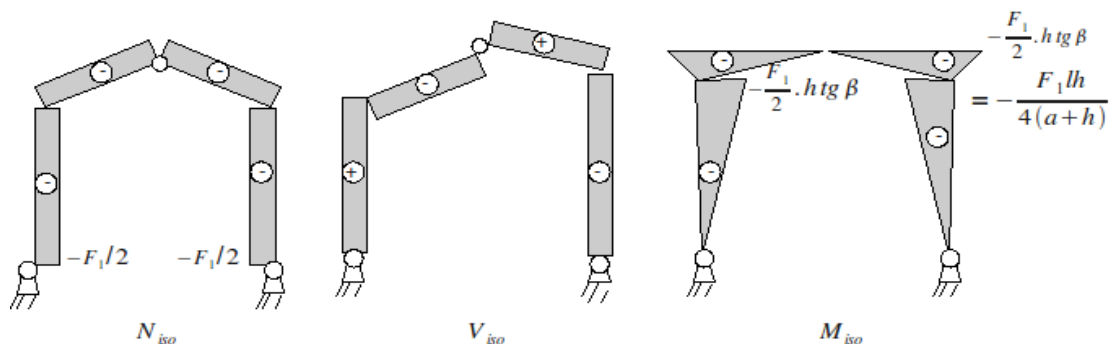
From where:

$$H_A = \frac{1}{2} F_1 \tan(\beta); \quad V_A = \frac{1}{2} F_1; \quad H_B = -\frac{1}{2} F_1 \tan(\beta); \quad V_B = \frac{1}{2} F_1$$

5.3.2 Requests

Beam AC_1 :	$N_{iso} = \frac{-1}{2} F_1$ $V_{iso} = \frac{1}{2} F_1 \tan(\beta)$ $M_{iso} = \frac{-1}{2} F_1 y \tan(\beta)$
Beam $C_2 B$:	$N_{iso} = \frac{-1}{2} F_1$ $V_{iso} = \frac{-1}{2} F_1 \tan(\beta)$ $M_{iso} = \frac{-1}{2} F_1 y \tan(\beta)$
Beam $C_1 C$:	$N_{iso} = \frac{-1}{2} F_1 (\tan(\beta) \cos(\alpha) + \sin(\alpha))$ $V_{iso} = \frac{1}{2} F_1 (\tan(\beta) \sin(\alpha) - \cos(\alpha))$ $M_{iso} = \frac{-1}{2} F_1 (y \tan(\beta) - x)$
Beam CC_2 :	$N_{iso} = \frac{-1}{2} F_1 (\tan(\beta) \cos(\alpha) + \sin(\alpha))$ $V_{iso} = \frac{1}{2} F_1 (\tan(\beta) \sin(\alpha) - \cos(\alpha))$ $M_{iso} = \frac{-1}{2} F_1 (y \tan(\beta) - (l - x))$

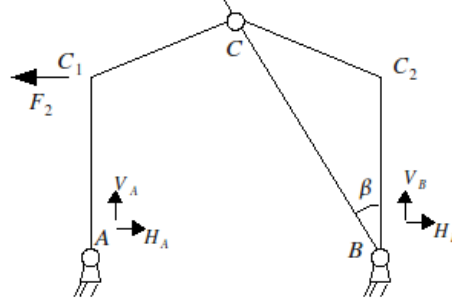
5.3.3 Diagrams (F_1 downwards)



5.4 Requests under the concentrated force F_2 (towards the left)

5.4.1 Reactions of support

$$\begin{aligned} H_A + H_B &= F_2; \\ V_A + V_B &= 0; \\ lV_B + hF_2 &= 0; \\ \begin{pmatrix} H_B \\ V_B \end{pmatrix} \text{ans } BC &= 0 \end{aligned}$$



From where:

$$H_A = F_2 \left(1 - \frac{h}{l} \tan(\beta)\right); V_A = F_2 \frac{h}{l}; H_B = F_2 \frac{h}{l} \tan(\beta); V_B = -F_2 \frac{h}{l};$$

Note:

$$\begin{cases} \frac{h}{l} \tan(\beta) = \frac{h}{2(a+h)} & \left(1 - \frac{h}{l} \tan(\beta)\right) = \frac{2a+h}{2(a+h)} \\ \tan(\beta) \sin(\alpha) - \cos(\alpha) = \frac{-hl}{2b(a+h)}, & \tan(\beta) \cos(\alpha) - \sin(\alpha) = \frac{l^2 - 4(a^2 + ah)}{4b(a+h)} \end{cases}$$

5.4.2 Requests

$$N_{iso} = -F_2 \frac{h}{l}$$

Beam AC_1 :

$$V_{iso} = F_2 \left(1 - \frac{h}{l} \tan(\beta)\right)$$

$$M_{iso} = -F_2 y \left(1 - \frac{h}{l} \tan(\beta)\right)$$

$$N_{iso} = F_2 \frac{h}{l}$$

Beam C_2B :

$$V_{iso} = F_2 \frac{h}{l} \tan(\beta)$$

$$M_{iso} = -F_2 y \frac{h}{l} \tan(\beta)$$

$$N_{iso} = F_2 \left(\left(1 - \frac{h}{l} \tan(\beta)\right) \cos(\alpha) - \frac{h}{l} \cos(\alpha) \right)$$

Beam C_1C :

$$V_{iso} = F_2 \left(\left(1 - \frac{h}{l} \tan(\beta)\right) - \frac{h}{l} \cos(\alpha) \right)$$

$$M_{iso} = F_2 \left(\frac{h}{l} x - \left(1 - \frac{h}{l} \tan(\beta)\right) y \right)$$

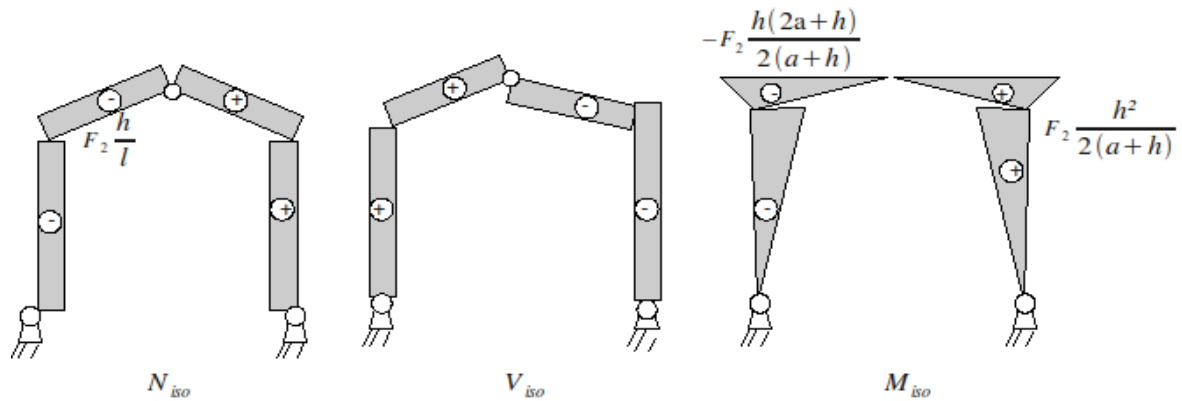
$$N_{iso} = F_2 \frac{h}{l} (\tan(\beta) \cos(\alpha) + \sin(\alpha))$$

Beam CC_2 :

$$V_{iso} = F_2 \frac{h}{l} (\tan(\beta) \sin(\alpha) - \cos(\alpha))$$

$$M_{iso} = F_2 \frac{h}{l} (y \tan(\beta) - (l - x))$$

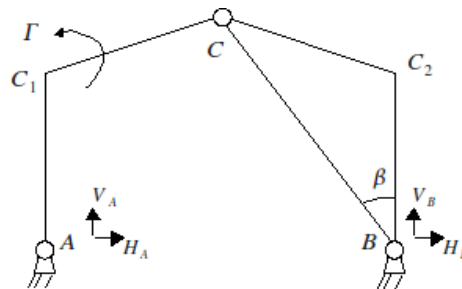
5.4.3 Diagrams



5.5 Requests under the concentrated couple Γ (positive)

5.5.1 Reactions of support

$$\begin{aligned} H_A + H_B &= 0; \\ V_A + V_B &= 0; \\ lV_B + \Gamma &= 0; \\ \begin{pmatrix} H_B \\ V_B \end{pmatrix} \wedge BC &= 0; \end{aligned}$$



From where: $H_A = -\Gamma \tan \frac{\beta}{l}$, $V_A = \frac{\Gamma}{l}$, $H_B = \Gamma \tan \frac{\beta}{l}$, $V_B = -\frac{\Gamma}{l}$

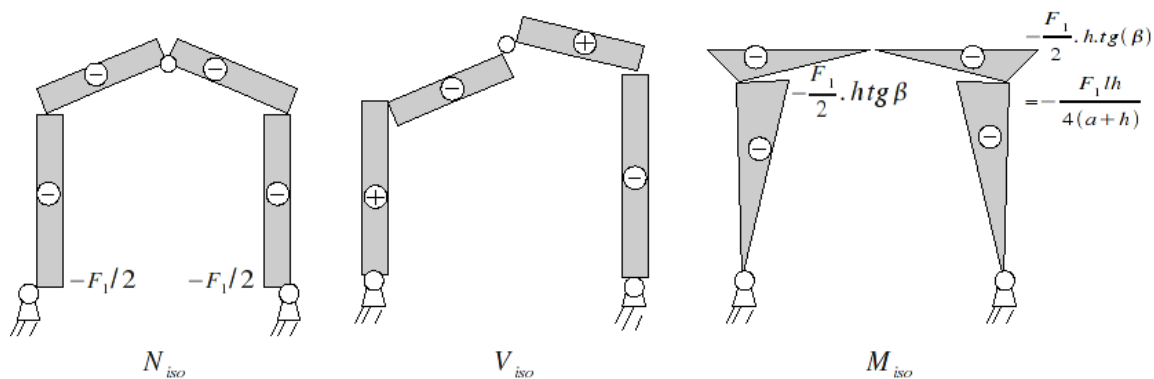
Note:

$$\left| \frac{\tan(\beta)}{l} = \frac{1}{2(a+h)} \right.$$

5.5.2 Requests

Beam AC_1 :	$N_{iso} = \frac{-\Gamma}{l}$ $V_{iso} = \frac{-\Gamma \tan(\beta)}{l}$ $M_{iso} = \frac{\Gamma y \tan(\beta)}{l}$
Beam C_2B :	$N_{iso} = \frac{-\Gamma}{l}$ $V_{iso} = \frac{\Gamma \tan(\beta)}{l}$ $M_{iso} = \frac{\Gamma y \tan(\beta)}{l}$
Beam C_1C :	$N_{iso} = \frac{\Gamma}{l} (\tan(\beta) \cos(\alpha) - \sin(\alpha))$ $N_{iso} = \frac{-\Gamma}{l} (\tan(\beta) \sin(\alpha) + \cos(\alpha))$ $N_{iso} = \frac{\Gamma}{l} (x + y \tan(\beta) - l)$
Beam CC_2 :	$N_{iso} = \frac{\Gamma}{l} (\tan(\beta) \cos(\alpha) + \sin(\alpha))$ $N_{iso} = \frac{\Gamma}{l} (\tan(\beta) \sin(\alpha) - \cos(\alpha))$ $N_{iso} = \frac{\Gamma}{l} (y \tan(\beta) - (l - x))$

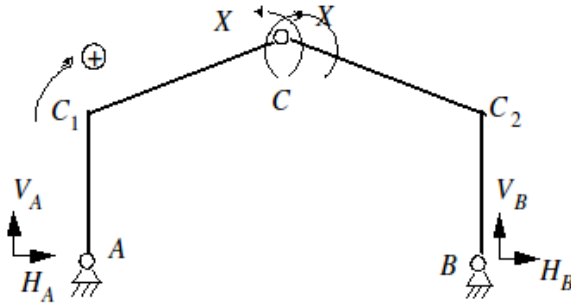
5.5.3 Diagrams (Γ positive)



5.6 Requests under the moment X hyperstatic

5.6.1 Reactions of support

$$\begin{aligned} H_A + H_B &= 0; \\ V_A + V_B &= 0; \\ IV_B &= 0; \\ H_B(a+h) - X &= 0; \end{aligned}$$

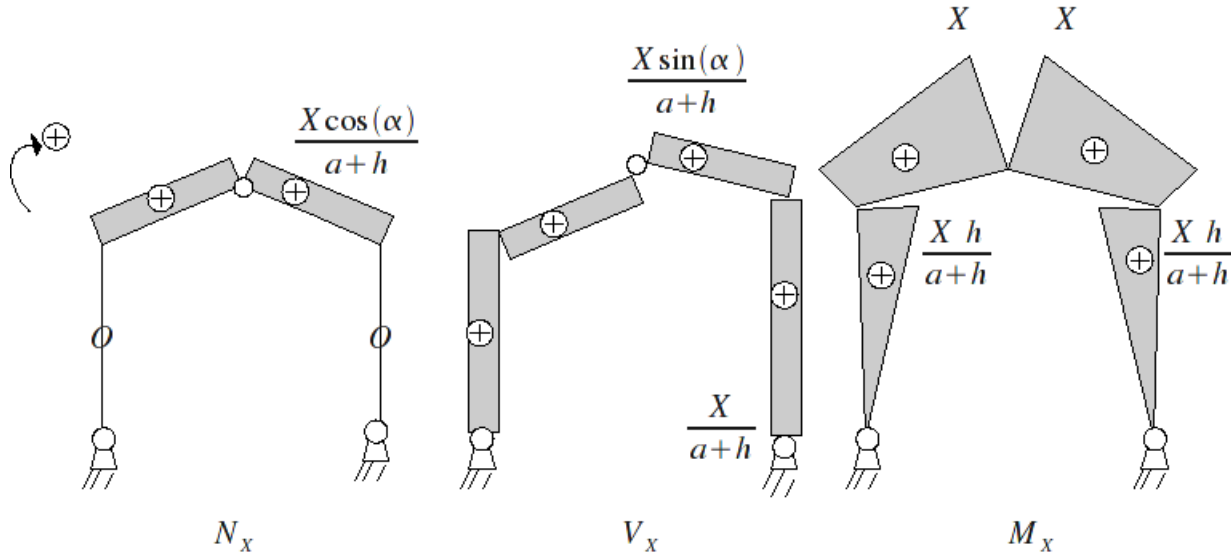


From where reactions: $H_A = \frac{-X}{a+h}$, $V_A = 0$, $H_B = \frac{X}{a+h}$, $V_B = 0$

5.6.2 Requests

Beam AC_1 :	$N_x = 0$ $V_x = \frac{-X}{a+h}$ $M_x = \frac{X}{a+h} y$
Beam C_2B :	$N_x = 0$ $V_x = \frac{X}{a+h}$ $M_x = \frac{X}{a+h} y$
Beam C_1C :	$N_x = \frac{X}{a+h} \cos(\alpha)$ $V_x = \frac{X}{a+h} \sin(\alpha)$ $M_x = \frac{X}{a+h} y = \frac{X}{a+h} (h + x \tan(\alpha))$
Beam CC_2 :	$N_x = \frac{X}{a+h} \cos(\alpha)$ $V_x = \frac{X}{a+h} \sin(\alpha)$ $M_x = \frac{X}{a+h} y$

5.6.3 Diagrams

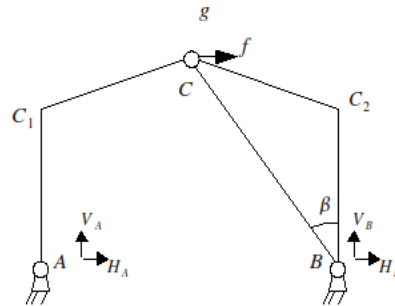


5.7 Requests under specific dummy loads in C

In order to calculate displacement in C , using the Principle of Virtual work (cf. the paragraph [§ 8]), it is necessary to establish the diagrams of requests under the action of two "fictitious" forces f and g applied in C .

5.7.1 Reactions of support

$$\begin{aligned} H_A + H_B &= -f; \\ V_A + V_B &= -g; \\ \begin{pmatrix} H_A \\ V_A \end{pmatrix} \wedge AC &= 0 = \begin{pmatrix} H_B \\ V_B \end{pmatrix} \wedge BC \end{aligned}$$



From where:

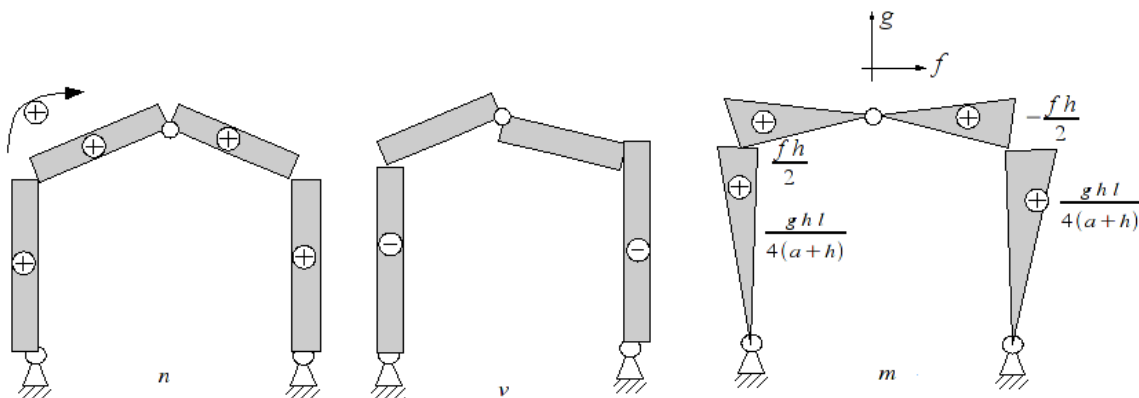
$$\begin{aligned} H_A &= \frac{-1}{2}(f + g \tan(\beta)), \quad V_A = \frac{-1}{2}(g + f \cot(\beta)) \\ H_B &= \frac{-1}{2}(f - g \tan(\beta)), \quad V_B = \frac{-1}{2}(g - f \cot(\beta)) \end{aligned}$$

5.7.2 Requests

	$n = \frac{1}{2}(g + f \cot(\beta))$
Beam AC_1 :	$v = \frac{-1}{2}(f + g \tan(\beta))$
	$m = \frac{1}{2}(f + g \tan(\beta))$
<hr/>	
	$n = \frac{1}{2}(g - f \cot(\beta))$
Beam C_2B :	$v = \frac{-1}{2}(f - g \tan(\beta))$
	$m = \frac{-1}{2}(f - g \tan(\beta))y$
<hr/>	
	$n = \frac{1}{2}(f + g \tan(\beta))\cos(\alpha) + \frac{1}{2}(g + f \cot(\beta))\sin(\alpha)$
Beam C_1C :	$v = \frac{-1}{2}(f + g \tan(\beta))\sin(\alpha) + \frac{1}{2}(g + f \cot(\beta))\cos(\alpha)$
	$m = \frac{1}{2}(f + g \tan(\beta))y - \frac{1}{2}(g + f \cot(\beta))x$
<hr/>	
	$n = \frac{-1}{2}(f - g \tan(\beta))\cos(\alpha) + \frac{1}{2}(g - f \cot(\beta))\sin(\alpha)$
Beam CC_2 :	$v = \frac{-1}{2}(f - g \tan(\beta))\sin(\alpha) - \frac{1}{2}(g - f \cot(\beta))\cos(\alpha)$
	$m = \frac{-1}{2}(f - g \tan(\beta))y - \frac{1}{2}(g - f \cot(\beta))(l - x)$

5.7.3 Diagrams

Here diagrams of requests under the action of the two "fictitious" forces f and g . One considers here: $f \geq 0, g \geq f \cot(\beta)$.



5.8 Determination of the moment X hyperstatic

One places oneself in elasticity; one considers only the energy of inflection, the beams being slim. The natural state is supposed to be virgin (not prestressings nor of displacement of support).

The complementary potential is then:

$$F^*(X) = \int_{\text{poteaux}} \frac{(M_{iso} + M_1 X)^2}{EI_1} + \int_{\text{charpentes}} \frac{(M_{iso} + M_1 X)^2}{EI_2}$$

It is stationary with balance, from where:

$$\delta \cdot X = \left[\int_{pot} \frac{M_1^2}{EI_1} + \int_{charp} \frac{M_1^2}{EI_2} \right] \cdot X = - \int_{pot} \frac{M_1 M_{iso}}{EI_1} - \int_{charp} \frac{M_1 M_{iso}}{EI_2} = S$$

The coefficient of flexibility δ is the sum of:

$$\int_{pot} \frac{M_1^2}{EI_1} = \frac{2h}{3EI_1} \left(\frac{h}{a+h} \right)^2$$

$$\int_{charp} \frac{M_1^2}{EI_2} = \frac{2b}{EI_2} \left[\left(\frac{h}{a+h} \right)^2 + \frac{1}{3} \left(\frac{a}{a+h} \right)^2 + \frac{ah}{(a+h)^2} \right]$$

that is to say:

$$E \cdot \delta = \frac{2}{(a+h)^2} \left[\frac{h^3}{3I_1} + \frac{b(3h^2 + a^2 + 3ah)}{3I_2} \right]$$

Digital application:

In the example considered:

$$I_1 = 2I_2 = 5.0 E - 4 m^4, \quad h = 2a = 8 m, \quad l = 20 m, \quad b = \frac{l}{2} \sqrt{1.16}$$

$$\text{From where: } \gamma = \frac{2}{E(a+h)^2} \frac{h^2}{3} \left(h + \frac{19b}{2} \right)$$

2353.45347 m³

One studies one after the other the various loadings to calculate the second members S .

5.8.1 Distributed load p on C_1C

The second member S had with f is:

$$- \int_{pot} \frac{M_1 M_{iso}}{EI_1} = \frac{3h}{3EI_1} \left(\frac{h}{a+h} \right) \left(\frac{pblh}{8(a+h)} \right) = \frac{2}{E(a+h)^2 I_1} \frac{ph^3 bl}{24}$$

$$- \int_{CC_2} \frac{M_1 M_{iso}}{EI_2} = \frac{pb^2 hl}{8(a+h)EI_2} \left(\frac{1}{2} \frac{h}{a+h} \right) + \left(\frac{a}{6} \frac{a}{a+h} \right) = \frac{2}{E(a+h)^2 I_2} \frac{phb^{21}(3h+a)}{48}$$

$$- \int_{C_1C} \frac{M_1 M_{iso}}{EI_2} = \frac{1}{EI_2} \frac{pl}{8(a+h)^2} \int_0^b \left[2s^2 \frac{a+h}{b} - s(2a+3h) + bh \right] \left[h + s \frac{a}{b} \right] ds$$

$$= \frac{1}{E(a+h)^2 I_2} \frac{plb^2}{48} (h^2 + 2ah + a^2)$$

From where:

$$S = \frac{2}{E(a+h)^2} \frac{plb}{96} \left[\frac{4h^3}{I_1} + \frac{hb(3h+a)}{I_2} + \frac{b(h^2 - 2ah - a^2)}{I_2} \right]$$

Digital application:

$$I_1 = 2I_2; \quad h = 2a; \quad p = 3000 N.m^{-1} \text{ (downwards)}$$

$$S = \frac{2}{E(a+h)^2 I_1} \frac{plbh^2}{96} \left[4h + \frac{13}{2} b \right]$$

43946021.89 N.m⁴

From where:

- moment in C :
 $X = 18672994 \text{ N.m}$
- reaction in A :

$$H_A = p \frac{bl}{8(a+h)} - \frac{X}{a+h} = \frac{pbl}{8-X} \frac{1}{a+h}, \quad H_A = 5175.37 \text{ N}$$

$$V_A = \frac{3pb}{4} - 0, \quad V_A = 24233.24 \text{ N}$$

5.8.2 Concentrated loading F_1 in C

The second member is obtained using:

$$-\int_{pot} \frac{M_1 M_{iso}}{EI_1} = \frac{2h}{3EI_1} \left(\frac{h}{a+h}\right) \left(\frac{F_1 lh}{4(a+h)}\right) = \frac{2}{E(a+h)^2 I_1} \frac{F_1 lh^3}{12}$$

$$-\int_{sharp} \frac{M_1 M_{iso}}{EI_2} = \frac{2b}{EI_2} \frac{F_1 lh}{4(a+h)} \left(\frac{1}{2} \left(\frac{h}{a+h}\right) + \frac{1}{6} \left(\frac{a}{a+h}\right)\right) = \frac{2}{E(a+h)^2 I_2} \frac{F_1 blh(3h+a)}{24}$$

From where:

$$S = \frac{2}{E(a+h)^2} \frac{F_1 lh}{24} \left[\frac{2h^2}{I_1} + \frac{b(3h+a)}{I_2} \right]$$

Digital application:

$$I_1 = 2I_2 ; \quad h = 2a ; \quad F_1 = 20000 \text{ N (downwards)}$$

$$S = \frac{2}{E(a+h)^2 I_1} \frac{F_1 lh^2}{24} [2h + 7b]$$

$\underbrace{\hspace{10em}}_{97485127.76 \text{ N.m}^4}$

From where:

- moment in C :
 $X = 41422.161 \text{ N.m}$
- reaction in A :

$$H_A = \frac{1}{4} F_1 \frac{l}{a+h} - \frac{X}{a+h} = \frac{F_1 l}{4-X} \frac{1}{a+h}, \quad H_A = 4881.4866 \text{ N}$$

$$V_A = \frac{1}{2} F_1 - 0, \quad V_A = 10000.0 \text{ N}$$

5.8.3 Concentrated loading F_2 in C_1

The second member is obtained using:

$$-\int_{AC_1} \frac{M_1 M_{iso}}{EI_1} = \frac{h}{3EI_1} \left(\frac{h}{a+h}\right) \frac{F_2 h(2a+h)}{2(a+h)} = \frac{2}{E(a+h)^2 I_1} \frac{F_2 h^3(2a+h)}{12}$$

$$-\int_{C_2B} \frac{M_1 M_{iso}}{EI_1} = \frac{h}{3EI_1} \left(\frac{h}{a+h}\right) \frac{(-F_2 h^2)}{2(a+h)} = \frac{2}{E(a+h)^2 I_1} \frac{F_2 h^3(2a+h)}{12}$$

$$-\int_{C_1C} \frac{M_1 M_{iso}}{EI_2} = \frac{b}{EI_2} \frac{F_2 h (2a+h)}{2(a+h)} \left[\frac{1}{2} \left(\frac{h}{a+h} \right) + \frac{1}{6} \left(\frac{a}{a+h} \right) \right] = \frac{2}{E(a+h)^2 I_2} \frac{F_2 b h (3h^2 + 7ah + 2a^2)}{24}$$

$$-\int_{CC_2} \frac{M_1 M_{iso}}{EI_2} = \frac{b}{EI_2} \frac{-F_2 h^2}{2(a+h)} \left[\frac{1}{2} \left(\frac{h}{a+h} \right) + \frac{1}{6} \left(\frac{a}{a+h} \right) \right] = \frac{2}{E(a+h)^2 I_2} \frac{-F_2 b h^2 (3h+a)}{24}$$

$$S = \frac{2}{E(a+h)^2} \frac{F_2 h a}{12} \left[\frac{2h^2}{I_1} + \frac{b(3h+a)}{I_2} \right]$$

Digital application:

$$I_1 = 2 I_2 \quad ; \quad h = 2a \quad ; \quad F_2 = 10\,000 \text{ N (towards the left)}$$

$$S = \frac{2}{E(a+h)^2 I_1} \frac{F_2 h^2 a}{12} [2h + 7b]$$

19 497 025.55 N.m⁴

From where:

- moment in C :
 $X = 8\,284.4321 \text{ N.m}$
- reaction in A :

$$H_A = F_2 \frac{2a+h}{2(a+h)} - \frac{X}{a+h} = \frac{F_2 \left(a + \frac{h}{2} \right) - X}{a+h}, \quad H_A = 5976.297 \text{ N}$$

$$V_A = \frac{F_2 h}{l}, \quad V_A = 4000.0 \text{ N}$$

5.8.4 Specific couple Γ in C_1

The second member is obtained using:

$$-\int_{pot} \frac{M_1 M_{iso}}{EI_1} = \frac{-2h}{3EI_1} \left(\frac{h}{a+h} \right) \frac{\Gamma h}{2(a+h)} = \frac{2}{E(a+h)^2 I_1} \frac{-\Gamma h^3}{6}$$

$$-\int_{C_1C} \frac{b}{EI_2} = \frac{\Gamma (h+2a)}{2(a+h)} \frac{b}{EI_2} \left[\frac{1}{2} \left(\frac{h}{a+h} + \frac{1}{6} \left(\frac{a}{a+h} \right) \right) \right] = \frac{2}{E(a+h)^2 I_2} \frac{\Gamma (h+2a)(3h+a)b}{24}$$

$$-\int_{CC_2} \frac{M_1 M_{iso}}{EI_2} = \frac{b}{EI_2} \frac{-\Gamma h}{2(a+h)} \left[\frac{1}{2} \left(\frac{h}{a+h} \right) + \frac{1}{6} \left(\frac{a}{a+h} \right) \right] = \frac{2}{E(a+h)^2 I_2} \frac{-\Gamma h b (3h+a)}{24}$$

From where:

$$S = \frac{2}{E(a+h)^2} \frac{-\Gamma}{12} \left[\frac{2h^3}{I_1} + \frac{ab(3h+a)}{I_2} \right]$$

Digital application:

$$I_1 = 2 I_2 \quad ; \quad h = 2a \quad ; \quad \Gamma = -100\,000 \text{ N.m (direction hands clock)}$$

$$S = \frac{2}{E(a+h)^2 I_1} \frac{-\Gamma}{6} [h^3 - ab(3h+a)]$$

11 571 281.93 N.m⁴

From where:

- moment in C :
 $X = 4916.7243 \text{ N.m}$

- reaction in A :

$$H_A = \frac{-\Gamma}{2(a+h)} - \frac{X}{a+h} = \frac{-\Gamma}{2} - X, \quad H_A = 4576.394 \text{ N}$$

$$V_A = \frac{\Gamma}{l}, \quad V_A = 5000.0 \text{ N}$$

5.8.5 Summary

CA	Moment in C ($N.m$)	Reactions in A (N)	
		H_A	V_A
p on C_1C	18672.994	5175.37	24233.240
F_1 in C	41422.161	4881.487	10000.000
F_2 in C_1	8284.432	5976.297	4000.000
Γ in C_1	4916.724	4576.394	5000.000
TOTAL	73296.311	22033.31	43233.24

Notice

Recall: in the post AC_1 : normal effort = $-V_A$, shearing action = H_A .

5.9 Calculation of displacement in C

One considers also only the elastic energy of inflection (slim beams). By applying the Principle of virtual Work to the structure subjected to the fictitious forces of the paragraph [§ 6], working in sought displacements, the numbers are calculated w and d depending linearly on f and g :

$$f u_c + g v_c = \int_{pot} \frac{m(M_{iso} + XM_1)}{EI_1} + \int_{charp} \frac{m(M_{iso} + XM_1)}{EI_2} = w + Xd, \quad \forall (f, g)$$

5.9.1 Distributed load p on C_1C

$$\int_{pot} \frac{m M_{iso}}{EI_1} = \frac{2h}{3EI_1} \frac{ghl}{4(a+h)} - \frac{pbhl}{8(a+h)} = \frac{2}{E(a+h)^2 I_1} - \frac{gpbh^3 l^2}{96}$$

$$\int_{C_1C} \frac{m M_{iso}}{EI_2} = \frac{2}{E(a+h)^2 I_2} - \frac{plhb^2}{384} (2f(a+h) + gl)(h-a)$$

$$\int_{CC_2} \frac{M_{iso}}{EI_2} = \frac{b}{3EI_2} \frac{pbhl}{8(a+h)} \left(\frac{fh}{2} - \frac{glh}{4(a+h)} \right) = \frac{2}{E(a+h)^2 I_2} - \frac{pb^2 lh^2 (gl - 2f(a+h))}{192}$$

From where:

$$w = \frac{2}{E(a+h)^2} - \frac{pbhl}{384} \left(\frac{4glh^2}{I_1} + \frac{glb(3h+a) - efb(a+h)^2}{I_2} \right)$$

Digital application:

$$I_1 = 2I_2 \quad ; \quad h = 2a \quad ; \quad p = 3000 \text{ N.m}^{-1} \quad (\text{downwards})$$

$$w = \frac{2}{E(a+h)^2 I_1} \left(gl \left(2h + \frac{5}{2}b \right) - \frac{9}{2}fbh \right) - \frac{pbh^2 l}{192}$$

-215 406.5922 N.m³

5.9.2 Concentrated loading F_1 in C

$$\int_{pot} \frac{m M_{iso}}{EI_1} = \frac{2h}{3EI_1} \frac{ghl}{4(a+h)} \frac{-F_1 hl}{4(a+h)} = \frac{2}{E(a+h)^2 I_1} \frac{-F_1 gh^3 l^2}{48}$$

$$\int_{charp} \frac{m M_{iso}}{EI_2} = \frac{2b}{3EI_2} \frac{ghl}{4(a+h)} \frac{-F_1 hl}{4(a+h)} = \frac{2}{E(a+h)^2 I_2} \frac{-F_1 gbh^2 l^2}{48}$$

From where (it is noted that w does not depend on f for this loading):

$$w = \frac{2}{E(a+h)^2} \frac{-F_1 gh^2 l^2}{48} \left(\frac{h}{I_1} + \frac{b}{I_2} \right)$$

Digital application:

$$I_1 = 2I_2, \quad h = 2a, \quad F_1 = 20\,000 \text{ N (downwards)}$$

$$w = \frac{2g}{E(a+h)^2 I_1} \frac{-F_1 h^2 l^2}{48} (h+2b)$$

-3155100365.0 N.m²

5.9.3 Concentrated loading F_2 in C_1

$$\int_{pot} \frac{m M_{iso}}{EI_1} = \frac{h}{3EI_1} \frac{F_2 h}{2(a+h)} \left[-(2a+h) \left(\frac{fh}{2} + \frac{ghl}{4(a+h)} \right) + h \left(\frac{-fh}{2} + \frac{ghl}{4(a+h)} \right) \right]$$

$$= \frac{2}{E(a+h)^2 I_1} \frac{-F_2 h^3}{24} (agl + 2f(a+h)^2)$$

$$\int_{charp} \frac{m M_{iso}}{EI_2} = \frac{2}{E(a+h)^2 I_2} \frac{-F_2 bh^2}{24} (agl + 2f(a+h)^2)$$

From where:

$$w = \frac{2}{E(a+h)^2} \frac{-F_2 h^2}{24} (agl + 2f(a+h)^2) \left(\frac{h}{I_1} + \frac{b}{I_2} \right)$$

Digital application:

$$I_1 = 2I_2, \quad h = 2a, \quad F_2 = 10\,000 \text{ N (towards the left)}$$

$$w = \frac{2}{E(a+h)^2 I_1} (gl + 9gh) \frac{-F_2 h^3 (h+2b)}{48}$$

-3151003.65 N.m⁴

5.9.4 Specific couple Γ in C_1

$$\int_{pot} \frac{m M_{iso}}{EI_1} = \frac{h}{3EI_1} \frac{\Gamma h}{2(a+h)} \left[\left(\frac{fh}{2} + \frac{glh}{4(a+h)} \right) + \left(\frac{-fh}{2} + \frac{glh}{4(a+h)} \right) \right]$$

$$= \frac{2}{E(a+h)^2 I_1} \frac{\Gamma h^3 lg}{24}$$

$$\int_{\text{charp}} \frac{m M_{iso}}{EI_2} = \frac{b}{3EI_2} \frac{\Gamma h}{2(a+h)} \left[-(2a+h) \left(\frac{fh}{2} + \frac{glh}{4(a+h)} \right) + h \left(\frac{-fh}{2} + \frac{glh}{4(a+h)} \right) \right]$$

$$= \frac{-2}{E(a+h)^2 I_2} \frac{\Gamma bh}{24} (agl + 2f(a+h)^2)$$

Digital application:

$$I_1 = 2I_2 \quad ; \quad h = 2a \quad ; \quad \Gamma = -100\,000 \text{ N.m}$$

$$w = \frac{2}{E(a+h)^2 I_1} \frac{\Gamma h^2}{24} (gl(h-b) - 9fhb)$$

-3151003.65 N.m⁴

5.9.5 Calculation of $d = \int \frac{m \cdot M_1}{EI}$

$$\int_{\text{pot}} \frac{m M_1}{EI_1} = \frac{2h}{3EI_1} \frac{glh}{4(a+h)} \frac{h}{a+h} = \frac{2}{E(a+h)^2 I_1} \frac{glh^3}{12}$$

$$\int_{\text{charp}} \frac{m M_1}{EI_2} = \frac{2b}{EI_2} \left[\frac{1}{2} \left(\frac{h}{a+h} \right) + \frac{1}{6} \left(\frac{a}{a+h} \right) \right] \frac{glh}{4(a+h)} = \frac{2}{E(a+h)^2 I_2} \frac{glbh(3h+a)}{24}$$

From where (it is noted that d does not depend on f):

$$d = \frac{2}{E(a+h)^2} \frac{glh}{24} \left(\frac{2h^2}{I_1} + \frac{b(3h+a)}{I_2} \right)$$

Digital application:

$$I_1 = 2I_2, \quad h = 2a$$

$$d = \frac{2}{E(a+h)^2 I_1} g \frac{lh^2}{24} (2h+7b)$$

-4874.2564 N.m⁴

5.9.6 Summary of displacements u_c and v_c

$$I_1 = 5.0 E - 4 m^4$$

$$E = 210\,000 \text{ MPA}$$

CA	X	$X \bar{d}$	w_v
pressure on $C_1 C$	18672.994	91016960.3	- 184930109.4
F_1 in C	41422.161	201902233.4	- 315100365.0
F_2 in C_1	8284.432	40380445.6	- 63020073.0
Γ in C_1	4916.724	23965373.4	14775091.25

CA	w_h	$u_c(m)$	$v_c(m)$
pressure on $C_1 C$	83519999.94	0.0110476	- 0.012422374
F_1 in C	0.00	0.00	- 0.01497330
F_2 in C_1	- 226872262.8	- 0.03000956	- 0.00299466
Γ in C_1	206790328.5	0.0273532	- 0.001215646

Note:

$$d = \frac{2}{E(a+h)^2 I_1} g \bar{d}, \text{ with: } \bar{d} = 4874.2564 m^4$$

$$w = \frac{2}{E(a+h)^2 I_1} (g w_v + f w_h) \text{ to see higher}$$

$$u_c = \frac{2}{E(a+h)^2 I_1} w_H \quad ; \quad v_c = \frac{2}{E(a+h)^2 I_1} (w_v + X \bar{d})$$

$$\frac{2}{E(a+h)^2 I_1} = 1.32275132 E-10 N^{-1} m^{-4}$$

Comparison Aster - analytical reference (R.)

CA		Moment in $C(N.m)$	Reaction $H_A(N)$	Reaction $V_A(N)$	Displacement $u_c(m)$	Displacement $v_c(m)$
P on $C_1 C$	R:	18672.994	5175.37	24233.24	0.0110476	-0.012422374
	Aster:	18673.20	5175.36	24233.2	0.0110472	-0.0124233
F_1 in C	R:	41422.161	4881.487	10000.00	0.00000	-0.01497330
	Aster:	41422.40	4881.47	10000.0	0.0000	-0.0
F_2 in C_1	R:	8284.432	5976.297	4000.00	-0.03000956	-0.00299466
	Aster:	8284.34	5976.31	4000.0	-0.0300098	-0.00299450
Γ in C_1	R:	4916.724	4576.394	5000.00	0.0273532	-0.001215646
	Aster:	4916.62	4576.38	5000.0	0.0273536	-0.00121583

Foot-note:

Calculation Aster was realized by taking very slim elements, so that: $S l^2 \ll I$. Thus, the energy of inflection is prevalent. Values of calculation Aster are resulting from the CAS-test VPCS called SSSL14, with the following data:

$$I_1 = 5.0 E - 4 m^4 \quad ; \quad I_2 = 2.5 E - 4 m^4 \quad ; \quad E = 210\,000 MPa$$

$$h = 2a = 8 m \quad ; \quad l = 20 m \quad ; \quad b = \frac{l}{2} \sqrt{1.16}$$

$p = 3000 N.m$ (downwards),

$F_1 = 20\,000 N$ (downwards),

$F_2 = 10\,000 N$ (towards the left),

$\Gamma = -100\,000 Nm$ (direction switches of watch).